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Commutators of parameter Marcinkiwicz integral with functions in Campanato spaces on Orlicz-Morrey spaces

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Abstract. In this paper, we consider the commutator $\mathcal{M}^{\rho}_{\Omega,b}$ generated by the locally integrable function b in generalized Campanato spaces and the parameter Marcinkiwicz integral with the rough kernel $\mathcal{M}^{\rho}_{\Omega}$, where the rough kernel Ω satisfies certain log-type regularity. Meanwhile, a necessary and sufficient condition for the boundedness of the commutator on Orlicz-Morrey spaces is established.

1. Introduction

Let $\mathbb{R}^n (n \geq 2)$ be the n-dimensional Euclidean space and $\mathbb{S}^{n-1}$ be the unit sphere in $\mathbb{R}^n$ equipped with the normalized Lebesgue measure $d\sigma = d\sigma(\cdot)$. Let Ω be a homogeneous function of degree zero and have mean value zero, namely,

$$\Omega(\lambda x') = \Omega(x') \text{ for any } \lambda \in (0,\infty) \text{ and } x' \in \mathbb{S}^{n-1}, \tag{1}$$

and

$$\int_{\mathbb{S}^{n-1}} \Omega\left(x'\right) d\sigma\left(x'\right) = 0. \tag{2}$$

In 1938, Marcinkiewicz[23] considered the operator $\mathcal{M}$ given by

$$\mathcal{M}(f)(x) = \left(\int_0^{2\pi} \frac{|F(x+t)+F(x-t)-2F(x)|^2}{t^3} dt\right)^{\frac{1}{2}}, \quad x \in [0,2\pi],$$

where $F(x) = \int_0^x f(t)\mathrm{d}t$. For convenience, the operator $\mathcal{M}$ is called the Marcinkiewicz integral. In 1958, Stein [38] generalized the preceding Marcinkiewicz integral $\mathcal{M}$ to the higher-dimensional case, which is defined as follows:

$$\mathcal{M}_{\Omega}(f)(x) = \left(\int_0^{\infty} \left|\int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy\right|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}}, \quad x \in \mathbb{R}^n. \tag{3}$$

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And Stein showed that if $\Omega \in \mathrm{Lip}_\alpha(\mathbb{S}^{n-1})$ $(0 < \alpha \le 1)$, then $\mathcal{M}_\Omega$ is bounded on the Lebesgue space $L^p(\mathbb{R}^n)$ for $1 < p \le 2$. Later, some mapping properties of the Marcinkiewicz integral $\mathcal{M}_\Omega$ are attentioned by many authors. For example, Benedeck, Calderón and Panzone [4] obtained that if $\Omega \in C^1(\mathbb{S}^{n-1})$, then $\mathcal{M}_\Omega$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. In 2002, Al-Salman, Al-Qassem, Cheng and Pan [2] proved that if $\Omega \in L(\log L)^{\frac{1}{2}}(\mathbb{S}^{n-1})$, $\mathcal{M}_\Omega$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. Furthermore, many prominent results about the Marcinkiewicz integral $\mathcal{M}_\Omega$ are also extensively investigated, we refer the readers to see [11, 35, 40, 42, 43] and therein references.

In 1965, Calderón [5] first introduced the definition of the commutator

$$[b, \mathcal{T}]f(x) = b\mathcal{T}f(x) - \mathcal{T}(bf)(x),$$

and proved that the commutator $[b, H]$ generated by the Hilbert transform H and the function $b \in BMO(\mathbb{R}^n)$ is bounded on the Lebesgue space $L^2(\mathbb{R}^n)$. In 1976, Coifman, Rochberg and Weiss[10] stated that the function $b \in BMO(\mathbb{R}^n)$ if and only if the commutator $[b, T]$ generated by the classical singular integral operator T and the locally integrable function b is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. After that, the properties of many commutators have received many attentions (see, for example,[6, 15, 31, 32]).

The parameter Marcinkiewicz integral was introduced by Hörmander [17]. For any $x \in \mathbb{R}^n$ and $\rho \in (0, \infty)$, the parameter Marcinkiewicz integral $\mathcal{M}^\rho_\Omega$ is defined by

$$\mathcal{M}^\rho_\Omega(f)(x) = \left(\int_0^\infty \left|F^\rho_{\Omega,t}(f)(x)\right|^2 \frac{dt}{t}\right)^{\frac{1}{2}}, \tag{4}$$

where

$$F^\rho_{\Omega,t}(f)(x) = \frac{1}{t^\rho}\int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}} f(y)dy.$$

The commutator $\mathcal{M}^\rho_{\Omega,b}$ generated by the parameter Marcinkiewicz integral $\mathcal{M}^\rho_\Omega$ and the function b is defined by

$$\mathcal{M}^\rho_{\Omega,b}(f)(x) = \left(\int_0^\infty \left|[b, F^\rho_{\Omega,t}](f)(x)\right|^2 \frac{dt}{t}\right)^{\frac{1}{2}},$$

where

$$[b, F^\rho_{\Omega,t}](f)(x) = \frac{1}{t^\rho}\int_{|x-y|\le t} [b(x) - b(y)]\frac{\Omega(x-y)}{|x-y|^{n-\rho}} f(y)dy.$$

If we take $\rho = 1$ as in (4), then the parameter Marcinkiewicz integral reduces to the classical Marcinkiewicz integral as in (3). We denote $\mathcal{M}^1_\Omega = \mathcal{M}_\Omega$, $\mathcal{M}^1_{\Omega,b} = \mathcal{M}_{\Omega,b}$.

In 1990, Torchinsky and Wang[39] proved that the boundedness of the commutator $\mathcal{M}_{\Omega,b}$ on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, if $\Omega \in Lip_\alpha(\mathbb{S}^{n-1})$ $(0 < \alpha < 1)$. In 2015, Chen and Ding [8] also showed that $b \in BMO(\mathbb{R}^n)$ is necessary for the boundedness of the commutator $\mathcal{M}_{\Omega,b}$ on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, if Ω satisfies the following logarithm type regularity:

$$\left|\Omega(x') - \Omega(y')\right| \lesssim \left(\log \frac{2}{\left|x' - y'\right|}\right)^{-\gamma} \text{ for any } x', y' \in S^{n-1}, \text{and some } \gamma > 1. \tag{5}$$

Recently, the parameter Marcinkiewicz integral and its commutator are still widely studied in various function spaces, we refer the readers to see [9, 12, 18, 21, 22, 33, 41] and therein references.

In 2012, Aliev and Guliyev [1] obtained that if $b \in BMO(\mathbb{R}^n)$ and $\Omega \in Lip_\alpha(\mathbb{S}^{n-1})$, then $\mathcal{M}_{\Omega,b}$ is bounded on generalized Morrey spaces. In 2021, Shi, Arai and Nakai [37] studied the commutator $[b, T]$ generated by the Calderón-Zygmund singular integral operator T and the function b on the Orlicz-Morrey spaces and showed that b is a function in generalized Campanato spaces (see Definition 1.7), which contain $BMO(\mathbb{R}^n)$ spaces and Lipschitz spaces as special examples, if and only if $[b, T]$ is bounded on Orlicz-Morrey spaces (see Definition 1.6). The corresponding result for the commutator of generalized fractional integrals was also obtained.

In this paper, our main purpose is to address the mapping properties of the commutator $\mathcal{M}^{\rho}_{\Omega,b}$ on Orlicz-Morrey spaces when b is a function in the generalized Campanato space. To state our main results, we first recall generalized Young functions. Later, we recall the definitions of Orlicz and Orlicz-Morrey spaces with generalized Young functions.

For an increasing (i.e. nondecreasing) function $\Phi : [0, \infty] \to [0, \infty]$, let

$$a(\Phi) = \sup\{t \geq 0 : \Phi(t) = 0\}, \quad b(\Phi) = \inf\{t \geq 0 : \Phi(t) = \infty\}, \tag{6}$$

with convention $\sup \emptyset = 0$ and $\inf \emptyset = \infty$. Then $0 \leq a(\Phi) \leq b(\Phi) \leq \infty$.

Let $\overline{\Phi}$ be the set of all increasing functions $\Phi : [0, \infty] \to [0, \infty]$ such that

$$0 \leq a(\Phi) < \infty, \quad 0 < b(\Phi) \leq \infty, \tag{7}$$

$$\lim_{t \to +0} \Phi(t) = \Phi(0) = 0, \tag{8}$$

$$\Phi \text{ is left continuous on } [0, b(\Phi)), \tag{9}$$

$$\text{if } b(\Phi) = \infty, \text{ then } \lim_{t \to \infty} \Phi(t) = \Phi(\infty) = \infty, \tag{10}$$

$$\text{if } b(\Phi) < \infty, \text{ then } \lim_{t \to b(\Phi)-0} \Phi(t) = \Phi(b(\Phi)) \ (\leq \infty). \tag{11}$$

In what follows, if an increasing and left continuous function $\Phi : [0, \infty) \to [0, \infty)$ satisfies (8) and $\lim_{t \to \infty} \Phi(t) = \infty$, then we always regard as $\Phi(\infty) = \infty$ and that $\Phi \in \overline{\Phi}$.

For $\Phi \in \overline{\Phi}$, we recall the generalized inverse of Φ in the sense of O'Neil [28, Definition 1.2].

Definition 1.1. *For $\Phi \in \overline{\Phi}$ and $u \in [0, \infty]$, let*

$$\Phi^{-1}(u) = \begin{cases} \inf\{t \geq 0 : \Phi(t) > u\}, & u \in [0, \infty), \\ \infty, & u = \infty. \end{cases}$$

Let $\Phi \in \overline{\Phi}$. Then Φ^{-1} is finite, increasing and right continuous on $[0, \infty)$ and positive on $(0, \infty)$. If Φ is bijective from $[0, \infty]$ to itself, then Φ^{-1} is the usual inverse function of Φ. Moreover, if $\Phi \in \overline{\Phi}$, then

$$\Phi(\Phi^{-1}(u)) \leq u \leq \Phi^{-1}(\Phi(u)) \quad \text{for all } u \in [0, \infty], \tag{12}$$

which is a generalization in [28, Property 1.3], see also [36, Proposition 2.2].

For $\Phi, \Psi \in \overline{\Phi}$, we write $\Phi \approx \Psi$ if there exists a positive constant C such that

$$\Phi(C^{-1}t) \leq \Psi(t) \leq \Phi(Ct) \quad \text{for all } t \in [0, \infty].$$

For functions $P, Q : [0, \infty] \to [0, \infty]$, we write $P \sim Q$ if there exists a positive constant C such that

$$C^{-1}P(t) \leq Q(t) \leq CP(t) \quad \text{for all } t \in [0, \infty].$$

Then, for $\Phi, \Psi \in \overline{\Phi}$,

$$\Phi \approx \Psi \quad \Leftrightarrow \quad \Phi^{-1} \sim \Psi^{-1}, \tag{13}$$

see [36, Lemma 2.3].

