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Existence of solutions for a delay singular high order fractional boundary value problem with sign-changing nonlinearity

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Abstract. This paper consider the existence of at least one positive solution of a Riemann-Liouville fractional delay singular boundary value problem with sign-changing nonlinerty. To establish sufficient conditions we use the Guo-Krasnosel'skii fixed point theorem.

1. Introduction

In the last decade, fractional analysis has become a growing field of study in mathematics due to its effective application in different scientific fields such as mathematical biology, statistics, dynamics, control theory, optimisation, and chaos theory. In recent years, fractional differential equations studies have been in the field of interest of many scientists and important studies have been carried out in this field see [1-9]. In particular, we would like to mention some results which has been done to the existence of positive solutions to Riemann -Liouvlle fractional boundary value problems involving singularity and delay terms [11-14] and the references therein.

In [10], Henderson and Luca considered the existence of positive solutions for the following multi-point Riemann-Liouville fractional boundary value problem with a sign-changing nonlinerity,

$$\begin{aligned}
&D^{\alpha}_{0^+}u(t)) + \lambda f(t, u(t)) = 0, \qquad t \in (0, 1),\\
&u(0) = u'(0) = u''(0) = ... = u^{(n-2)}(0) = 0,\\
&D^{p}_{0^+}u(1) = \sum_{i=1}^{m} a_i D^{q}_{0^+}u(\xi_i),
\end{aligned}$$

where $\lambda > 0$ is a positive parameter, $\alpha \in \mathbb{R}$, $\alpha \in (n-1, n]$, $n \in \mathbb{N}$, $n \geq 3$, $\xi_i \in \mathbb{R}$ for all $i = 1, ..., m$, $m \in \mathbb{N}$, $0 < \xi_1 < \xi_2 < ... < \xi_m < 1$, $p, q \in \mathbb{R}$, $p \in [1, n-2]$, $q \in [0, p]$ and the nonlinearity f may change sign and may be singular at $t = 0$ or $t = 1$. By using the Guo-Krasnosel'skii fixed point theorem, some new results on the existence of one positive solution were obtained.

In [11], Su was concerned with existence of one positive solution of the following boundary value problem

$$\begin{aligned}
&D^{\alpha}x(t)) + f(t, x(t-\tau) = 0, \qquad t \in (0, 1) \setminus \{\tau\},\\
&x(t) = \eta(t), \qquad t \in [-\tau, 0]\\
&x(1) = 0,
\end{aligned}$$

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where $1 < \alpha \leq 2$, $0 < \tau < 1, \eta \in C([-\tau, 0])$, $\eta(t) > 0$ for $t \in [-\tau, 0)$ and $\eta(0) = 0$ and f may be singular at $t = 0$, $t = 1$ and $x = 0$ and may take negative values. By using Krasnosel'skii fixed point theorem, the existence of at least one positive solution of the model was presented. In [12], Mu et al. studied the following singular Reimann-Liouville fractional boundary value problem with delay

$$\begin{aligned}
&D^{\alpha}x(t)) + \lambda f(t, x(t-\tau) = 0, \qquad t \in (0,1) \setminus \{\tau\},\\
&x(t) = \eta(t), \qquad t \in [-\tau, 0]\\
&x(1) = x'(0) = 0,
\end{aligned}$$

where $2 < \alpha \leq 3$, $\lambda > 0$, $0 < \tau < 1, \eta \in C([-\tau, 0])$, $\eta(t) > 0$ for $t \in [-\tau, 0)$ and $\eta(0) = 0$ and f may be singular at $t = 0$, $t = 1$ and $x = 0$ and may change sign. By using Guo-Krasnosel'skii fixed point theorem, the eigenvalue intervals for the existence of at least one positive solution were obtained.

In [13], Liu and Zhang obtained the existence of at least one positive solution for the following high order fractional boundary value problem with delay and singularities including changing sign nonlinearity

$$\begin{aligned}
&D^{\alpha}_{0^+}x(t)) + f(t, x(t-\tau) = 0, \qquad t \in (0,1),\\
&x(t) = \eta(t), \qquad t \in [-\tau, 0]\\
&x'(0) = x''(0) = ... = x^{(n-2)}(0) = 0, \ n \geq 3\\
&x^{(n-2)}(1) = 0
\end{aligned}$$

where $n - 1 < \alpha \leq n$ and f may be singular at $t = 0$, $t = 1$ and $x = 0$ and may change sign. By the means of the Guo-Krasnosel'skii fixed point theorem, Leray-Schauder's nonlinear alternative theorem, existence results of positive solutions were given.

Recently, numerous studies have been considered in nonlinear fractional differential equations (see [14-28]).

