Filomat 37:21 (2023), 7287–7302
https://doi.org/10.2298/FIL2321287B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Self-inversive polynomials and quasi-orthogonality on the unit circle

Kiran Kumar Behera[a]

[a] *School of Mathematical Sciences, NISER Bhubaneswar, India*

Abstract. In this paper we study quasi-orthogonality on the unit circle based on the structural and orthogonal properties of a class of self-invariant polynomials. We discuss a special case in which these polynomials are represented in terms of the reversed Szegő polynomials of consecutive degrees and illustrate the results using contiguous relations of hypergeometric functions. This work is motivated partly by the fact that recently cases have been made to establish para-orthogonal polynomials as the unit circle analogues of quasi-orthogonal polynomials on the real line so far as spectral properties are concerned. We show that structure wise too there is great analogy when self-inversive polynomials are used to study quasi-orthogonality on the unit circle.

1. Introduction

The concept of quasi-orthogonality of polynomials was introduced by Riesz and later studied, among others, by Fejèr, Shohat and Chihara in relation to moment problems and associated quadrature formulae. If $\{P_n(x)\}_{n=0}^{\infty}$ is a sequence of polynomials orthogonal with respect to a positive weight function $w(x)$ on the real line $\mathbb{R}$, then a necessary and sufficient condition [7] for a polynomial $Q_n(x)$ to be quasi-orthogonal of order r with respect to $w(x)$ is

$$Q_n(x) = c_{n,0}P_n(x) + c_{n,1}P_{n-1}(x) + \cdots + c_{n,r}P_{n-r}(x), \quad n \geq r, \tag{1}$$

where $c_{n,i} \in \mathbb{R}$ and $c_{n,0}c_{n,r} \neq 0$. The relation (1) justifies the name quasi-orthogonality: $Q_n(x)$ is orthogonal to every polynomial of degree not exceeding $n-r$ with respect to $w(x)$. This can be equivalently stated as

$$\int_a^b x^k Q_n(x)w(x)dx = \begin{cases} 0 & \text{if } k = 0, \cdots, n-r-1, \\ h_n \neq 0 & \text{if } k = n-r, \end{cases} \tag{2}$$

where we have the standard orthogonality on $\mathbb{R}$ for $r = 0$. We refer to [7, 11, 19, 20, 23, 27, 29] and references therein for this classical theory and its applications.

There have been attempts to generalize the concept of quasi-orthogonality from $\mathbb{R}$ to the unit circle $\partial\mathbb{D}$ via the relations (1) and (2). However, it was proved that if $\{P_n(z)\}$ is a sequence of polynomials orthogonal on the unit circle, also called Szegő polynomials [30, 33], then the necessary and sufficient condition [28,

2020 *Mathematics Subject Classification*. Primary 42C05; Secondary 33C45
Keywords. Self-inversive polynomials, quasi-orthogonality
Received: 21 December 2022; Revised: 09 March 2023; Accepted: 17 March 2023
Communicated by Fuad Kittaneh
Research supported by the Institute postdoctoral fellowship provided by NISER Bhubaneswar, India
Email address: kkb@niser.ac.in (Kiran Kumar Behera)

Theorem 1] (see also [1, 5]) for the polynomial $Q_n(z)$ given by (1) to be orthogonal with respect to a non-trivial positive measure on the unit circle is that $Q_n(z)$ belongs to the class of Bernstein-Szegő polynomials. Hence, a new concept of quasi-orthogonality is defined in [1] which is dependent on the structure of the Szegő polynomials and related semi-classical forms are discussed.

The Szegő polynomials find many applications in areas like approximation on the complex plane $\mathbb{C}$, prediction theory and signal processing [15, 18, 22, 24**?**]. However, they suffer from the major drawback that their zeros lie outside the support of the measure and hence cannot be used in interpolation processes on the unit circle. This was overcome by introducing para-orthogonal polynomials which are self-invariant polynomials with symmetric orthogonality conditions. The polynomial $\mathcal{P}_n(z)$ is a self-invariant polynomial if and only if it satisfies $\mathcal{P}_n^*(z) = \tau_n \mathcal{P}_n(z)$, where $\tau_n \in \partial\mathbb{D}$ and $\mathcal{P}_n^*(z) = z^n \overline{\mathcal{P}_n(1/\bar{z})}$. This invariance property leads to the symmetry that $\mathcal{P}_n(z)$ is orthogonal to the monomial z^k if and only if $\mathcal{P}_n(z)$ is orthogonal to the monomial z^{n-k} for $k = 0, \cdots, n$. Hence, if one defines the spaces

$$\Lambda_{n,2l+1} = \text{span}\{z^k : k = l+1, \cdots, n-l-1\}, \quad 0 \le l \le \left\lfloor \frac{n}{2} \right\rfloor - 1,$$

then the invariant polynomials lying in $\Lambda_{n,1}^{\perp}$ ($l = 0$) are precisely the para-orthogonal polynomials. These are introduced in [25] to study quadrature formula on the unit circle and in the solution of the trigonometric moment problem. They are also used to adapt many spectral properties developed on the real line to the unit circle [6, 9, 21]

In view of this, cases have been made, see for instance [8], to establish para-orthogonal polynomials as the counterpart of quasi-orthogonal polynomials on $\mathbb{R}$ so far as spectral properties are concerned. In particular, the classical concept of para-orthogonality is generalized [8] to the concept of quasi para-orthogonal polynomials of order $2l+1$, which is precisely the set of invariant polynomials of degree n lying in the space $\Lambda_{n,2l+1}^{\perp}$. The prefix quasi refers to the fact that for fixed degree n, these polynomials satisfy fewer (symmetric) orthogonality conditions.

Motivated by the recent assertions [8] that invariance of polynomials should have a role in the unit circle analogue of quasi-orthogonality on $\mathbb{R}$, we study quasi-orthogonality on $\partial\mathbb{D}$ based on the structure of a class of self-invariant polynomials. Such polynomials satisfy a three term recurrence relation of the form

$$\tilde{R}_{n+1}(z) = (\bar{\beta}_n z + \beta_n)\tilde{R}_n(z) - z\tilde{R}_{n-1}(z), \quad n \ge 0, \tag{3}$$

and are introduced by Delsarte and Genin [13–17] in order to solve certain problems in digital signal processing. Hence, we will refer to this class as the $\mathcal{DG}$ class of invariant polynomials. This class is also used to characterize a family of non-trivial measures on the unit circle along with the corresponding families of para-orthogonal polynomials and the Szegő polynomials [10, 12]. Orthogonality of quasi-orthogonal polynomials on the real line is recently studied [4]. In a way, the following results may be viewed as exploring the quasi-orthogonality properties, of order one, of para-orthogonal polynomials. The key to our study is the following representation

$$(z-1)\mathcal{P}_n(z) = \varphi_{n+1}^*(z) + c_{n+1}\varphi_n^*(z), \quad n \ge 1, \tag{4}$$

that we obtain for any polynomial $\mathcal{P}_n(z)$ in $\mathcal{DG}$ class, where $\varphi_n^*(z)$ is the monic reversed Szegő polynomial of degree n. We present an overview of the primary results that are obtained in this context. We show that given any complex parameter ζ lying on the unit circle, one can begin with the polynomials $\mathcal{B}_n(z)$, $n \ge 0$, such that

- $\mathcal{B}_n(z)$, $n \ge 0$, satisfies a three term recurrence relation with appropriately chosen parameters.
- The polynomials $\{\mathcal{P}_n(z)\}_{n=0}^{\infty}$ defined by

$$\mathcal{P}_0^{(\zeta)}(z) := 1, \quad \mathcal{P}_n^{(\zeta)}(z) = \frac{1}{z-\zeta}\left(\mathcal{B}_{n+1}(z) + \omega_{n+1}(\zeta)\mathcal{B}_n(z)\right), \quad n \ge 1, \tag{5}$$

  for some $\omega_n(\zeta) \in \mathbb{C}$ belong to the $\mathcal{DG}$ class of invariant polynomials.

The polynomials $\mathcal{P}_n(z)$ given by (4) are obtained from $\mathcal{P}_n^{(\zeta)}(z)$ for $\zeta = 1$ and have been studied as a class of para-orthogonal polynomials. Our use of $\mathcal{DG}$ class serves two purposes. First, the representation (4) is similar to (1) for $r = 1$ and hence $\mathcal{P}_n(z)$ is at least structurally similar to a quasi-orthogonal polynomial of order 1 on the real line. Second, we show that $\mathcal{P}_n^{(\zeta)}(z)$, $n \geq 1$, satisfy a three term recurrence relation, a result that is known for a quasi-orthogonal polynomial of order 1 on the real line [11, 20]. In addition, we prove that if $\mathcal{P}_n(z)$ is quasi-orthogonal with respect to the functional $\mathfrak{v}$ and satisfy symmetric orthogonality conditions with respect to the functional $\mathfrak{u}$, then

$$\mathfrak{v} = [z^{-1}\mathcal{U}(z) + z\bar{\mathcal{U}}(z^{-1})]\mathfrak{u}, \quad \mathcal{U}(z) = \frac{1}{2}(c_{s-1}z^s + \cdots + c_1z^2 + c_0z + \bar{c}_1), \tag{6}$$

where $c_{s-1} \neq 0$, that is $\mathcal{U}(z)$ is a polynomial of exact degree s.

The manuscript is organized as follows. In Section 2, we obtain preliminary structural relations that lead to the proof of (5). In particular, we give an algorithm to generate the sequence $\omega_n(\zeta)$, $n \geq 2$, with a special choice of expression for $\omega_1(\zeta)$. In Section 3, we identify that the class of reversed Szegő polynomials can be used as $\mathcal{B}_n(z)$ in the representation (5). We prove a Szegő type relation that any polynomial in the $\mathcal{DG}$ class satisfies. We state our definition of quasi-orthogonality on $\partial\mathbb{D}$ in Section 4 and use this to obtain the characterization (6). We conclude with Section 5 in which we illustrate the theory presented in the paper using contiguous hypergeometric relations.