Furthermore, we recall the definition of the Young function and its generalization in [24].

Definition 1.2. *A function $\Phi \in \overline{\Phi}$ is called the Young function (or sometimes also called Orlicz function) if Φ is convex on $[0, b(\Phi))$. Let Φ_Y be the set of all Young functions. Let $\overline{\Phi}_Y$ be the set of all $\Phi \in \overline{\Phi}$ such that $\Phi \approx \Psi$ for some $\Psi \in \Phi_Y$.*

Definition 1.3. *1. A function $\Phi \in \overline{\Phi}$ is said to satisfy the Δ_2-condition, denoted by $\Phi \in \overline{\Delta}_2$, if there exists a constant $C > 0$ such that*

$$\Phi(2t) \le C\Phi(t) \quad \text{for all } t > 0.$$

2. A function $\Phi \in \overline{\Phi}$ is said to satisfy the ∇_2-condition, denoted by $\Phi \in \overline{\nabla}_2$, if there exists a constant $k > 1$ such that

$$\Phi(t) \le \frac{1}{2k}\Phi(kt) \quad \text{for all } t > 0.$$

3. Let $\Delta_2 = \Phi_Y \cap \overline{\Delta}_2$ and $\nabla_2 = \Phi_Y \cap \overline{\nabla}_2$.

Remark 1.4. *1. Let $\Phi \in \overline{\Phi}_Y$. Then $\Phi \in \overline{\Delta}_2$ if and only if $\Phi \approx \Psi$ for some $\Psi \in \Delta_2$, and, $\Phi \in \overline{\nabla}_2$ if and only if $\Phi \approx \Psi$ for some $\Psi \in \nabla_2$.*

2. Let $\Phi \in \Phi_Y$. Then Φ^{-1} satisfies the doubling condition by its concavity, that is,

$$\Phi^{-1}(u) \le \Phi^{-1}(2u) \le 2\Phi^{-1}(u) \quad \text{for all } u \in [0, \infty].$$

3. Let $\Phi \in \Phi_Y$. Then $\Phi \in \Delta_2$ if and only if $t \mapsto \frac{\Phi(t)}{t^p}$ is almost decreasing for some $p \in [1, \infty)$.

Let (X, μ) be a measure space, and let $L^0(X)$ be the set of all measurable functions on X. The Orlicz space is first introduced by [29, 30], which is defined by the following.

Definition 1.5. *(Orlicz spaces) For $\Phi \in \overline{\Phi}_Y$, let*

$$L^{\Phi}(X) = \left\{ f \in L^0(X) : \int_X \Phi(\epsilon|f(x)|)d\mu(x) < \infty \text{ for some } \epsilon > 0 \right\},$$
$$\|f\|_{L^{\Phi}(X)} = \inf\left\{ \lambda > 0 : \int_X \Phi\left(\frac{|f(x)|}{\lambda}\right) d\mu(x) \le 1 \right\}.$$

Then $\|\cdot\|_{L^{\Phi}(X)}$ is a quasi-norm, thereby $L^{\Phi}(X)$ is a quasi-Banach space. If $\Phi \in \Phi_Y$, then $\|\cdot\|_{L^{\Phi}(X)}$ is a norm and thereby $L^{\Phi}(X)$ is a Banach space. For $\Phi, \Psi \in \overline{\Phi}_Y$, if $\Phi \approx \Psi$, then $L^{\Phi}(X) = L^{\Psi}(X)$ with equivalent quasi-norms. In the case $X = \mathbb{R}^n$, we always write $\|\cdot\|_{L^{\Phi}}$ instead of $\|\cdot\|_{L^{\Phi}(\mathbb{R}^n)}$.

We denote by $B(x, r)$ the open ball centered at $x \in \mathbb{R}^n$ and of radius r. For the locally integrable function f and a ball B, let

$$f_B = \fint_B f(y)dy = \frac{1}{|B|}\int_B f(y)dy,$$

where $|B|$ is the Lebesgue measure of the ball B. Then, Orlicz-Morrey spaces are defined as follows.

Definition 1.6. *[25, Definition 3.1] (Orlicz-Morrey spaces) For a Young function $\Phi : [0, \infty] \to [0, \infty]$, a function $\varphi : (0, \infty) \to (0, \infty)$ and a ball $B = B(x, r)$, let*

$$\|f\|_{\Phi,\varphi,B} = \inf\left\{ \lambda > 0 : \frac{1}{\varphi(r)} \fint_B \Phi\left(\frac{|f(x)|}{\lambda}\right) \mathrm{d}x \le 1 \right\}.$$

Let $L^{(\Phi,\varphi)}(\mathbb{R}^n)$ be the set of all functions f such that the following norm is finite:

$$\|f\|_{L^{(\Phi,\varphi)}} = \sup_B \|f\|_{\Phi,\varphi,B}, \tag{14}$$

where the supremum is taken over all balls B in $\mathbb{R}^n$.

Then $\|f\|_{L^{(\Phi,\varphi)}}$ is a norm and thereby $L^{(\Phi,\varphi)}(\mathbb{R}^n)$ is a Banach space. If $\varphi(r)=\frac{1}{r^n}$, then $L^{(\Phi,\varphi)}(\mathbb{R}^n)$ coincides with the Orlicz space $L^{\Phi}(\mathbb{R}^n)$ equipped with the norm

$$\|f\|_{L^{\Phi}}=\inf\left\{\lambda>0:\int_{\mathbb{R}^n}\Phi\left(\frac{|f(x)|}{\lambda}\right)\mathrm{d}x\leq 1\right\}.$$

If $\Phi(t)=t^p, 1\leq p<\infty$, then $L^{(\Phi,\varphi)}(\mathbb{R}^n)$ coincides with the generalized Morrey space $L^{(p,\varphi)}(\mathbb{R}^n)$ equipped with the norm

$$\|f\|_{L^{(p,\varphi)}}=\sup_{B=B(x,r)}\left(\frac{1}{\varphi(r)}⨍_B|f(x)|^p\,\mathrm{d}x\right)^{\frac{1}{p}}.$$

The Orlicz-Morrey space $L^{(\Phi,\varphi)}(\mathbb{R}^n)$ were investigated in [24–26]. For other kinds of Orlicz-Morrey spaces(see, for example, [13, 14, 16, 34, 44]).

For $\Phi\in\overline{\Phi}_Y$ and $\varphi:(0,\infty)\to(0,\infty)$, we define the Orlicz-Morrey space $L^{(\Phi,\varphi)}(\mathbb{R}^n)$ together with $\|\cdot\|_{L^{(\Phi,\varphi)}}$ by (14). Let $\mu_B=\frac{\mathrm{d}x}{|B|\varphi(r)}$. Then we have the following relation:

$$\|f\|_{\Phi,\varphi,B}=\|f\|_{L^{\Phi}(B,\mu_B)}. \tag{15}$$

Since the relation (15), $\|\cdot\|_{L^{(\Phi,\varphi)}}$ is a quasi-norm, thereby $L^{(\Phi,\varphi)}(\mathbb{R}^n)$ is a quasi-Banach space. If $\Phi\in\Phi_Y$, then $\|\cdot\|_{L^{(\Phi,\varphi)}}$ is a norm and thereby $L^{(\Phi,\varphi)}(\mathbb{R}^n)$ is a Banach space. If $\Phi\approx\Psi$ and $\varphi\sim\psi$, then $L^{(\Phi,\varphi)}(\mathbb{R}^n)=L^{(\Psi,\psi)}(\mathbb{R}^n)$ with equivalent quasi-norms.

Next, we recall the definition of generalized Campanato spaces (see, for example, [3, 37]).

Definition 1.7. *For $p\in[1,\infty)$ and a function $\psi:(0,\infty)\to(0,\infty)$, let $\mathcal{L}_{p,\psi}(\mathbb{R}^n)$ be the set of all functions f such that the following functional is finite:*

$$\|f\|_{\mathcal{L}_{p,\psi}}=\sup_{B=B(x,r)}\frac{1}{\psi(r)}\left(⨍_B\left|f(y)-f_B\right|^p dy\right)^{\frac{1}{p}},$$

where the supremum is taken over all balls $B(x,r)$ in $\mathbb{R}^n$.

Then $\|f\|_{\mathcal{L}_{p,\psi}(\mathbb{R}^n)}$ is a norm modulo constant function and thereby $\mathcal{L}_{p,\psi}(\mathbb{R}^n)$ is a Banach space. If $p=1$ and $\psi\equiv 1$, then $\mathcal{L}_{p,\psi}(\mathbb{R}^n)=\mathrm{BMO}(\mathbb{R}^n)$. If $p=1$ and $\psi(r)=r^{\alpha}(0<\alpha\leq 1)$, then $\mathcal{L}_{p,\psi}(\mathbb{R}^n)$ coincides with $\mathrm{Lip}_{\alpha}(\mathbb{R}^n)$. If ψ is almost increasing, then $\mathcal{L}_{p,\psi}(\mathbb{R}^n)=\mathcal{L}_{1,\psi}(\mathbb{R}^n)$.

We say that a function $\theta:(0,\infty)\to(0,\infty)$ satisfies the doubling condition if there exists a positive constant C such that, for all $r,s\in(0,\infty)$,

$$\frac{1}{C}\leq\frac{\theta(r)}{\theta(s)}\leq C,\quad \text{if } \frac{1}{2}\leq\frac{r}{s}\leq 2. \tag{16}$$

We say that θ is almost increasing (resp. almost decreasing) if there exists a positive constant C such that, for all $r,s\in(0,\infty)$,

$$\theta(r)\leq C\theta(s)\quad (\text{resp. } \theta(s)\leq C\theta(r)),\quad \text{if } r<s. \tag{17}$$

In this paper, we also need the following condition (see [24, Definition 2.5]).

