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Fixed points theorems for (φ, ψ, p)-weakly contractive mappings via w-distance in relational metric spaces with applications

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Abstract. In this paper, we establish certain fixed point theorems for (φ, ψ, p)-weakly contractive mappings on relational metric spaces by using the notion of w-distance and locally S-transitivity of binary relation. Our results generalize the results of Alam and Imdad *(J. Fixed Point Theory Appl., 17(4), 693–702, 2015)*, Senapati and Dey *(J. Fixed Point Theory Appl., 19, 2945–2961, 2017)* and many others of the existing literature. Moreover, we have an application to the nonlinear Fredholm integral equations and some illustrative examples to reveal the usability of our findings.

1. Introduction

The classical Banach contraction principle is one of the pivotal result of analysis, which states that every contraction mapping on a complete metric space (X, d) has a unique fixed point. In 1997, Alber and Guerre-Delabariere [4] generalized the Banach contraction principle in setting of Hilbert spaces, which was further extended by Rhoades [21] for arbitrary complete metric spaces. A self mapping $S : M \to M$ is said to be weakly contractive mapping, if for all $\nu, \mu \in M$

$$d(S\nu, S\mu) \leq d(\nu, \mu) - \psi d(\nu, \mu), \tag{1}$$

where $\psi : [0, \infty) \to [0, \infty)$ is a continuous and non-decreasing function such that $\psi(t) = 0$ if and only if $t = 0$.

Remark 1.1. *Notice that the condition* (1) *is weaker than the Banach contraction condition, that is* $d(S\nu, S\mu) \leq \lambda d(\nu, \mu)$ *for* $\lambda \in [0, 1)$. *One can take* $\psi(t) = (1 - \lambda)t$, *where* $\lambda \in [0, 1)$ *in condition* (1) *then this condition reduces to the Banach contraction condition.*

Motivated by the work of Alber and Guerre-Delabariere [4], and Rhoades [21], Dutta and Chaudhary [8] generalized the classical Banach contraction principle for weak contractive mappings on complete metric spaces.

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Theorem 1.2 (Dutta and Chaudhary). *Let (M, d) be a complete metric space and let $S : M \to M$ be a self-mapping satisfying the inequality*

$$\varphi(d(S\nu, S\mu) \leq \varphi(d(\nu, \mu)) - \psi(d(\nu, \mu)), \tag{2}$$

where $\psi, \varphi : [0, \infty) \to [0, \infty)$ are both continuous and monotonic non-decreasing functions with $\psi(t) = 0 = \varphi(t)$ if and only if $t = 0$. Then S has a unique fixed point.

On the other hand, Nieto and Rodríguez-López [14, 15], and Ran and Reurings [20] extended the Banach contraction principle to partially ordered metric spaces, which was further extended by Alam and Imdad [1] for a metric space endowed with amorphous binary relation. Now, many researchers have extended and generalized the result of Alam and Imdad in different ways (see, [2], [5]-[7], [11], [16], [17] and reference therein). Subsequently, Senapati and Dey [23] improved the result of Alam and Imdad [1] by using the notion of w-distance, which was recently refined by Gopi and Khantwal [18] for nonlinear contractions on relational metric spaces.

Our aim in this paper is to establish some fixed point theorems in relational metric spaces by utilizing the notions of w-distance, altering distance function and employing locally S-transitivity of binary relation. Our results extend and generalize the results of Alam and Imdad [1], Senapati and Dey [23] and many others in the existing literature. Moreover, we extend our findings for the existence and uniqueness of solution for nonlinear Fredholm integral equations and also give some illustrative examples to support our results.

2. Preliminaries

Throughout the paper, we follow that M, $\mathfrak{R}$, $\mathbb{N}$ and $\mathbb{N}_0$ stand for a non-empty set, non-empty binary relation on M, the set of non-zero positive integers and the set of whole numbers, respectively. Let us recall, the following notations and relevant results.

Definition 2.1. *[13] Let $\mathfrak{R}$ be a binary relation defined on M.*

(i) If $\mathfrak{R}$ is a subset of $M \times M$. We say that ν is $\mathfrak{R}$-related to μ if and only if $(\nu, \mu) \in \mathfrak{R}$.

(ii) ν and μ are $\mathfrak{R}$-comparable if either $(\nu, \mu) \in \mathfrak{R}$ or $(\mu, \nu) \in \mathfrak{R}$. We denote it by $[\nu, \mu] \in \mathfrak{R}$.