2. Structural Relations

Let $\{\mathcal{B}_n(z)\}_{n=0}^{\infty}$ be a sequence of polynomials given by the three term recurrence relation

$$\mathcal{B}_{n+1}(z) = (z + \sigma_{n+1})\mathcal{B}_n(z) - \lambda_{n+1}z\mathcal{B}_{n-1}(z), \quad n \geq 0, \tag{7}$$

with $\mathcal{B}_{-1}(z) = 0$ and $\mathcal{B}_0(z) = 1$. Even though λ_1 does not affect (7), we fix $\lambda_1 \in \mathbb{C} \setminus \{0\}$ and consider the complex parameters σ_n and λ_n, $n \geq 1$. A characterization of such polynomials is that if $\sigma_n \neq 0$ and $\lambda_{n+1} \neq 0$, $n \geq 1$, then there exists [10, 12] a quasi-definite moment functional $\mathcal{M}$ defined on the space of Laurent polynomials such that

$$\mathcal{M}[z^{-k} \cdot \mathcal{B}_n(z)] = \delta_{n,k}\frac{\lambda_2 \cdots \lambda_{n+1}}{\sigma_2 \cdots \sigma_{n+1}}, \quad k = 0, 1, \cdots, n, \quad n \geq 1.$$

Further, with specific choices for σ_n and λ_{n+1}, the recurrence relation (7) can be transformed into the form (3) satisfied by $\mathcal{B}_n(z)$ after a scaling.

Let $\zeta \in \mathbb{C} \setminus \{0\}$ be fixed and $z \in \mathbb{C} \setminus \{\zeta\}$. We consider the polynomial sequence $\{\mathcal{Q}_n^{(\zeta)}(z)\}_{n=0}^{\infty}$ defined by

$$\mathcal{Q}_0^{(\zeta)}(z) := 1, \quad \mathcal{Q}_n^{(\zeta)}(z) := \mathcal{B}_n(z) + \omega_n(\zeta)\mathcal{B}_{n-1}(z), \quad n \geq 1, \tag{8}$$

where $\{\omega_n(\zeta)\}_{n=1}^{\infty}$ is a sequence of complex numbers depending on ζ.

The next result is partly motivated by [20], precisely the fact that quasi-orthogonal polynomials on $\mathbb{R}$ satisfy a three term recurrence relation with polynomial coefficients.

Lemma 2.1. *Suppose $\omega_n(\zeta)$, $n \geq 2$, is defined recursively as*

$$\omega_n(\zeta) = -\sigma_n - \zeta\left(1 + \frac{\lambda_n}{\omega_{n-1}(\zeta)}\right) \quad n \geq 2, \tag{9}$$

where $\omega_1(\zeta)$ is arbitrary. Then $\{\mathcal{Q}_{n+1}^{(\zeta)}(z)\}_{n=2}^{\infty}$ satisfy a three term recurrence relation with initial conditions defined for $\mathcal{Q}_1^{(\zeta)}(z)$ and $\mathcal{Q}_2^{(\zeta)}(z)$.

Proof. For $n \geq 1$, consider the system

$$\begin{pmatrix} 0 & 0 & 0 & \omega_{n+1}(\zeta) & 1 \\ 0 & 0 & \omega_n(\zeta) & 1 & 0 \\ -1 & \omega_{n-1}(\zeta) & 1 & 0 & 0 \\ 0 & 0 & -\lambda_{n+1}z & z+\sigma_{n+1} & -1 \\ 0 & -\lambda_n z & z+\sigma_n & -1 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{Q}^{(\zeta)}_{n-1}(z) \\ \mathcal{B}_{n-2}(z) \\ \mathcal{B}_{n-1}(z) \\ \mathcal{B}_n(z) \\ \mathcal{B}_{n+1}(z) \end{pmatrix} = \begin{pmatrix} \mathcal{Q}^{(\zeta)}_{n+1}(z) \\ \mathcal{Q}^{(\zeta)}_{n}(z) \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{10}$$

Solving for the first variable $\mathcal{Q}^{(\zeta)}_{n-1}(z)$, we obtain

$$(u_n z + v_n)\mathcal{Q}^{(\zeta)}_{n+1}(z) = (u_n z^2 + s_n z - t_n)\mathcal{Q}^{(\zeta)}_{n}(z) - \lambda_n(u_{n+1}z + v_{n+1})z\mathcal{Q}^{(\zeta)}_{n-1}(z), \tag{11}$$

for $n \geq 1$, with $\mathcal{Q}^{(\zeta)}_0(z) = 1$, $\mathcal{Q}^{(\zeta)}_1(z) = z + \sigma_1 + \omega_1(\zeta)$. The parameters involved in (11) are given by

$$u_n := u_n(\zeta) = \omega_{n-1}(\zeta) + \lambda_n, \quad s_n := s_n(\zeta) = \omega_n^{-1}(\zeta)u_n v_{n+1} + v_n - \omega_{n-1}(\zeta)u_{n+1},$$
$$v_n := v_n(\zeta) = \omega_{n-1}(\zeta)(\omega_n(\zeta) + \sigma_n), \quad t_n := t_n(\zeta) = \omega_n^{-1}(\zeta)\omega_{n-1}(\zeta)\sigma_n v_{n+1}.$$

However, with $\omega_n(\zeta)$ defined as in (9), it can be seen that $u_n\zeta + v_n = 0$, $n \geq 2$, which also yields that $z - \zeta$ is a factor of $u_n z^2 + s_n z - t_n$. Hence cancelling the common factor $z - \zeta$ from (11) (which is possible since $z \in \mathbb{C} \setminus \{\zeta\}$), we obtain the simplified form

$$\mathcal{Q}^{(\zeta)}_{n+1}(z) = \left(z + \frac{u_{n+1}\sigma_n\omega_{n-1}(\zeta)}{u_n\omega_n(\zeta)}\right)\mathcal{Q}^{(\zeta)}_{n}(z) - \frac{u_{n+1}}{u_n}\lambda_n z\mathcal{Q}^{(\zeta)}_{n-1}(z), \quad n \geq 2, \tag{12}$$

with the initial conditions

$$\mathcal{Q}^{(\zeta)}_1(z) = z + \sigma_1 + \omega_1(\zeta) \quad \text{and} \quad \mathcal{Q}^{(\zeta)}_2(z) = (z + \sigma_2 + \omega_2(\zeta))(z + \sigma_1) - \lambda_2 z, \tag{13}$$

found by direct computations from (8). □

We emphasize that the parameters appearing in (11) depend on the way $\omega_n(\zeta)$ is defined for $n \geq 2$. Let us now choose $\omega_1(\zeta) = -(\sigma_1 + \zeta)$. With this initial choice $\omega_n(\zeta)$, $n \geq 2$, can be uniquely generated from (9). Further, $\mathcal{Q}^{(\zeta)}_1(z) = z - \zeta$ and $\mathcal{Q}^{(\zeta)}_2(z)$ vanishes at $z = \zeta$. Writing $\mathcal{Q}^{(\zeta)}_2(z) = (z - \zeta)(z + y)$, we find that $y = \sigma_1 + \sigma_2 + \lambda_2 + \omega_2(\zeta) - \zeta$ and $-\zeta y = \sigma_1(\sigma_2 + \omega_2(\zeta))$. Thus we have

$$\mathcal{Q}^{(\zeta)}_2(z) = (z - \zeta)\left(z - \frac{\sigma_1(\sigma_2 + \omega_2(\zeta))}{\zeta}\right).$$

We now show that $\mathcal{Q}^{(\zeta)}_i(z)$, $i = 0, 1, 2$, also satisfy a three term recurrence relation but different from (12). There are various ways to do this. We choose $\omega_0(\zeta) = -\lambda_1$ so that $u_1 = 0$ and then divide (11) by $v_1 = \lambda_1\zeta$. Or, with y_1, y_2, y_3 to be determined, we write

$$\mathcal{Q}^{(\zeta)}_2(z) = y_1(z + y_2)\mathcal{Q}^{(\zeta)}_1(z) - y_3 z(z - \zeta)\mathcal{Q}^{(\zeta)}_0(z).$$

Canceling the factor $z - \zeta$, we find $y_1 - y_3 = 1$ and $y_1 y_2 = -\sigma_1(\sigma_2 + \omega_2(\zeta))\zeta^{-1}$. We put $y_1 = \pi_2 \in \mathbb{C} \setminus \{0\}$ so that $y_3 = \pi_2 - 1$ and $y_2 = -\sigma_1(\sigma_2 + \omega_2(\zeta))\zeta^{-1}x_2^{-1}$. With this we have

$$\mathcal{Q}^{(\zeta)}_2(z) = \pi_2\left(z - \frac{\sigma_1(\sigma_2 + \omega(\zeta))}{\pi_2\zeta}\right)\mathcal{Q}^{(\zeta)}_1(z) - (\pi_2 - 1)z(z - \zeta)\mathcal{Q}^{(\zeta)}_0(z).$$

Remark 2.2. *Since $\mathcal{Q}^{(\zeta)}_1(z) = \mathcal{Q}^{(\zeta)}_2(z) = 0$ at $z = \zeta$, from (12) we find that $\mathcal{Q}^{(\zeta)}_n(z) = 0$ at the excluded point $z = \zeta$ for $n \geq 1$. This construction depends on the unique choice of $\omega_1(\zeta)$ which makes both $\mathcal{Q}^{(\zeta)}_1(z)$ and $\mathcal{Q}^{(\zeta)}_2(z)$ vanish at $z = \zeta$. We would like to add that such mixed recurrence relations were also obtained for $\mathcal{Q}^{(\zeta)}_n(z)$ in [3], though for $\zeta = 1$.*

This motivates us to define a new polynomial sequence $\{\mathcal{P}_n^{(\zeta)}(z)\}_{n=0}^{\infty}$, where

$$\mathcal{P}_n^{(\zeta)}(z) = \frac{\mathcal{Q}_{n+1}^{(\zeta)}(z)}{z-\zeta} = \frac{\mathcal{B}_{n+1}(z) + \omega_{n+1}(\zeta)\mathcal{B}_n(z)}{z-\zeta}, \quad n \geq 0,$$

so that $\mathcal{P}_0^{(\zeta)}(z) = 1$ and $\mathcal{P}_1^{(\zeta)}(z) = z - \sigma_1(\sigma_2 + \omega_2(\zeta))\zeta^{-1}$. Then, (12) gives

$$\mathcal{P}_{n+1}^{(\zeta)}(z) = \left(z + \frac{u_{n+2}\sigma_{n+1}\omega_n(\zeta)}{u_{n+1}\omega_{n+1}(\zeta)}\right)\mathcal{P}_n^{(\zeta)}(z) - \lambda_{n+1}\frac{u_{n+2}}{u_{n+1}}z\mathcal{P}_{n-1}^{(\zeta)}(z), \quad n \geq 1, \tag{14}$$

with the initial conditions as mentioned above. The form (14) is more convenient to work with since it removes the ambiguity associated with the recurrence relation involving $\mathcal{Q}_i^{(\zeta)}(z)$, $i = 0, 1, 2$, as observed above.