Definition 1.8. *(1) Let $\mathcal{G}^{\mathrm{dec}}$ be the set of all functions $\varphi:(0,\infty)\to(0,\infty)$ such that φ is almost decreasing and that $r\mapsto\varphi(r)r^n$ is almost increasing. That is, there exists a positive constant C such that, for all $r,s\in(0,\infty)$,*

$$C\varphi(r)\geq\varphi(s),\quad \varphi(r)r^n\leq C\varphi(s)s^n,\quad \text{if } r<s.$$

(2) Let $\mathcal{G}^{inc}$ be the set of all functions $\varphi:(0,\infty)\to(0,\infty)$ such that φ is almost increasing and that $r\mapsto\varphi(r)/r$ is almost decreasing. That is, there exists a positive constant C such that, for all $r,s\in(0,\infty)$,

$$\varphi(r)\leq C\varphi(s),\quad C\varphi(r)/r\geq\varphi(s)/s,\quad \text{if } r<s.$$

If $\varphi \in \mathcal{G}^{\text{dec}}$ or $\varphi \in \mathcal{G}^{\text{inc}}$, then φ satisfies the doubling condition (16). Let $\psi : (0,\infty) \to (0,\infty)$. If $\psi \sim \varphi$ for some $\varphi \in \mathcal{G}^{\text{dec}}$ (*resp.*$\mathcal{G}^{\text{inc}}$), then $\psi \in \mathcal{G}^{\text{dec}}$ (*resp.*$\mathcal{G}^{\text{inc}}$)

Remark 1.9. *Let $\varphi \in \mathcal{G}^{dec}$. Then there exists $\tilde{\varphi} \in \mathcal{G}^{dec}$ such that $\varphi \sim \tilde{\varphi}$ and that $\tilde{\varphi}$ is continuous and strictly decreasing, see [25, Proposition 3.4]. Moreover, if*

$$\lim_{r\to 0}\varphi(r) = \infty, \quad \lim_{r\to\infty}\varphi(r) = 0, \tag{18}$$

then $\tilde{\varphi}$ is bijective from $(0,\infty)$ to itself.

Let $\Phi \in \Delta_2 \cap \nabla_2$. For $f \in L^{(\Phi,\varphi)}(\mathbb{R}^n)$, we define $\mathcal{M}^{\rho}_{\Omega}(f)$ on each ball B by

$$\mathcal{M}^{\rho}_{\Omega}(f)(x) = \left(\int_0^{\infty}\left|F^{\rho}_{\Omega,t}(f\chi_{2B})(x) + F^{\rho}_{\Omega,t}(f\chi_{(2B)^{\complement}})(x)\right|^2 dt\right)^{\frac{1}{2}}, \quad x \in B. \tag{19}$$

Here, and in what follows, $E^{\complement} = \mathbb{R}^n\backslash E$ denotes the complementary set of any measurable subset E of $\mathbb{R}^n$. Then,

$$\mathcal{M}^{\rho}_{\Omega}(f)(x) \le \mathcal{M}^{\rho}_{\Omega}(f\chi_{2B})(x) + \mathcal{M}^{\rho}_{\Omega}(f\chi_{(2B)^{\complement}})(x).$$

Note that $\mathcal{M}^{\rho}_{\Omega}(f\chi_{2B})$ is well defined since $f\chi_{2B} \in L^{\Phi}(\mathbb{R}^n)$, and it easy to check that

$$\mathcal{M}^{\rho}_{\Omega}(f\chi_{(2B)^{\complement}})(x) \le \int_{(2B)^{\complement}}\frac{\Omega(x-y)}{|x-y|^n}|f(y)|dy,$$

which converges absolutely. Moreover, $\mathcal{M}^{\rho}_{\Omega}(f)(x)$ defined in (19) is independent of the choice of the ball containing x. Furthermore, we can show that $\mathcal{M}^{\rho}_{\Omega}$ is bounded on $L^{(\Phi,\varphi)}(\mathbb{R}^n)$. See Lemma 2.7 for the details. For $f \in L^{(\Phi,\varphi)}(\mathbb{R}^n)$, $1 < p < \infty$, we define $\mathcal{M}^{\rho}_{\Omega,b}(f)$ on each ball B by

$$\mathcal{M}^{\rho}_{\Omega,b}(f)(x) = \left(\int_0^{\infty}\left|[b,F^{\rho}_{\Omega,t}](f\chi_{2B})(x) + [b,F^{\rho}_{\Omega,t}](f\chi_{(2B)^{\complement}})(x)\right|^2 dt\right)^{\frac{1}{2}}, \quad x \in B. \tag{20}$$

See Remark 2.11 for its well definedness. Then we have the following theorem.

Theorem 1.10. *Let $\Phi, \Psi \in \overline{\Phi}_Y, \varphi \in \mathcal{G}^{\text{dec}}$ and $\psi \in \mathcal{G}^{inc}$. Let $\mathcal{M}^{\rho}_{\Omega}$ be a parameter Marcinkiwicz integral with the rough kernel satisfying (1), (2) and (5).*

1. *Let $\Phi, \Psi \in \overline{\Delta}_2 \cap \overline{\nabla}_2$. Assume that φ satisfies*

$$\int_r^{\infty}\frac{\varphi(t)}{t}dt \le C\varphi(r) \tag{21}$$

and there exists a positive constant C_0 such that, for all $r \in (0,\infty)$,

$$\psi(r)\Phi^{-1}(\varphi(r)) \le C_0\Psi^{-1}(\varphi(r)). \tag{22}$$

If $b \in \mathcal{L}_{1,\psi}(\mathbb{R}^n)$, then $\mathcal{M}^{\rho}_{\Omega,b}(f)$ in (20) is well defined for all $f \in L^{(\Phi,\varphi)}(\mathbb{R}^n)$ and there exists a positive constant C, independent of b and f, such that

$$\|\mathcal{M}^{\rho}_{\Omega,b}(f)\|_{L^{(\Psi,\varphi)}} \le C\|b\|_{\mathcal{L}_{1,\psi}}\|f\|_{L^{(\Phi,\varphi)}}.$$

2. *Conversely, assume that there exists a positive constant C_0 such that, for all $r \in (0,\infty)$*

$$C_0\psi(r)\Phi^{-1}(\varphi(r)) \ge \Psi^{-1}(\varphi(r)). \tag{23}$$

If $\mathcal{M}^{\rho}_{\Omega,b}(f)$ is bounded from $L^{(\Phi,\varphi)}(\mathbb{R}^n)$ to $L^{(\Psi,\varphi)}(\mathbb{R}^n)$, then b is in $\mathcal{L}_{1,\psi}(\mathbb{R}^n)$ and there exists a positive constant C, independent of b and f, such that

$$\|b\|_{\mathcal{L}_{1,\psi}} \le C\|\mathcal{M}^{\rho}_{\Omega,b}\|_{L^{(\Phi,\varphi)}\to L^{(\Psi,\varphi)}},$$

where $\|\mathcal{M}^{\rho}_{\Omega,b}\|_{L^{(\Phi,\varphi)}\to L^{(\Psi,\varphi)}}$ is the operator norm of $\mathcal{M}^{\rho}_{\Omega,b}$ from $L^{(\Phi,\varphi)}(\mathbb{R}^n)$ to $L^{(\Psi,\varphi)}(\mathbb{R}^n)$.

Remark 1.11. *For $b \in L^1_{loc}(\mathbb{R}^n)$ and $\rho = 1$, Chen and Ding [7] showed that, if $\mathcal{M}^{\rho}_{\Omega,b}$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, then $b \in BMO(\mathbb{R}^n)$, under the assumption of that Ω satisfies the logarithm type regularity condition (5). For $b \in L^1_{loc}(\mathbb{R}^n)$ and $\rho = 1$, Ku and Wu [20] obtained that, if $b \in \mathcal{L}^{(1,\psi)}(\mathbb{R}^n)$, then $\mathcal{M}^{\rho}_{\Omega,b}$ is bounded from $L^{(p,\varphi)}(\mathbb{R}^n)$ to $L^{(q,\varphi)}(\mathbb{R}^n)$, under the same assumptions of Theorem 1.10. It is also interesting whether or not the corresponding conclusions are still true if the regularity of Ω is weakened or removed. In addition, for $b \in \mathcal{L}^{(1,\psi)}(\mathbb{R}^n)$, it is also worth exploring the mapping properties of $\mathcal{M}^{\rho}_{\Omega,b}$ on weighted Orlicz-Morrey spaces, etc.*

Finally, we make some conventions on notation. Throughout this paper, we always use C to denote a positive constant that is independent of the main parameters involved but whose value may differ from line to line. Constants with subscripts, such as C_p, are dependent on the subscripts. We denote by $f \lesssim g$ if $f \le Cg$, and $f \sim g$ if $f \lesssim g \lesssim f$. For $1 \le p \le \infty$, p' is the conjugate index of p, that is $\frac{1}{p} + \frac{1}{p'} = 1$.

2. Preliminaries

To prove our results, some necessary lemmas and defnitions are given in this section.

For a function $\varrho : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$, the generalized Hardy-Littlewood maximal operator is defined by

$$\mathcal{M}_{\varrho}(f)(x) = \sup_{B \ni x} \varrho(B) \fint_B |f(y)|dy,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$.

If $\varrho \equiv 1$, then $\mathcal{M}_{\varrho}(f)(x)$ is the Hardy-Littlewood maximal operator M, and if $\varrho(B) = |B|^{\alpha/n}$, then $\mathcal{M}_{\varrho}(f)$ is the fraction maximal operator M_α defined by

$$M_\alpha(f)(x) = \sup_{B \ni x} |B|^{\alpha/n} \fint_B |f(y)|dy.$$

For the generalized Hardy-Littlewood maximal operator $\mathcal{M}_{\varrho}$, we have the following lemma.

Lemma 2.1. *[37, Theorem 5.1] Let $\Phi, \Psi \in \overline{\Phi}_Y, \varphi \in \mathcal{G}^{dec}$ and $\varrho : (0,\infty) \to (0,\infty)$. Assume that $\lim_{r\to\infty} \varphi(r) = 0$ or $\Psi^{-1}(t)/\Phi^{-1}(t)$ is almost decreasing on $(0,\infty)$. If there exists a positive constant A such that, for all $r \in (0,\infty)$,*

$$\left(\sup_{0<t\le r} \rho(t)\right)\Phi^{-1}(\varphi(r)) \le A\Psi^{-1}(\varphi(r)), \tag{24}$$

then, for any positive constant C_0, there exists a positive constant C_1 such that, for all $f \in L^{(\Phi,\varphi)}(\mathbb{R}^n)$ with $f \not\equiv 0$

$$\Psi\left(\frac{M_{\varrho} f(x)}{C_1\|f\|_{L^{(\Phi,\varphi)}}}\right) \le \Phi\left(\frac{Mf(x)}{C_0\|f\|_{L^{(\Phi,\varphi)}}}\right), \quad x \in \mathbb{R}^n.$$

Consequently, if $\Phi \in \overline{\nabla}_2$, then M_{ϱ} is bounded from $L^{(\Phi,\varphi)}(\mathbb{R}^n)$ to $L^{(\Psi,\varphi)}(\mathbb{R}^n)$.