Definition 2.2. *[3] For a binary relation $\mathfrak{R}$ defined on M.*

(i) The inverse or transpose or dual relation of $\mathfrak{R}$, denoted by $\mathfrak{R}^{-1}$ is defined by $\mathfrak{R}^{-1} = \{(\nu, \mu) \in M^2 : (\mu, \nu) \in \mathfrak{R}\}$.
(ii) The symmetric closure $\mathfrak{R}^s$ is the smallest symmetric relation containing $\mathfrak{R}$, i.e., $\mathfrak{R}^s = \mathfrak{R} \cup \mathfrak{R}^{-1}$.

Definition 2.3. *[1] For a binary relation $\mathfrak{R}$ defined on M.*

(i) $(\nu, \mu) \in \mathfrak{R}^s \iff [\nu, \mu] \in \mathfrak{R}$.
(ii) $\mathfrak{R}$ is called S-closed if for any $\nu, \mu \in M$, $(\nu, \mu) \in \mathfrak{R} \Rightarrow (S\nu, S\mu) \in \mathfrak{R}$.
(iii) If $\mathfrak{R}$ is S-closed, then $\mathfrak{R}^s$ is also S-closed.
(iv) If $\mathfrak{R}$ is S-closed, then $\forall\, n \in \mathbb{N}_0$, $\mathfrak{R}$ is also S^n-closed where S^n denotes nth iterate of S.
(iv) A sequence $\{\nu_n\} \subset M$ is called $\mathfrak{R}$-preserving if $(\nu_n, \nu_{n+1}) \in \mathfrak{R}$, $n \in \mathbb{N}_0$.
(iv) $\mathfrak{R}$ is called d-self-closed if whenever $\{\nu_n\}$ is an $\mathfrak{R}$-preserving sequence and $\nu_n \xrightarrow{d} \nu$ as $n \to \infty$, then there exists a subsequences $\{\nu_{n_k}\}$ of $\{\nu_n\}$ with $[\nu_{n_k}, \nu] \in \mathfrak{R}$, for all $k \in \mathbb{N}$.

Definition 2.4. *[3] A subset D of a non-empty set M is called $\mathfrak{R}$-connected if for each pair $\nu, \mu \in D$, there exists a path in $\mathfrak{R}$ from ν to μ.*

Definition 2.5. *[22] A subset D of a non-empty set M is called $\mathfrak{R}$-directed if for each pair $\nu, \mu \in D$, there exists $\varrho \in M$ such that $(\nu, \varrho) \in \mathfrak{R}$ and $(\mu, \varrho) \in \mathfrak{R}$.*

Inspired by Turinici [24], Alam and Imdad [3] introduce the notion of locally transitive by localizing the notion of transitivity. Further, for the notion of $\mathfrak{R}$-completeness and $\mathfrak{R}$-continuity, we may refer to [2].

Definition 2.6. *[12] Let $\mathfrak{R}$ be a binary relation on a non-empty set M. For $\nu, \mu \in M$, a path of length $k \in \mathbb{N}$ in $\mathfrak{R}$ from ν to μ is a finite sequence $\{\varrho_0, \varrho_1, \dots, \varrho_k\} \subseteq M$ satisfying:*

(i) $\varrho_0 = \nu$ *and* $\varrho_k = \mu$;

(ii) $(\varrho_i, \varrho_{i+1}) \in \mathfrak{R}\ \forall\ i \in \{0, 1, 2, \dots, k-1\}$.

Definition 2.7. *[10] A function $\varphi : [0, \infty) \to [0, \infty)$ is said to be an altering distance function if it satisfies the following conditions:*

(i) φ is monotone increasing and continuous,

(ii) $\varphi(t) = 0 \iff t = 0$.

The notion of $\mathfrak{R}$-lower semi-continuity (briefly, $\mathfrak{R}$-LSC) of a function is defined by Senapati and Dey [23]. The respective authors explained by giving examples that the $\mathfrak{R}$-LSC is weaker than both the $\mathfrak{R}$-continuity and the lower semi-continuity (see for details [23]) and modified the definition of w-distance (Definition 2.8) and the related Lemma 1 given in [9] in relation to the metric spaces endowed with an arbitrary binary relation $\mathfrak{R}$.

Definition 2.8. *[23] Let (M, d) be a metric space and $\mathfrak{R}$ be a binary relation on M. A function $p : M \times M \to [0, \infty)$ is said to be a w-distance on M if*

(w_1) $p(\nu, \varrho) \le p(\nu, \mu) + p(\mu, \varrho)$, for any $\nu, \mu, \varrho \in M$;
(w_2) for any $\nu \in M$, $p(\nu, .) : M \to [0, \infty)$ is $\mathfrak{R}$-lower semi-continuous;
(w_3) for any $\epsilon > 0$, there exists $\delta > 0$, such that $p(\varrho, \nu) \le \delta$ and $p(\varrho, \mu) \le \delta$ implies $d(\nu, \mu) \le \epsilon$.

