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L^p-moment mixed quermassintegrals

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Abstract. L^p-moment mixed quermassintegrals of convex bodies in $\mathbb{R}^n$ are introduced. The Brunn-Minkowski type inequality and Aleksandrov-Fenchel type inequality are established for L^p-moment mixed quermassintegrals that imply affine mixed quermassintegrals inequality, Lutwak's mixed polar projection inequality, and isoperimetric inequality for L^p-moment mixed quermassintegrals. Inequalities of L^p-moment mixed quermassintegrals of polar bodies are proved.

1. Introduction

The combination of Minkowski addition and volume leads to the rich and powerful classical Brunn-Minkowski theory for compact convex sets, which constitutes the core of modern convex geometry. As the first milestone of the Brunn-Minkowski theory, the Brunn-Minkowski inequality plays a fundamental role in attacking problems in analysis, geometry, information theory, and many other fields, which states that if K, L are convex bodies (compact convex subsets with nonempty interiors) in Euclidean n-space $\mathbb{R}^n$, then

$$V(K+L)^{1/n} \geq V(K)^{1/n} + V(L)^{1/n}.$$

Here V and $+$ denote volume and Minkowski sum. Equality holds if and only if K and L are homothetic. It is a far-reaching generalization of the isoperimetric inequality. The classic treatise of Schneider [25] provides a detailed survey of the Brunn-Minkowski theory and a host of references. For later developments, we refer to [2, 4, 6, 15, 26].

The classical Brunn-Minkowski theory is also known as the theory of mixed volumes. The notion of mixed volumes, which forms a central part of the Brunn-Minkowski theory of convex bodies, was created by Minkowski [23, 24] and subsequently attracted the attentions of many scholars, see e.g., [3, 8, 12, 18–20]. Around 1935, Aleksandrov [1] and Fenchel [7] discovered the relation between mixed volumes independently, which is called by Aleksandrov-Fenchel inequality, that is, if $K_1, \ldots, K_n$ are compact convex subsets in $\mathbb{R}^n$, for $1 \leq m \leq n$, then

$$V(K_1, \ldots, K_n)^m \geq \prod_{i=1}^{m} V(\underbrace{K_i, \ldots, K_i}_{m}, K_{m+1}, \ldots, K_n). \tag{1}$$

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Here $V(K_1, \dots, K_n)$ is the mixed volume of $K_1, \dots, K_n$. The complete equality conditions for the Aleksandrov-Fenchel inequality are not known. The following special case is useful: If $K_{m+1}, \dots, K_n$ are smooth convex bodies, and the dimensions of $K_1, \dots, K_m$ are more than or equal to m, then the equality in (1) holds if and only if $K_1, \dots, K_m$ are homothetic. It is not difficult to find that the Aleksandrov-Fenchel inequality implies Minkowski's first inequality [25].

One of the most fundamental concepts in convex geometry is *quermassintegrals* of a compact convex subsets in $\mathbb{R}^n$, which have an intimate connection with the mixed volumes. It can be shown that if K is a convex body in $\mathbb{R}^n$, for $1 \le i \le n-1$, then the $(n-i)$-th quermassintegral $W_{n-i}(K)$ of K is defined by

$$W_{n-i}(K) = \frac{\omega_n}{\omega_i} \int_{G_{n,i}} V_i(K|\xi) \mathrm{d}\mu_i(\xi), \tag{2}$$

where define $W_n(K) = \omega_n$, $W_0(K) = V(K)$ and ω_n is volume of unit ball in $\mathbb{R}^n$. Let $G_{n,i}$ denote the Grassmann manifold of all i-dimensional linear subspaces in $\mathbb{R}^n$. For $\xi \in G_{n,i}$, $V_i(K|\xi)$ denotes the i-dimensional volume of the orthogonal projection of K onto ξ, and the integral with respect to Haar probability measure μ_i over $G_{n,i}$. For more information, we refer to [9, 16, 25, 27, 28].

However, quermassintegrals of a convex body K are not invariant under volume-preserving affine transformations, so it is tempting to find an analogous notion which is invariant under such transformations. By replacing the L^1-norm in (2) by the L^{-n}-norm, Lutwak [16] proposed to define *affine quermassintegrals* for a convex body K by taking $\Phi_0(K) = V(K), \Phi_n(K) = \omega_n$, and for $1 \le i \le n-1$,

$$\Phi_{n-i}(K) = \frac{\omega_n}{\omega_i} \left(\int_{G_{n,i}} V_i(K|\xi)^{-n} \mathrm{d}\mu_i(\xi) \right)^{-1/n}. \tag{3}$$

It was showed by Grinberg [11] that these geometric quantities are invariant under volume-preserving affine transformations. Consequently, the affine quermassintegrals have become a central pillar of affine convex geometry.

