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Some new Hardy inequalities in probability

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Abstract. Hardy et al. (1934) came up with Hardy's inequality in their book. Klaassen and Wellner (2021) gave the probability version of the Hardy inequality when the parameter $p > 1$. Based on their work, in this paper, we assign the randomness to variables as well. When $p > 1$, we give some extensions of Hardy's inequality. When $0 < p < 1$, we provide the corresponding Hardy inequality in probability language. Also, we show that in some circumstances, our results contain the integral form of Hardy's inequality. We give a reversed Hardy inequality for random variables as well.

1. Introduction

Considering the maximum of sums of bilinear forms with complex variables and coefficients $\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}x_iy_j$, a lot of work has been done in different forms and in different settings. Let $a_{ij} = \frac{1}{i+j}$, Hilbert first studied this type and got his conclusions. Borwein (2008) applied them to various multiple zeta values. Pogany (2008) derived the extensions in terms of ℓ_p norms. By using the Euler-Maclaurin's summation formula, Yang and Debnathy (2005) gave some new results which involved the Beta function as constant factor. Combining weighted coefficients and a complex integral formula, Chen and Yang (2018) gave an extended reversed Hardy-Hilbert inequality through the Hermite-Hadamard inequality.

To simplify the proof of Hilbert's results, Hardy et al. (1934) pointed out Hardy's inequality as follows.
(i) (Theorem 327 in Hardy et al. (1934)) The integral version: if $p > 1$, $h(t) \geq 0$, then

$$\int_0^\infty \left(\frac{\int_0^x h(t)dt}{x}\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty h^p(x)dx.$$

(ii) (Theorem 326 in Hardy et al. (1934)) The series version: if $p > 1$, $a_n \geq 0$, then

$$\sum_{n=1}^{\infty}\left(\frac{\sum_{k=1}^{n} a_k}{n}\right)^p \leq \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} a_n^p.$$

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The above two inequalities have been widely studied and many generalizations have been shown. Ruzhansky and Verma (2019) gave several characterisations of weights for Hardy's inequalities on metric measure spaces.

Klaassen and Wellner (2021) proposed a Hardy inequality by means of probability theory. Let X and Y be independent and identically distributed variables with right continuous distribution function F on $(\mathbb{R}, \mathfrak{B})$, and h be a nonnegative measurable function on $(\mathbb{R}, \mathfrak{B})$. For $p > 1$, they provided a bilateral inequality which is

$$\mathrm{E}h^p(Y) \leq \mathrm{E}\left[\frac{\mathrm{E}(h(Y)\mathbb{I}_{\{Y\leq X\}}|X)}{\mathrm{F}(X)}\right]^p \leq \left(\frac{p}{p-1}\right)^p \mathrm{E}h^p(Y),$$

where h also needs to be nonincreasing in the left inequality. However, they did not consider the situation of $0 < p < 1$. In fact, when $0 < p < 1$, Hardy et al. (1934) did derive some inequalities in the integral and series form as listed below.
(i) (Theorem 337 in Hardy et al. (1934)) The integral version: if $0 < p < 1$, $h(x) \geq 0$, $\int_0^\infty h^p(t)dt < \infty$, then

$$\int_0^\infty \left(\frac{\int_x^\infty h(t)dt}{x}\right)^p dx \geq \left(\frac{p}{1-p}\right)^p \int_0^\infty h^p(x)dx. \tag{1.1}$$

(ii) (Theorem 338 in Hardy et al. (1934)) The series version: if $0 < p < 1$, $a_n \geq 0$, $\Sigma_0^\infty a_n^p < \infty$, then

$$\left(1+\frac{1}{1-p}\right)(a_1+a_2+\ldots)^p + \sum_{n=2}^\infty \left(\frac{a_n+a_{n+1}+\ldots}{n}\right)^p \geq \left(\frac{p}{1-p}\right)^p \Sigma_{n=1}^\infty a_n^p.$$