Motivated by the articles mentioned above, we concentrate on the existence of positive solutions for the following boundary value problem (BVP) of a fractional differential equation with delay term

$$D^{\alpha}_{0^+}y(t) + \lambda h(t) f(t, y(t-\tau)) = 0, \qquad t \in (0,1) - \{\tau\}, \tag{1}$$

$$\begin{cases} y(t) = \eta(t), \qquad t \in [-\tau, 0],\\ y(0) = y'(0) = y''(0) = ... = y^{(n-2)}(0) = 0, \quad n \geq 3,\\ D^{p}_{0^+}y(1) = \sum_{i=1}^{m} a_i D^{q}_{0^+}y(\xi_i),, \end{cases} \tag{2}$$

where λ is a positive parameter, $\tau \in (0,1)$, $\alpha \in \mathbb{R}$, $\alpha \in (n-1, n]$, $n \in \mathbb{N}$, $n \geq 3$, $\xi_i \in \mathbb{R}$ for all $i = 1, ..., m$, $m \in \mathbb{N}$, $0 < \xi_1 < \xi_2 < ... < \xi_m < 1$ and $\xi_i \neq \tau$, $p, q \in \mathbb{R}$, $p \in [1, n-2]$, $q \in [0, p]$, $h \in C([0,1], \mathbb{R}^+)$, $f(t, y) \in C((0,1) \times \mathbb{R}^+, \mathbb{R})$, $f(t, y)$ may change sign and be singular at $t = 0$, $t = 1$, $y = 0$ and $\eta(t) > 0$ for $t \in [-\tau, 0)$, $\eta(t) = 0$ for $t = 0$. D^{α}, D^{p} and D^{q} are the standard Riemann-Liouville fractional derivatives.

The paper is structured in such a manner, in Section 2, we will give some necessary definitions and lemmas which are used in the main results. We present the associated Green's function with its properties. For clarity, we also state Guo-Krasnosel'skii fixed point theorem. In Section 3, we present the existence theorems for the positive solutions of the boundary value problem (1)-(2) with respect to a cone.

2. Basic Definitions and Preliminaries

We first introduce some necessary definitions and lemmas in this section. The following auxiliary Lemmas are necessary to illustrate the existence of solutions for problem (1)-(2).

Definition 2.1. [29,30] *The integral*

$$I^{\beta}g(t) = \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g(s) ds,$$

where $\beta > 0$, is the fractional integral of order β for a function $g(t)$.

Definition 2.2. [29,30] *For a function $g(t)$ the expression*

$$D^{\beta}_{0^+}g(t) = \frac{1}{\Gamma(n-\beta)}\left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\beta-1}g(s)ds,$$

is called the Riemann-Liouville fractional derivative of order β, where $n = [\beta] + 1$, and $[\beta]$ denotes the integer part of number β.

Lemma 2.3. [31] *Assume that $g \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\beta > 0$ that belongs to $C(0,1) \cap L(0,1)$. Then*

$$I^{\beta}D^{\beta}g(t) = g(t) + c_1t^{\beta-1} + c_2t^{\beta-2} + \cdots + c_Nt^{\beta-N},$$

for some $c_i \in \mathbb{R},\ i = 1,2,\cdots,N$, where N is the smallest integer greater than or equal to β.

Lemma 2.4. [10] *Let $g \in C[0,1]$. Then the fractional differential equation*

$$\begin{aligned}
&D^{\alpha}y(t) + g(t) = 0,\ t \in (0,1)\\
&y(0) = y'(0) = y''(0) = \ldots = y^{(n-2)}(0) = 0, \quad n \geq 3\\
&D^{p}_{0^+}y(1) = \sum_{i=1}^{m} a_iD^{q}_{0^+}y(\xi_i),
\end{aligned}$$

has a unique solution which is given by

$$y(t) = \int_0^1 G(t,s)g(s)ds,$$

where

$$G(t,s) = g_1(t,s) + \frac{t^{\alpha-1}}{\Delta}\sum_{i=1}^{m} a_ig_2(\xi_i,s), \tag{3}$$

in which

$$\begin{aligned}
g_1(t,s) &= \frac{1}{\Gamma(\alpha)}\begin{cases} t^{\alpha-1}(1-s)^{\alpha-p-1} - (t-s)^{\alpha-1}, & 0 \leq s \leq t \leq 1,\\ t^{\alpha-1}(1-s)^{\alpha-p-1}, & 0 \leq t \leq s \leq 1,\end{cases}\\
g_2(t,s) &= \frac{1}{\Gamma(\alpha-q)}\begin{cases} t^{\alpha-q-1}(1-s)^{\alpha-p-1} - (t-s)^{\alpha-q-1}, & 0 \leq s \leq t \leq 1,\\ t^{\alpha-q-1}(1-s)^{\alpha-p-1}, & 0 \leq t \leq s \leq 1,\end{cases}
\end{aligned}$$

$\Delta = \frac{\Gamma(\alpha)}{\Gamma(\alpha-p)} - \frac{\Gamma(\alpha)}{\Gamma(\alpha-q)}\sum_{i=1}^{m} a_i\xi_i^{\alpha-q-1} \neq 0.$

Lemma 2.5. [10] *Let $\Delta > 0$ and $a_i \geq 0$ for all $i = 1,2,...,m$. Then the function $G(t,s)$ given by (3) is continuous on $[0,1] \times [0,1]$ and have the following inequalities:*