The following lemma is required in our subsequent discussion.

Lemma 2.9. *[19, 24] Let p be a w-distance on metric space (M, d) and $\{\nu_n\}$ a sequence in M. If $\{\nu_n\}$ is not a Cauchy, then there exist $\epsilon > 0$ and subsequences $\{\nu_{n_k}\}$ and $\{\nu_{m_k}\}$ of $\{\nu_n\}$ such that*

(i) $k \le n_k \le m_k$, for all $k \in \mathbb{N}$,
(ii) $p(\nu_{n_k}, \nu_{m_k}) > \epsilon$, for all $k \in \mathbb{N}$,
(iii) $p(\nu_{n_k}, \nu_{m_{k-1}}) \le \epsilon$, for all $k \in \mathbb{N}$.

Moreover, suppose that $\lim_{n\to\infty} p(\nu_{n_k}, \nu_{m_k}) = 0$, then

(iv) $\lim_{n\to\infty} p(\nu_{n_k}, \nu_{m_k}) = \epsilon$,
(v) $\lim_{n\to\infty} p(\nu_{n_{k+1}}, \nu_{m_{k+1}}) = \epsilon$.

3. Main Results

Now, we drive our main result on the existence of fixed points for the class of nonlinear contractions by using the notion of w-distance and employing the S-transitive binary relation on metric spaces.

Theorem 3.1. *Let (M, d) be a metric space endowed with an arbitrary binary relation $\mathfrak{R}$ and p be a w-distance on M. Assume that $S : M \to M$ be a mapping and the following conditions are hold:*

(a) there exists $Y \subseteq M$, $S(M) \subseteq Y \subset M$ such that (Y, d) is $\mathfrak{R}$-complete,
(b) $\mathfrak{R}$ is S-closed and locally S-transitivity,
(c) either $\mathfrak{R}|_Y$ is d-self closed or S is $\mathfrak{R}$-continuous,
(d) $M(S, \mathfrak{R})$ is non-empty,
(e) for all $\nu, \mu \in M$ with $(\nu, \mu) \in \mathfrak{R}$,

$$\varphi(p(S\nu, S\mu)) \le \varphi(p(\nu, \mu)) - \psi(p(\nu, \mu)), \tag{3}$$

where $\varphi : [0, \infty) \to [0, \infty)$ is an altering distance function and $\psi : [0, \infty) \to [0, \infty)$ is a lower semi-continuous with $\psi(t) > 0$, for $t \in (0, \infty)$ and $\psi(t) = 0$, for $t = 0$. Then S has a fixed point.

Proof. We have $M(S,\mathfrak{R})$ is non-empty. Let $\nu_0 \in M(S,\mathfrak{R})$ then $(\nu_0, S\nu_0) \in \mathfrak{R}$. Construct a sequence $\{\nu_n\}$ of Picard iterates with initial point ν_0 and

$$\nu_n = S^n(\nu_0), \text{ for all } n \in \mathbb{N}_0. \tag{4}$$

Since $(\nu_0, S\nu_0) \in \mathfrak{R}$ and $\mathfrak{R}$ is S-closed therefore

$$(S\nu_0, S^2\nu_0), (S^2\nu_0, S^3\nu_0), \ldots, (S^n\nu_0, S^{n+1}\nu_0), \cdots \in \mathfrak{R},$$

and

$$(\nu_n, \nu_{n+1}) \in \mathfrak{R}, \quad \text{for all } n \in \mathbb{N}_0, \tag{5}$$

which yields $\{\nu_n\}$ is $\mathfrak{R}$-preserving sequence. From (3.1), we have

$$\varphi(p(\nu_n, \nu_{n+1})) \leq \varphi(p(\nu_{n-1}, \nu_n)) - \psi(p(\nu_{n-1}, \nu_n)) \leq \varphi(p(\nu_{n-1}, \nu_n)), \tag{6}$$

and by monotonicity of function φ, we get

$$p(\nu_n, \nu_{n+1}) \leq p(\nu_{n-1}, \nu_n).$$

It follows that $\{p(\nu_n, \nu_{n+1})\}$ is bounded monotonic decreasing sequence of positive numbers, so there exists $r \geq 0$ such that

$$\lim_{n\to\infty} p(\nu_n, \nu_{n+1}) = r.$$

Letting $n \to \infty$ in (6) and using continuity of functions ψ and φ, we obtain

$$\varphi(r) \leq \varphi(r) - \psi(r) \leq \varphi(r),$$

which implies that $\psi(r) = 0$. Thus $r = 0$ and

$$\lim_{n\to\infty} p(\nu_n, \nu_{n+1}) = 0. \tag{7}$$

Now, we shall show that $\{\nu_n\}$ is a Cauchy sequence. On contrary, we suppose that the sequence $\{\nu_n\}$ is not Cauchy. Then, in view of Lemma 2.9, there exist $\epsilon > 0$ and sub-sequences $\{\nu_{n_k}\}$, $\{\nu_{m_k}\}$ of $\{\nu_n\}$ such that