In order to obtain the sharp lower bound of $\Phi_i(K)$, Lutwak [21] put forward the following insightful conjecture as

$$\Phi_i(K) \ge \Phi_i(B_K), \quad i = 1, \dots, n-1, \tag{4}$$

where B_K denotes the Euclidean ball having the same volume as K, and equality holds if and only if K is an ellipsoid. Zou and Xiong [29] posed another lower bound for $\Phi_i(K)$ by the $(n-i)$-th projection mean ellipsoid.

By Jensen's inequality, the affine inequality (4) is stronger than the classical isoperimetric inequality. Two nontrivial cases of $i = 1$ and $i = n-1$ in (4) are true, they follow, respectively, from the Petty projection inequality and the Blaschke-Santaló inequality. For $i = 2, \dots, n-2$, the Lutwak's conjecture (4) is recently confirmed by Milman and Yehudayoff [22]. In [22], they extended affine quermassintegrals to more general *L^p-moment quermassintegrals* and obtained the isoperimetric inequalities for them. For $1 \le i \le n$ and $p \in \mathbb{R} \setminus \{0\}$, the $(n-i)$-th L^p-moment quermassintegrals of a convex body K are defined by

$$Q_{n-i,p}(K) = \frac{\omega_n}{\omega_i} \left(\int_{G_{n,i}} V_i(K|\xi)^p \mathrm{d}\mu_i(\xi) \right)^{1/p}.$$

The case $p = 0$ is interpreted in the limiting sense as

$$Q_{n-i,0}(K) = \frac{\omega_n}{\omega_i} \exp\left(\int_{G_{n,i}} \log V_i(K|\xi) \mathrm{d}\mu_i(\xi) \right).$$

Notice that $p = -n$ is the unique value of $p \in \mathbb{R}$ for which $Q_{i,p}(K)$ is invariant under volume-preserving affine transformations [11]. Some special cases such as $Q_{i,-n}(K) = \Phi_i(K), Q_{i,1}(K) = W_i(K)$, and $Q_{i,-1}(K) = \hat{W}_i(K)$ (

the harmonic quermassintegral introduced by Hadwiger [13]) show that L^p-moment quermassintegrals are the generalization of classical quermassintegrals.

Based on the importance of mixed volume in the Brunn-Minkowski theory and motivated by the excellent paper [22], we're going to consider the mixed form of L^p-moment quermassintegrals, namely *L^p-moment mixed quermassintegrals*. Let $K_1, \ldots, K_i$ are convex bodies in $\mathbb{R}^n$ and $\xi \in G_{n,i}$. We use $V_\xi(K_1|\xi, K_2|\xi, \ldots, K_i|\xi)$ to denote the mixed volume of $K_1|\xi, K_2|\xi, \ldots, K_i|\xi$ in the subspace ξ. Suppose $0 \le i \le n$ and $p \in \mathbb{R}\backslash\{0\}$, then the $(n-i)$-th L^p-moment mixed quermassintegrals $Q_{n-i,p}(K_1, K_2, \ldots, K_i)$ for convex bodies $K_1, \ldots, K_i$ are defined, by letting $Q_{0,p}(K_1, \ldots, K_n) = V(K_1, \ldots, K_n)$, $Q_{n,p}(K_1, \ldots, K_i) = \omega_n$ and, for $1 \le i \le n-1$,

$$Q_{n-i,p}(K_1, \ldots, K_i) = \frac{\omega_n}{\omega_i}\left(\int_{G_{n,i}} V_\xi(K_1|\xi, \ldots, K_i|\xi)^p \mathrm{d}\mu_i(\xi)\right)^{1/p}. \tag{5}$$

The case of $p = 0$ is interpreted in the limiting sense as

$$Q_{n-i,0}(K_1, \ldots, K_i) = \frac{\omega_n}{\omega_i}\exp\left(\int_{G_{n,i}} \log V_\xi(K_1|\xi, \ldots, K_i|\xi)\mathrm{d}\mu_i(\xi)\right). \tag{6}$$

In some special cases, we can get $Q_{n-i,p}(K, \ldots, K) = Q_{n-i,p}(K)$ when $K_1 = \cdots = K_i = K$ and $Q_{n-i,-n}(K, \ldots, K) = \Phi_{n-i}(K)$. In Section 3, some fundamental properties for L^p-moment mixed quermassintegrals are introduced. In Section 4, we prove that the functional $Q_{n-i,p}^{1/i}$ from $\mathcal{K}^n$ to $[0, \infty)$ is concave, this is an analogous Brunn-Minkowski inequality. The Aleksandrov-Fenchel type inequality for L^p-moment mixed quermassintegrals is established as following.