What's more, Hardy et al. (1934) also proposed other inequalities, a part of them is selected and reorganised as follows.
(iii) (Theorem 329 in Hardy et al. (1934)) If $p > 1$, $h(t) \geq 0$, $r > 0$, define

$$h_r(x) = \int_0^x (x-t)^{r-1}h(t)dt.$$

Then

$$\int_0^\infty \left(\frac{h_r(x)}{x^r}\right)^p dx \leq B^p\left(r, 1-\frac{1}{p}\right)\int_0^\infty h^p(x)dx. \tag{1.2}$$

The B means Beta function, and its expression is $B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt$. Define

$$g_r(x) = \int_x^\infty (t-x)^{r-1}h(t)dt,$$

then

$$\int_0^\infty g_r^p(x)dx \leq B^p\left(r, \frac{1}{p}\right)\int_0^\infty (x^r h(x))^p\, dx. \tag{1.3}$$

(iv) (Theorem 330 in Hardy et al. (1934)) If $p > 1$, $r \neq 1$, $h(x) \geq 0$, then

$$\int_0^\infty x^{-r}H^p(x)dx \leq \left(\frac{p}{|r-1|}\right)^p \int_0^\infty x^{-r}(xh(x))^p dx, \tag{1.4}$$

where $H(x)$ is defined by

$$H(x) = \begin{cases} \int_0^x h(t)dt, & r > 1; \\ \int_x^\infty h(t)dt, & r < 1. \end{cases}$$

It is their work that motivates our study. In this paper, based on the technology in Hardy et al. (1934) and the ideas of Klaassen and Wellner (2021), we give the corresponding probability inequalities from (1.1) to (1.4).

The rest of the paper is organized as follows. We present our main results in Section 2, and complete proofs are given in Section 3.

2. Main results

When $0 < p < 1$, using Hölder's inequality, we can get the probability version of Hardy's inequality (1.1).

Theorem 2.1. *Let X and Y be independent and identically distributed random variables with absolutely continuous distribution function* F *with respect to Lebesgue measure on* $(\mathbb{R}, \mathfrak{B})$*, and let h be a nonnegative measurable function on* $(\mathbb{R}, \mathfrak{B})$, $\mathrm{E}h(Y) < \infty$*. For $0 < p < 1$, we have*

$$\mathrm{E}\left[\frac{\mathrm{E}(h(Y)\mathbb{I}_{\{Y>X\}}|X)}{\mathrm{F}(X)}\right]^p \geq \left(\frac{p}{1-p}\right)^p \mathrm{E}h^p(Y).$$

In fact, recently a more general version of Theorem 2.1 has been published as Theorem 3.1 in Klaassen (2023). Using Theorem 2.1, under stronger conditions, we can get Hardy's inequality (1.1).

Corollary 2.2. *For $0 < p < 1$, $h(x) \geq 0$, $\int_0^\infty h(t)dt < \infty$, (1.1) holds.*

Proof. Similar to the proof in Klaassen and Wellner (2021), we take X and Y with the uniform distribution on $[0, K]$, where K is a large positive number, which means that their distribution function is $\mathrm{F}(x) = x/K$. When $K \to \infty$, using Theorem 2.1, we can get (1.1). □

The preceding narration has given a lower bound of $\mathrm{E}\left[\frac{\mathrm{E}(h(Y)\mathbb{I}_{\{Y>X\}}|X)}{\mathrm{F}(X)}\right]^p$. With some conditions, we also show an upper bound of it.

Theorem 2.3. *Let X and Y be independent and identically distributed random variables with absolutely continuous distribution function* F *with respect to Lebesgue measure on* $(\mathbb{R}, \mathfrak{B})$*, and let h be a nonnegative, nondecreasing measurable function on* $(\mathbb{R}, \mathfrak{B})$*. For $0 < p < 1$, we have*

$$\mathrm{E}\left[\frac{\mathrm{E}(h(Y)\mathbb{I}_{\{Y>X\}}|X)}{\mathrm{F}(X)}\right]^p \leq \frac{p}{1-p}\mathrm{E}\left[h^p(Y)\left(\frac{\mathrm{F}(Y)}{1-\mathrm{F}(Y)}\right)^{1-p}\right].$$

Corresponding to (1.2) and (1.3), we provide the probability form as listed below.