The next result is crucial in the sense that it helps us introduce invariant polynomials into our analysis. However, as is the case, we need to impose conditions on the parameters $\{\sigma_n\}$ and $\{\lambda_n\}$ appearing in (7). We do so by first choosing σ_n such that $|\sigma_1 \cdots \sigma_k| > 1$ for $k \geq 1$. Thereafter, we choose λ_n such that

$$\frac{\lambda_{n+1}}{\sigma_n} = 1 - \left|\frac{\zeta}{\sigma_1 \cdots \sigma_n}\right|^2, \quad n \geq 1.$$

We also restrict $\zeta \in \mathbb{C}$ such that $|\zeta| = 1$.

Theorem 2.3. *Define the sequences* $\{\hat{\tau}_n(\zeta)\}_{n=0}^{\infty}$ *and* $\{\hat{\alpha}_{n-1}\}_{n=1}^{\infty}$ *where* $\tau_0(\zeta) = 1$,

$$\hat{\tau}_n(\zeta) = \frac{u_{n+1}}{\omega_n(\zeta)}\sigma_1 \cdots \sigma_n \quad \text{and} \quad \hat{\alpha}_{n-1} = -\frac{u_{n+1}}{\hat{\tau}_n(\zeta)\omega_n(\zeta)}, \quad n \geq 1. \tag{15}$$

Then the polynomials $\mathcal{P}_n^{(\zeta)}(z)$, $n \geq 1$, *satisfy a three term recurrence relation of the form*

$$\mathcal{P}_{n+1}^{(\zeta)}(z) = (z + b_{n+1}(\zeta))\mathcal{P}_n^{(\zeta)}(z) - a_{n+1}(\zeta)z\mathcal{P}_{n-1}^{(\zeta)}(z), \quad n \geq 1, \tag{16}$$

with $\mathcal{P}_0^{(\zeta)}(z) = 1$, $\mathcal{P}_1^{(\zeta)}(z) = z + b_1(\zeta)$ *and where*

$$b_n(\zeta) = \frac{\hat{\tau}_n(\zeta)}{\hat{\tau}_{n-1}(\zeta)} \quad \text{and} \quad a_{n+1}(\zeta) = [1 + \hat{\tau}_n(\zeta)\hat{\alpha}_{n-1}][1 - \overline{\zeta\hat{\tau}_n(\zeta)\hat{\alpha}_n}]\zeta, \quad n \geq 1.$$

Proof. With the expression for u_{n+1} obtained in Lemma 2.1, we can also write

$$\hat{\tau}_n(\zeta) = -\frac{\omega_{n+1}(\zeta) + \sigma_{n+1}}{\zeta}\sigma_1 \cdots \sigma_n \quad \text{and} \quad \hat{\alpha}_{n-1} = -\frac{1}{\sigma_1 \cdots \sigma_n}, \quad n \geq 1.$$

This gives $\lambda_{n+1} = \sigma_n(1 - |\hat{\alpha}_{n-1}|^2)$, $n \geq 1$. Further, from (15)

$$\hat{\tau}_n(\zeta)\hat{\alpha}_{n-1} = -\frac{u_{n+1}}{\omega_n(\zeta)} \implies 1 + \hat{\tau}_n(\zeta)\hat{\alpha}_{n-1} = 1 - \frac{\omega_n(\zeta) + \lambda_{n+1}}{\omega_n(\zeta)} = -\frac{\lambda_{n+1}}{\omega_n(\zeta)},$$

while with similar computations

$$1 - \zeta\hat{\tau}_{n-1}(\zeta)\hat{\alpha}_{n-1} = 1 + \frac{\zeta}{\sigma_n}\left(1 + \frac{\lambda_n}{\omega_{n-1}(\zeta)}\right) = -\frac{\omega_n(\zeta)}{\sigma_n}$$

Thus, we arrive at

$$[1 + \hat{\tau}_n(\zeta)\hat{\alpha}_{n-1}][1 - \zeta\hat{\tau}_{n-1}(\zeta)\hat{\alpha}_{n-1}] = \frac{\lambda_{n+1}}{\sigma_n} = 1 - |\hat{\alpha}_{n-1}|^2, \quad n \geq 1,$$

which can be simplified to give the relation

$$\hat{\tau}_{n+1}(\zeta) = \frac{\zeta\hat{\tau}_n(\zeta) - \bar{\hat{\alpha}}_n}{1 - \zeta\hat{\tau}_n(\zeta)\hat{\alpha}_n}, \quad n \geq 0, \quad \text{with} \quad \hat{\tau}_0(\zeta) = 1. \tag{17}$$

Since $|\zeta| = 1$ and $|\hat{\alpha}_n| < 1$, we have $|\hat{\tau}_n| = 1$ for $n \geq 0$. The following relation

$$[1 + \hat{\tau}_n(\zeta)\hat{\alpha}_{n-1}][1 - \zeta\hat{\tau}_n(\zeta)\hat{\alpha}_n] = \frac{\lambda_{n+1}}{\sigma_{n+1}}\frac{\omega_{n+1}(\zeta)}{\omega_n(\zeta)}, \quad n \geq 1, \tag{18}$$

can also be easily derived, on either side of which we multiply

$$\frac{\hat{\tau}_{n+1}(\zeta)}{\hat{\tau}_n(\zeta)} = \frac{\omega_n(\zeta)}{\omega_{n+1}(\zeta)}\frac{u_{n+2}}{u_{n+1}}\sigma_{n+1}, \quad n \geq 1.$$

The right hand side of (18) is $\frac{\lambda_{n+1}u_{n+2}}{u_{n+1}}$. Further, from (17), we have

$$\frac{\hat{\tau}_{n+1}(\zeta)}{\hat{\tau}_n(\zeta)}[1 - \zeta\hat{\tau}_n(\zeta)\hat{\alpha}_n] = [1 - \overline{\zeta\hat{\tau}_n(\zeta)\hat{\alpha}_n}]\zeta,$$

which used in the left hand side of (18) gives

$$\frac{\lambda_{n+1}u_{n+2}}{u_{n+1}} = [1 + \hat{\tau}_n(\zeta)\hat{\alpha}_{n-1}][1 - \overline{\zeta\hat{\tau}_n(\zeta)\hat{\alpha}_n}], \quad n \geq 1.$$

The proof is complete by comparing the above expressions with (14). □

That $\mathcal{P}_n^{(\zeta)}(z)$ is self-inversive and satisfies $\hat{\tau}_n\mathcal{P}_n^{(\zeta)*}(z) = \mathcal{P}_n^{(\zeta)}(z)$ can be seen by inverting the relation (16) and using (17). Further, the recurrence relation (16) appears in the theory of para-orthogonal polynomials that are obtained from the Christoffel-Darboux (CD) kernels $K_n(w; z)$ for $|w| = 1$ and is satisfied by the monic form of $K_n(w; z)$ [9, 12]. We refer the reader to [31] for a recent survey of applications of the CD kernels in the spectral theory of orthogonal polynomials.

3. A special case: the reversed Szegő polynomials

Our goal in this section is to identify a class of polynomials that satisfies the recurrence relation (7) along with the conditions that we have imposed on σ_n and λ_{n+1}, $n \geq 1$. To begin with, we are concerned only with the recurrence relation (16) and not how we arrived at it. Then, we use the theory that associates with (16), a non trivial measure on the unit circle and the corresponding orthogonal polynomials.

Restricting the value of ζ to be 1, a key role is played by the scaled polynomials

$$R_n(z) = \frac{\prod_{j=0}^{n-1}[1 - \hat{\tau}_j\hat{\alpha}_j]}{\prod_{j=0}^{n-1}[1 - \mathrm{Re}(\hat{\tau}_j\hat{\alpha}_j)]}\mathcal{P}_n^{(1)}(z), \quad \hat{\tau}_n := \hat{\tau}_n(1), \quad n \geq 1,$$

which, among other things, relates to a continued fraction transformation observed by Wall [34, Section 78] leading to the recurrence relation

$$R_{n+1}(z) = [(1 + ic_{n+1})z + (1 - ic_{n+1})]R_n(z) - 4d_{n+1}zR_{n-1}(z), \quad n \geq 1, \tag{19}$$

with $R_0(z) = 1$ and $R_1(z) = (1 + ic_1)z + (1 - ic_1)$. Here $\{c_n\}_{n=1}^{\infty}$ and $\{d_{n+1}\}_{n=1}^{\infty}$ are real sequences expressed in terms of $\hat{\tau}_n$ and $\hat{\alpha}_{n-1}$. In addition, when $d_n := (1 - m_{n-1})m_n$, $n \geq 1$ (d_n in such a case is called a chain sequence with m_n, $n \geq 0$, the minimal parameters if $m_0 = 0$), a Favard type theorem is also established [10, Theorem 4.1] for (19) which proves the existence of a non-trivial probability measure $\hat{\mu}$ on the unit circle such that

$$\int_{\partial\mathbb{D}} z^{-n+k}R_n(z)(1 - z)d\hat{\mu}(z) = 0, \quad k = 0, 1, \cdots, n - 1, \quad n \geq 1.$$