(a) *$G(t,s) \leq J(s)$ for all $t,s \in [0,1]$ where $J(s) = h_1(s) + \dfrac{1}{\Delta}\displaystyle\sum_{i=1}^{m} a_ig_2(\xi_i,s)$ and*

$h_1(s) = \frac{(1-s)^{\alpha-p-1}(1-(1-s)^{p})}{\Gamma(\alpha)}$, $s \in [0,1]$;

(b) *$G(t,s) \geq t^{\alpha-1}J(s)$ for all $t,s \in [0,1]$;*

(c) *$G(t,s) \leq \sigma t^{\alpha-1}$ for all $t,s \in [0,1]$, where $\sigma = \dfrac{1}{\Gamma(\alpha)} + \dfrac{\sum_{i=1}^{m} a_i\xi_i^{\alpha-q-1}}{\Delta\Gamma(\alpha-q)}$.*

Remark 2.6. [15] *The function* $G^*(t,s) = t^{2n-1-\alpha}G(t,s)$ *satisfies the following condition:*

$$J(s)t^{2n-2} \le G^*(t,s) \le \sigma t^{2n-2}, \quad t,s \in [0,1].$$

The main tool used is the following well-known Guo-Krasnosel'skii fixed point theorem [32,33].

Theorem 2.7. *Let* $\mathbb{X}$ *be a Banach space,* $P \subseteq \mathbb{X}$ *be a cone, and* Ω_1, Ω_2 *be two bounded open balls of* $\mathbb{X}$ *centred at the origin with* $\overline{\Omega}_1 \subset \Omega_2$. *Suppose that* $T : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ *is a completely continuous operator such that either*

i. $\|Ty\| \le \|y\|,\ y \in P \cap \partial\Omega_1$ *and* $\|Ty\| \ge \|y\|,\ y \in P \cap \partial\Omega_2$ *or*

ii. $\|Ty\| \ge \|y\|,\ y \in P \cap \partial\Omega_1$ *and* $\|Ty\| \le \|y\|,\ y \in P \cap \partial\Omega_2$ *holds.*

Then T *has a fixed point in* $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Main Results

In this section, we will try our best to discuss the existence of positive solutions for BVP (1)-(2). For convenience, we will give some conditions, which will play important roles in this paper.
Throughout this paper, we always assume that $\Delta > 0$ and the following conditions hold:

(H1) There exists a nonnegative function $\rho \in C(0,1) \cap L(0,1)$ such that

$$f(t,y) > -\rho(t)$$

and

$$\varphi_2(t)l_2(y) \le f(t, \vartheta(t)y) + \rho(t) \le \varphi_1(t)(k(y) + l_1(y))$$

for $\forall(t,y) \in (0,1) \times \mathbb{R}^+$, where $\varphi_1, \varphi_2 \in L(0,1)$ are nonnegative for $t \in (0,1)$, $l_1, l_2 \in C(\mathbb{R}_0^+, \mathbb{R}^+)$ are nondecreasing, $k \in C(\mathbb{R}_0^+, \mathbb{R}^+)$ is nonincreasing where $\mathbb{R}_0^+ = [0, +\infty)$ and

$$\vartheta(t) = \begin{cases} 1, & t \in (0,\tau]; \\ (t-\tau)^{\alpha-2n+1}, & t \in (\tau, 1). \end{cases}$$

When $s \in [0,\tau]$, we have $-\tau \le s-\tau \le 0$. Suppose there is a positive number $A > 0$, such that $\max\limits_{-\tau \le s-\tau \le 0} \eta(s-\tau) = A$, so $\eta(s-\tau) \le A$ and $0 < k(A) \le k(\eta(s-\tau)) \le k(0)$.
Let $\Upsilon = \{y : y \in C[-\tau, 1]\}$, then $(\Upsilon, \|\cdot\|)$ is a Banach space with the maximum norm

$$\|y\| = \max_{-\tau \le t \le 1} |y(t)|, \text{ for } y \in \Upsilon.$$

And we set a cone $K \subset \Upsilon$ by

$$K = \{y \in \Upsilon | y(t) = 0 \text{ for } t \in [-\tau, 0],\ y(t) \ge t^{\alpha-1}\|y\| \text{ for } t \in [0,1]\}.$$

Define

$$\overline{\eta}(t) = \begin{cases} \eta(t), & t \in [-\tau, 0]; \\ 0, & t \in (0,1], \end{cases}$$

and

$$w(t) = \begin{cases} 0, & t \in [-\tau, 0]; \\ \int_0^1 G(t,s)\lambda h(s)\rho(s)ds, & t \in (0,1], \end{cases}$$

and a nonnegative function

$$\begin{aligned} y^*(t) &= [y(t) + \overline{\eta}(t) - w(t)]^+ \\ &= \max\{y(t) + \overline{\eta}(t) - w(t), 0\} \\ &= \begin{cases} \eta(t), & t \in [-\tau, 0]; \\ \max\{y(t) - w(t), 0\}, & t \in (0,1], \end{cases} \end{aligned}$$

for any $y \in K$.