$$k \leq n_k \leq m_k, \;\; \vartheta_k = p(\nu_{n_k}, \nu_{m_k}) > \epsilon, \;\; p(\nu_{n_k}, \nu_{m_{k-1}}) \leq \epsilon, \;\; \text{for all } k, m, n \in \mathbb{N},$$

and

$$\lim_{k\to\infty} \vartheta_k = \lim_{k\to\infty} p(\nu_{n_k}, \nu_{m_k}) = \epsilon, \quad \lim_{k\to\infty} \vartheta_{k+1} = \lim_{k\to\infty} p(\nu_{n_{k+1}}, \nu_{m_{k+1}}) = \epsilon. \tag{8}$$

Applying the assumption of locally S-transitivity of binary relation $\mathfrak{R}$ yields $(\nu_{n_k}, \nu_{m_k}) \in \mathfrak{R}$ and from assumption (e), we have

$$\varphi(\vartheta_{k+1}) \leq \varphi(\vartheta_k) - \psi(\vartheta_k). \tag{9}$$

Letting $k \to \infty$ in (9) and using (8), we get

$$\varphi(\epsilon) \leq \varphi(\epsilon) - \psi(\epsilon),$$

which is possible unless $\psi(\epsilon) = 0$ implies $\epsilon = 0$. Therefore, by (L_3) of Lemma 1.19 in [23], we have the sequence $\{\nu_n\}$ is $\mathfrak{R}$-preserving Cauchy in Y. As (Y, d) is $\mathfrak{R}$-complete, we must have a point $\nu \in Y$ such that $\nu_n \to \nu$ as $n \to \infty$.

Now, we claim that ν is a fixed point of S. Since $\{\nu_n\}$ is $\mathfrak{R}$-preserving with $\nu_n \xrightarrow{p} \nu$, then $\mathfrak{R}$-continuity of S implies $\nu_{n+1} = S\nu_n \xrightarrow{p} S\nu$ and by unicity of the limit, we get $S\nu = \nu$ that is, ν is a fixed point of S.

Alternately, we assume that $\mathfrak{R}|_Y$ is d-self closed. So there exists a sub-sequence $\{\nu_{n_k}\}$ of $\{\nu_n\}$ with $[\nu_{n_k}, \nu] \in \mathfrak{R}$ and using condition (3.1), we have

$$\varphi(p(\nu_{n_{k+1}}, S\nu)) = \varphi(p(S\nu_{n_k}, S\nu)) \leq \varphi(p(\nu_{n_k}, \nu)) - \psi(p(\nu_{n_k}, \nu)), \tag{10}$$

for all $k \in \mathbb{N}_0$. By virtue of monotonicity of ψ, we have

$$\varphi(p(S\nu_{n_k}, S\nu)) \leq \varphi(p(\nu_{n_k}, \nu)), \quad \text{for all } k \in \mathbb{N}_0. \tag{11}$$

Next, we claim that

$$p(S\nu_{n_k}, S\nu) \leq p(\nu_{n_k}, \nu), \quad \text{for all } k \in \mathbb{N}. \tag{12}$$

On account of different values of $p(\nu_{n_k}, \nu)$, we get a partition $\{\mathbb{N}^0, \mathbb{N}^+\}$ of $\mathbb{N}$ (that is, $\mathbb{N}^0 \cup \mathbb{N}^+ = \mathbb{N}$ and $\mathbb{N}^0 \cap \mathbb{N}^+ = \emptyset$) such that $p(\nu_{n_k}, \nu) = 0$, for all $n \in \mathbb{N}^0$ and $p(\nu_{n_k}, \nu) > 0$, for all $n \in \mathbb{N}^+$.

If $p(\nu_{n_k}, \nu) = 0$, for all $k \in \mathbb{N}^0$ that is, $p(S\nu_{n_k}, S\nu) = 0$, for all $k \in \mathbb{N}^0$ then condition (12) holds. Otherwise, by monotonicity of functions φ, ψ and from (10), we have $p(S\nu_{n_k}, S\nu) < p(\nu_{n_k}, \nu)$, for all $k \in \mathbb{N}^+$, which concludes condition (12) also holds for all $n \in \mathbb{N}^+$. Finally, the condition (12) holds for all $n \in \mathbb{N}$. Making $k \to \infty$ in (12) and using $\nu_{n_k} \xrightarrow{p} \nu$, we have $\nu_{n_{k+1}} \xrightarrow{p} S\nu$. Thus, by uniqueness of the limit, we get $S\nu = \nu$ that is, ν is a fixed point of S. □