Further, the polynomials $\hat{S}_n(z)$ defined by

$$\hat{S}_0(z) = 1, \quad \hat{S}_n(z) = \frac{R_n(z) - 2(1-m_n)R_{n-1}(z)}{\prod_{k=1}^{n}(1+ic_k)}, \quad n \ge 1, \tag{20}$$

are the monic orthogonal polynomials on the unit circle with respect to the measure $\hat{\mu}$ [10, Theorem 5.2], where the associated Verblunsky coefficients are given by

$$-\overline{\hat{S}_n(0)} = \frac{1-2m_n-ic_n}{1+ic_n}\prod_{k=1}^{n}\frac{1+ic_k}{1-ic_k} \quad n \ge 1.$$

Lemma 3.1 ([12]**).** *Let us choose the real sequence $\{c_n\}_{n=1}^{\infty}$ and the positive chain sequence $\{d_n\}_{n=1}^{\infty}$ having the minimal parameter sequence $\{m_n\}_{n=0}^{\infty}$ where*

$$c_k = \frac{-\mathrm{Im}(\hat{\tau}_{k-1}\hat{\alpha}_{k-1})}{1-\mathrm{Re}(\hat{\tau}_{k-1}\hat{\alpha}_{k-1})} \quad \text{and} \quad m_0 = 0, \quad m_k = \frac{1}{2}\frac{|1-\hat{\tau}_{k-1}\hat{\alpha}_{k-1}|^2}{1-\mathrm{Re}(\hat{\tau}_{k-1}\hat{\alpha}_{k-1})}, \quad k \ge 1,$$

where $\{\hat{\tau}_k\}$ and $\{\hat{\alpha}_{k-1}\}$ are as defined in (15). *Then, $\hat{\alpha}_{n-1} = -\overline{\hat{S}_n(0)}$, $n \ge 1$.*

Proof. We observe that with our choice for σ_n, $|\hat{\alpha}_{n-1}| < 1$, $n \ge 1$, and hence $\{\hat{\alpha}_{n-1}\}_{n=1}^{\infty}$ is eligible to constitute a sequence of Verblunsky coefficients [30, Theorem 3.1.3]. The rest of the proof follows from simple computations using (17) for $\zeta = 1$. □

With $\hat{\tau}_n = \prod_{j=1}^{n}\frac{1-ic_j}{1+ic_j}$, the following relation

$$\mathcal{P}_n^{(1)}(z) = \frac{1}{z-1}(z\hat{S}_n(z) - \hat{\tau}_n\hat{S}_n^*(z)), \quad n \ge 1, \tag{21}$$

can also be derived from $\hat{S}_n(z)$ and $\hat{S}_n^*(z)$ given by (20), using which it can be shown that $\mathcal{P}_n^{(1)}(z)$ satisfies the orthogonality relations

$$\int_{\partial\mathbb{D}} z^{-n+k}\mathcal{P}_n^{(1)}(z)(1-z)d\hat{\mu}(z) = \begin{cases} \tilde{\eta}_n \neq 0, & k=-1, \\ 0, & 0 \le k \le n-1, \\ \hat{\eta}_n \neq 0, & k = n, \end{cases} \tag{22}$$

for any $n \ge 1$. Moreover, that $\mathcal{P}_n^{(1)*}(z) = \bar{\hat{\tau}}_n\mathcal{P}_n^{(1)}(z)$, $n \ge 1$, can also be verified from (21).

Remark 3.2. *The essence of the above discussion is that starting with any polynomial sequence $\mathcal{B}_n(z)$, $n \ge 0$, satisfying* (7) *with σ_n and λ_n appropriately chosen, we are able to arrive at the recurrence relation* (16) *satisfied by monic CD kernels associated with orthogonal polynomials on the unit circle. Thus, with $\mathcal{P}_0^{(1)}(z) = 1$, we have on one hand*

$$\mathcal{P}_n^{(1)}(z) = \frac{1}{z-1}\left(\mathcal{B}_{n+1}(z) + \omega_{n+1}\mathcal{B}_n(z)\right), \quad \omega_n := \omega_n(1), \quad n \ge 1,$$

while on the other

$$\mathcal{P}_n^{(1)}(z) = \frac{1}{z-1}(z\hat{S}_n(z) - \hat{\tau}_n\hat{S}_n^*(z)), \quad n \ge 1.$$

We note that the crucial link is the expression (15) *of the parameters $\hat{\tau}_n$ and $\hat{\alpha}_{n-1}$ in terms of σ_n, λ_{n+1} and ω_n, $n \ge 1$.*

In view of Remark 3.2, the special class of polynomials that we will use are the reversed Szegő polynomials. For any sequence $\{S_n(z)\}_{n=0}^{\infty}$ of polynomials orthogonal on the unit circle with respect to the non-trivial positive measure μ, consider the Szegő recurrences

$$\begin{aligned} S_n(z) &= zS_{n-1}(z) - \bar{\alpha}_{n-1}S^*_{n-1}(z), \\ S_n(z) &= (1-|\alpha_{n-1}|^2)zS_{n-1}(z) - \bar{\alpha}_{n-1}S^*_n(z), \quad n \geq 1. \end{aligned} \tag{23}$$

It can be shown that the sequence of polynomials $\{S^*_n(z)\}_{n=0}^{\infty}$ satisfies the recurrence relation (normalized to monic form)

$$\frac{S^*_{n+1}(z)}{-\alpha_n} = \left(z + \frac{\alpha_{n-1}}{\alpha_n}\right)\frac{S^*_n(z)}{-\alpha_{n-1}} - \frac{\alpha_{n-2}}{\alpha_{n-1}}(1-|\alpha_{n-1}|^2)z\frac{S^*_{n-1}(z)}{-\alpha_{n-2}}, \quad n \geq 1,$$

with $S^*_0(z) = 1$ and $\frac{S^*_1(z)}{-\alpha_0} = z - \frac{1}{\alpha_0}$. Comparing with (7), we obtain $\mathcal{B}_0(z) = S^*_0(z) = 1$,

$$\mathcal{B}_n(z) = \frac{S^*_n(z)}{-\alpha_{n-1}}, \quad \sigma_{n+1} = \frac{\alpha_{n-1}}{\alpha_n} \quad \text{and} \quad \lambda_{n+1} = \frac{\alpha_{n-2}}{\alpha_{n-1}}(1-|\alpha_{n-1}|^2), \quad n \geq 1,$$

with $|\sigma_1 \cdots \sigma_k| = |\frac{1}{\alpha_{k-1}}| > 1$ and $\lambda_{k+1} = \sigma_k(1 - \frac{1}{|\sigma_1\cdots\sigma_k|^2})$, $k \geq 1$. Hence, we have the following expression

$$\mathcal{P}^{(1)}_n(z) = \frac{1}{z-1}\left(\frac{S^*_{n+1}(z)}{-\alpha_n} + \omega_{n+1}\frac{S^*_n(z)}{-\alpha_{n-1}},\right) \quad n \geq 1. \tag{24}$$

which was the crux of Section 2. Further, from the orthogonality relations for $S^*_n(z)$ with respect to μ, it follows that

$$\int_{\partial\mathbb{D}} z^{-n+k}\mathcal{P}^{(1)}_n(z)d\mu(z) = 0, \quad k = 0, 1\cdots, n-1, \quad n \geq 1. \tag{25}$$

The relation (25) may be compared with (22), something that we will generalize in Section 4. Moreover, in Section 5 we will illustrate the case when $\hat{\alpha}_{n-1}$ given by (15) is used to generate the Szegő polynomials via the recurrences (23). This will explain why the class of reversed Szegő polynomials is special. Before that, we present a kind of Szegő relation for the polynomials $\mathcal{P}^{(1)}_n(z)$, $n \geq 0$.

Lemma 3.3. *Consider the polynomial sequence $\{\hat{\mathcal{P}}_n(z)\}_{n=0}^{\infty}$ where*

$$\hat{\mathcal{P}}_0(z) = 1, \quad \hat{\tau}_{n+1}\hat{\mathcal{P}}_n(z) = \mathcal{P}^{(1)}_{n+1}(z) - z\mathcal{P}^{(1)}_n(z), n \geq 1.$$

Then the following relation

$$\mathcal{P}^{(1)}_n(z) = (1+\hat{\tau}_n\hat{\alpha}_{n-1})(1-\hat{\tau}_n\hat{\alpha}_n)z\mathcal{P}^{(1)}_{n-1}(z) + \hat{\tau}_n\hat{\mathcal{P}}_n(z), \quad n \geq 1, \tag{26}$$

holds.

Proof. From (17) for $\zeta = 1$, (21) and the Szegő relations (23) satisfied by $\hat{S}_n(z)$ we obtain

$$\mathcal{P}^{(1)}_n(z) = \frac{1}{z-1}\frac{\hat{S}_{n+1}(z) - \hat{\tau}_{n+1}\hat{S}^*_{n+1}(z)}{1+\hat{\tau}_{n+1}\hat{\alpha}_n}, \quad n \geq 0. \tag{27}$$

This gives $z(1+\hat{\tau}_n\hat{\alpha}_{n-1})\mathcal{P}^{(1)}_{n-1}(z) = \mathcal{P}^{(1)}_n(z) - \hat{\tau}_n\hat{S}^*_n(z)$, which upon multiplication of the factor $(1-\hat{\tau}_n\hat{\alpha}_n)$ either side leads to

$$\mathcal{P}^{(1)}_n(z) = z(1+\hat{\tau}_n\hat{\alpha}_{n-1})(1-\hat{\tau}_n\hat{\alpha}_n)\mathcal{P}^{(1)}_{n-1}(z) + \hat{\tau}_n[\hat{\alpha}_n\mathcal{P}^{(1)}_n(z) + (1-\hat{\tau}_n\hat{\alpha}_n)\hat{S}^*_n(z)]$$

We show that the last term above is $\hat{\tau}_n\hat{\mathcal{P}}_n(z)$. For this, we note that $\hat{\mathcal{P}}_n(z)$ has an alternative expression from the relation

$$\mathcal{P}^{(1)}_{n+1}(z) - z\mathcal{P}^{(1)}_n(z) = \hat{\tau}_{n+1}[z\hat{\alpha}_n\mathcal{P}^{(1)}_n(z) + \hat{S}^*_{n+1}(z)],$$

where equality follows from (21). The claim is that the relation $(z-1)\hat{\alpha}_n\mathcal{P}^{(1)}_n(z) = (1-\hat{\tau}_n\hat{\alpha}_n)\hat{S}^*_n(z) - \hat{S}^*_{n+1}(z)$ holds. That this is true is seen from the fact that

$$(z-1)\mathcal{P}_n(z) = zS_n(z) - \tau_n S^*_n(z) = S_{n+1}(z) - \tau_{n+1}(1-\tau_n\alpha_n)S^*_n(z),$$

in which we eliminate $S_{n+1}(z)$ with its expression obtained from (27). □

We conclude this section with the observation that while $\mathcal{P}^{(1)}_n(0) = \hat{\tau}_n$, $n \geq 1$, as can be seen from (21) or (27), $\hat{\tau}_{n+1}\hat{\mathcal{P}}_n(0) = \mathcal{P}^{(1)}_{n+1}(0) = \hat{\tau}_{n+1}$ which implies $\hat{\mathcal{P}}_n(0) = 1$, $n \geq 1$.