Theorem 2.4. *Let X and Y be independent and identically distributed random variables with absolutely continuous distribution function* F *with respect to Lebesgue measure on* $(\mathbb{R}, \mathfrak{B})$*, and let h be a nonnegative measurable function on* $(\mathbb{R}, \mathfrak{B})$*, $r > 0$, then for $p > 1$, we have*

$$\mathrm{E}\left[\frac{\mathrm{E}\left(h_r(X,Y)\mathbb{I}_{\{\mathrm{F}(X)\leq\mathrm{F}(Y)\}}|Y\right)}{\mathrm{F}^r(Y)}\right]^p \leq B^p\left(r, 1-\frac{1}{p}\right)\mathrm{E}h^p(X),$$

where

$$h_r(x,y) = h(x)\left(\mathrm{F}(y) - \mathrm{F}(x)\right)^{r-1}.$$

We also have

$$\mathrm{E}\left[\mathrm{E}\left(\frac{g_r(X,Y)}{\mathrm{F}^r(X)}\mathbb{I}_{\{\mathrm{F}(X)\geq\mathrm{F}(Y)\}}\middle|Y\right)\right]^p \leq B^p\left(r, \frac{1}{p}\right)\mathrm{E}h^p(X),$$

where

$$g_r(x,y) = h(x)\left(\mathrm{F}(x) - \mathrm{F}(y)\right)^{r-1}.$$

Similarly, for a large positive number K, take X and Y with the uniform distribution on $[0, K]$. Using Theorem 2.4, when $K \to \infty$, we can get (1.2) and (1.3).

We also give the corresponding version of (1.4) in probability language.

Theorem 2.5. *Let X and Y be independent and identically distributed random variables with absolutely continuous distribution function F with respect to Lebesgue measure on $(\mathbb{R}, \mathfrak{B})$, and let h be a nonnegative measurable function on $(\mathbb{R}, \mathfrak{B})$, $r \neq 1$. Then for $p > 1$, we get*

$$\mathrm{E}\left[\mathrm{F}^{-r}(Y)G_r^p(Y)\right] \leq \left(\frac{p}{|r-1|}\right)^p \mathrm{E}\left[\mathrm{F}^{-r}(Y)(\mathrm{F}(Y)h(Y))^p\right],$$

where $G_r(y)$ is defined by

$$G_r(y) = \mathrm{E}[h(X)(\mathbb{I}_{\{\mathrm{F}(X)\leq \mathrm{F}(y), r>1\}} + \mathbb{I}_{\{\mathrm{F}(X)\geq \mathrm{F}(y), r<1\}})].$$

Similarly, for a large positive number K, let the distribution function of X and Y be $\mathrm{F}(x) = x/K$. When $K \to \infty$, we can get (1.4) through Theorem 2.5.

For proving Theorem 2.4 and Theorem 2.5, we need the following results.

Lemma 2.6. *Let X and Y be independent and identically distributed random variables with absolutely continuous distribution function F with respect to Lebesgue measure on $(\mathbb{R}, \mathfrak{B})$, and let h and g be nonnegative measurable functions on $(\mathbb{R}, \mathfrak{B})$. Nonzero constants p and q satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Let $K(x,y)$ be a bivariate function on $(\mathbb{R} \times \mathbb{R}, \mathfrak{B})$, which satisfies the following properties:*
(a) $K(x,y)$ is nonnegative measurable and homogeneous of degree -1, which means for arbitrary constant $c > 0$, $K(cx, cy) = \frac{1}{c}K(x, y)$ holds;
(b)

$$\int_0^{+\infty} K(x, 1)x^{-1/p}dx = \int_0^{+\infty} K(1, y)y^{-1/q}dy = k.$$

When $p > 1$, we have

$$\mathrm{E}(K(\mathrm{F}(X), \mathrm{F}(Y))h(X)g(Y)) \leq k(\mathrm{E}h^p(X))^{1/p}(\mathrm{E}g^q(Y))^{1/q}.$$