Remark 3.2. *If we take $\varphi(t) = t$ and $\psi(t) = (1 - \lambda)\varphi(t)$ in the Theorem 3.1, then we get the Theorem 2.1 of Senapati and Dey [23]. Similarly, if we take $p(\nu, \mu) = d(\nu, \mu)$, $\varphi(t) = t$ and $\psi(t) = (1 - \lambda)\varphi(t)$ in our main result then we obtain the Theorem 2.1 of Alam and Imdad [1]. Hence, our main result improves and generalizes many well known results in relation-theoretic metrical fixed-point theory.*

Remark 3.3. *Theorem 3.1 remains valid if replace the assumption of locally S-transitivity of $\mathfrak{R}$ by any one of the assumptions: (a) $\mathfrak{R}$-transitivity, (b) S-transitivity of $\mathfrak{R}$ and (c) locally transitivity of $\mathfrak{R}$.*

Theorem 3.4. *In addition to the hypotheses of Theorem 3.1, if*

(U) $S(M)$ is $\mathfrak{R}^s$-connected,

then S has a unique fixed point.

Proof. Let ν and μ be two fixed points of S in relation metric space (M, d). Then,

$$S^n\nu = \nu \quad \text{and} \quad S^n\mu = \mu, \quad \text{for all } n \in \mathbb{N},$$

and clearly $\nu, \mu \in S(M)$. Assume that assumption (U) also holds in addition to the hypotheses of Theorem 3.1. Then, there exists a path, say $(\varrho_0, \varrho_1, \varrho_2, \ldots, \varrho_k)$ of some finite length k in $\mathfrak{R}^s$, from ν to μ such that

$$\varrho_0 = \nu, \qquad \varrho_k = \mu \text{ and } [\varrho_i, \varrho_{i+1}] \in \mathfrak{R}, \quad \text{for each } i \ (0 \leq i \leq k-1). \tag{13}$$

and by S-closedness of $\mathfrak{R}$, we have

$$[S^n\varrho_i, S^n\varrho_{i+1}] \in \mathfrak{R}, \quad \text{for all } n \in \mathbb{N}_0 \text{ and } i \ (0 \leq i \leq k-1). \tag{14}$$

If $(\nu, \mu) \in \mathfrak{R}^s$ then either $(S^n\nu, S^n\mu) \in \mathfrak{R}$ or $(S^n\mu, S^n\nu) \in \mathfrak{R}$ and by S-closedness of $\mathfrak{R}$, we have $(S^n\nu, S^n\mu) \in \mathfrak{R}$, for $n = 0, 1, 2, \ldots$. From condition (3.1), we get

$$\begin{aligned} \varphi(p(\nu, \mu)) = \varphi(p(S^n\nu, S^n\mu)) &\leq \varphi(p(S^{n-1}\nu, S^{n-1}\mu)) - \psi(p(S^{n-1}\nu, S^{n-1}\mu)) \\ &= \varphi(p(\nu, \mu)) - \psi(p(\nu, \mu)), \end{aligned} \tag{15}$$

which leads to a contradiction. Thus, we obtain that $\psi(p(\nu,\mu)) = 0$ implies $p(\nu,\mu) = 0$, that is $\nu = \mu$. Hence S has a unique fixed point.

On the other hand, if $(\nu,\mu) \neq \mathfrak{R}^s$, then there exists a path of length $k > 1$ in $\mathfrak{R}^s$. We define $t_n^i = p(S^n\varrho_i, S^n_{\varrho_{i+1}}) \in \mathfrak{R}^s$, for $i(0 \leq i \leq (k-1))$ and $n \in \mathbb{N}_0$. Moreover, form (3.1) and for any fix i, we have

$$\begin{aligned} \varphi(t_n^i) = \varphi(p(S^n\varrho_i, S^n\varrho_{i+1})) &\leq \varphi(p(S^{n-1}\varrho_i, S^{n-1}\varrho_{i+1})) - \psi(p(S^{n-1}\varrho_i, S^{n-1}\varrho_{i+1})) \\ &\leq \varphi(p(S^{n-1}\varrho_i, S^{n-1}\varrho_{i+1})) = \varphi(t_{n-1}^i). \end{aligned} \tag{16}$$