Remark 2.2. *If ϱ is almost increasing or if $\frac{\Psi^{-1}(t)}{\Phi^{-1}(t)}$ is almost decreasing, then the inequality $\varrho(r)\Phi^{-1}(\varphi(r)) \lesssim \Psi^{-1}(\varphi(r))$ implies (24).*

Lemma 2.3. *[37, Lemma 4.1]Let $\Phi \in \Phi_Y$ and $\varphi \in \mathcal{G}^{\mathrm{dec}}$. Then there exists a constant $C \ge 1$ such that, for any ball $B = B(x,r)$,*

$$\frac{1}{\Phi^{-1}(\varphi(r))} \le \|\chi_B\|_{L^{(\Phi,\varphi)}} \le \frac{C}{\Phi^{-1}(\varphi(r))}.$$

Lemma 2.4. *[36, Lemma 4.4] If $\Phi \in \overline{\nabla}_2$, then, for some $\theta \in (0,1)$, $\Phi((\cdot)^{\theta}) \in \overline{\nabla}_2$.*

Lemma 2.5. *[37, Lemma 4.3] Let $\Phi \in \Phi_Y$, $\varphi : (0, \infty) \to (0, \infty)$ and $B = B(x, r) \subset \mathbb{R}^n$. Then*

$$\fint_B |f(x)|dx \leq 2\Phi^{-1}(\varphi(r))\|f\|_{\Phi,\varphi,B}.$$

Moreover, if $\Phi \in \nabla_2$, then there exists $p \in (1, \infty)$ such that

$$\left(\fint_B |f(y)|^p dy\right)^{\frac{1}{p}} \leq C\Phi^{-1}(\varphi(r))\|f\|_{\Phi,\varphi,B},$$

where the constant C is independent of f and $B = B(x, r)$.

Lemma 2.6. *[37, Lemma 4.4] Let $\Phi \in \Delta_2$ and $\varphi \in \mathcal{G}^{\mathrm{dec}}$. If φ satisfies (21), then there exists a positive constant C such that, for all $r \in (0, \infty)$,*

$$\int_r^\infty \frac{\Phi^{-1}(\varphi(t))}{t}\, \mathrm{d}t \leq C\Phi^{-1}(\varphi(r)). \tag{25}$$

Lemma 2.7. *Let $\Phi \in \Delta_2 \cap \nabla_2$, $\varphi \in \mathcal{G}^{dec}$. There exists a positive constant C such that, for all $r \in (0, \infty)$,satisfy*

$$\int_r^\infty \frac{\varphi(t)}{t} dt \leq C\varphi(r).$$

Suppose that $\Omega \in L^\infty(S^{n-1})$. Then, $\mathcal{M}^\rho_\Omega$ defined in (19) is bounded on $L^{(\Phi,\varphi)}(\mathbb{R}^n)$. That is, there exists a positive constant C such that, for all $f \in L^{(\Phi,\varphi)}(\mathbb{R}^n)$,

$$\|\mathcal{M}^\rho_\Omega(f)\|_{L^{(\Phi,\varphi)}(\mathbb{R}^n)} \leq C\|f\|_{L^{(\Phi,\varphi)}(\mathbb{R}^n)}.$$

Proof. For any $x \in \mathbb{R}^n$, applying (1), (2) and Minkowski's inequality, we obtain that

$$\begin{aligned}
\mathcal{M}^\rho_\Omega(f)(x) &\leq \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\rho}}|f(y)|\left(\int_{|x-y|}^\infty \frac{dt}{t^{1+2\rho}}\right)^{\frac{1}{2}} dy \\
&\leq \int_{\mathbb{R}^n} \frac{\left|\Omega\left((x-y)'\right)\right|}{|x-y|^{n-\rho}}|f(y)|\left(\int_{|x-y|}^\infty \frac{dt}{t^{1+2\rho}}\right)^{\frac{1}{2}} dy \\
&\leq C\int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-\rho}} \frac{1}{|x-y|^\rho} dy \\
&\leq C\int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n} dy \\
&:= CT(f)(x).
\end{aligned}$$

With the boundedness of the operator T on Olicz-Morrey spaces by Nakai [26], we have the desired result. □

Next, we will prove that the commutator $\mathcal{M}^\rho_{\Omega,b}$ is well defined for all $b \in \mathcal{L}_{1,\psi}(\mathbb{R}^n)$ and $f \in L^{(\Phi,\varphi)}(\mathbb{R}^n)$. The following lemma is well known, we can see [3, 27] for example.

Lemma 2.8. *Let $\psi \in \mathcal{G}^{inc}$. Then, for each $p \in (1, \infty)$, $\mathcal{L}_{p,\psi}(\mathbb{R}^n) = \mathcal{L}_{1,\psi}(\mathbb{R}^n)$ with equivalent norms.*

Lemma 2.9. *[3, Lemma 4.7] Let $p \in [1, \infty)$ and $\psi \in \mathcal{G}^{inc}$. Then there exists a positive constant C dependent only on n, p and ψ such that, for all $f \in \mathcal{L}_{1,\psi}(\mathbb{R}^n)$ and for all $x \in \mathbb{R}^n$ and $r, s \in (0, \infty)$,*

$$\left(\fint_{B(x,s)} \left|f(y) - f_{B(x,r)}\right|^p dy\right)^{\frac{1}{p}} \leq C\int_r^s \frac{\psi(t)}{t}\, \mathrm{d}t\|f\|_{\mathcal{L}_{1,\psi}}, \quad \text{if } 2r < s,$$

and

$$\left(\fint_{B(x,s)}\left|f(y)-f_{B(x,r)}\right|^p dy\right)^{\frac{1}{p}} \le C\left(\log_2\frac{s}{r}\right)\psi(s)\|f\|_{\mathcal{L}_{1,\psi}}, \quad \textit{if } 2r<s.$$

Lemma 2.10. *Under the assumption of the part 1 of Theorem 3.1, there exists a positive constant C such that, for all* $b\in\mathcal{L}_{1,\psi}(\mathbb{R}^n)$*, all* $f\in L^{(\Phi,\varphi)}(\mathbb{R}^n)$ *and all balls* $B=B(z,r)$*,*

$$\left|\fint_B\left(\int_0^\infty\left|[b,F^\rho_{\Omega,t}](f\chi_{(2B)^\complement})(x)\right|^2 dt\right)^{\frac12}dx\right|\le C\Psi^{-1}(\varphi(B))\|b\|_{\mathcal{L}_{1,\psi}}\|f\|_{L^{(\Phi,\varphi)}}.$$

Proof. For $x\in B$, we have

$$\begin{aligned}\left(\int_0^\infty\left|[b,F^\rho_{\Omega,t}](f\chi_{(2B)^\complement})(x)\right|^2\frac{dt}{t^3}\right)^{\frac12} &\lesssim \int_{(2B)^\complement}\frac{|\Omega(x-y)|}{|x-y|^n}|b(x)-b_B+b_B-b(y)||f(y)|dy\\ &\le |b(x)-b_B|\int_{(2B)^\complement}\frac{|\Omega(x-y)|}{|x-y|^n}|f(y)|dy\\ &\quad+\int_{(2B)^\complement}\frac{|\Omega(x-y)|}{|x-y|^n}|b(y)-b_B||f(y)|dy\\ &=G_1(x)+G_2(x).\end{aligned}$$

If $x\in B$ and $y\notin 2B$, we have $\frac{|z-y|}{2}\le|x-y|\le\frac{3|z-y|}{2}$. Then

$$\begin{aligned}G_1(x)&\le|b(x)-b_B|\int_{\mathbb{R}^n\setminus 2B}\frac{|\Omega(x-y)|}{|x-y|^n}|f(y)|dy\\ &\lesssim|b(x)-b_B|\int_{\mathbb{R}^n\setminus 2B}\frac{|f(y)|}{|z-y|^n}dy=|b(x)-b_B|\sum_{j=1}^\infty\int_{2^{j+1}B\setminus 2^jB}\frac{|f(y)|}{|z-y|^n}dy.\end{aligned}$$

By Lemma 2.5, Holder's inequality and the doubling condition of φ, we can see that

$$\begin{aligned}\int_{2^{j+1}B\setminus 2^jB}\frac{|f(y)|}{|z-y|^n}dy\lesssim\fint_{2^{j+1}B}|f(y)|dy&\lesssim\Phi^{-1}(\varphi(2^{j+1}r))\|f\|_{L^{(\Phi,\varphi)}}\\ &\lesssim\int_{2^jr}^{2^{j+1}r}\frac{\Phi^{-1}(\varphi(t))}{t}\,\mathrm{d}t\|f\|_{L^{(\Phi,\varphi)}}.\end{aligned}$$

Therefore,

$$G_1(x)\lesssim|b(x)-b_B|\int_{2r}^\infty\frac{\Phi^{-1}(\varphi(t))}{t}dt\|f\|_{L^{(\Phi,\varphi)}}.$$

Then, using (22) and (25), we have

$$\begin{aligned}\fint_B G_1(x)\mathrm{d}x&\lesssim\fint_B|b(x)-b_B|\,dx\Phi^{-1}(\varphi(r))\|f\|_{L^{(\Phi,\varphi)}}\\ &\lesssim\psi(r)\Phi^{-1}(\varphi(r))\|b\|_{\mathcal{L}_{1,\psi}}\|f\|_{L^{(\Phi,\varphi)}}\\ &\lesssim\Psi^{-1}(\varphi(r))\|b\|_{\mathcal{L}_{1,\psi}}\|f\|_{L^{(\Phi,\varphi)}}.\end{aligned}$$

If $x\in B$ and $y\notin 2B$, we have $\frac{|z-y|}{2}\le|x-y|\le\frac{3|z-y|}{2}$. Then

$$\begin{aligned}\int_{\mathbb{R}^n\setminus 2B}(b(y)-b_B)\frac{|\Omega(x-y)|}{|x-y|^n}|f(y)|dy&\lesssim\int_{\mathbb{R}^n\setminus 2B}\frac{|(b(y)-b_B)f(y)|}{|z-y|^n}dy\\ &=\sum_{j=1}^\infty\int_{2^{j+1}B\setminus 2^jB}\frac{|(b(y)-b_B)f(y)|}{|z-y|^n}dy.\end{aligned}$$

Using Lemma 2.5, we know that there exists $p \in (1, \infty)$ such that

$$\left(\fint_{2^{j+1}B} |f(y)|^p dy\right)^{\frac{1}{p}} \lesssim \Phi^{-1}(\varphi(2^{j+1}r))\|f\|_{L^{(\Phi,\varphi)}}.$$