Remark 2.7. *Obviously, based on Lemma 2.6, if we let $g(y) = (\mathrm{E}(K(F(X), F(y))h(X)))^{p-1} = (\mathrm{E}(K(F(X), F(Y))h(X)|Y = y))^{p-1}$, we can get the following results*

$$\mathrm{E}[\mathrm{E}(K(\mathrm{F}(X), \mathrm{F}(Y))h(X)|Y)]^p \leq k^p \mathrm{E}h^p(X) \tag{2.1}$$

and

$$\mathrm{E}[\mathrm{E}(K(\mathrm{F}(X), \mathrm{F}(Y))g(Y)|X)]^q \leq k^q \mathrm{E}g^q(Y). \tag{2.2}$$

Using the above results, we can prove Theorem 2.4 and Theorem 2.5.

3. Proof of main results

3.1. *Proof of Theorem 2.1*

Define $G(x) = \int_x^{+\infty} h(t)d\mathrm{F}(t)$, then we can get

$$\begin{aligned}
\int_{-\infty}^{+\infty} \left(\frac{G(x)}{\mathrm{F}(x)}\right)^p d\mathrm{F}(x) &= \int_{-\infty}^{+\infty}\int_x^{+\infty} pG^{p-1}(y)h(y)d\mathrm{F}(y)\mathrm{F}^{-p}(x)d\mathrm{F}(x) \\
&= p\int_{-\infty}^{+\infty}\int_{-\infty}^{y} \mathrm{F}^{-p}(x)d\mathrm{F}(x)G^{p-1}(y)h(y)d\mathrm{F}(y) \\
&= \frac{p}{1-p}\int_{-\infty}^{+\infty}\left(\frac{G(y)}{\mathrm{F}(y)}\right)^{p-1} h(y)d\mathrm{F}(y).
\end{aligned}$$

Using Hölder's inequality for $0 < p < 1$, we have

$$\int_{-\infty}^{+\infty} \left(\frac{G(x)}{F(x)}\right)^{p-1} h(x)dF(x) \geq \left(\int_{-\infty}^{+\infty} \left(\frac{G(x)}{F(x)}\right)^{p} dF(x)\right)^{1/q} \cdot \left(\int_{-\infty}^{+\infty} h^p(x)dF(x)\right)^{1/p},$$

where q satisfies $\frac{1}{p} + \frac{1}{q} = 1$. Combining the above two results, the proof is finished.

3.2. *Proof of Theorem 2.3*

Let f be the density of F. Since $0 < p < 1$, from the monotonicity of h and x^{p-1}, for Lebesgue almost $x \in \mathbb{R}$, we have

$$\begin{aligned}
\frac{d}{dx}\left(\int_{x}^{+\infty} h(y)dF(y)\right)^p &= -p\left(\int_{x}^{+\infty} h(y)dF(y)\right)^{p-1} h(x)f(x) \\
&\geq -ph^p(x)(1-F(x))^{p-1}f(x).
\end{aligned}$$

Take integration over the left hand side and by the definition of conditional expectation, we can get

$$\begin{aligned}
E\left[\frac{E(h(Y)\mathbb{I}_{\{Y>X\}}|X)}{F(X)}\right]^p &= \int_{-\infty}^{+\infty}\left[\frac{\int_{x}^{+\infty} h(y)dF(y)}{F(x)}\right]^p dF(x) \\
&\leq p\int_{-\infty}^{+\infty}\left(\int_{x}^{+\infty} h^p(y)(1-F(y))^{p-1}dF(y)\right)F^{-p}(x)dF(x) \\
&= p\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{y} F^{-p}(x)dF(x)\right)h^p(y)(1-F(y))^{p-1}dF(y) \\
&= \frac{p}{1-p}\int_{-\infty}^{+\infty} h^p(y)\left(\frac{F(y)}{1-F(y)}\right)^{1-p} dF(y) \\
&= \frac{p}{1-p}E\left[h^p(Y)\left(\frac{F(Y)}{1-F(Y)}\right)^{1-p}\right].
\end{aligned}$$

In this way, we finish our proof.