Theorem 1.1. *Suppose $K_1, \ldots, K_m$ are convex bodies in $\mathbb{R}^n$, $K_{m+1}, \ldots, K_i$ are smooth convex bodies in $\mathbb{R}^n$ and $1 \le m \le i$, for $p \le 0$, then*

$$Q_{n-i,mp}(K_1, \ldots, K_i)^m \ge \prod_{j=1}^{m} Q_{n-i,mp}(\underbrace{K_j, \ldots, K_j}_{m}, K_{m+1}, \ldots, K_i),$$

with equality if and only if $K_1, \ldots, K_m$ are homothetic.

Theorem 1.1 implies affine mixed quermassintegrals inequality (Corollary 4.7) and Lutwak's mixed polar projection inequality (Corollary 4.8). For convex bodies $K_1, \ldots, K_i$ in $\mathbb{R}^n$ and $p \ge -n$, Theorem 1.1 together with (4) yields the isoperimetric inequality for L^p-moment mixed quermassintegrals

$$Q_{n-i,p}(K_1, \ldots, K_i) \ge Q_{n-i,p}(B_{K_1}, \ldots, B_{K_i}), \tag{7}$$

with equality if and only if $K_1, \ldots, K_i$ are balls.

In Section 5, we consider inequalities of L^p-moment mixed quermassintegrals of polar bodies and establish the following inequality.

Theorem 1.2. *Suppose that K_1 is a smooth convex body containing the origin in its interior in $\mathbb{R}^n$, $K_2, \ldots, K_i$ are convex bodies in $\mathbb{R}^n$. For $0 \le i \le n$ and $p \ge 0$, then*

$$Q_{n-i,p}(K_1, K_2, \ldots, K_i)Q_{n-i,p}(K_1^*, K_2, \ldots, K_i) \ge Q_{n-i,p}(B, K_2, \ldots, K_i)^2,$$

with equality if and only if K_1 is a ball.

2. Preliminaries

As usual, S^{n-1} denotes the unit sphere, B the unit ball and o the origin in n-dimensional Euclidean space $\mathbb{R}^n$. A *convex body* is a compact convex subset of $\mathbb{R}^n$ with non-empty interior. The set of convex bodies in

$\mathbb{R}^n$ is denoted by $\mathcal{K}^n$ and the set of convex bodies in $\mathbb{R}^n$ that contain the origin in their interiors is denoted by $\mathcal{K}_o^n$. And $\mathcal{K}_s^n$ denotes the set of centrally symmetric convex bodies in $\mathcal{K}^n$. For real number $c > 0$ and $K \in \mathcal{K}^n$, we have $V(cK) = c^n V(K)$. For $K, L \in \mathcal{K}^n$ are said to be *homothetic* if there exists a real number $c > 0$ and a vector $x \in \mathbb{R}^n$ such that $K = cL + x$. For K is a subset of $\mathbb{R}^n$, its *polar set* K^* is defined by

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \le 1 \ for\ all\ y \in K\},$$

where $x \cdot y$ is the standard inner product of x and y in $\mathbb{R}^n$. In particular, if $K \in \mathcal{K}_o^n$, we have $K^{**} = K$.

Let $K \in \mathcal{K}^n$, then its *support function* $h(K, \cdot)$ is defined by

$$h(K, u) = \max\{u \cdot x : x \in K, u \in S^{n-1}\}.$$

The *projection body* of a convex body K is the centered convex body ΠK, which is defined by

$$h(\Pi K, u) = V_{n-1}(K|u^\perp),$$

for each $u \in S^{n-1}$, where $K|u^\perp$ is the orthogonal projection of K on $u^\perp$.