By Holder's inequality and the doubling condition of ψ and φ, we obtain that

$$\begin{aligned}\int_{2^{j+1}B\backslash 2^jB} \frac{|(b(y)-b_B)f(y)|}{|z-y|^n} dy &\sim \frac{1}{(2^{j+1}r)^n}\int_{2^{j+1}B\backslash 2^jB} |(b(y)-b_B)f(y)|dy \\ &\lesssim \left(\fint_{2^{j+1}B} |b(y)-b_B|^{p'}dy\right)^{\frac{1}{p'}}\left(\fint_{2^{j+1}B} |f(y)|^p dy\right)^{\frac{1}{p}} \\ &\lesssim \int_r^{2^{j+1}r} \frac{\psi(t)}{t} dt \Phi^{-1}(\varphi(2^{j+1}r))\|b\|_{\mathcal{L}_{1,\psi}}\|f\|_{L^{(\Phi,\varphi)}} \\ &\lesssim \int_{2^jr}^{2^{j+1}r}\left(\int_r^u \frac{\psi(t)}{t}\,\mathrm{d}t\right)\frac{\Phi^{-1}(\varphi(u))}{u}\,\mathrm{d}u\|b\|_{\mathcal{L}_{1,\psi}}\|f\|_{L^{(\Phi,\varphi)}}.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{\mathbb{R}^n\backslash 2B}(b(y)-b_B)\frac{|\Omega(x-y)|}{|x-y|^n}|f(y)|dy &\lesssim \int_r^\infty\left(\int_r^u \frac{\psi(t)}{t}dt\right)\frac{\Phi^{-1}(\varphi(u))}{u}du\|b\|_{\mathcal{L}_{1,\psi}}\|f\|_{L^{(\Phi,\varphi)}} \\ &= \int_r^\infty \frac{\psi(t)}{t}\left(\int_t^\infty \frac{\Phi^{-1}(\varphi(u))}{u}\,\mathrm{d}u\right)\mathrm{d}t\|b\|_{\mathcal{L}_{1,\psi}}\|f\|_{L^{(\Phi,\varphi)}}.\end{aligned} \tag{26}$$

By Lemma(21), (22) and (25), we have

$$\int_r^\infty \frac{\psi(t)}{t}\left(\int_t^\infty \frac{\Phi^{-1}(\varphi(u))}{u}\,\mathrm{d}u\right)\mathrm{d}t \lesssim \int_r^\infty \frac{\psi(t)\Phi^{-1}(\varphi(t))}{t}\,\mathrm{d}t \lesssim \int_r^\infty \frac{\Psi^{-1}(\varphi(t))}{t}\,\mathrm{d}t \lesssim \Psi^{-1}(\varphi(r)). \tag{27}$$

Using (26) and (27), it is easy to see that

$$\fint_B G_2(x)\mathrm{d}x \lesssim \Psi^{-1}(\varphi(r))\|b\|_{\mathcal{L}_{1,\psi}}\|f\|_{L^{(\Phi,\varphi)}}.$$

Thus, we complete the proof of Lemma 2.10. □

Remark 2.11. *Under the assumption in the part 1 of Theorem 1.10, let $b \in \mathcal{L}_{1,\psi}(\mathbb{R}^n)$ and $f \in L^{(\Phi,\varphi)}(\mathbb{R}^n)$. Since $\Phi \in \overline{\Delta}_2$, there exists $p \in (1,\infty)$ such that $t^p \lesssim \Phi(t)$ for $t \geq 1$, see the part 3 of Remark 1.4. Then $L^{(\Phi,\varphi)}(\mathbb{R}^n) \subset L^{\Phi}_{loc}(\mathbb{R}^n) \subset L^p_{loc}(\mathbb{R}^n)$, which implies $f \in L^p_{loc}(\mathbb{R}^n)$ and $bf \in L^{p_1}_{loc}(\mathbb{R}^n)$ for all $p_1 \in (1,p)$ by Lemma 2.8. Hence, $\mathcal{M}^\rho_\Omega(f\chi_{2B})$ and $\mathcal{M}^\rho_\Omega(bf\chi_{2B})$ are well defined for any ball $B = B(z,r)$. That is, $\mathcal{M}^\rho_{\Omega,b}(f\chi_{2B})$ is well defined for any ball $B = B(z,r)$.*

In addition, it follows from the proof of Lemma 2.10 that $\mathcal{M}^\rho_{\Omega,b}(f\chi_{(2B)^\complement})$ is well defined for any ball $B = B(z,r)$. By Minkowski's inequality, we have

$$\begin{aligned}&\left(\int_0^\infty \left|[b,F^\rho_{\Omega,t}](f\chi_{2B})(x) + [b,F^\rho_{\Omega,b}](f\chi_{(2B)^\complement})(x)\right|^2\frac{dt}{t}\right)^{\frac{1}{2}} \\ &\leq \mathcal{M}^\rho_{\Omega,b}(f\chi_{2B})(x) + \mathcal{M}^\rho_{\Omega,b}(f\chi_{(2B)^\complement})(x), \quad x \in B.\end{aligned}$$

Therefore, we can write

$$\mathcal{M}^\rho_{\Omega,b}(f)(x) = \left(\int_0^\infty \left|[b,F^\rho_{\Omega,t}](f\chi_{2B})(x) + [b,F^\rho_{\Omega,b}](f\chi_{(2B)^\complement})(x)\right|^2\frac{dt}{t}\right)^{\frac{1}{2}}, \quad x \in B.$$

Moreover, if $x \in B_1 \cap B_2$, then, taking B_3 such that $B_1 \cup B_2 \subset B_3$, we have

$$\begin{aligned}&\left([b, F^{\rho}_{\Omega,t}]\ (f\chi_{2B_i})\,(x) + [b, F^{\rho}_{\Omega,t}](f\chi_{(2B_i)^{\complement}})(x)\right) - \left([b, F^{\rho}_{\Omega,t}]\,(f\chi_{2B_3})\,(x) + [b, F^{\rho}_{\Omega,t}](f\chi_{(2B_3)^{\complement}})(x)\right)\\ &= -[b, F^{\rho}_{\Omega,t}](f\chi_{2B_3\setminus 2B_i})(x) + [b, F^{\rho}_{\Omega,t}]\,(f\chi_{2B_3\setminus 2B_i})\,(x) = 0, \quad i = 1, 2,\end{aligned}$$

which implies that

$$\left([b, F^{\rho}_{\Omega,t}]\,(f\chi_{2B_1})\,(x) + [b, F^{\rho}_{\Omega,t}](f\chi_{(2B_1)^{\complement}})(x)\right) = \left([b, F^{\rho}_{\Omega,t}]\,(f\chi_{2B_2})\,(x) + [b, F^{\rho}_{\Omega,t}](f\chi_{(2B_2)^{\complement}})(x)\right).$$

This shows that $\mathcal{M}^{\rho}_{\Omega,b}(f)$ in (20) is independent of the choice of the ball B containing x.

3. Sharp maximal operator and pointwise estimate

In this section, we will establish a sharp maximal inequality of the commutator $\mathcal{M}^{\rho}_{\Omega,b}$. For the locally integrable function f, let

$$M^{\sharp}f(x) = \sup_{B\ni x} \fint_B |f(y) - f_B|dy, \quad x \in \mathbb{R}^n,$$

where the supremum is taken over all balls B containing x.

For sharp maximal operator, the following lemma is known.

Lemma 3.1. *[37, Corollary 6.3] Let $\Phi \in \overline{\Phi}_Y$ and $\varphi \in \mathcal{G}^{dec}$. Assume that $\Phi \in \overline{\Delta}_2$ and that φ satisfies (21). Then there exist a positive constant C such that, for any $f \in L^{(\Phi,\varphi)}(\mathbb{R}^n)$ satisfying $\lim_{r\to\infty} f_{B(0,r)} = 0$,*

$$\|f\|_{L^{(\Phi,\varphi)}} \le C\|M^{\sharp}f\|_{L^{(\Phi,\varphi)}}.$$

Moreover, if $\Phi \in \overline{\nabla}_2$, then

$$C^{-1}\|f\|_{L^{(\Phi,\varphi)}} \le \|M^{\sharp}f\|_{L^{(\Phi,\varphi)}} \le C\|f\|_{L^{(\Phi,\varphi)}}.$$

Proposition 3.2. *Let $\mathcal{M}^{\rho}_{\Omega}$ be a parameter Marcinkiewicz integral. Let $\psi \in \mathcal{G}^{\mathrm{inc}}$. Assume that ψ satisfy the same assumption in Theorem 1.10. Then, for any $\eta \in (1, \infty)$, there exists a positive constant C such that, for all $b \in \mathcal{L}_{1,\psi}(\mathbb{R}^n)$, $f \in L^{(\Phi,\varphi)}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,*

$$M^{\sharp}(\mathcal{M}^{\rho}_{\Omega,b}(f))(x) \le C\|b\|_{\mathcal{L}_{1,\psi}} \left(\left(M_{\psi^{\eta}}\left(|\mathcal{M}^{\rho}_{\Omega}(f)|^{\eta}\right)(x)\right)^{\frac{1}{\eta}} + \left(M_{\psi^{\eta}}\left(|f|^{\eta}\right)(x)\right)^{\frac{1}{\eta}}\right),$$

where C is a positive constant independent of b and f.

Proof. Using the vector-valued singular integral notation of Benedek, Calderón and Panzone in [4], let $\mathcal{H}$ be the Hilbert space defined by

$$\mathcal{H} = \left\{h : \|h\|_{\mathcal{H}} = \left(\int_0^{\infty} \frac{|h(t)|^2}{t}dt\right)^{\frac{1}{2}} < \infty\right\}$$

and $F^{\rho}_{\Omega,t}(f)(x)$, $[b, F^{\rho}_{\Omega,t}](f)(x)$ be as before. Then, we can write

$$\mathcal{M}^{\rho}_{\Omega}(f)(x) = \|F^{\rho}_{\Omega,t}(f)(x)\|_{\mathcal{H}}, \quad \mathcal{M}^{\rho}_{\Omega,b}(f)(x) = \|[b, F^{\rho}_{\Omega,t}](f)(x)\|_{\mathcal{H}}.$$

For $x \in \mathbb{R}^n$, let B be a ball centered at x. Take $B^* = 2B$. We decompose $f = f\chi_{B^*} + f\chi_{(B^*)^{\complement}} := f_1 + f_2$ and write

$$\begin{aligned}\mathcal{M}^{\rho}_{\Omega,b}(f)(y) &= \mu_{\Omega,b-b_{B^*}}(f)(y) = \|[b - b_{B^*}, F^{\rho}_{\Omega,t}](f)(y)\|_{\mathcal{H}} := \|F^{\rho,b-b_{B^*}}_{\Omega,t}(f)(y)\|_{\mathcal{H}}\\ &= \|(b(y) - b_{B^*})F^{\rho}_{\Omega,t}(f)(y) - F^{\rho}_{\Omega,t}((b - b_{B^*})f_1)(y) - F^{\rho}_{\Omega,t}((b - b_{B^*})f_2)(y)\|_{\mathcal{H}}.\end{aligned}$$