4. Quasi-orthogonality on the unit circle

Motivated by the representation (24) that expresses $\mathcal{P}^{(1)}_n(z)$, $n \geq 1$, as a linear combination of the reversed Szegő polynomials, we present the following definition of quasi-orthogonality on the unit circle.

Definition 4.1. *Let $\mathfrak{v}$ be a Hermitian linear functional defined on the space of Laurent polynomials and $s \in \mathbb{N}$. Let $\{\psi_n(z)\}_{n=0}^{\infty}$ be a sequence of monic polynomials. We say that $\psi_n(z)$ is quasi-orthogonal of order s with respect to $\mathfrak{v}$ if*

(i) $\mathfrak{v}[z^{-n+k}\cdot\psi_n(z)] = 0$, *for $k = s-1, s, \cdots, n-s$ and for every $n \geq 2s-1$.*
(ii) *There exists $n_0 \geq 2s-2$ such that $\mathfrak{v}[z^{-s+1}\cdot\psi_{n_0}(z)] \neq 0$.*

We emphasize the fact that invariant polynomials in the $\mathcal{DG}$ class are not orthogonal to constants (see (22)) plays a fundamental role in Definition 4.1. To see that $\mathcal{P}^{(1)}_n(z)$, $n \geq 1$, satisfies our definition, suppose the moment functional $\mathfrak{v}$ is given by the integral representation

$$\mathfrak{v}[z^{-n+k}\cdot\psi_n(z)] = c\int_{\partial\mathbb{D}} z^{-n+k}\psi_n(z)(z-1)d\mu(z), \quad c \in \mathbb{R}\setminus\{0\}.$$

From (25), the monic polynomials $\mathcal{P}^{(1)}_n(z)$, $n \geq 0$, satisfy part (i) of Definition 4.1 for $s = 1$. Further, inverting the relation (24), we have

$$\bar{\hat{\tau}}_n\int_{\partial\mathbb{D}}\mathcal{P}^{(1)}_n(z)(z-1)d\mu(z) = \frac{1}{-\bar{\alpha}_n}\int_{\partial\mathbb{D}}S_{n+1}(z)d\mu(z) + \frac{\bar{\omega}_{n+1}}{-\bar{\alpha}_{n-1}}\int_{\partial\mathbb{D}}zS_n(z)d\mu(z),$$

which, upon using the Szegő relations (23) for the last term above gives

$$\int_{\partial\mathbb{D}}\mathcal{P}^{(1)}_n(z)(z-1)d\mu(z) = -\frac{\tau_n\bar{\omega}_{n+1}}{c}\frac{\bar{\alpha}_n}{\bar{\alpha}_{n-1}}||S_n(z)||^2 \neq 0.$$

Hence, the sequence $\{\mathcal{P}^{(1)}_n(z)\}$ given by (24) is quasi-orthogonal of order $s = 1$ with respect to $\mathfrak{v}$ (where we have put $n_0 = n$).

Let $\mathfrak{u} \in \mathcal{H}(\mathcal{DG})$, the class of moment functionals given by the integral representation

$$\mathfrak{u}[z^{-n+k}\cdot\psi_n(z)] = \int_{\partial\mathbb{D}} z^{-n+k}\psi_n(z)(1-z)d\xi(z) = \begin{cases}\tilde{h}_n \neq 0, & k=-1,\\ 0, & 0 \leq k \leq n-1,\\ \hat{h}_n \neq 0, & k = n.\end{cases} \tag{28}$$

The sequence of monic polynomials $\{\psi_n(z)\}_{n=0}^{\infty}$ satisfying (28) is said to be associated with $\mathfrak{u}$ if $\xi(z)$ is a non-trivial positive measure on $\partial\mathbb{D}$. Assuming $\psi_n(z)$, $n \geq 0$, is quasi-orthogonal with respect to $\mathfrak{v}$, our goal in the remainder of this section is to characterize $\mathfrak{v}$ in terms of $\mathfrak{u} \in \mathcal{H}(\mathcal{DG})$.

We begin with the following orthogonality properties. If $\mathcal{P}_n^{(1)}(z)$, $n \geq 1$, is quasi-orthogonal with respect to $\mathfrak{v}$, then from Lemma 3.3 we have $\mathfrak{v}[z^{-n+k} \cdot \hat{\mathcal{P}}_n(z)] = 0$ for $k = s-2, \cdots, n-s-1$. Further, $\mathfrak{u}[z^{-n+k} \cdot \hat{\mathcal{P}}_n(z)] = 0$ for $k = -1, 0, \cdots, n-2$. Hence, from (26) it follows that

$$\mathfrak{u}[z^{-n-1} \cdot \mathcal{P}_n^{(1)}(z)] = \prod_{j=1}^{n} (1 + \hat{\tau}_j \hat{\alpha}_{j-1})(1 - \hat{\tau}_j \hat{\alpha}_j), \quad n \geq 1. \tag{29}$$

Remark 4.2. *We invert the relation* (26) *to obtain*

$$\mathcal{P}_n^{(1)}(z) = \overline{(1 + \hat{\tau}_n \hat{\alpha}_{n-1})(1 - \hat{\tau}_n \hat{\alpha}_n)} \frac{\hat{\tau}_n}{\hat{\tau}_{n-1}} \mathcal{P}_{n-1}^{(1)}(z) + \hat{\mathcal{P}}_n^*(z), \quad n \geq 1.$$

Since $\mathfrak{v}$ *is Hermitian*

$$\mathfrak{v}[z^{-s+1} \cdot \hat{\mathcal{P}}_n^*(z)] = \mathfrak{v}[z^{-s+1} \cdot z^n \overline{\hat{\mathcal{P}}}_n(z^{-1})] = \overline{\mathfrak{v}[z^{-n+s-1} \cdot \hat{\mathcal{P}}_n(z)]} = 0,$$

which, from Lemma 3.3, leads to the following relation

$$\mathfrak{v}[z^{-s+1} \cdot \mathcal{P}_n^{(1)}(z)] = \prod_{j=2s-1}^{n} \overline{(1 + \hat{\tau}_j \hat{\alpha}_{j-1})(1 - \hat{\tau}_j \hat{\alpha}_j)} \frac{\hat{\tau}_n}{\hat{\tau}_{2s-2}} \mathfrak{v}[z^{-s+1} \cdot \mathcal{P}_{2s-2}^{(1)}(z)],$$

for $n \geq 2s-1$. *If in Definition 4.1 we assume* $n_0 = 2s-2$, *then the above relation implies*

$$\mathfrak{v}[z^{-s+1} \cdot \mathcal{P}_{n_0}^{(1)}(z)] \neq 0 \Longleftrightarrow \mathfrak{v}[z^{-s+1} \cdot \mathcal{P}_n^{(1)}(z)] \neq 0, \quad \forall n \geq 2s-1.$$

It may be noted that this notion is referred to as strict quasi-orthogonality of order s in [1, Definition 2.3].

Consider a polynomial $\mathcal{U}(z) = \hat{c}_0^{(n)} + \hat{c}_1^{(n)} z + \cdots + \hat{c}_s^{(n)} z^s$ with $\hat{c}_s^{(n)} \neq 0$. Let $\mathcal{U}_*(z^{-1})$ be obtained from $\mathcal{U}(z)$ by replacing z with z^{-1} and (for notational purposes) $\hat{c}_j^{(n)}$ with $\hat{c}_{-j}^{(n)}$. Here, the superscript (n) signifies that the coefficients are associated with a polynomial of degree n. Consider the following representation

$$\begin{aligned} \mathfrak{v}[z^{-n+k} \cdot \mathcal{P}_n^{(1)}(z)] &= [z^{-1}\mathcal{U}(z) + z\mathcal{U}_*(z^{-1})]\mathfrak{u}[z^{-n+k} \cdot \mathcal{P}_n^{(1)}(z)] \\ &= \int_{\partial\mathbb{D}} z^{-n+k} \mathcal{P}_n^{(1)}(z)[z^{-1}\mathcal{U}(z) + z\mathcal{U}_*(z^{-1})](1-z) d\hat{\mu}(z). \end{aligned} \tag{30}$$

Then, using the orthogonality properties (28) we have

$$\mathfrak{v}[z^{-n+k} \cdot \mathcal{P}_n^{(1)}(z)] = \mathfrak{u}[z^{-n+k-1}\mathcal{U}(z) \cdot \mathcal{P}_n^{(1)}(z)] + \mathfrak{u}[z^{-n+k+1}\mathcal{U}_*(z^{-1}) \cdot \mathcal{P}_n^{(1)}(z)] = 0,$$

for $k = s-1, s, \cdots, n-s$ and $\mathfrak{v}[z^{-s+1} \cdot \mathcal{P}_{2s-2}^{(1)}(z)] \neq 0$. Hence, $\mathcal{P}_n^{(1)}(z)$, $n \geq 1$, is quasi-orthogonal of order s with respect to $\mathfrak{v}$. The following puts this in a formal setting.