Consequently, $\{\varphi(t_n^i)\} = \{\varphi(p(S^n\varrho_i, S^n\varrho_{i+1}))\}$ is a non-negative decreasing sequence. Monotonicity of φ gives us the sequence $\{t_n^i\}$ is also a decreasing and consequently, there exists $t \geq 0$ such that

$$\lim_{n\to\infty} t_n^i = t.$$

Letting $n \to \infty$ in (16) and utilizing the monotonicity of functions φ and ψ, we get

$$\varphi(t) \leq \varphi(t) - \psi(t) \leq \varphi(t).$$

This implies $\psi(t) = 0$ and consequently $t = 0$, for each i $(0 \leq i \leq k-1)$. Finally, from the above conclusion and using triangular inequality, we obtain

$$p(\nu,\mu) = p(S^n\varrho_0, S^n\varrho_k) \leq t_n^0 + t_n^1 + \cdots + t_n^{k-1} \to 0, \text{ as } n \to \infty.$$

Hence S has a unique fixed point. □

Remark 3.5. *Theorem 3.4 remains true if assumption (U) is replaced by the assumption that either $\mathfrak{R}|_{S(M)}$ is complete or $S(M)$ is $\mathfrak{R}^s$-directed.*

Example 3.6. *Let $M = [0,1]$ be a complete metric space equipped with usual metric d. We define a binary relation $(\nu,\mu) \in \mathfrak{R}$ implies $\nu < \mu$ on M and a self mapping S on M such that*

$$S(\nu) = \frac{\nu}{1+\nu^2}, \text{ for all } \nu \in M,$$

Then, it is easy to verify that $\mathfrak{R}$ is S-closed and mapping S is $\mathfrak{R}$-continuous. Again, we define two mappings $\varphi, \psi : [0,\infty) \to [0,\infty)$ by

$$\psi(t) = \begin{cases} \dfrac{t^3}{1+t^2}, & \text{for } t > 0, \\ 0, & \text{for } t = 0, \end{cases}$$

$\varphi(t) = t$, for all $t \in [0,\infty)$ and a w-distance $p : M \times M \to M$ by $p(\nu,\mu) = \mu$. For all $\nu,\mu \in M$ with $(\nu,\mu) \in \mathfrak{R}$, we have

$$\varphi\left(p(S(\nu), S(\mu))\right) = \varphi(S\mu) = \frac{\mu}{1+\mu^2} = \mu - \frac{\mu^3}{1+\mu^2} = \varphi(p(\nu,\mu)) - \psi(p(\nu,\mu)).$$

Thus, all the assumptions of Theorem 3.1 are satisfied and S has a fixed points in M (namely at $\nu = 0$).

Remark 3.7. *In the above example, if we take $\nu = 0$ and $\mu = \epsilon$, where $\epsilon > 0$ is an arbitrary point, very close to 0. Then $(0,\epsilon) \in \mathfrak{R}$ and S does not satisfy the contractive condition of Senapati and Dey [23], that is $p(S(\nu), S(\mu)) \leq \lambda p(\nu,\mu)$, where $\lambda \in [0,1)$. Since $p(S(\nu), S(\mu)) \leq \lambda p(\nu,\mu)$ implies that $\frac{\epsilon}{1+\epsilon^2} \leq \lambda\epsilon$ or $\lambda \geq \frac{1}{1+\epsilon^2}$. Hence $\lambda \notin [0,1)$ and the above example shows, the utility of Theorem 3.1 over the result of Sanapati and Dey [23] and many others.*

For different setting of functions ψ, φ, p and contractive condition (3.1), we may obtained several metric fixed point results from Theorem 3.1 in relational metric spaces.

Let Λ be the set of functions $\xi : [0,\infty) \to [0,\infty)$, such that
(h_1) ξ is Lebesgue integrable on each compact subset of $[0,\infty)$,
(h_2) $\int_0^\epsilon \xi(z)dz > 0$, for each $\epsilon > 0$.

Theorem 3.8. *Theorem 3.1 is valid even if contraction condition (e) is replaced by*
(e_1) $\int_0^{p(S\nu,S\mu)} \xi(z)dz \leq \int_0^{p(\nu,\mu)} \xi(z)dz - \int_0^{p(\nu,\mu)} \delta(z)dz,$
for all $\nu, \mu \in M$ *with* $(\nu, \mu) \in \mathfrak{R}$*, where* $\xi, \delta \in \Lambda$.

Proof. The proof follows from Theorem 3.1 by taking the function $\varphi(t) = \int_0^t \xi(z)dz$ and $\psi(t) = \int_0^t \delta(z)dz$, for all $t \in [0, \infty)$. □

Further, if we take $\delta(z) = (1 - \lambda)\xi(z)$, $\lambda \in [0, 1)$ in Theorem 3.8, we get the following result.

Corollary 3.9. *Theorem 3.8 is valid even if contraction condition* (e_1) *is replaced by*
(e_2) $\int_0^{p(S\nu,S\mu)} \xi(z)dz \leq \lambda \int_0^{p(\nu,\mu)} \xi(z)dz,$
for all $\nu, \mu \in M$ *with* $(\nu, \mu) \in \mathfrak{R}$*, where* $\xi \in \Lambda$.