3.3. *The proof of Lemma 2.6*

Similar to the proof of Theorem 319 in Hardy et al. (1934), when $p > 1$, using Hölder's inequality, we can get

$$\begin{aligned}
& E(K(F(X), F(Y))h(X)g(Y)) \\
= & \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} K(F(x), F(y))h(x)g(y)dF(x)dF(y) \\
= & \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(x)K^{1/p}(F(x), F(y))\left(\frac{F(x)}{F(y)}\right)^{1/pq} \\
\times & \; g(y)K^{1/q}(F(x), F(y))\left(\frac{F(y)}{F(x)}\right)^{1/pq} dF(x)dF(y) \\
\leq & \left(\int_{-\infty}^{+\infty} h^p(x)\int_{-\infty}^{+\infty} K(F(x), F(y))\left(\frac{F(x)}{F(y)}\right)^{1/q} dF(y)dF(x)\right)^{1/p} \\
\times & \left(\int_{-\infty}^{+\infty} g^q(y)\int_{-\infty}^{+\infty} K(F(x), F(y))\left(\frac{F(y)}{F(x)}\right)^{1/p} dF(x)dF(y)\right)^{1/q}
\end{aligned}$$

$$
\begin{aligned}
&= \left(\int_{-\infty}^{+\infty} h^p(x) \int_{-\infty}^{+\infty} K\left(1, \frac{F(y)}{F(x)}\right) \frac{1}{F(x)} \left(\frac{F(y)}{F(x)}\right)^{-1/q} dF(y)dF(x)\right)^{1/p} \\
&\times \left(\int_{-\infty}^{+\infty} g^q(y) \int_{-\infty}^{+\infty} K\left(\frac{F(x)}{F(y)}, 1\right) \frac{1}{F(y)} \left(\frac{F(x)}{F(y)}\right)^{-1/p} dF(x)dF(y)\right)^{1/q} \\
&= \left(\int_{-\infty}^{+\infty} h^p(x) \int_{0}^{1/F(x)} K(1, v)\, v^{-1/q} dv dF(x)\right)^{1/p} \\
&\times \left(\int_{-\infty}^{+\infty} g^q(y) \int_{0}^{1/F(y)} K(u, 1)\, u^{-1/p} du dF(y)\right)^{1/q} \\
&\leq \left(\int_{-\infty}^{+\infty} h^p(x) dF(x)\right)^{1/p} \left(\int_{0}^{+\infty} K(1, v) v^{-1/q} dv\right)^{1/p} \\
&\times \left(\int_{-\infty}^{+\infty} g^q(y) dF(y)\right)^{1/q} \left(\int_{0}^{+\infty} K(u, 1) u^{-1/p} du\right)^{1/q} \\
&= k\left(\int_{-\infty}^{+\infty} h^p(x) dF(x)\right)^{1/p} \left(\int_{-\infty}^{+\infty} g^q(y) dF(y)\right)^{1/q} \\
&= k(Eh^p(X))^{1/p}(Eg^q(Y))^{1/q}.
\end{aligned}
$$

Here, we let $u = F(x)/F(y)$ and $v = F(y)/F(x)$ in the third equality from the bottom. Thus we finish our proof.

3.4. *Proof of Theorem 2.4*

When $h_r(x, y) = h(x)\left(F(y) - F(x)\right)^{r-1}$, let the K function in Lemma 2.6 be

$$
K(x, y) = \begin{cases} \frac{(y-x)^{r-1}}{y^r}, & y \geq x \geq 0; \\ 0, & else. \end{cases}
$$