Theorem 4.3. *Let* $\mathfrak{u} \in \mathcal{H}(\mathcal{DG})$ *and* $\{\mathcal{P}_n^{(1)}(z)\}_0^\infty$ *be the sequence of polynomials associated with* $\mathfrak{u}$. *Then,* $\{\mathcal{P}_n^{(1)}(z)\}_0^\infty$ *is quasi-orthogonal of order s with respect to* $\mathfrak{v}$ *if and only if*

$$\mathfrak{v} = \sum_{j=-s+1}^{j=s-1} c_j^{(n)} z^j \mathfrak{u}, \tag{31}$$

where $c_j^{(n)}$ *are complex scalars.*

Proof. The necessity for existence of the scalars $c_j^{(n)}$ is proved in the discussion preceding Theorem 4.3 and follows from the fact that $[z^{-1}\mathcal{U}(z) + z\mathcal{U}_*(z^{-1})]$ can in fact be written in the required form (31). Hence, in order to prove the sufficient part, define the linear functional

$$\mathfrak{w} = \mathfrak{v} - \sum_{j=-s+1}^{s-1} c_j^{(n)} z^j \mathfrak{u}, \quad c_j^{(n)} \in \mathbb{C}.$$

Since $\mathcal{P}_n^{(1)}(z)$ is quasi-orthogonal with respect to $\mathfrak{v}$ and satisfies the orthogonality conditions (28), it follows that

$$\mathfrak{w}[z^{-n+k} \cdot \mathcal{P}_n^{(1)}(z)] = 0, \quad k = s-1, s, \cdots, n-s,$$

and for every $n \geq 2s-1$. We now prove that $\mathfrak{w}[z^{-n+k} \cdot \mathcal{P}_n^{(1)}(z)] = 0$, separately, for the values $k = n-s+1, n-s+2, \cdots, n$ and $k = 0, 1, \cdots, s-2$. The first case holds if

$$\mathfrak{v}[z^{-n+k} \cdot \mathcal{P}_n^{(1)}(z)] = \sum_{j=-s+1}^{s-1} c_j^{(n)} \mathfrak{u}[z^{-n+k+j} \cdot \mathcal{P}_n^{(1)}(z)], \quad k = n-s+1, \cdots, n,$$

which has a unique solution for $c_0^{(n)}, \cdots, c_{s-1}^{(n)}$ since $\mathfrak{u}[\psi_n(z) \cdot 1] \neq 0$. Similarly, the second case holds since $c_{-s+1}^{(n)}, c_{-s+2}^{(n)}, \cdots, c_{-1}^{(n)}$ are uniquely determined owing to the fact that $\mathfrak{u}[z^{-n-1} \cdot \mathcal{P}_n^{(1)}(z)] \neq 0$. Thus, we have shown that $\mathfrak{w}[z^{-n+k} \cdot \mathcal{P}_n^{(1)}(z)] = 0$ for $k = 0, 1, \cdots, n$ and for every $n \geq 2s-1$. Hence, arguing as in [1, Proposition 2.6], we obtain $\mathfrak{w} \equiv 0$ which implies $\mathfrak{v} = \sum_{j=-s+1}^{s-1} c_j^{(n)} z^j \mathfrak{u}$. $\square$

The next result provides the characterization between $\mathfrak{v}$ and $\mathfrak{u} \in \mathcal{H}(\mathcal{DG})$.

Theorem 4.4. *With the conditions of Theorem* (4.3), $\{\mathcal{P}_n^{(1)}(z)\}$ *is quasi-orthogonal of order s with respect to* $\mathfrak{v}$ *if and only if there exists an unique polynomial* $\mathcal{U}(z)$ *such that*

$$\mathfrak{v} = [z^{-1}\mathcal{U}(z) + z\bar{\mathcal{U}}(z^{-1})]\mathfrak{u}, \tag{32}$$

where $\deg \mathcal{U}(z) = s$ *and* $\bar{\mathcal{U}}(z^{-1})$ *is obtained from* $\mathcal{U}(z)$ *by replacing z with* z^{-1} *and* c_j *with* $\bar{c}_j$, $j = 0, 1, \cdots, s-1$.

Proof. Since by Theorem 4.3 the representation (31) holds, the existence part of the proof follows if we show that the functional $\sum_{j=-s+1}^{s-1} c_j^{(n)} z^j \mathfrak{u}$ is independent of n and $\bar{c}_{-j} = c_j$ for $j = 0, 1, \cdots, s-1$. We show that the coefficients are independent of n by proving that $c_j^{(n)} = c_j^{(n-1)}$ for $j = -s+1, \cdots, s-1$.

Using the orthogonality of $\hat{\mathcal{P}}_n(z)$, we operate the functional $\mathfrak{v}$ on (26) to get

$$\mathfrak{v}[z^{-n+s-2} \cdot \mathcal{P}_n^{(1)}(z)] = (1 + \hat{\tau}_n \hat{\alpha}_{n-1})(1 - \hat{\tau}_n \hat{\alpha}_n)\mathfrak{v}[z^{-n+s-1} \cdot \mathcal{P}_n^{(1)}(z)]. \tag{33}$$

On the other hand, from the system of equations involving $c_{-j}^{(n)}$, $j = s-1, \cdots, 1$, obtained in the proof of Theorem 4.3, we have

$$\mathfrak{v}[z^{-n+s-2} \cdot \mathcal{P}_n^{(1)}(z)] = c_{-s+1}^{(n)} \mathfrak{u}[z^{-n-1} \cdot \mathcal{P}_n^{(1)}(z)].$$

which used with (33) sets up an iteration leading to

$$c_{-s+1}^{(n)} \mathfrak{u}[z^{-n-1} \cdot \mathcal{P}_n^{(1)}(z)] = (1 + \hat{\tau}_n \hat{\alpha}_{n-1})(1 - \hat{\tau}_n \hat{\alpha}_n) \prod_{j=1}^{n-1} (1 + \hat{\tau}_j \hat{\alpha}_{j-1})(1 - \hat{\tau}_j \hat{\alpha}_j) c_{-s+1}^{(n-1)}.$$

That $c_{-s+1}^{(n)} = c_{-s+1}^{(n-1)}$ immediately follows from (29). We now assume $c_j^{(n)} = c_j^{(n-1)}$ for $j = -s+1, \cdots, t-1$, where $t = -s+1, -s+2, \cdots, -1$. We have

$$\mathfrak{v}[z^{-n-t-1} \cdot \mathcal{P}_n^{(1)}(z)] = \sum_{j=-s+1}^{t} c_j^{(n)} \mathfrak{u}[z^{-n-1+j-t} \cdot \mathcal{P}_n^{(1)}(z)],$$

while Lemma 3.3 gives $\tau_{n+1}\mathfrak{v}[z^{-n-t-1} \cdot \hat{\mathcal{P}}_n^{(1)}(z)]$

$$= \sum_{j=-s+1}^{t-1} c_j^{(n+1)} \mathfrak{u}[z^{-n-1+j-t} \cdot \mathcal{P}_{n+1}^{(1)}(z)] - \sum_{j=-s+1}^{t-1} c_j^{(n)} \mathfrak{u}[z^{-n+j-t} \cdot \mathcal{P}_n^{(1)}(z)].$$

Using the above two relations in (26) and separating the case $j = t$, it follows from the induction hypothesis that

$$\begin{aligned} c_t^{(n)} \mathfrak{u}[z^{-n-1} \cdot \mathcal{P}_n^{(1)}(z)] &- (1+\hat{\tau}_n\hat{\alpha}_{n-1})(1-\hat{\tau}_n\hat{\alpha}_n)\mathfrak{u}[z^{-n} \cdot \mathcal{P}_{n-1}^{(1)}(z)]c_t^{(n-1)} \\ &= \sum_{j=-s+1}^{t-1} \Big[c_j^{(n-1)}(1+\hat{\tau}_n\hat{\alpha}_{n-1})(1-\hat{\tau}_n\hat{\alpha}_n)\mathfrak{u}[z^{-n+j-t} \cdot \mathcal{P}_{n-1}^{(1)}(z)] - c_j^{(n)}\mathfrak{u}[z^{-n-1+j-t} \cdot \mathcal{P}_n^{(1)}(z)]\Big] \\ &\quad + \frac{\hat{\tau}_n}{\hat{\tau}_{n+1}} \sum_{j=-s+1}^{t-1} c_j^{(n+1)} \mathfrak{u}[z^{-n-1+j-t} \cdot \mathcal{P}_{n+1}^{(1)}(z)] - \sum_{j=-s+1}^{t-1} \frac{\hat{\tau}_n}{\hat{\tau}_{n+1}} c_j^{(n)} \mathfrak{u}[z^{-n+j-t} \cdot \mathcal{P}_n^{(1)}(z)]. \end{aligned}$$

The induction is complete if the right hand side above vanishes. This is shown using the three term recurrence relation (16) for $\zeta = 1$ and observing that

$$b_{n+1} = \frac{\hat{\tau}_{n+1}}{\hat{\tau}_n} \quad \text{and} \quad \frac{\hat{\tau}_n}{\hat{\tau}_{n+1}} a_{n+1}(1) = (1+\hat{\tau}_n\hat{\alpha}_{n-1})(1-\hat{\tau}_n\hat{\alpha}_n).$$

Similar computations prove $c_j^{(n)} = c_j^{(n-1)}$ for $j = 0, \cdots, s-1$. Hence, we put

$$c_j^{(n)} = c_j^{(n-1)} = c_j, \quad j = -s+1, \cdots, s-1,$$

which we use to prove the next claim in the theorem.