Taking $\varphi(t) = t$ in Theorem 3.1, we get the following corollary as a direct consequence of our Theorem 3.1.

Corollary 3.10. *Let* (M, d) *be a metric space endowed with binary relation* $\mathfrak{R}$ *and p be a w-distance on M. Assume that* $S : M \to M$ *be a mapping and the following hypotheses hold:*
(a) there exists $Y \subseteq M$, $S(M) \subseteq Y \subset M$ *such that* (Y, d) *is* $\mathfrak{R}$*-complete,*
(b) $\mathfrak{R}$ *is S-closed and locally S-transitivity,*
(c) either $\mathfrak{R}|_Y$ *is d-self closed or S is* $\mathfrak{R}$*-continuous,*
(d) $M(S, \mathfrak{R})$ *is non-empty,*
(e) there exists a lower semi-continuous function $\psi : [0, \infty) \to [0, \infty)$*, with* $\psi(t) > 0$*, for all* $t \in (0, \infty)$ *and* $\psi(t) = 0$*, for* $t = 0$*, such that*

$$p(S\nu, S\mu) \leq p(\nu, \mu) - \psi(p(\nu, \mu)), \tag{17}$$

for all $\nu, \mu \in M$ *with* $(\nu, \mu) \in \mathfrak{R}$*. Then S has a fixed point.*

Remark 3.11. *Notice that by taking p = d in Corollary 3.10, our result extends the result of Dutta and Rhoades [8], and Prasad et al. [17] as the notion of lower semi continuity is weaker than the notion of continuity.*

Similarly, if we take $\psi(t) = t - \phi(t)$, where $\phi : [0, \infty) \to [0, \infty)$ such that $\phi(t) < t$ and $\sum_{n=1}^{\infty} \phi^n(t) < \infty$, in Corollary 3.10 then we get the following result.

Corollary 3.12. *Let* (M, d) *be a metric space endowed with arbitrary binary relation* $\mathfrak{R}$ *and p be a w-distance on metric space M. Suppose that* $S : M \to M$ *be a mapping and the following conditions hold:*
(a) there exists $Y \subseteq M$, $S(M) \subseteq Y \subset M$ *such that* (Y, d) *is* $\mathfrak{R}$*-complete,*
(b) $\mathfrak{R}$ *is S-closed and locally S-transitivity,*
(c) either $\mathfrak{R}|_Y$ *is d-self closed or S is* $\mathfrak{R}$*-continuous,*
(d) $M(S, \mathfrak{R})$ *is non-empty,*
(e) there exists $\nu, \mu \in M$ *such that*

$$p(S\nu, S\mu) \leq \phi(p(\nu, \mu)), \tag{18}$$

for all $\nu, \mu \in M$ *with* $(\nu, \mu) \in \mathfrak{R}$*. Then S has a fixed point.*

4. Application to nonlinear Fredholm equations

Consider the nonlinear Fredholm integral equation

$$\nu(t) = g(t) + \int_\alpha^\beta k(t, s, \nu(s))ds, \tag{19}$$

where $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$, and $\nu \in C[\alpha, \beta]$, $g : [\alpha, \beta] \to \mathbb{R}$ and $k : [\alpha, \beta] \times [\alpha, \beta] \times \mathbb{R} \to \mathbb{R}$ are given continuous mappings.

Theorem 4.1. *Let $\varphi, \psi : [0,\infty) \to [0,\infty)$ with φ is an altering distance function and ψ is lower semi-continuous function with $\psi(t) > 0$, for all $t \in (0,\infty)$ and $\psi(t) = 0$ for $t = 0$. If*

$$\sup_{t\in[\alpha,\beta]} |k(t,s,\mu(t))| \leq \frac{\varphi(\sup_{t\in[\alpha,\beta]} |\mu(t)|) - \psi(\sup_{t\in[\alpha,\beta]} |\mu(t)|) - \|\psi\|}{\beta - \alpha},$$

for all $t, s \in [\alpha, \beta]$, then inequality (19) has a solution.

Proof. We define a function $S : C[\alpha,\beta] \to C[\alpha,\beta]$ by

$$Sv(t) = \psi(t) + \int_{\alpha}^{\beta} k(t,s,v(s))ds \tag{20}$$

and a binary relation

$$\mathfrak{R} = \Big\{(v,\mu) \in C[\alpha,\beta] \times C[\alpha,\beta] : v(t) \leq \mu(t),\ \forall\, t \in [\alpha,\beta]\Big\}.$$

on $C[\alpha,\beta]$. Let $p : M \times M \to [0,\infty)$ given by

$$p(v,\mu) = \|\mu\|_\infty = \sup_{t\in[\alpha,\beta]} |\mu(t)|, \text{ for all } v,\mu \in M. \tag{21}$$

Then, clearly p is a w-distance on M.