It is easy to verify that $K(x, y)$ satisfies the conditions in Lemma 2.6 and $k = B\left(r, 1 - \frac{1}{p}\right)$. Using this $K(x, y)$, we have

$$
\begin{aligned}
&E[E(K(F(X), F(Y))h(X)|Y)]^p \\
&= \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} h(x) \frac{(F(y) - F(x))^{r-1}}{F^r(y)} \mathbb{I}_{\{F(x) \leq F(y)\}} dF(x)\right]^p dF(y) \\
&= \int_{-\infty}^{+\infty} \left[\frac{1}{F^r(y)} \int_{-\infty}^{+\infty} h(x)(F(y) - F(x))^{r-1} \mathbb{I}_{\{F(x) \leq F(y)\}} dF(x)\right]^p dF(y) \\
&= E\left[\frac{E\left(h_r(X, Y)\mathbb{I}_{\{F(X) \leq F(Y)\}}|Y\right)}{F^r(Y)}\right]^p.
\end{aligned}
$$

Then from (2.1), we finish the proof of the first result.

When $g_r(x, y) = h(x)\left(F(x) - F(y)\right)^{r-1}$, let the K function in Lemma 2.6 be

$$
K(x, y) = \begin{cases} \frac{(x-y)^{r-1}}{x^r}, & x \geq y \geq 0; \\ 0, & else. \end{cases}
$$

It is easy to verify that $K(x, y)$ satisfies the conditions in Lemma 2.6 and $k = B\left(r, \frac{1}{p}\right)$. Using this $K(x, y)$, we have

$$\begin{aligned}
& \mathrm{E}[\mathrm{E}(K(\mathrm{F}(X), \mathrm{F}(Y))h(X)|Y)]^p \\
= & \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} h(x)\frac{(\mathrm{F}(x) - \mathrm{F}(y))^{r-1}}{\mathrm{F}^r(x)}\mathbb{I}_{\{\mathrm{F}(x)\geq \mathrm{F}(y)\}}d\mathrm{F}(x)\right]^p d\mathrm{F}(y) \\
= & \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} \frac{h(x)}{\mathrm{F}^r(x)}(\mathrm{F}(x) - \mathrm{F}(y))^{r-1}\mathbb{I}_{\{\mathrm{F}(x)\geq \mathrm{F}(y)\}}d\mathrm{F}(x)\right]^p d\mathrm{F}(y) \\
= & \mathrm{E}\left[\mathrm{E}\left(\frac{g_r(X, Y)}{\mathrm{F}^r(X)}\mathbb{I}_{\{\mathrm{F}(X)\geq \mathrm{F}(Y)\}}\middle| Y\right)\right]^p.
\end{aligned}$$

Then from (2.1), the proof of the second result is finished.

3.5. Proof of Theorem 2.5

When $r > 1$, let the constant $\alpha = 1 - r/p$ and $h_\alpha(x) = h(x)/\mathrm{F}^\alpha(x)$. Let the K function in Lemma 2.6 be

$$K(x, y) = \begin{cases} \frac{y^{\alpha-1}}{x^\alpha}, & y \geq x \geq 0; \\ 0, & else. \end{cases}$$

It is easy to verify that $K(x, y)$ satisfies the conditions in Lemma 2.6 and $k = \frac{p}{(1-\alpha)p-1}$. Using this $K(x, y)$, we have

$$\begin{aligned}
& \mathrm{E}[\mathrm{E}(K(\mathrm{F}(X), \mathrm{F}(Y))h(X)|Y)]^p \\
= & \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} \frac{h(x)\mathrm{F}^{\alpha-1}(y)}{\mathrm{F}^\alpha(x)}\mathbb{I}_{\{\mathrm{F}(x)\leq \mathrm{F}(y)\}}d\mathrm{F}(x)\right]^p d\mathrm{F}(y) \\
= & \int_{-\infty}^{+\infty} \left[\mathrm{F}^{\alpha-1}(y)\int_{-\infty}^{+\infty} \frac{h(x)}{\mathrm{F}^\alpha(x)}\mathbb{I}_{\{\mathrm{F}(x)\leq \mathrm{F}(y)\}}d\mathrm{F}(x)\right]^p d\mathrm{F}(y) \\
= & \mathrm{E}\left[\mathrm{F}^{\alpha-1}(Y)\mathrm{E}(h_\alpha(X)\mathbb{I}_{\{\mathrm{F}(X)\leq \mathrm{F}(Y)\}}|Y)\right]^p.
\end{aligned}$$