Remark 3.1. *We can derive from Lemma 2.4 that the restriction $w|_{[0,1]}$ of w on $[0,1]$ is the solution of*

$$\begin{aligned}
&D^{\alpha}_{0^+}y(t)) + \lambda h(t)\rho(t) = 0, \qquad t \in (0,1),\\
&y(0) = y'(0) = y''(0) = \ldots = y^{(n-2)}(0) = 0, \quad n \geq 3\\
&D^{p}_{0^+}y(1) = \sum_{i=1}^{m} a_i D^{q}_{0^+}y(\xi_i)
\end{aligned}$$

As $f : [0,1] \times \mathbb{R}^+ \to \mathbb{R}$ is a continuous function, we can know that the function y is a solution of BVP (1)-(2) if and only if it satisfies

$$y(t) = \begin{cases} \displaystyle\int_0^1 G(t,s)\lambda h(s) f(s, y(s-\tau))ds, & t \in (0,1];\\ \eta(t), & t \in [-\tau, 0]. \end{cases}$$

Considering the following operator;

$$Ty(t) = \begin{cases} \displaystyle\int_0^1 G(t,s)\lambda h(s)(f(s, y^*(s-\tau)) + \rho(s))ds, & t \in (0,1];\\ 0, & t \in [-\tau, 0]. \end{cases} \tag{4}$$

Let

$$z(t) = \begin{cases} t^{2n-1-\alpha}y(t), & t \in (0,1];\\ 0, & t \in [-\tau, 0] \end{cases}$$

and

$$z^*(t) = \begin{cases} \max\{t^{2n-1-\alpha}z(t) - w(t), 0\}, & t \in (0,1];\\ \eta(t), & t \in [-\tau, 0]. \end{cases}$$

Next (4) is equivalent to

$$Tz(t) = \begin{cases} \displaystyle\int_0^1 G^*(t,s)\lambda h(s)(f(s, z^*(s-\tau)) + \rho(s))ds, & t \in (0,1];\\ 0, & t \in [-\tau, 0]. \end{cases} \tag{5}$$

Clearly, if $\widetilde{z}$ is a fixed point of the operator T in (5), then

$$\widetilde{y}(t) = \begin{cases} t^{\alpha-2n-1}\widetilde{z}(t), & t \in (0,1];\\ 0, & t \in [-\tau, 0] \end{cases}$$

is a fixed point of the operator T defined in (4). By Lemma 2.4, we can obtain that

$$D^{\alpha}_{0^+}\widetilde{y}(t) + \lambda h(t)(f(t, \widetilde{y}^*(t-\tau)) + \rho(t)) = 0, \qquad t \in (0,1) - \{\tau\}, \tag{6}$$

$$\widetilde{y}(t) = 0, \qquad t \in [-\tau, 0],$$

$$\widetilde{y}'(0) = \widetilde{y}''(0) = \ldots = \widetilde{y}^{(n-2)}(0) = 0, \quad n \geq 3, \tag{7}$$

$$D^{p}_{0^+}\widetilde{y}(1) = \sum_{i=1}^{m} a_i D^{q}_{0^+}\widetilde{y}(\xi_i).$$

Therefore if

$$\widetilde{y}(t-\tau) + \overline{\eta}(t-\tau) - w(t-\tau) \geq 0 \text{ for } t \in [0,1]$$

then $\widetilde{y}^*(t-\tau) = \widetilde{y}(t-\tau) + \overline{\eta}(t-\tau) - w(t-\tau)$ and $y(t) = \widetilde{y}(t) + \overline{\eta}(t) - w(t)$.

Lemma 3.2. *y is a positive solution of the BVP (1)-(2) if and only if $\widetilde{y}(t) = y(t) + w(t) - \overline{\eta}(t)$ is a positive solution of the BVP (6)-(7) and inequality $\widetilde{y}(t) + w(t) - \overline{\eta}(t) \geq 0$ holds up when $t \in (0,1) - \{\tau\}$.*

Proof. If y is a positive solution of the BVP (1)-(2), we shall prove it in two cases.
For $t \in [-\tau, 0]$

$$\begin{aligned}\widetilde{y}(t) &= y(t) + w(t) - \overline{\eta}(t)\\ &= y(t) - \overline{\eta}(t)\\ &= \eta(t) - \eta(t)\\ &= 0\end{aligned}$$

which implies that $\widetilde{y}(t) = 0$. It is easy to show that $\widetilde{y}(t)$ satisfies the rest boundary conditions (7) when $t \in [-\tau, 0]$.
For $t \in (0,1) - \{\tau\}$

$$\begin{aligned}D^{\alpha}_{0^+}(y(t) + w(t) - \overline{\eta}(t)) &= D^{\alpha}_{0^+}y(t) + D^{\alpha}_{0^+}w(t) - D^{\alpha}_{0^+}\overline{\eta}(t)\\ &= D^{\alpha}_{0^+}y(t) + D^{\alpha}_{0^+}w(t)\\ &= -\lambda h(t)f(t, y(t-\tau)) - \lambda h(t)\rho(t)\\ &= -\lambda h(t)[f(t, y(t-\tau)) + \rho(t)]\\ &= -\lambda h(t)[f(t, \widetilde{y}^{*}(t-\tau)) + \rho(t)]\end{aligned}$$

which implies that

$$D^{\alpha}_{0^+}\widetilde{y}(t) = -\lambda h(t)[f(t, \widetilde{y}^{*}(t-\tau)) + \rho(t)].$$