If $K_1, \dots, K_{n-1} \in \mathcal{K}^n$, Lutwak [17] introduced the *mixed projection body* of $K_1, \dots, K_{n-1}$ denoted by $\Pi(K_1, \dots, K_{n-1})$, and defined by

$$h(\Pi(K_1, \dots, K_{n-1}), u) = V_{n-1}(K_1|u^\perp, \dots, K_{n-1}|u^\perp), \tag{8}$$

where $V_{n-1}(K_1|u^\perp, \dots, K_{n-1}|u^\perp)$ is the mixed volume of the compact convex sets $K_1|u^\perp, \dots, K_{n-1}|u^\perp$ in the $(n-1)$-dimensional space $u^\perp$. We use $\Pi^*(K_1, \dots, K_{n-1})$ to denote the polar body of $\Pi(K_1, \dots, K_{n-1})$.

Suppose $K_1, \dots, K_n \in \mathcal{K}^n$, the mixed volume of $K_1, \dots, K_n$ is denoted by $V(K_1, \dots, K_n)$. In general, for $r_1 + \cdots + r_k = n$, we introduce the abbreviation

$$V(\underbrace{K_1, \dots, K_1}_{r_1}, \dots, \underbrace{K_k, \dots, K_k}_{r_k}) := V(K_1, r_1; \dots; K_k, r_k).$$

Similarly, for $\xi \in G_{n,i}$, $V_\xi(K_1|\xi, \dots, K_i|\xi)$ denotes the i-dimensional mixed volume of body $K_1|\xi, \dots, K_i|\xi$ in subspace ξ.

Let $M \in \mathcal{K}_s^n$ and $c > 0$. A body $K \in \mathcal{K}^n$ is said to have *constant relative i-brightness* with respect to M [5], for $0 < i < n-1$, if

$$V_i(K|\xi) = cV_i(M|\xi), \ for\ all\ \xi \in G_{n,i}.$$

In general, $K_1, \dots, K_i \in \mathcal{K}^n$ are said to have *constant relative mixed i-brightness* with repsect to M [5], for $0 < i < n-1$, if

$$V_\xi(K_1|\xi, \dots, K_i|\xi) = cV_i(M|\xi), \ for\ all\ \xi \in G_{n,i}.$$

If $K_1, \dots, K_i \in \mathcal{K}^n$ and $0 \le i \le n-1$, the *affine mixed quermassintegral* of $K_1, \dots, K_i$ is defined by

$$\Phi_{n-i}(K_1, \dots, K_i) = \frac{\omega_n}{\omega_i}\left(\int_{G_{n,i}} V_\xi(K_1|\xi, \dots, K_i|\xi)^{-n} \mathrm{d}\mu_i(\xi)\right)^{-1/n}$$

and letting by $\Phi_0(K_1, \dots, K_n) = V(K_1, \dots, K_n)$. It is clear to notice that $Q_{n-i,-n}(K_1, \dots, K_i)$ is equivalent to $\Phi_{n-i}(K_1, \dots, K_i)$.

The following Lemma, will be needed several times, shows that i-dimensional mixed volumes of orthogonal projections to ξ can be expressed by mixed volumes in $\mathbb{R}^n$ [25].

Lemma 2.1. *Suppose $K_1, \dots, K_i \in \mathcal{K}^n$ and $\xi \in G_{n,i}$, for $0 \le i \le n$, then*

$$V_\xi(K_1|\xi, \dots, K_i|\xi) = c_{n,i} V(K_1, \dots, K_i; B \cap \xi^\perp, n-i), \tag{9}$$

where $c_{n,i} = \frac{\binom{n}{i}}{\omega_{n-i}}$.

The following inequality for mixed volumes of polar bodies which proved by Ghandehari [10] will be applied to set up inequality for L^p-moment mixed quermassintegrals of polar bodies.

Lemma 2.2. *Suppose $K_1 \in \mathcal{K}_o^n$ and K_1 is smooth, $K_2, \ldots, K_n \in \mathcal{K}^n$, then*

$$V(K_1, K_2, \ldots, K_n)V(K_1^*, K_2, \ldots, K_n) \geq V(B, K_2, \ldots, K_n)^2, \tag{10}$$

with equality if and only if K_1 is a ball.

3. L^p-moment mixed quermassintegrals

In this section, some fundamental properties for L^p-moment mixed quermassintegrals are introduced.

By the translation invariance and positive homogeneity of mixed volume $V_\xi(K_1|\xi, \ldots, K_i|\xi)$, then such properties of L^p-moment mixed quermassintegrals can be obtained immediately.