Let $C_B = \mathcal{M}_\Omega^\rho\left(\left(b - b_{B^*}\right) f_2\right)(x) = \|F_{\Omega,t}^\rho\left(\left(b - b_{B^*}\right) f_2\right)(x)\|_{\mathcal{H}}$. Then, for $y \in B$, we can see that

$$\begin{aligned}\left|\mathcal{M}_{\Omega,b}(f)(y) - C_B\right| &= \left|\|F_{\Omega,t}^{\rho,b-b_{B^*}}(f)(y)\|_{\mathcal{H}} - \|F_{\Omega,t}^\rho\left(\left(b - b_{B^*}\right) f_2\right)(x)\|_{\mathcal{H}}\right| \\ &\leq |b(y) - b_{B^*}|\|F_{\Omega,t}^\rho(f)(y)\|_{\mathcal{H}} + \|F_{\Omega,t}^\rho\left(\left(b - b_{B^*}\right) f_1\right)(y)\|_{\mathcal{H}} \\ &\quad + \|F_{\Omega,t}^\rho\left(\left(b - b_{B^*}\right) f_2(y) - F_{\Omega,t}^\rho\left(\left(b - b_{B^*}\right) f_2\right)(x)\|_{\mathcal{H}}\right. \\ &= I_1(y) + I_2(y) + I_3(y).\end{aligned}$$

Next, we estimate each term separately. For $1 < \eta < \infty$, by Hölder's inequality and Lemma 2.8, we have

$$\begin{aligned}\fint_{B(x,r)} |I_1(y)| dy &= \fint_{B(x,r)} |(b(y) - b_{B^*}) \mathcal{M}_\Omega^\rho(f)(y)| dy \\ &\leq \frac{1}{\psi(B)}\left(\fint_{B(x,r)} |b(y) - b_{B^*}|^{\eta'} dy\right)^{\frac{1}{\eta'}}\left(\psi(B)^\eta \fint_{B(x,r)} |\mathcal{M}_\Omega^\rho(f)(y)|^\eta dy\right)^{\frac{1}{\eta}} \\ &\lesssim \|b\|_{\mathcal{L}^{(1,\psi)}}\left(\mathcal{M}_{\psi^\eta}\left(|\mathcal{M}_\Omega^\rho(f)|^\eta\right)(x)\right)^{\frac{1}{\eta}}.\end{aligned}$$

For the second term $I_2(y)$, choose $v \in (1, \eta)$ and let $\frac{1}{v} = \frac{1}{u} + \frac{1}{\eta}$. Then, by the boundedness of $\mathcal{M}_\Omega^\rho$ on $L^v(\mathbb{R}^n)$, Hölder's inequality and Lemma 2.8, we obtain that

$$\begin{aligned}\fint_{B(x,r)} I_2(y) dy &= \fint_B \mathcal{M}_\Omega^\rho\left(\left(b - b_{B^*}\right) f_1\right)(y) dy \\ &\leq \left(\fint_B |\mathcal{M}_\Omega^\rho\left(\left(b - b_{B^*}\right) f_1\right)(y)|^v dy\right)^{\frac{1}{v}} \\ &\lesssim \left(\frac{1}{|B|}\int_{B^*} |\left(b(y) - b_{B^*}\right) f(y)|^v dy\right)^{\frac{1}{v}} \\ &\lesssim \frac{1}{\psi(B^*)}\left(\fint_{B^*} |b(y) - b_{B^*}|^u\right)^{\frac{1}{u}}\left(\psi(B^*)^\eta \fint_{B^*} |f(y)|^\eta dy\right)^{\frac{1}{\eta}} \\ &\lesssim \|b\|_{\mathcal{L}^{(1,\psi)}} \mathcal{M}_{\psi^\eta}\left(|f|^\eta\right)(x)^{\frac{1}{\eta}}.\end{aligned}$$

Finally, for $I_3(y)$, it is easy to see that

$$\begin{aligned}I_3(y) &= \left(\int_0^\infty \left|\frac{1}{t^\rho}\int_{|y-z|\leq t} \left(b(z) - b_{B^*}\right) f_2(z) \frac{\Omega(y-z)}{|y-z|^{n-\rho}} dz - \frac{1}{t^\rho}\int_{|x-z|\leq t} \left(b(z) - b_{B^*}\right) f_2(z) \frac{\Omega(y-z)}{|y-z|^{n-\rho}} dz\right|^2 \frac{dt}{t}\right)^{\frac{1}{2}} \\ &\leq \left(\int_0^\infty \left|\int_{|y-z|<t\leq|x-z|} \left(b(z) - b_{B^*}\right) f_2(z) \frac{\Omega(y-z)}{|y-z|^{n-\rho}} dz\right|^2 \frac{dt}{t}\right)^{\frac{1}{2}} \\ &\quad + \left(\int_0^\infty \left|\int_{|x-z|<t\leq|y-z|} \left(b(z) - b_{B^*}\right) f_2(z) \frac{\Omega(x-z)}{|x-z|^{n-\rho}} dz\right|^2 \frac{dt}{t}\right)^{\frac{1}{2}} \\ &\quad + \left(\int_0^\infty \left|\int_{|x-z|\leq t, |y-z|\leq t} \left(b(z) - b_{B^*}\right) f_2(z)\left[\frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(x-z)}{|x-z|^{n-\rho}}\right] dz\right|^2 \frac{dt}{t}\right)^{\frac{1}{2}} \\ &:= II_1(y) + II_2(y) + II_3(y).\end{aligned}$$

In what follows, we estimate $II_1(y)$, $II_2(y)$ and $II_3(y)$, respectively. Note that, for $x, y \in B$, $z \in (B^*)^{\complement}$, we have

$|x-z| \sim |y-z|$. By Hölder's inequality and Lemma 2.8,

$$\begin{aligned}
II_1(y) &\leq \int_{(B^*)^{\complement}} |b(z)-b_{B^*}||f(z)| \frac{|\Omega(y-z)|}{|y-z|^{n-1}} \left| \frac{1}{|y-z|^2} - \frac{1}{|x-z|^2} \right|^{\frac{1}{2}} dz \\
&\leq \int_{(B^*)^{\complement}} |b(z)-b_{B^*}||f(z)| \frac{1}{|y-z|^{n-1}} \frac{|x-y|^{\frac{1}{2}}}{|x-z|^{\frac{3}{2}}} dz \\
&\leq \sum_{j=1}^{\infty} \int_{2^{j+1}B \setminus 2^j B} |b(z)-b_{B^*}||f(z)| \frac{|x-y|^{\frac{1}{2}}}{|x-z|^{n+\frac{1}{2}}} dz \\
&\lesssim \sum_{j=1}^{\infty} 2^{-\frac{j}{2}} \fint_{2^{j+1}B} |b(z)-b_{B^*}||f(z)| dz \\
&\lesssim \sum_{j=1}^{\infty} \frac{j}{2^{\frac{j}{2}}} \|b\|_{\mathcal{L}^{(1,\psi)}} M_{\psi^\eta}\left(|f|^\eta\right)(x)^{\frac{1}{\eta}} \\
&\lesssim \|b\|_{\mathcal{L}^{(1,\psi)}} \mathcal{M}_{\psi^\eta}\left(|f|^\eta\right)(x)^{\frac{1}{\eta}}.
\end{aligned}$$

By the similar arguments as in estimating $II_1(y)$, we obtain that

$$II_2(y) \lesssim \|b\|_{\mathcal{L}^{(1,\psi)}} \mathcal{M}_{\psi^\eta}\left(|f|^\eta\right)(x)^{\frac{1}{\eta}}.$$

For $II_3(y)$, by general Minkowski's inequality, we have

$$\begin{aligned}
II_3(y) &\lesssim \int_{(B^*)^{\complement}} |b(z)-b_{B^*}||f(z)| \left| \frac{\Omega(y-z)}{|y-z|^{n-1}} - \frac{\Omega(x-z)}{|x-z|^{n-1}} \right| \frac{1}{|x-z|} dz \\
&\leq \int_{(B^*)^{\complement}} |b(z)-b_{B^*}||f(z)| \frac{|\Omega(x-z)|}{|x-z|} \left| \frac{1}{|y-z|^{n-1}} - \frac{1}{|x-z|^{n-1}} \right| dz \\
&\quad + \int_{(B^*)^{\complement}} |b(z)-b_{B^*}||f(z)| \frac{|\Omega(y-z)-\Omega(x-z)|}{|x-z|^n} \\
&:= III_1(y) + III_2(y).
\end{aligned}$$

As in estimating $II_1(y)$, we have

$$\begin{aligned}
III_1(y) &\lesssim \int_{(B^*)^{\complement}} |b(z)-b_{B^*}||f(z)| \frac{|x-y|}{|x-z|^{n+1}} dz \\
&\lesssim \sum_{j=1}^{\infty} 2^{-j} \fint_{2^{j+1}B} |b(z)-b_{B^*}||f(z)| dz \\
&\lesssim \sum_{j=1}^{\infty} \frac{j}{2^j} \|b\|_{\mathcal{L}^{(1,\psi)}} M_{\psi^\eta}\left(|f|^\eta\right)(x)^{\frac{1}{\eta}} \\
&\lesssim \|b\|_{\mathcal{L}^{(1,\psi)}} \mathcal{M}_{\psi^\eta}\left(|f|^\eta\right)(x)^{\frac{1}{\eta}}.
\end{aligned}$$

For $III_2(y)$, invoking the condition (5), we obtain that

$$III_2(y) \lesssim \int_{(B^*)^{C}} |b(z)-b_{B^*}| \frac{|f(z)|}{|x-z|^n} \left(\log \frac{2|x-z|}{|x-y|} \right)^{-\gamma} dz$$

$$\leq \sum_{j=1}^{\infty} \int_{2^{j+1}B\setminus 2^jB} |b(z) - b_{B^*}| \frac{|f(z)|}{|x-z|^n} \left(\log \frac{2|x-z|}{|x-y|}\right)^{-\gamma} dz$$

$$\lesssim \sum_{j=1}^{\infty} \frac{j}{(j+1)^{\gamma}} \|b\|_{\mathcal{L}^{(1,\psi)}} \mathcal{M}_{\psi^{\eta}}\left(|f|^{\eta}\right)(x)^{\frac{1}{\eta}}$$

$$\lesssim \|b\|_{\mathcal{L}^{(1,\psi)}} \mathcal{M}_{\psi^{\eta}}\left(|f|^{\eta}\right)(x)^{\frac{1}{\eta}}.$$

Summing up the estimates of $II_1(y)$, $II_2(y)$, $III_1(y)$ and $III_2(y)$, we can see that

$$⨍_{B(x,r)} I_3(y)dy \lesssim \|b\|_{\mathcal{L}^{(1,\psi)}} \mathcal{M}_{\psi^{\eta}}\left(|f|^{\eta}\right)(x)^{\frac{1}{\eta}}.$$