The key point in this part of the proof is that $\mathfrak{u}, \mathfrak{v}$ are Hermitian and $\mathcal{P}_n^{(1)}(z)$ is a self-inversive polynomial. Hence

$$\mathfrak{v}[z^{-n+s-2} \cdot \mathcal{P}_n^{(1)}(z)] = c_{-s+1}^{(n)} \mathfrak{u}[z^{-n-1} \cdot \mathcal{P}_n^{(1)}(z)] \Longrightarrow \mathfrak{v}[z^{-s+2} \cdot \mathcal{P}_n^{(1)}(z)] = \overline{c_{-s+1}^{(n)}} \mathfrak{u}[z \cdot \mathcal{P}_n^{(1)}(z)].$$

With $\mathfrak{v}[z^{-s+1} \cdot \mathcal{P}_n^{(1)}(z)] = c_{s-1}^{(n)} \mathfrak{u}[1 \cdot \mathcal{P}_n^{(1)}(z)]$, the relation (26) yields

$$\begin{aligned} c_{s-1}^{(n)} &\left[\frac{\hat{\tau}_n}{\hat{\tau}_{n+1}} \mathfrak{u}[1 \cdot \mathcal{P}_{n+1}^{(1)}(z)] - \mathfrak{u}[1 \cdot \mathcal{P}_n^{(1)}(z)]\right] \\ &= \overline{c_{-s+1}^{(n)}} \left[\frac{\hat{\tau}_n}{\hat{\tau}_{n+1}} \mathfrak{u}[z \cdot \mathcal{P}_n^{(1)}(z)] - (1+\hat{\tau}_n\hat{\alpha}_{n-1})(1-\hat{\tau}_n\hat{\alpha}_n)\mathfrak{u}[z \cdot \mathcal{P}_{n-1}^{(1)}(z)]\right]. \end{aligned}$$

We let $\mathfrak{u}$ operate on (16) to conclude that $c_{s-1} = \bar{c}_{-s+1}$. We now assume $c_j = \bar{c}_{-j}$ for $j = -t+1, \cdots, s-1$ where $t = -s+1, -s+2, \cdots, -1$. The induction is completed by using the relations

$$\mathfrak{v}[z^t \cdot \mathcal{P}_n^{(1)}(z)] = \sum_{j=-t}^{s-1} c_j \mathfrak{u}[z^{j+t} \cdot \mathcal{P}_n^{(1)}(z)],$$

$$\mathfrak{v}[z^{t+1} \cdot \mathcal{P}_{n-1}^{(1)}(z)] = \sum_{j=-t}^{s-1} \bar{c}_{-j} \mathfrak{u}[z^{j+t+1} \cdot \mathcal{P}_{n-1}^{(1)}(z)],$$

in (26), separating the case $j = -t$ and letting $\mathfrak{u}$ operate on (16). Finally, from (31), since $\mathfrak{v}[1] \in \mathbb{R}$, it follows that $c_0 \in \mathbb{R}$. Thus writing

$$\mathcal{U}(z) = \frac{1}{2}(c_{s-1}z^s + \cdots + c_2z^3 + c_1z^2 + c_0z + \bar{c}_1),$$

the expression (32) follows.

To prove uniqueness of $\mathcal{U}(z)$, let $\mathcal{U}_1(z)$ and $\mathcal{U}_2(z)$ be any two polynomial solutions of (32), where $\deg \mathcal{U}_1(z) = s_1$ and $\deg \mathcal{U}_2(z) = s_2$. Let $\mathcal{U}_3(z) = \mathcal{U}_2(z) - \mathcal{U}_1(z)$, with $\deg \mathcal{U}_3(z) = r = \max\{s_1, s_2\}$. Then from (32), we have

$$[z^{-1}\mathcal{U}_3(z) + z\bar{\mathcal{U}}_3(z^{-1})]\mathfrak{u} = \sum_{j=-r+1}^{r-1} u_j z^j \mathfrak{u} = 0.$$

We let the above functional act on $[z^{-n+k} \cdot \mathcal{P}_n^{(1)}(z)]$ for $n \geq 2r - 1$ and $k \geq -1$ to have

$$\sum_{j=-r+1}^{r-1} u_j \mathfrak{u}[z^{-n+k+j} \cdot \mathcal{P}_n^{(1)}(z)] = 0,$$

which leads to a homogeneous system of equations for the choice of $k = -1, 0, \cdots, r-2$. The solution is trivial with $u_{-r+1} = u_{-r+2} = \cdots = u_0 = 0$, so that $\mathcal{U}_1(z) = \mathcal{U}_2(z)$, thus establishing uniqueness. □

We now present an illustration for the preceding theory using hypergeometric functions.

5. Illustration

Explicit representations of orthogonal polynomials on the unit circle are available in the literature [30, 33]. In this section, through a judicious use of the contiguous relations satisfied by hypergeometric functions, we will illustrate the theory presented so far. The Gaussian hypergeometric functions are denoted as

$$F(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \quad |z| < 1,$$

where $a, b \in \mathbb{C}$ and $c \in \mathbb{C} \setminus \{0\}$. For the theory of hypergeometric functions, the contiguous relations satisfied by them and the Pochhammer symbol $(a)_k$, we refer to [2].

A family of Szegő polynomials in terms of hypergeometric functions is given by [32]

$$\begin{aligned} S_n(z) &= \frac{(b + \bar{b} + 1)_n}{(b+1)_n} F(-n, b+1; b + \bar{b} + 1; 1 - z), \quad n \geq 1, \\ S_n^*(z) &= \frac{(b + \bar{b} + 1)_n}{(\bar{b}+1)_n} F(-n, b; b + \bar{b} + 1; 1 - z), \quad n \geq 1, \end{aligned} \tag{34}$$

which are orthogonal on the unit circle with respect to the measure [32, Theorem 4.1] $d\mu(b; e^{i\theta}) = \phi(b; \theta)d\theta$, where for $0 \leq \theta \leq 2\pi$

$$\phi(b; \theta) = \tau^b e^{(\pi-\theta)\mathrm{Im} b}[\sin(\theta/2)]^{2\mathrm{Re} b}, \quad \tau^b = \frac{2^{b+\bar{b}}|\Gamma(b+1)|^2}{2\pi\Gamma(b + \bar{b} + 1)}.$$

We first express the Szegő recurrences (23) satisfied by $S_n(z)$ and $S_n^*(z)$ using contiguous relations. Consider

$$(c - a - b)F(a, b; c; 1 - z) + azF(a + 1, b; c; 1 - z) = (c - b)F(a, b - 1; c; 1 - z),$$

in which we substitute $a = -n$, $b = b+1$, $c = b+\bar{b}+1$ and multiply $\frac{(b+\bar{b}+1)_n}{(b+1)_n}$ to get

$$S_n(z) = \frac{n(b+\bar{b}+n)}{|b+n|^2} zS_{n-1}(z) + \frac{(\bar{b})_n}{(b+1)_n} S_n^*(z), \quad n \geq 1. \tag{35}$$

Next, in the contiguous relation

$$F(a, b+1; c; 1-z) = \frac{z(a-b)}{a-c+1} F(a+1, b+1; c; 1-z) - \frac{(c-b-1)}{(a-c+1)} F(a+1, b; c; 1-z),$$

we substitute $a = -n$, $c = b+\bar{b}+1$ and multiply $\frac{(b+\bar{b}+1)_n}{(b+1)_n}$ both sides to get

$$S_n(z) = zS_{n-1} + \frac{(\bar{b})_n}{(b+1)_n} S_{n-1}^*(z). \tag{36}$$

We observe that the relations (35) and (36) constitute the Szegő recurrences (23) if we identify $\bar{\alpha}_{n-1} = -\frac{(\bar{b})_n}{(b+1)_n}$ so that $1 - |\alpha_{n-1}|^2 = \frac{n(b+\bar{b}+1)}{|b+n|^2}$. The first equality follows from [2, Corollary 2.2.3]

$$-\bar{\alpha}_{n-1} = S_n(0) = \frac{(b+\bar{b}+1)_n}{(b+1)_n} F(-n, b+1; b+\bar{b}+1; 1),$$

while the second equality follows from simple computations. We now illustrate the characterization (24) as discussed in Section 3. Using the Szegő relations (35) and (36) along with the expressions for α_{n-1}, we begin with the recurrence relation

$$\frac{S_{n+1}^*(z)}{-\alpha_n} = \left(z + \frac{\bar{b}+n+1}{b+n}\right) \frac{S_n^*(z)}{-\alpha_{n-1}} - \frac{n(b+\bar{b}+n)}{(b+n-1)(b+n)} z \frac{S_{n-1}^*(z)}{-\alpha_{n-2}}, \quad n \geq 1,$$

with $S_0^*(z) = 1$ and $\frac{S_1^*(z)}{-\alpha_0} = z + \frac{\bar{b}+1}{b}$, which is satisfied by the monic polynomials

$$\frac{S_n^*(z)}{-\alpha_{n-1}} = \frac{(b+\bar{b}+1)_n}{(b)_n} F(-n, b; b+\bar{b}+1; 1-z), \quad n \geq 1.$$

It can be easily verified from (9) that

$$\sigma_n = \frac{\bar{b}+n}{b+n-1} \text{ and } \lambda_{n+1} = \frac{n(b+\bar{b}+n)}{(b+n-1)(b+n)} \Longrightarrow \omega_n = -\frac{b+\bar{b}+n}{b+n-1}, \quad n \geq 1.$$

Thus, we have

$$\mathcal{P}_n^{(1)}(z) = \frac{1}{z-1}\left(\frac{S_{n+1}^*(z)}{-\alpha_n} - \frac{b+\bar{b}+n+1}{b+n} \frac{S_n^*(z)}{-\alpha_{n-1}}\right), \quad n \geq 1. \tag{37}$$

The sequence of polynomials $\mathcal{P}_n^{(1)}(z)$, $n \geq 0$, is thus quasi-orthogonal of order $s = 1$ on the unit circle with respect to $\phi(b;\theta)d\theta$ since using the orthogonality properties of $S_n^*(z)$ and [32, Theorem 3.1] we have

$$\int_{\partial\mathbb{D}} z^{-n+k} \mathcal{P}_n^{(1)}(z)(1-z)\phi(b;\theta)d\theta = \begin{cases} 0 & \text{if } k = 0, \cdots, n-1, \\ (h_n^{(b)})^{-2} & \text{if } k = n, \end{cases}$$

where the orthogonality constant is given by

$$h_n^{(b)} = ||S_n(z)||^{-1} = \sqrt{\frac{|(b+1)_n|^2}{(b+\bar{b}+1)_n} n!}, \quad n \geq 1.$$