Proposition 3.1. *Suppose $K_1, \ldots, K_i \in \mathcal{K}^n$ and $p \in \mathbb{R}$, then*

$$Q_{n-i,p}(K_1 + x_1, \ldots, K_i + x_i) = Q_{n-i,p}(K_1, \ldots, K_i),$$

for $x_1, \ldots, x_i \in \mathbb{R}^n$, and

$$Q_{n-i,p}(\lambda_1 K_1, \ldots, \lambda_i K_i) = \lambda_1 \cdots \lambda_i Q_{n-i,p}(K_1, \ldots, K_i), \tag{11}$$

for $p \in \mathbb{R}$ and $\lambda_1, \ldots, \lambda_i > 0$.

Since the classical mixed volume in $\mathbb{R}^n$ is multilinear, it is obvious to notice that the L^p-moment mixed quermassintegrals are multilinear when $p = 1$. The following proposition will state L^p-moment mixed quermassintegrals are not multilinear when $p \neq 1$.

Proposition 3.2. *Suppose $K_0, K_1, \ldots, K_i \in \mathcal{K}^n$ and $a, b \geq 0$ for $p > 1$, then*

$$Q_{n-i,p}(aK_0 + bK_1, K_2, \ldots, K_i) \leq aQ_{n-i,p}(K_0, K_2, \ldots, K_i) + bQ_{n-i,p}(K_1, K_2, \ldots, K_i). \tag{12}$$

If $0 \neq p < 1$, then (12) holds with the inequality sign reversed. Equality in (12) holds if K_0 and K_1 are homothetic.

Proof. Combining the fact that the classical mixed volume in $\mathbb{R}^n$ is multilinear and Lemma 2.1, for all $\xi \in G_{n,i}$, then

$$\begin{aligned} &V_\xi(aK_0 + bK_1|\xi, K_2|\xi, \ldots, K_i|\xi) \\ &= c_{n,i} V(aK_0 + bK_1, K_2, \ldots, K_i; B \cap \xi^\perp, n-i) \\ &= c_{n,i} \left(aV(K_0, K_2, \ldots, K_i; B \cap \xi^\perp, n-i) + bV(K_1, K_2, \ldots, K_i; B \cap \xi^\perp, n-i)\right) \\ &= aV_\xi(K_0|\xi, K_2|\xi, \ldots, K_i|\xi) + bV_\xi(K_1|\xi, K_2|\xi, \ldots, K_i|\xi). \end{aligned} \tag{13}$$

Applying Minkowski's inequality for $p > 1$ and (13), we have

$$\begin{aligned} &Q_{n-i,p}(aK_0 + bK_1, K_2, \ldots, K_i) \\ &\quad = \frac{\omega_n}{\omega_i}\left(\int_{G_{n,i}} (aV_\xi(K_0|\xi, K_2|\xi, \ldots, K_i|\xi) + bV_\xi(K_1|\xi, K_2|\xi, \ldots, K_i|\xi))^p \mathrm{d}\mu_i(\xi)\right)^{1/p} \\ &\quad \leq \frac{\omega_n}{\omega_i}\left(\int_{G_{n,i}} a^p V_\xi(K_0|\xi, \ldots, K_i|\xi)^p \mathrm{d}\mu_i(\xi)\right)^{1/p} + \frac{\omega_n}{\omega_i}\left(\int_{G_{n,i}} b^p V_\xi(K_1|\xi, \ldots, K_i|\xi)^p \mathrm{d}\mu_i(\xi)\right)^{1/p} \\ &\quad = aQ_{n-i,p}(K_0, K_2, \ldots, K_i) + bQ_{n-i,p}(K_1, K_2, \ldots, K_i). \end{aligned}$$

The inequality is reversed if $p < 1$ and $p \neq 0$. If K_0 and K_1 are homothetic, then $V_\xi(K_0|\xi, \ldots, K_i|\xi)$ and $V_\xi(K_1|\xi, \ldots, K_i|\xi)$ are proportional, that is, equality in (12) holds. □

Proposition 3.3. *Suppose $K, L, K_1, \dots, K_{i-2}$ are compact convex subsets in $\mathbb{R}^n$ such that $K \cup L$ is compact convex in $\mathbb{R}^n$, for $2 \le i \le n$, then*

$$Q_{n-i,p}(K, L, K_1, \dots, K_{i-2}) = Q_{n-i,p}(K \cup L, K \cap L, K_1, \dots, K_{i-2}). \tag{14}$$

Proof. By the theorem proved by Groemer [12]: If $K, L, K_1, \dots, K_{i-2}$ are compact convex subsets in $\mathbb{R}^n$ such that $K \cup L$ is compact convex in $\mathbb{R}^n$, then

$$V(K, L, K_1, \dots, K_{i-2}) = V(K \cup L, K \cap L, K_1, \dots, K_{i-2}). \tag{15}$$