Since $y(t)$ is a positive solution, then $\widetilde{y}(t) + \overline{\eta}(t) - w(t) \geq 0$ holds when $t \in (0,1) - \{\tau\}$. It is easy to show that $\widetilde{y}(t)$ satisfies the boundary conditions (7). Therefore, $\widetilde{y}(t)$ is a positive solution of BVP (6)-(7). On the other hand, if $\widetilde{y}(t) = y(t) + w(t) - \overline{\eta}(t)$ is a positive solution of the BVP (6)-(7) and $\widetilde{y}(t) + \overline{\eta}(t) - w(t) \geq 0$ holds when $t \in (0,1) - \{\tau\}$, as similar as the proof above, we can easily prove that $y(t)$ is a positive solution of the BVP (1)-(2).
As a result, we will concentrate our mind on finding the fixed points of operator T defined by (5). For the existence results we need the following assumptions, too.

(H2) $0 < \int_0^{\tau} J(s)\varphi_1(s)k(\eta(s-\tau))ds < +\infty,$

and there exists a constant $l > 0$ such that $\int_{\tau}^{1} J(s)\varphi_1(s)k(\frac{l}{2}(s-\tau)^{\alpha-1})ds < +\infty$.

(H3) Let

$$\lim_{z\to+\infty} \frac{l_1(z)}{z} \leq \phi, \quad \text{for } \phi > 0$$

such that ϕ satisfies $\lambda\phi \int_{\tau}^{1} \sigma h(s)\varphi_1(s)ds < 1$.

In view of (H3), there exists an $M > 0$ such that $l_1(z) \leq \phi z$ for all $z > M$.

(H4) There exists a subinterval $[d_1, d_2] \subset (\tau, 1)$ and a constant $r_1 \geq \max\{l, 2c\}$ such that

$$\lambda\xi_2 l_2(\frac{r_1\xi_1}{2}) \int_{d_1}^{d_2} J(s)h(s)\varphi_2(s)ds > r_1$$

where $\xi_1 := \min_{t\in[d_1,d_2]} (t-\tau)^{\alpha-1} = (d_1-\tau)^{\alpha-1}$, $\xi_2 := \min_{t\in[d_1,d_2]} t^{2n-2} = {d_1}^{2n-2}$ and
$c = \lambda \int_0^1 J(s)h(s)\rho(s)ds < +\infty$.

Let define

$$\delta_1 := \frac{\int_0^\tau \sigma h(s)\varphi_1(s)(k(\eta(s-\tau)) + l_1(\eta(s-\tau)))ds + \int_\tau^1 \sigma h(s)\varphi_1(s)(k(\frac{l}{2}(s-\tau)^{\alpha-1}))ds}{\frac{1}{\lambda} - \phi\int_\tau^1 \sigma h(s)\varphi_1(s)ds} > 0$$

and choose an $r_2 > \{M+1, r_1+1, \delta_1\}$. Define the open balls

$$\Omega_1 = \{y \in K : \|y\| < r_1\},$$
$$\Omega_2 = \{y \in K : \|y\| < r_2\}.$$

Lemma 3.3. *Let (H1)-(H2) hold. Then the operator $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ is completely continuous.*

Proof. First we show that T is well defined on $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$. For any $z \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$, we know that $r_1 \leq \| z \| \leq r_2$ and $z(t) \geq t^{\alpha-1}\|z\| \geq t^{\alpha-1}r_1$ for $t \in [0,1]$. Then, for $t \in [0,1]$, we get

$$\begin{aligned}
t^{2n-1-\alpha}w(t) &= t^{2n-1-\alpha}\int_0^1 G(t,s)\lambda h(s)\rho(s)ds \\
&\leq t^{2n-1-\alpha}\int_0^1 \lambda J(s)h(s)\rho(s)ds \\
&\leq t^{\alpha-1}\int_0^1 \lambda J(s)h(s)\rho(s)ds \\
&= t^{\alpha-1}c
\end{aligned}$$

where $c = \int_0^1 \lambda J(s)h(s)\rho(s)ds < +\infty$. Thus, for $t \in [0,1]$,

$$z(t) - t^{2n-1-\alpha}w(t) \geq t^{\alpha-1}(r_1 - c) \geq \frac{r_1}{2}t^{\alpha-1}.$$