Further, an expression for $\mathcal{P}_n^{(1)}(z)$ given by (37) can be found from the contiguous relation

$$\frac{b}{c}(z-1)F(a,b+1;c+1;1-z) = F(a-1,b;c;1-z) - F(a,b;c;1-z),$$

in which we substitute $a=-n$, $c=b+\bar{b}+1$ and multiply $\frac{(b+\bar{b}+1)_{n+1}}{(b)_{n+1}}$ both sides to get

$$\mathcal{P}_n^{(1)}(z) = \frac{(b+\bar{b}+2)_n}{(b+1)_n}F(-n,b+1,b+\bar{b}+2;1-z), \quad n \geq 1. \tag{38}$$

We now find the Szegő polynomials $\hat{S}_n(z)$, $n \geq 1$, which appear in (21) or (27) by first finding the scaled polynomials $R_n(z)$. From the definitions (15) we have

$$\hat{\tau}_n = -(\omega_{n+1}+\sigma_{n+1})\sigma_1\cdots\sigma_n = \frac{(\bar{b}+1)_n}{(b+1)_n} \text{ and } \hat{\alpha}_{n-1} = -\frac{1}{\sigma_1\cdots\sigma_n} = -\frac{(b)_n}{(\bar{b}+1)_n}$$

which are the same as obtained in (34) earlier. With $\lambda = \mathrm{Re}(b)$, the scaled polynomials are given by [10, 12]

$$R_n(z) = \frac{(2\lambda+2)_n}{(\lambda+1)_n}F(-n,b+1;b+\bar{b}+2;1-z), \quad n \geq 1,$$

and the corresponding Szegő polynomials $\hat{S}_n(z)$ obtained in (20) are the same as given by (34). Further, with $1+\hat{\tau}_n\hat{\alpha}_{n-1} = \frac{n}{b+n}$, $n \geq 1$, we find an expression for (27), for instance, using the contiguous relation

$$F(a,b;c;1-z) - F(a,b+1;c;1-z) = \frac{a}{c}(z-1)F(a+1,b+1;c+1;1-z).$$

We substitute $a=-n$, $c=b+\bar{b}+1$ and multiply $\frac{(b+\bar{b}+1)_n}{(b+1)_n}$ both sides to get

$$S_n(z) - \frac{(\bar{b}+1)_n}{(b+1)_n}S_n^*(z) = \frac{n}{b+n}(z-1)\frac{(b+\bar{b}+2)_{n-1}}{(b+1)_{n-1}}F(-n+1,b+1;b+\bar{b}+2;1-z),$$

which yields an expression for $\mathcal{P}_n^{(1)}(z)$ as obtained in (38). The expression (21) can also be found on similar lines from the contiguous relation

$$c(1-z)F(a,b;c;z) - cF(a-1,b;c;z) + (c-b)zF(a,b;c+1;z) = 0.$$

We conclude with the final remark that we have focused on quasi-orthogonality of order one and to begin with the class of reversed Szegő polynomials is a special case because it allowed us to express members of the $\mathcal{DG}$ class of invariant polynomials in two different ways using the same sequence of Szegő polynomials. The key feature we have used is that these invariant polynomials satisfy a three term recurrence relation, which is also a key feature of quasi-orthogonal polynomials of order one on the real line.

References

[1] M. Alfaro and L. Moral, *Quasi-orthogonality on the unit circle and semi-classical forms*, Portugal. Math. **51** (1994), no. 1, 47–62.

[2] G. E. Andrews, R. Askey and R. Roy, *Special functions*, Encyclopedia of Mathematics and its Applications, 71, Cambridge Univ. Press, Cambridge, 1999.

[3] K. K. Behera and A. Swaminathan, *Biorthogonality and para-orthogonality of R_I polynomials*, Calcolo **55** (2018), no. 4, Art. 41, 22 pp.

[4] C. F. Bracciali, F. Marcellán and S. Varma *Orthogonality of quasi-orthogonal polynomials*, Filomat **32(20)** (2018), 6953–6977.

[5] A. Branquinho and F. Marcellán, *Generating new classes of orthogonal polynomials*, Internat. J. Math. Math. Sci. **19** (1996), no. 4, 643–656.

[6] J. Breuer and E. Seelig, *On the spacing of zeros of paraorthogonal polynomials for singular measures*, J. Approx. Theory **259** (2020), 105482, 20 pp.

[7] C. Brezinski, K. A. Driver and M. Redivo-Zaglia, *Quasi-orthogonality with applications to some families of classical orthogonal polynomials*, Appl. Numer. Math. **48** (2004), no. 2, 157–168.

[8] A. Bultheel, R. Cruz-Barroso and C. Díaz Mendoza, Zeros of quasi-paraorthogonal polynomials and positive quadrature, J. Comput. Appl. Math. **407** (2022), Paper No. 114039.
[9] M. J. Cantero, L. Moral and L. Velázquez, *Measures and para-orthogonal polynomials on the unit circle*, East J. Approx. **8** (2002), no. 4, 447–464.
[10] K. Castillo, M.S. Costa, A. Sri Ranga and D.O. Veronese, *A Favard type theorem for orthogonal polynomials on the unit circle from a three term recurrence formula*, J. Approx. Theory **184** (2014), 146–162.
[11] T. S. Chihara, *On quasi-orthogonal polynomials*, Proc. Amer. Math. Soc. **8** (1957), 765–767.
[12] M. S. Costa, H. M. Felix and A. Sri Ranga, *Orthogonal polynomials on the unit circle and chain sequences*, J. Approx. Theory **173** (2013), 14–32.
[13] P. Delsarte and Y. V. Genin, *The split Levinson algorithm*, IEEE Trans. Acoust. Speech Signal Process. **34** (1986), no. 3, 470–478.
[14] P. Delsarte and Y. Genin, *The tridiagonal approach to Szegő's orthogonal polynomials, Toeplitz linear systems, and related interpolation problems*, SIAM J. Math. Anal. **19** (1988), no. 3, 718–735.
[15] P. Delsarte and Y. Genin, On the role of orthogonal polynomials on the unit circle in digital signal processing applications, in *Orthogonal polynomials (Columbus, OH, 1989)*, 115–133, NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci., 294, Kluwer Acad. Publ., Dordrecht.
[16] P. Delsarte and Y. Genin, *Tridiagonal approach to the algebraic environment of Toeplitz matrices. I. Basic results*, SIAM J. Matrix Anal. Appl. **12** (1991), no. 2, 220–238.
[17] P. Delsarte and Y. Genin, *Tridiagonal approach to the algebraic environment of Toeplitz matrices. II. Zero and eigenvalue problems*, SIAM J. Matrix Anal. Appl. **12** (1991), no. 3, 432–448.
[18] P. Dewilde and H. Dym, *Lossless chain scattering matrices and optimum linear prediction: the vector case*, Internat. J. Circuit Theory Appl. **9** (1981), no. 2, 135–175.
[19] D. Dickinson, *On quasi-orthogonal polynomials*, Proc. Amer. Math. Soc. 12 (1961) 185–194.
[20] A. Draux, *On quasi-orthogonal polynomials*, J. Approx. Theory **62** (1990), no. 1, 1–14.
[21] L. Golinskii, *Quadrature formula and zeros of para-orthogonal polynomials on the unit circle*, Acta Math. Hungar. **96** (2002), no. 3, 169–186.
[22] U. Grenander and G. Szegő, *Toeplitz forms and their applications*, second edition, Chelsea Publishing Co., New York, 1984.
[23] M. E. H. Ismail and X.-S. Wang, *On quasi-orthogonal polynomials: their differential equations, discriminants and electrostatics*, J. Math. Anal. Appl. **474** (2019), no. 2, 1178–1197.
[24] W. B. Jones and O. Njåstad, *Applications of Szegő polynomials to digital signal processing*, Rocky Mountain J. Math. **21** (1991), no. 1, 387–436.
[25] W. B. Jones, O. Njåstad and W. J. Thron, *Moment theory, orthogonal polynomials, quadrature, and continued fractions associated with the unit circle*, Bull. London Math. Soc. **21** (1989), no. 2, 113–152.
[26] W. B. Jones, O. Njåstad and H. Waadeland, *Application of Szegő polynomials to frequency analysis*, SIAM J. Math. Anal. **25** (1994), no. 2, 491–512.
[27] H. Joulak, *A contribution to quasi-orthogonal polynomials and associated polynomials*, Appl. Numer. Math. 54 (2005) 65–78.
[28] F. Marcellán, F. Peherstorfer and R. Steinbauer, *Orthogonality properties of linear combinations of orthogonal polynomials*, Adv. Comput. Math. **5** (1996), no. 4, 281–295.
[29] J. Shohat, *On mechanical quadratures, in particular, with positive coefficients*, Trans. Amer. Math. Soc. **42** (1937), no. 3, 461–496.
[30] B. Simon, *Orthogonal polynomials on the unit circle. Part 1*, American Mathematical Society Colloquium Publications, 54, Part 1, American Mathematical Society, Providence, RI, 2005.
[31] B. Simon, The Christoffel-Darboux kernel, in *Perspectives in partial differential equations, harmonic analysis and applications*, 295–335, Proc. Sympos. Pure Math., 79, Amer. Math. Soc., Providence, RI, 2008.
[32] A. Sri Ranga, *Szegő polynomials from hypergeometric functions*, Proc. Amer. Math. Soc. **138** (2010), no. 12, 4259–4270.
[33] G. Szegő, *Orthogonal polynomials*, fourth edition, American Mathematical Society Colloquium Publications, Vol. XXIII, American Mathematical Society, Providence, RI, 1975.
[34] H. S. Wall, *Analytic Theory of Continued Fractions*, D. Van Nostrand Co., Inc., New York, NY, 1948.