Applying Lemma 2.1 and integrating over $G_{n,i}$ give that

$$\begin{aligned} Q_{n-i,p}(K, L, &K_1, \dots, K_{i-2}) \\ &= \frac{\omega_n}{\omega_i}\left(\int_{G_{n,i}} V_\xi(K|\xi, L|\xi, K_1|\xi, \dots, K_{i-2}|\xi)^p \mathrm{d}\mu_i(\xi)\right)^{1/p} \\ &= \frac{\omega_n}{\omega_i}\left(\int_{G_{n,i}} c_{n,i}^p V(K, L, K_1, \dots, K_{i-2}; B \cap \xi^\perp, n-i)^p \mathrm{d}\mu_i(\xi)\right)^{1/p} \\ &= \frac{\omega_n}{\omega_i}\left(\int_{G_{n,i}} c_{n,i}^p V(K \cup L, K \cap L, K_1, \dots, K_{i-2}; B \cap \xi^\perp, n-i)^p \mathrm{d}\mu_i(\xi)\right)^{1/p} \\ &= \frac{\omega_n}{\omega_i}\left(\int_{G_{n,i}} V_\xi((K \cup L)|\xi, (K \cap L)|\xi, K_1|\xi, \dots, K_{i-2}|\xi)^p \mathrm{d}\mu_i(\xi)\right)^{1/p} \\ &= Q_{n-i,p}(K \cup L, K \cap L, K_1, \dots, K_{i-2}). \end{aligned}$$

□

Proposition 3.4. *Suppose $K_1, \dots, K_i \in \mathcal{K}^n$ and $p, q \in \mathbb{R}$ satisfied $p < q$, then*

$$Q_{n-i,p}(K_1, \dots, K_i) \le Q_{n-i,q}(K_1, \dots, K_i), \tag{16}$$

with equality if and only if $K_1, \dots, K_i$ have constant relative mixed i-brightness with respect to B.

Proof. It can be deduced follows Jensen's inequality, and equality holds if and only if $V_\xi(K_1|\xi, \dots, K_i|\xi)$ is a constant for all $\xi \in G_{n,i}$, that is, equality holds if and only if $K_1, \dots, K_i$ have constant relative mixed i-brightness with respect to B. □

L^p-moment mixed quermassintegrals have the following monotone property.

Proposition 3.5. *Suppose $K, L, K_2, \dots, K_i \in \mathcal{K}^n$ and $K \subset L$, for $p \in \mathbb{R}$, then*

$$Q_{n-i,p}(K, K_2, \dots, K_i) \le Q_{n-i,p}(L, K_2, \dots, K_i). \tag{17}$$

Proof. From Lemma 2.1, for all $\xi \in G_{n,i}$, we have

$$\begin{aligned} V_\xi(K|\xi, K_2|\xi, \dots, K_i|\xi) &= c_{n,i} V(K, K_2, \dots, K_i; B \cap \xi^\perp, n-i) \\ &\le c_{n,i} V(L, K_2, \dots, K_i; B \cap \xi^\perp, n-i) \\ &= V_\xi(L|\xi, K_2|\xi, \dots, K_i|\xi). \end{aligned}$$

Then, for $p \in \mathbb{R}$, we get

$$\left(\int_{G_{n,i}} V_\xi(K|\xi, K_2|\xi, \dots, K_i|\xi)^p \mathrm{d}\mu_i(\xi)\right)^{1/p} \le \left(\int_{G_{n,i}} V_\xi(L|\xi, K_2|\xi, \dots, K_i|\xi)^p \mathrm{d}\mu_i(\xi)\right)^{1/p},$$

that is the desired (17).

It is easily to verify (17) is true for $p = 0$. □

4. L^p-moment mixed quermassintegrals inequalities

The following theorem shows that the functional $Q_{n-i,p}^{1/i}$ from $\mathcal{K}^n$ to $[0,\infty)$ is concave.

Theorem 4.1. *Given $p \in \mathbb{R}$ satisfied $ip \leq 1$, suppose $K_1, \dots, K_m \in \mathcal{K}^n$, then*

$$Q_{n-i,p}(K_1 + \cdots + K_m)^{1/i} \geq Q_{n-i,p}(K_1)^{1/i} + \cdots + Q_{n-i,p}(K_m)^{1/i}, \tag{18}$$

with equality if and only if $K_1, \dots, K_m$ are homothetic.