In view of (H1), (H2) and Remark 3.1, we show

$$\begin{aligned}
Tz(t) &= \int_0^\tau G^*(t,s)\lambda h(s)(f(s,\eta(s-\tau)) + \rho(s))ds \\
&+ \int_\tau^1 G^*(t,s)\lambda h(s)(f(s,(s-\tau)^{\alpha-2n+1}z(s-\tau) - w(s-\tau)) + \rho(s))ds \\
&\leq \lambda t^{2n-1-\alpha}\int_0^\tau J(s)h(s)\varphi_1(s)[k(\eta(s-\tau)) + l_1(\eta(s-\tau))]ds \\
&+ \lambda t^{2n-1-\alpha}\int_\tau^1 J(s)h(s)\varphi_1(s)[k(\frac{r_1}{2}(s-\tau)^{\alpha-1}) + l_1(z(s-\tau) - (s-\tau)^{2n-1-\alpha}w(s-\tau))]ds \\
&\leq \lambda t^{2n-1-\alpha}\int_0^\tau J(s)h(s)\varphi_1(s)[k(\eta(s-\tau)) + l_1(\eta(s-\tau))]ds \\
&+ \lambda t^{2n-1-\alpha}\int_\tau^1 J(s)h(s)\varphi_1(s)[k(\frac{r_1}{2}(s-\tau)^{\alpha-1}) + l_1(z(s-\tau)]ds \\
&\leq \lambda t^{2n-1-\alpha}\int_0^\tau J(s)h(s)\varphi_1(s)[k(\eta(s-\tau)) + l_1(\eta(s-\tau))]ds \\
&+ \lambda t^{2n-1-\alpha}\int_\tau^1 J(s)h(s)\varphi_1(s)[k(\frac{l}{2}(s-\tau)^{\alpha-1}) + l_1(r_2)]ds \\
&\leq \lambda\int_0^\tau J(s)h(s)\varphi_1(s)[k(\eta(s-\tau)) + l_1(\eta(s-\tau))]ds \\
&+ \lambda\int_\tau^1 J(s)h(s)\varphi_1(s)[k(\frac{l}{2}(s-\tau)^{\alpha-1}) + l_1(r_2)]ds < +\infty.
\end{aligned}$$

Hence T is well defined and uniformly bounded.
In fact, for $z \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$, $t \in [0,1]$ in view of Remark 3.1, we have

$$Tz(t) \le \lambda t^{2n-1-\alpha} \int_0^1 J(s)h(s)(f(s, z^*(s-\tau)) + \rho(s))ds$$

and

$$\begin{aligned} Tz(t) &= \int_0^1 G^*(t,s)\lambda h(s)(f(s, z^*(s-\tau)) + \rho(s))ds \\ &\ge \lambda t^{2n-2} \int_0^1 J(s)h(s)(f(s, z^*(s-\tau)) + \rho(s))ds \\ &= \lambda t^{2n-\alpha-1} t^{\alpha-1} \int_0^1 J(s)h(s)(f(s, z^*(s-\tau)) + \rho(s))ds \\ &= t^{\alpha-1}\|Tz\|. \end{aligned}$$

Hence $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$.
Next we show $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ is continuous and compact. For any $z_n, z \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$, $n = 1, 2, ...$ with $\|z_n - z\|_{[-\tau,1]} \to 0$ as $n \to +\infty$. Since $r_1 \le \|z_n\| \le r_2$ and $r_1 \le \|z\| \le r_2$ for $t \in [0,1]$, we know

$$z_n(t) - t^{2n-\alpha-1}w(t) \ge \frac{r_1}{2} t^{\alpha-1}$$

and

$$z(t) - t^{2n-\alpha-1}w(t) \ge \frac{r_1}{2} t^{\alpha-1}.$$

Then for $t \in [0,1]$, we know

$$\begin{aligned} |Tz_n(t) - Tz(t)| &= \Big| \int_\tau^1 \lambda G^*(t,s)h(s)(f(s, (s-\tau)^{\alpha-2n+1}z_n(s-\tau) - w(s-\tau)) \\ &\quad -f(s, (s-\tau)^{\alpha-2n+1}z(s-\tau) - w(s-\tau)))ds \Big| \\ &\le \lambda \int_\tau^1 J(s)h(s)|f(s, (s-\tau)^{\alpha-2n+1}z_n(s-\tau) - w(s-\tau)) \\ &\quad - f(s, (s-\tau)^{\alpha-2n+1}z(s-\tau) - w(s-\tau)))|ds \\ &\le \lambda \int_\tau^1 J(s)h(s)\varphi(s)(k(\frac{l}{2}(s-\tau)^{\alpha-1}) + l_1(r_2))ds \\ &< +\infty. \end{aligned}$$

This implies that $|Tz_n(t) - Tz(t)|_{[-\tau,1]} \to 0$ as $n \to +\infty$. Therefore T is continuous. Since G^* is uniformly continuous for $t \in (0,1)$, that is, for any $\epsilon > 0$, there exists $\delta > 0$, when $t_1, t_2 \in [0,1]$ and $|t_1 - t_2| < \delta$ we have

$$\begin{aligned} |G^*(t_1,s) - G^*(t_2,s)| < \frac{\epsilon}{2} \Bigg(&\int_0^\tau \lambda h(s)\varphi_1(s)(k(\eta(s-\tau)) + l_1(\eta(s-\tau)))ds \\ &+ \int_\tau^1 \lambda h(s)\varphi_1(s)(k(\frac{l}{2}(s-\tau)^{\alpha-1}) + l_1(r_2))ds \Bigg)^{-1}. \end{aligned}$$