This, together with the estimates for $I_1(y)$, $I_2(y)$, immediately yields that

$$M^{\#}(\mathcal{M}^{\rho}_{\Omega,b}(f))(x) \lesssim \|b\|_{\mathcal{L}^{(1,\psi)}} \left(\left(\mathcal{M}_{\psi^{\eta}}\left(|\mathcal{M}^{\rho}_{\Omega}(f)|^{\eta}\right)(x)\right)^{\frac{1}{\eta}} + \left(\mathcal{M}_{\psi^{\eta}}\left(|f|^{\eta}\right)(x)\right)^{\frac{1}{\eta}}\right),$$

which completes the proof of Proposition 3.2. □

4. Proof of main results

We note that, for $\theta \in (0, \infty)$

$$\left\||g|^{\theta}\right\|_{L^{(\Phi,\varphi)}} = \left(\|g\|_{L^{(\Phi((\cdot)^{\theta}),\varphi))}}\right)^{\theta}. \tag{28}$$

Proof of the part 1 of Theorem 1.10. First, we know that $\mathcal{M}^{\rho}_{\Omega}$ is bounded on $L^{(\Phi,\varphi)}(\mathbb{R}^n)$ by Lemma 2.7. We can take $\eta \in (1, \infty)$ such that $\Phi\left((\cdot)^{\frac{1}{\eta}}\right) \in \overline{\nabla}_2$ by Lemma 2.4. Then, using (22), it is easy to see that

$$\psi(r)^{\eta}\Phi^{-1}(\varphi(r))^{\eta} \leq C_0{}^{\eta}\Psi^{-1}(\varphi(r))^{\eta}.$$

By Lemma 2.1 with this condition we have the boundedness of $M_{\psi^{\eta}}$ from $L^{(\Phi((\cdot)^{\frac{1}{\eta}}),\varphi)}(\mathbb{R}^n)$ to $L^{(\Psi((\cdot)^{\frac{1}{\eta}}),\varphi)}(\mathbb{R}^n)$. Using this boundedness and (28), we have

$$\begin{aligned}
\left\|\left(M_{\psi^{\eta}}\left(|\mathcal{M}^{\rho}_{\Omega}(f)|^{\eta}\right)\right)^{\frac{1}{\eta}}\right\|_{L^{(\Psi,\varphi)}} &= \left(\left\|M_{\psi^{\eta}}\left(|\mathcal{M}^{\rho}_{\Omega}(f)|^{\eta}\right)\right\|_{L^{(\Psi((\cdot)^{\frac{1}{\eta}},\varphi)}}\right)^{\frac{1}{\eta}} \\
&\lesssim \left(\left\||\mathcal{M}^{\rho}_{\Omega}(f)|^{\eta}\right\|_{L^{(\Phi((\cdot)^{\frac{1}{\eta}},\varphi)}}\right)^{\frac{1}{\eta}} \\
&= \|\mathcal{M}^{\rho}_{\Omega}(f)\|_{L^{(\Phi,\varphi)}} \lesssim \|f\|_{L^{(\Phi,\varphi)}},
\end{aligned}$$

and

$$\left\|\left(M_{\psi^{\eta}}\left(|f|^{\eta}\right)\right)^{\frac{1}{\eta}}\right\|_{L^{(\Psi,\varphi)}} = \left(\left\|M_{\psi^{\eta}}\left(|f|^{\eta}\right)\right\|_{L^{(\Psi((\cdot)^{\frac{1}{\eta}},\varphi)}}\right)^{\frac{1}{\eta}} \lesssim \left(\left\||f|^{\eta}\right\|_{L^{(\Phi((\cdot)^{\frac{1}{\eta}}),\varphi)}}\right)^{\frac{1}{\eta}} = \|f\|_{L^{(\Phi,\varphi)}}.$$

Then, using Proposition 3.2 , we obtain that

$$\|M^{\sharp}(\mathcal{M}^{\rho}_{\Omega,b}(f))\|_{L^{(\Psi,\varphi)}} \lesssim \|b\|_{\mathcal{L}_{1,\psi}} \|f\|_{L^{(\Phi,\varphi)}}.$$

Therefore, if we show that, for $B_r = B(0, r)$,

$$⨍_{B_r} \mathcal{M}^{\rho}_{\Omega,b}(f) \to 0 \quad \text{as} \quad r \to \infty. \tag{29}$$

Then, by Lemma 3.2, we have

$$\|\mathcal{M}^{\rho}_{\Omega,b}(f)\|_{L^{(\Psi,\varphi)}} \lesssim \|b\|_{\mathcal{L}_{1,\psi}}\|f\|_{L^{(\Phi,\varphi)}},$$

which is the conclusion.

It remains to show that (29) holds. Notice that

$$\mathcal{M}^{\rho}_{\Omega,b}(f)(x) \le |b(x)|\mathcal{M}^{\rho}_{\Omega}(f)(x) + \mathcal{M}^{\rho}_{\Omega}(bf)(x) =: \mathcal{M}^1_b(f)(x) + \mathcal{M}^2_b(f)(x).$$

To prove (29), it suffices to show that

$$⨍_{B_r} \mathcal{M}^1_b(f)(x)dx \to 0 \text{ and } ⨍_{B_r} \mathcal{M}^2_b(f)(x)dx \to 0 \text{ as } r \to \infty.$$

In the following we show (29).

Case 1 First we show (29) for all $f \in L^{(\Phi,\varphi)}(\mathbb{R}^n)$ with compact support. Let $\operatorname{supp} f \subset B_s = B(0,s)$ with $s \ge 1$. Then $f \in L^p(\mathbb{R}^n)$ and $bf \in L^{p_1}(\mathbb{R}^n)$ for some $1 < p_1 < p < \infty$ (see Remark 2.11). Since $\mathcal{M}^{\rho}_{\Omega}$ is bounded on Lebesgue spaces, we see that both $\mathcal{M}^1_b(f)\chi_{B_{2s}}$ and $\mathcal{M}^2_b(f)\chi_{B_{2s}}$ are in $L^1(\mathbb{R}^n)$ and that

$$⨍_{B_r} \mathcal{M}^1_b(f)(x)\chi_{B_{2s}}(x)dx \to 0 \text{ and } ⨍_{B_r} \mathcal{M}^2_b(f)(x)\chi_{B_{2s}}(x)dx \to 0 \text{ as } r \to \infty.$$

If $x \notin B_{2s}$ and $y \in B(0,s)$, then $|x|/2 \le |x-y| \le 3|x|/2$. For $x \notin B_{2s}$, we have

$$\begin{aligned}|\mathcal{M}^{\rho}_{\Omega}(f)(x)| &\le \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\rho}}|f(y)|\left(\int_{|x-y|}^{\infty} \frac{dt}{t^{1+2\rho}}\right)^{\frac{1}{2}} dy \\ &\lesssim \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^n}|f(y)|dy \lesssim \frac{1}{|x|^n}\|f\|_{L^1(\mathbb{R}^n)},\end{aligned}$$

and

$$\begin{aligned}\mathcal{M}^{\rho}_{\Omega}(bf)(x) &\le \int_{B_s} \frac{|\Omega(x-y)|}{|x-y|^{n-\rho}}|b(y)f(y)|\left(\int_{|x-y|}^{\infty} \frac{dt}{t^{1+2\rho}}\right)^{\frac{1}{2}} \\ &\lesssim \int_{B_s} \frac{|\Omega(x-y)|}{|x-y|^n}|b(y)f(y)|dy \lesssim \frac{1}{|x|^n}\|bf\|_{L^1(\mathbb{R}^n)},\end{aligned}$$

which yields that

$$\begin{aligned}⨍_{B_r} \mathcal{M}^2_b(f)(x)\chi_{(B_{2s})^{\complement}}(x)dx &\lesssim ⨍_{B_r} \frac{1}{|x|^n}\chi_{(B_{2s})^{\complement}}(x)dx\|bf\|_{L^1(\mathbb{R}^n)} \\ &\lesssim \frac{1}{r^n}\left(\log\frac{r}{2s}\right)\|bf\|_{L^1(\mathbb{R}^n)} \to 0 \text{ as } r \to \infty,\end{aligned}$$

and

$$\begin{aligned}⨍_{B_r} \mathcal{M}^1_b(f)(x)\chi_{(B_{2s})^{\complement}}(x)dx &\lesssim ⨍_{B_r} \frac{|b(x)-b_{B_{2s}}|}{|x|^n}\chi_{(B_{2s})^{\complement}}(x)dx\|f\|_{L^1(\mathbb{R}^n)} \\ &\quad + ⨍_{B_r} \frac{|b_{B_{2s}}|}{|x|^n}\chi_{(B_{2s})^{\complement}}dx\|f\|_{L^1(\mathbb{R}^n)} =: F_1 + F_2.\end{aligned}$$

For F_2, we can see that

$$F_2 = |b_{B_{2s}}| ⨍_{B_r} \frac{1}{|x|^n}\chi_{(B_{2s})^{\complement}}(x)dx\|f\|_{L^1(\mathbb{R}^n)} \lesssim |b_{B_{2s}}|\frac{1}{r^n}\left(\log\frac{r}{2s}\right)\|f\|_{L^1(\mathbb{R}^n)} \to 0 \text{ as } r \to \infty.$$

To estimate F_1 , we take $\varepsilon \in (0,1)$ such that $1 + 1/q - 1/p > \varepsilon$ and let $v = 1/(1-\varepsilon)$. Then, for $r > 4s$, Hölder's inequality and Lemma 2.9 tell us that

$$\begin{aligned}
F_1 &\le \left(\fint_{B_r} |b(x) - b_{B_{2s}}|^{v'} dx \right)^{\frac{1}{v'}} \left(\fint_{B_r} \frac{1}{|x|^{nv}} \chi_{(B_{2s})^{\complement}}(x) dx \right)^{\frac{1}{v}} \|f\|_{L^1(\mathbb{R}^n)} \\
&\lesssim \int_{2s}^{r} \frac{\psi(t)}{t} dt \|b\|_{\mathcal{L}^{(1,\psi)}(\mathbb{R}^n)} \frac{1}{r^{\frac{n}{v}}} \|f\|_{L^1(\mathbb{R}^n)} \\
&\lesssim \psi(r) \log r \|b\|_{\mathcal{L}_{1,\psi}} \frac{1}{r^{\frac{n}{v}}} \|f\|_{L^1(\mathbb{R}^n)} \\
&\lesssim \frac{\Psi^{-1}(\varphi(r))}{\Phi^{-1}(\varphi(r))} \log r \|b\|_{\mathcal{L}_{1,\psi}} \frac{1}{r^{\frac{n}{v}}} \|f\|_{L^1(\mathbb{R}^n)} \\
&= \frac{\log r}{r^{\frac{n}{v}}} \frac{\Psi^{-1}(\varphi(r))}{\Phi^{-1}(\varphi(r))} \|b\|_{\mathcal{L}_{1,\psi}} \|f\|_{L^1(\mathbb{R}^n)} \\
&\lesssim \frac{\log r}{r^{\frac{n}{v}}} \frac{\varphi(r)^{1/p}}{\varphi(r)} \|b\|_{\mathcal{L}_{1,\psi}} \|f\|_{L^1(\mathbb{R}^n)} \\
&= \frac{\log r}{r^{\frac{n}{2p}} (r^n \varphi(r))^{1-\frac{1}{p}}} \|b\|_{\mathcal{L}_{1,\psi}} \|f\|_{L^1(\mathbb{R}^n)} \\
&\to 0 \text{ as } r \to \infty.
\end{aligned}$$

Summing up the estimates of F_1 and F_2, we obtain that

$$\fint_{B_r} \mathcal{M}_b^1(f)(x) dx \to 0 \text{ as } r \to \infty.$$

This completes the proof of Case 1 .