(i) Let $M = C[\alpha,\beta]$ endowed with metric $d : M \times M \to [0,\infty)$ given by $d(v,\mu) = \sup_{t\in[\alpha,\beta]} |v(t) - \mu(t)|$, for all $v,\mu \in M$. Then, (M,d) forms a complete metric space and so (M,d) is $\mathfrak{R}$-complete.

(ii) Choose an $\mathfrak{R}$-preserving sequence $\{v_n\}$ such that $v_n \xrightarrow{d} v$. Then, for all $t \in [\alpha,\beta]$, we get

$$v_0(t) \leq v_1(t) \leq v_2(t) \leq \cdots \leq v_n(t) \leq v_{n+1} \leq \ldots$$

and converges to $v(t)$ implies that $v_n(t) \leq v(t)$, for all $t \in [\alpha,\beta]$, $n \in \mathbb{N}_0$, which amount of saying that $[v_n, v] \in \mathfrak{R}$, for all $n \in \mathbb{N}_0$. Hence, $\mathfrak{R}$ is d-self closed.

(iii) From (20) and for any $(v,\mu) \in \mathfrak{R}$, that is $v(t) \leq \mu(t)$, we have

$$\begin{aligned}(Sv)(t) &= \psi(t) + \int_{\alpha}^{\beta} k(t,s,v(s))ds \\ &\leq \psi(t) + \int_{\alpha}^{\beta} k(t,s,\mu(s))ds \\ &= (S\mu)(t),\end{aligned}$$

which shows that $(Sv, S\mu) \in \mathfrak{R}$, that is $\mathfrak{R}$ is S-closed.

(iv) Let $v \in C([\alpha,\beta],\mathfrak{R})$ be a solution of (19), that is

$$v(t) \leq \psi(t) + \int_{\alpha}^{\beta} k(t,s,v(s))ds = (Sv)(t),$$

implies that $(v, Sv) \in \mathfrak{R}$ and $M(S,\mathfrak{R}) \neq \emptyset$.

(v) Now, for $(\nu,\mu)\in\mathfrak{R}$,

$$\begin{aligned} |S\mu(t)| &= \sup_{t\in[\alpha,\beta]}\left|\psi(t)+\int_\alpha^\beta k(t,s,\mu(s))ds\right| \\ &\le \|\psi\|+\sup_{t\in[\alpha,\beta]}\int_\alpha^\beta |k(t,s,\mu(s))|ds \\ &\le \|\psi\|+\int_\alpha^\beta\Big(\frac{\varphi(\sup_{t\in[\alpha,\beta]}|\mu(t)|)-\psi(\sup_{t\in[\alpha,\beta]}|\mu(t)|)-\|\psi\|}{\beta-\alpha}\Big)ds \\ &= \|\psi\|+\frac{1}{\beta-\alpha}\int_\alpha^\beta\Big(\varphi(p(\nu,\mu))-\psi(p(\nu,\mu))-\|\psi\|\Big)ds \\ &= \varphi(p(\nu,\mu))-\psi(p(\nu,\mu)), \end{aligned}$$

which implies that

$$\sup_{t\in[\alpha,\beta]}|(S\mu)(t)|\le\varphi(p(\nu,\mu))-\psi(p(\nu,\mu))$$

and so

$$p(S\nu,S\mu)\le\varphi(p(\nu,\mu))-\psi(p(\nu,\mu)),$$

for all $(\nu,\mu)\in\mathfrak{R}$. Hence, we have

$$\psi(p(S\nu,S\mu))\le p(S\nu,S\mu)\le\varphi(p(\nu,\mu))-\psi(p(\nu,\mu)),$$

for all $\nu,\mu\in M$ with $(\nu,\mu)\in\mathfrak{R}$. This prove that S satisfies all the hypotheses of Theorem 3.1 and so the inequality (19) has a solution.

□

Conclusion. We have established non-unique fixed points for (φ,ψ,p)-weakly contractive mappings in non-complete relational metric spaces for a discontinuous single-valued map using the notion of w-distance and locally S-transitivity of binary relation. For the uniqueness of the fixed point, we have assumed range space to be $\mathfrak{R}^s$-connected. Our conclusions are sharpened versions of the existing conclusions, wherein continuity and completeness have been substituted by comparatively weaker notions (their $\mathfrak{R}$ analogs). Finally, we have given an application to solve the nonlinear Fredholm integral equation which demonstrates the usability of our conclusion.

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