Thus for any $z \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$, we get

$$\begin{aligned}|Tz(t_1) - Tz(t_2)| \leq &\int_0^\tau \lambda|G^*(t_1,s) - G^*(t_2,s)|h(s)\varphi_1(s)(k(\eta(s-\tau)) + l_1(\eta(s-\tau)))ds \\ &+ \int_\tau^1 \lambda|G^*(t_1,s) - G^*(t_2,s)|h(s)\varphi_1(s)(k(\frac{l}{2}(s-\tau)^{\alpha-1}) + l_1(r_2))ds \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

Thus T is equicontinuous. According to Arzela- Ascoli Theorem, T is completely continuous.

Now, we will prove the existence of positive solutions for the BVP (1)-(2) by using the Guo-Krasnoselskii fixed point theorem.

Theorem 3.4. *Assume that the conditions (H1)-(H4) hold. Then the BVP (1)-(2) has at least one positive solution.*

Proof. Since $r_2 \geq \max\{M+1, r_1+1, \delta_1\}$, then for $z \in \partial\Omega_2$, for $t \in [0,1]$ we obtain

$$z(t) - t^{2n-\alpha-1}w(t) \geq t^{\alpha-1}(r_2 - c) \geq \frac{r_2}{2}t^{\alpha-1} \tag{8}$$

because of $z(t) \geq t^{\alpha-1}\|z\| = t^{\alpha-1}r_2$. Then from (H1)-(H3), (8) and Remark 2.4, we get

$$\begin{aligned}Tz(t) &\leq \lambda t^{2n-2}\int_0^\tau \sigma h(s)\varphi_1(s)(k(\eta(s-\tau)) + l_1(\eta(s-\tau)))ds \\ &+ \lambda t^{2n-2}\int_\tau^1 \sigma h(s)\varphi_1(s)(k(\frac{r_2}{2}(s-\tau)^{\alpha-1}) + l_1(z(s-\tau)))ds \\ &\leq \lambda\int_0^\tau \sigma h(s)\varphi_1(s)(k(\eta(s-\tau)) + l_1(\eta(s-\tau)))ds \\ &+ \lambda\int_\tau^1 \sigma h(s)\varphi_1(s)(k(\frac{l}{2}(s-\tau)^{\alpha-1}) + l_1(r_2))ds \\ &\leq \lambda\int_0^\tau \sigma h(s)\varphi_1(s)(k(\eta(s-\tau)) + l_1(\eta(s-\tau)))ds \\ &+ \lambda\int_\tau^1 \sigma h(s)\varphi_1(s)(k(\frac{l}{2}(s-\tau)^{\alpha-1}) + \phi r_2)ds \\ &< r_2 = \|z\|.\end{aligned}$$

Therefore, for $z \in \partial\Omega_2$, we have $\|Tz\| \leq \|z\|$. On the other hand, for $z \in \partial\Omega_1$, and $t \in [0,1]$, we have

$$z(t) - t^{2n-\alpha-1}w(t) \geq t^{\alpha-1}(r_1 - c) \geq \frac{r_1}{2}t^{\alpha-1}. \tag{9}$$

Thus from (H1), (H4), (9) and Remark 2.4, we get

$$\begin{aligned}Tz(t) &\geq \lambda\int_{d_1}^{d_2} \min_{t\in[d_1,d_2]} G^*(t,s)h(s)(f(s,(s-\tau)^{\alpha-2n+1}z(s-\tau) - w(s-\tau)) + \rho(s))ds \\ &\geq \lambda\int_{d_1}^{d_2} \min_{t\in[d_1,d_2]} G^*(t,s)h(s)\varphi_2(s)l_2(\frac{r_1}{2}(s-\tau)^{\alpha-1})ds \\ &\geq \lambda l_2(\frac{r_1\xi_1}{2})\xi_2\int_{d_1}^{d_2} J(s)h(s)\varphi_2(s)ds \\ &> r_1 = \|z\|.\end{aligned}$$

Therefore, for $z \in \partial\Omega_1$, we have $\|Tz\| \geq \|z\|$.
Then T defined by (5) has a fixed point $\widetilde{z} \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$ from Theorem 2.7. In view of (9) we have

$$t^{\alpha-2n+1}\widetilde{z}(t) - w(t) = t^{\alpha-2n+1}(\widetilde{z}(t) - t^{2n-\alpha-1}w(t)) \geq t^{\alpha-2n+1}\frac{r_1}{2}t^{\alpha-1} \geq \frac{r_1}{2}t^{2\alpha-2n} = \frac{r_1}{2}t^{2(\alpha-n)} > 0.$$

The proof is completed.
Now, we'll present another result of this work, we give the following conditions:

(H5) There exists a subinterval $[d_3, d_4] \subset (\tau, 1)$ such that $\lambda \int_{d_3}^{d_4} J(s)h(s)\varphi_2(s)ds > 0$ and let

$$\lim_{z\to+\infty} \frac{l_2(z)}{z} \geq N, \text{ for } N > 0$$

such that N satisfies

$$\frac{\lambda N}{2}\mu_1\mu_2 \int_{d_3}^{d_4} J(s)h(s)\varphi_2(s)ds > 1$$

where $\mu_1 := \min_{t\in[d_3,d_4]}(t-\tau)^{\alpha-1} = (d_3-\tau)^{\alpha-1}$, $\mu_2 := \min_{t\in[d_3,d_4]} t^{2n-2} = {d_3}^{2n-2}$.