Case 2 For any $f \in L^{(\Phi,\varphi)}(\mathbb{R}^n)$, using the result of Case 1, we have

$$\|\mathcal{M}^{\rho}_{\Omega,b}(f\chi_{B_{2r}})\|_{L^{(\Psi,\varphi)}} \lesssim \|b\|_{\mathcal{L}_{1,\psi}} \|f\chi_{B_{2r}}\|_{L^{(\Phi,\varphi)}} \le \|b\|_{\mathcal{L}_{1,\psi}} \|f\|_{L^{(\Phi,\varphi)}}.$$

Then, by Lemma 2.5,we can see that

$$\fint_{B_r} \mathcal{M}^{\rho}_{\Omega,b}(f\chi_{B_{2r}}) \le \Psi^{-1}(\varphi(r)) \|\mathcal{M}^{\rho}_{\Omega,b}(f\chi_{B_{2r}})\|_{L^{(\Psi,\varphi)}} \lesssim \Psi^{-1}(\varphi(r)) \|b\|_{\mathcal{L}_{1,\psi}} \|f\|_{L^{(\Phi,\varphi)}}.$$

Combining this with Lemma 2.10, we have

$$\fint_{B_r} \mathcal{M}^{\rho}_{\Omega,b} f \lesssim \Psi^{-1}(\varphi(r)) \|b\|_{\mathcal{L}_{1,\psi}} \|f\|_{L^{(\Phi,\varphi)}},$$

which implies (29). Therefore, we have (29) for all $f \in L^{(\Phi,\varphi)}(\mathbb{R}^n)$. The proof is completed. □

Proof of the part 2 of Theorem 1.10. We use the method given by Janson in [19]. Let $K(x) := \frac{1}{|x|^n}$. Choose $0 \neq z_0 \in \mathbb{R}^n$ and $\delta > 0$, such that $\frac{1}{K(z)}$ can be expressed in the neighborhood $\left\{ z : |z - z_0| < \sqrt{n}\delta \right\}$ as a Fourier series which is absolutely convergent, that is

$$\frac{1}{K(z)} = \sum_{n=0}^{\infty} a_n e^{i v_n \cdot z},$$

with $\sum_{n=0}^{\infty} |a_n| < \infty$. Let $z_1 = \frac{z_0}{\delta}$. If $|z - z_1| < 2\sqrt{n}$, we have

$$\frac{1}{K(x)} = \frac{\delta^{-n}}{K(x\delta)} = \delta^{-n} \sum_{n=0}^{\infty} a_n e^{i v_n \cdot \delta z}.$$

For any ball $B = B(x_0, r)$. Set $y_0 = x_0 - rz_1$ and $B' = B(y_0, r)$. Then for $x \in B$ and $y \in B'$, we have

$$\left|\frac{x-y}{r} - z_1\right| = \left|\frac{x-y}{r} - \frac{x_0-y_0}{r}\right| \leq \left|\frac{x-x_0}{r}\right| + \left|\frac{y-y_0}{r}\right| \leq 2\sqrt{n}.$$

We set $s(x) = \operatorname{sgn}(b(x) - b_{B'})$, then

$$\begin{aligned}
\int_B |b(x) - b_{B'}|\,dx &= \int_B (b(x) - b_{B'})\,s(x)dx \\
&= \frac{1}{|B'|}\int_B \int_{B'} (b(x) - b(y))s(x)dydx \\
&= C\int_B \int_{\mathbb{R}^n} (b(x) - b(y))K(x-y)\sum_{n=0}^{\infty} a_n e^{iv_n\cdot\delta\frac{x-y}{r}} s(x)\chi_B(x)\chi_{B'}(y)dydx \\
&= C\sum_{n=0}^{\infty} a_n \int_B \int_{\mathbb{R}^n} (b(x) - b(y))K(x-y)f_n(y)g_n(x)dydx \\
&\leq C\sum_{n=0}^{\infty} |a_n| \int_B \left|\int_{\mathbb{R}^n} (b(x) - b(y))K(x-y)f_n(y)dy\right| dx \\
&= C\sum_{n=0}^{\infty} |a_n| \int_B \left|\int_{\mathbb{R}^n} (b(x) - b(y))\frac{1}{|x-y|^n}f_n(y)dy\right| dx,
\end{aligned}$$

where

$$f_n(y) = e^{-i\frac{\delta}{r}v_n\cdot y}\chi_{B'}(y) \quad \text{and} \quad g_n(x) = e^{i\frac{\delta}{r}v_n\cdot x}s(x)\chi_B(x).$$

In addition, since Ω satisfies (1), (2), (5), then there exists a positive constant A with $0 < A < 1$, for $x, y \in \mathbb{R}^n$ with $x \neq y$, we have

$$\Omega(x-y) = \Omega\left(\frac{x-y}{|x-y|}\right) = \Omega((x-y)') \geq \frac{C}{(\log(\frac{2}{A}))^{\gamma}}. \tag{30}$$

Using Minkowski's inequality, Hölder's inequality and (30), we obtain that

$$\begin{aligned}
&\int_B \left|\int_{\mathbb{R}^n} (b(x) - b(y))\frac{1}{|x-y|^n}f_n(y)dy\right| dx \\
&= \int_B \left|\int_{\mathbb{R}^n} (b(x) - b(y))\frac{1}{|x-y|^{n-\rho}}\frac{1}{|x-y|^{\rho}}f_n(y)dy\right| dx \\
&\lesssim \int_B \frac{1}{|x-y_0|^{\rho}}\left|\int_{\mathbb{R}^n} (b(x) - b(y))\frac{1}{|x-y|^{n-\rho}}f_n(y)dy\right| dx \\
&= \frac{(\log(\frac{2}{A}))^{\gamma}}{(\log(\frac{2}{A}))^{\gamma}}\int_B \frac{1}{|x-y_0|^{\rho}}\left|\int_{\mathbb{R}^n} (b(x) - b(y))\frac{1}{|x-y|^{n-\rho}}f_n(y)dy\right| dx \\
&\lesssim \int_B \frac{1}{|x-y_0|^{\rho}}\left|\int_{\mathbb{R}^n} \frac{1}{(\log(\frac{2}{A}))^{\gamma}}(b(x) - b(y))\frac{1}{|x-y|^{n-\rho}}f_n(y)dy\right| dx
\end{aligned}$$

$$\begin{aligned}
&\lesssim \int_B \frac{1}{|x-y_0|^\rho}\left|\int_{\mathbb{R}^n}(b(x)-b(y))\frac{\Omega(x-y)}{|x-y|^{n-\rho}}f_n(y)dy\right|dx\\
&\lesssim \int_B \left|\int_{\mathbb{R}^n}(b(x)-b(y))\frac{\Omega(x-y)}{|x-y|^{n-\rho}}f_n(y)dy\right|\left(\int_{|x-y_0|}^\infty \frac{dt}{t^{1+2\rho}}\right)\left(\int_{|x-y_0|}^\infty \frac{dt}{t^{1+2\rho}}\right)^{-\frac{1}{2}}dx\\
&\lesssim \int_B \left(\int_{|x-y_0|}^\infty \left|\int_{\mathbb{R}^n}(b(x)-b(y))\frac{\Omega(x-y)}{|x-y|^{n-\rho}}f_n(y)\chi_{\{y\in\mathbb{R}^n:|x-y|\le t\}}(y)dy\right|\frac{dt}{t^{1+2\rho}}\right)\left(\int_{|x-y_0|}^\infty \frac{dt}{t^{1+2\rho}}\right)^{-\frac{1}{2}}dx\\
&\lesssim \int_B \left(\int_0^\infty \left|\frac{1}{t^\rho}\int_{|x-y|\le t}(b(x)-b(y))\frac{\Omega(x-y)}{|x-y|^{n-\rho}}f_n(y)dy\right|^2\frac{dt}{t}\right)^{\frac{1}{2}}dx\\
&= \int_B \mathcal{M}^\rho_{\Omega,b}(f_n)(x)dx.
\end{aligned}$$

Combining preceding estimates, we have

$$\int_B |b(x)-b_{B'}|dx \lesssim \sum_{n=0}^\infty |a_n|\int_B \mathcal{M}^\rho_{\Omega,b}(f_n)(x)dx.$$

By Lemma 2.3, we know that $\|f_n\|_{L^{(\Phi,\varphi)}} = \|\chi_{B'}\|_{L^{(\Phi,\varphi)}} \sim \frac{1}{\Phi^{-1}(\varphi(B'))}$. Then

$$\begin{aligned}
\int_B |b(x)-b_{B'}|dx &\le C\sum_{n=0}^\infty |a_n||B|\Psi^{-1}(\varphi(r))\|\mathcal{M}^\rho_{\Omega,b}(f_n)\|_{L^{(\Psi,\varphi)}}\\
&\le C\|\mathcal{M}^\rho_{\Omega,b}\|_{L^{(\Phi,\varphi)}\to L^{(\Psi,\varphi)}}|B|\Psi^{-1}(\varphi(r))\sum_{n=0}^\infty |a_n|\|f_n\|_{L^{(\Phi,\varphi)}}\\
&\lesssim \|\mathcal{M}^\rho_{\Omega,b}\|_{L^{(\Phi,\varphi)}\to L^{(\Psi,\varphi)}}|B|\frac{\Psi^{-1}(\varphi(B))}{\Phi^{-1}(\varphi(B))}.
\end{aligned}$$

By (23), we obtain that

$$\frac{1}{\psi(B)}⨍_B |b(x)-b_B|\,dx \le \frac{2}{\psi(B)}⨍_B |b(x)-b_{B'}|\,dx \lesssim \|\mathcal{M}^\rho_{\Omega,b}\|_{L^{(\Phi,\varphi)}\to L^{(\Psi,\varphi)}}.$$

That is $\|b\|_{\mathcal{L}_{1,\psi}} \lesssim \|\mathcal{M}^\rho_{\Omega,b}\|_{L^{(\Phi,\varphi)}\to L^{(\Psi,\varphi)}}$. Thus, we have the conclusion. □

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