Proof. By applying the Brunn-Minkowski inequality in subspace $\xi \in G_{n,i}$ and combining the fact $(\sum_{j=1}^m K_j)|\xi = \sum_{j=1}^m (K_j|\xi)$, we have

$$V_i((K_1 + \cdots + K_m)|\xi)^{1/i} \geq V_i(K_1|\xi)^{1/i} + \cdots + V_i(K_m|\xi)^{1/i}, \tag{19}$$

with equality if and only if $K_1|\xi, \dots, K_m|\xi$ are homothetic for all $\xi \in G_{n,i}$, and therefore $K_1, \dots, K_m$ are homothetic follows [9, Theorem 3.1.3].

Combining (19) with the reverse Minkowski's inequality for $ip \leq 1$ and $p \neq 0$, then

$$\begin{aligned} Q_{n-i,p}(\sum_{j=1}^m K_j)^{1/i} &= \left(\frac{\omega_n}{\omega_i}\right)^{1/i} \left(\int_{G_{n,i}} V_i((K_1 + \cdots + K_m)|\xi)^p \mathrm{d}\mu_i(\xi)\right)^{1/ip} \\ &= \left(\frac{\omega_n}{\omega_i}\right)^{1/i} \left(\int_{G_{n,i}} V_i(K_1|\xi + \cdots + K_m|\xi)^{ip/i} \mathrm{d}\mu_i(\xi)\right)^{1/ip} \\ &\geq \left(\frac{\omega_n}{\omega_i}\right)^{1/i} \left(\int_{G_{n,i}} (V_i(K_1|\xi)^{1/i} + \cdots + V_i(K_m|\xi)^{1/i})^{ip} \mathrm{d}\mu_i(\xi)\right)^{1/ip} \\ &\geq \sum_{j=1}^m Q_{n-i,p}(K_j)^{1/i}. \end{aligned}$$

When $ip = 1$, the equality condition follows (19) immediately. When $ip < 1$, assume the equality holds in (18), then we have equality in both inequalities above. Equality in the third line implies by (19) that $K_1, \dots, K_m$ are homothetic. Equality in the fourth line implies $V_i(K_k|\xi)$ and $V_i(K_j|\xi)$ are proportional for $1 \leq k, j \leq m$. Therefore, they are also homothetic. On the other hand, if $K_1, \dots, K_m$ are homothetic, it is obvious that the equality holds in (18).

When $p = 0$, we will prove (18). For this aim, [14, Theorem 184] will turn out to be the key to finish that, which says that: For $K_1, \dots, K_m \in \mathcal{K}^n$, it follows that

$$\begin{aligned} &\exp\left(\int_{G_{n,i}} \log\left(V_i(K_1|\xi)^{1/i} + \cdots + V_i(K_m|\xi)^{1/i}\right) \mathrm{d}\mu_i(\xi)\right) \\ &\geq \exp\left(\int_{G_{n,i}} \log V_i(K_1|\xi)^{1/i} \mathrm{d}\mu_i(\xi)\right) + \cdots + \exp\left(\int_{G_{n,i}} \log V_i(K_m|\xi)^{1/i} \mathrm{d}\mu_i(\xi)\right), \end{aligned}$$

with equality if and only if $V_i(K_k|\xi)$ and $V_i(K_j|\xi)$ are proportional for $1 \le k, j \le m$. By (5) and (19), we have

$$\begin{aligned} &Q_{n-i,0}(K_1 + \cdots + K_m)^{1/i} \\ &= \left(\frac{\omega_n}{\omega_i}\right)^{1/i} \left(\exp\left(\int_{G_{n,i}} \log V_i((K_1 + \cdots + K_m)|\xi) \mathrm{d}\mu_i(\xi)\right)\right)^{1/i} \\ &= \left(\frac{\omega_n}{\omega_i}\right)^{1/i} \exp\left(\int_{G_{n,i}} \log V_i(K_1|\xi + \cdots + K_m|\xi)^{1/i} \mathrm{d}\mu_i(\xi)\right) \\ &\ge \left(\frac{\omega_n}{\omega_i}\right)^{1/i} \exp\left(\int_{G_{n,i}} \log\left(V_i(K_1|\xi)^{1/i} + \cdots + V_i(K_m|\xi)^{1/i}\right) \mathrm{d}\mu_i(\xi)\right) \\ &\ge \left(\frac{\omega_n}{\omega_i}\right)^{1/i} \sum_{j=1}^{m} \exp\left(\int_{G_{n,i}} \log V_i(K_j|\xi)^{1/i} \mathrm{d}\mu_i(\xi)\right) \\ &= \sum_{j=1}^{m} Q_{n-i,0}(K_j)^{1/i}. \end{aligned}$$

The equality condition is obtained for $p = 0$ in the same way as above. □

Theorem 4.2. *Suppose $K, L \in \mathcal{K}^n$ and $0 < \lambda < 1$, for $ip \le 1$, then*

$$Q_{n-i,p}((1-\lambda)K + \lambda L) \ge Q_{n-i,p}(K)^{1-\lambda} Q_{n-i,p}(L)^{\lambda}, \tag{20}$$

with equality if and only if K and L are translates.