(H6) There exists an $R_1 \geq \max\{l, 2c\}$ such that

$$\lambda \int_0^{\tau} \sigma h(s)\varphi_1(s)(k(\eta(s-\tau)) + l_1(\eta(s-\tau)))ds + \lambda \int_{\tau}^{1} \sigma h(s)\varphi_1(s)(k(\frac{l}{2}(s-\tau)^{\alpha-1})) + l_1(R_1)ds \leq R_1$$

where l is in (H2) and c is in (H4).

Next we choose an $R_2 \geq \max\{R_1+1, N+1\}$ and define the open balls

$$\Omega_3 = \{y \in K : \|y\| < R_1\},$$
$$\Omega_4 = \{y \in K : \|y\| < R_2\}.$$

Lemma 3.5. *Let (H1)-(H2) hold. Then the operator $T : K \cap (\overline{\Omega}_4 \setminus \Omega_3) \to K$ is completely continuous.*

Proof. Similarly, as in Lemma 3.3, we can easily see that T is well defined and equicontinuous on $K\cap(\overline{\Omega}_4\setminus\Omega_3)$. Thus, according to Arzela-Ascoli Theorem, T is completely continuous.

Theorem 3.6. *Assume that the conditions (H1)-(H2), (H5) and (H6) hold. Then the BVP (1)-(2) has at least one positive solution.*

Proof. Since $R_2 \geq \max\{R_1+1, N+1\}$, then for $z \in \partial\Omega_4$, for $t \in [0,1]$ we obtain

$$z(t) - t^{2n-\alpha-1}w(t) \geq t^{\alpha-1}(R_2 - c) \geq \frac{R_2}{2}t^{\alpha-1}$$

because of $z(t) \geq t^{\alpha-1}\|z\| = t^{\alpha-1}R_2$.

As a result of this and (H1), (H2), (H5) and Remark 2.4, we have

$$\begin{aligned}
Tz(t) &\geq \lambda \int_{d_3}^{d_4} \min_{t\in[d_3,d_4]} G^*(t,s)h(s)(f(s,(s-\tau)^{\alpha-2n+1}z(s-\tau) - w(s-\tau)) + \rho(s))ds \\
&\geq \lambda \int_{d_3}^{d_4} \min_{t\in[d_3,d_4]} G^*(t,s)h(s)\varphi_2(s)l_2(\frac{R_2}{2}(s-\tau)^{\alpha-1})ds \\
&\geq \lambda \int_{d_3}^{d_4} \min_{t\in[d_3,d_4]} G^*(t,s)h(s)\varphi_2(s)l_2(\frac{R_2}{2}\mu_1)ds \\
&\geq \lambda N\frac{R_2}{2}\mu_1\mu_2 \int_{d_3}^{d_4} J(s)h(s)\varphi_2(s)ds \\
&> R_2 = \|z\|.
\end{aligned}$$

Therefore, for $z \in \partial\Omega_4$, we have $\|Tz\| \geq \|z\|$.
On the other hand, for $z \in \partial\Omega_4$, for $t \in [0,1]$ we obtain

$$z(t) - t^{2n-\alpha-1}w(t) \geq t^{\alpha-1}(R_1 - c) \geq \frac{R_1}{2}t^{\alpha-1}. \tag{10}$$

With condition (H6) and (10), we get

$$\begin{aligned} Tz(t) &\leq \lambda t^{2n-2}\int_0^\tau \sigma h(s)\varphi_1(s)(k(\eta(s-\tau)) + l_1(\eta(s-\tau)))ds \\ &+ \lambda t^{2n-2}\int_\tau^1 \sigma h(s)\varphi_1(s)(k(\frac{R_1}{2}(s-\tau)^{\alpha-1}) + l_1(z(s-\tau)))ds \\ &\leq \lambda \int_0^\tau \sigma h(s)\varphi_1(s)(k(\eta(s-\tau)) + l_1(\eta(s-\tau)))ds \\ &+ \lambda \int_\tau^1 \sigma h(s)\varphi_1(s)(k(\frac{l}{2}(s-\tau)^{\alpha-1}) + l_1(R_1))ds \\ &< R_1 = \|z\|. \end{aligned}$$

Therefore, for $z \in \partial\Omega_3$, we have $\|Tz\| \leq \|z\|$. Thus, T has a fixed point in $K \cap (\overline{\Omega}_4 \setminus \Omega_3)$ by Theorem 2.7. Arguments similar to those at the end of the proof of Theorem 3.4 show that BVP (1)-(2) has a positive solution.

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