Then from (2.1), we can get the following result:

$$\mathrm{E}\left[\mathrm{F}^{-(1-\alpha)p}(Y)\mathrm{E}^p(h_\alpha(X)\mathbb{I}_{\{\mathrm{F}(X)\leq \mathrm{F}(Y)\}}|Y)\right] \leq \left(\frac{p}{(1-\alpha)p-1}\right)^p \mathrm{E}h^p(X).$$

If we replace $h_\alpha(x)$ by $h(x)$, we can get

$$\mathrm{E}\left[\mathrm{F}^{-(1-\alpha)p}(Y)\mathrm{E}^p(h(X)\mathbb{I}_{\{\mathrm{F}(X)\leq \mathrm{F}(Y)\}}|Y)\right] \leq \left(\frac{p}{(1-\alpha)p-1}\right)^p \mathrm{E}\left[\mathrm{F}^{-(1-\alpha)p}(X)(\mathrm{F}(X)h(X))^p\right].$$

Note that $r = (1 - \alpha)p > 1$, we finish our proof when $r > 1$.

When $r < 1$, let the constant $\beta = r/q$ and $g_\beta(y) = g(y)\mathrm{F}^{\beta-1}(y)$. Let the K function in Lemma 2.6 be

$$K(x, y) = \begin{cases} \frac{y^{\beta-1}}{x^\beta}, & y \geq x \geq 0; \\ 0, & else. \end{cases}$$

It is easy to verify that $K(x, y)$ satisfies the conditions in Lemma 2.6 and $k = \frac{q}{1-\beta q}$. Using this $K(x, y)$, we can get

$$\begin{aligned}
& \mathrm{E}[\mathrm{E}(K(\mathrm{F}(X), \mathrm{F}(Y))g(Y)|X)]^q \\
= & \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} \frac{g(y)\mathrm{F}^{\beta-1}(y)}{\mathrm{F}^\beta(x)}\mathbb{I}_{\{\mathrm{F}(y)\geq \mathrm{F}(x)\}}d\mathrm{F}(y)\right]^q d\mathrm{F}(x) \\
= & \int_{-\infty}^{+\infty} \left[\mathrm{F}^{-\beta}(x)\int_{-\infty}^{+\infty} g(y)\mathrm{F}^{\beta-1}(y)\mathbb{I}_{\{\mathrm{F}(y)\geq \mathrm{F}(x)\}}d\mathrm{F}(y)\right]^q d\mathrm{F}(x) \\
= & \mathrm{E}\left[\mathrm{F}^{-\beta}(X)\mathrm{E}(g_\beta(Y)\mathbb{I}_{\{\mathrm{F}(Y)\geq \mathrm{F}(X)\}}|X)\right]^q.
\end{aligned}$$

Then from (2.2), we can get the following result:

$$\mathrm{E}\left[\mathrm{F}^{-\beta q}(X)\mathrm{E}^{q}(g_{\beta}(Y)\mathbb{I}_{\{\mathrm{F}(Y)\geq \mathrm{F}(X)\}}|X)\right] \leq \left(\frac{q}{1-\beta q}\right)^{q} \mathrm{E}g^{q}(Y).$$

If we replace $g_{\beta}(y)$ by $g(y)$ and exchange X and Y, we can get

$$\mathrm{E}\left[\mathrm{F}^{-\beta q}(Y)\mathrm{E}^{q}(g(X)\mathbb{I}_{\{\mathrm{F}(X)\geq \mathrm{F}(Y)\}}|Y)\right] \leq \left(\frac{q}{1-\beta q}\right)^{q} \mathrm{E}\left[\mathrm{F}^{-\beta q}(X)(\mathrm{F}(X)g(X))^{q}\right].$$

Note that $r = \beta q < 1$ and the equivalence of p and q, we can finish the proof when $r < 1$.

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