Proof. Let $K_2 = \cdots = K_i = cK$ in (11), we have $Q_{n-i,p}(cK) = c^i Q_{n-i,p}(K)$ for $p \in \mathbb{R}$. From this and (18), we get

$$\begin{aligned} Q_{n-i,p}((1-\lambda)K + \lambda L)^{1/i} &\ge Q_{n-i,p}((1-\lambda)K)^{1/i} + Q_{n-i,p}(\lambda L)^{1/i} \\ &= (1-\lambda) Q_{n-i,p}(K)^{1/i} + \lambda Q_{n-i,p}(L)^{1/i}. \end{aligned} \tag{21}$$

Then apply the arithmetic-geometric inequality to (21), that is

$$Q_{n-i,p}((1-\lambda)K + \lambda L)^{1/i} \ge Q_{n-i,p}(K)^{(1-\lambda)/i} Q_{i,p}(L)^{\lambda/i}.$$

The equality in (21) holds if and only if K, L are homothetic and the equality condition of arithmetic-geometric inequality is $Q_{i,p}(K) = Q_{i,p}(L)$. Thus, equality in (20) holds if and only if K and L are translates. □

Lemma 4.3. *Suppose $K_1, \dots, K_i \in \mathcal{K}^n$, $K_{m+1}, \dots, K_i$ are smooth, $1 \le m \le i$, then*

$$V_\xi(K_1|\xi, \dots, K_i|\xi)^m \ge \prod_{j=1}^{m} V_\xi(\underbrace{K_j|\xi, \dots, K_j|\xi}_{m}, K_{m+1}|\xi, \dots, K_i|\xi), \tag{22}$$

with equality if and only if $K_1, \dots, K_m$ are homothetic.

Moreover, if $m = i$ in (22)*, then*

$$V_\xi(K_1|\xi, \dots, K_i|\xi)^i \ge V_i(K_1|\xi) \cdots V_i(K_i|\xi), \tag{23}$$

with equality if and only if $K_1, \dots, K_i$ are homothetic.

Proof. For all $\xi \in G_{n,i}$ and $K_1, \dots, K_i \in \mathcal{K}^n$, the Aleksandrov-Fenchel inequality (1) for compact convex sets $K_1|\xi, \dots, K_i|\xi$ in subspace ξ yields (22). In order to show the equality condition in (1), we need to prove $K_{m+1}|\xi, \dots, K_i|\xi$ are smooth convex bodies in subspace ξ. For the sake of simplicity, we assume that the smooth convex body K is of (at least) C^2 and has positive curvature, which is equivalent to its support function h_K is of C^2. Since $\rho_{K^*} = h_K^{-1}$, that h_K is C^2 is equivalent to ρ_{K^*} is C^2, that is, ∂K^* is C^2. Then $\partial K^* \cap \xi$ is C^2 for all $\xi \in G_{n,i}$, and therefore we obtain $\partial (K^* \cap \xi)^*$ is C^2, where the latter polar operation is taken in ξ. By the relationship (see (0.38) in [9])

$$K|\xi = (K^* \cap \xi)^*,$$

and thus, $K|\xi$ is a smooth convex body in ξ. Therefore, $K_{m+1}|\xi, \dots, K_i|\xi$ are smooth convex bodies in subspace ξ.

Then the equality condition in Aleksandrov-Fenchel inequality implies that equality holds in (22) if and only if $K_1|\xi, \dots, K_m|\xi$ are homothetic for $\xi \in G_{n,i}$. [9, Theorem 3.1.3] shows that such equality condition is equivalent to that $K_1, \dots, K_m$ are homothetic. □

The following theorem is the Aleksandrov-Fenchel type inequality for L^p-moment mixed quermassintegrals.

Theorem 4.4. *Suppose $K_1, \dots, K_i \in \mathcal{K}^n$, $K_{m+1}, \dots, K_i$ are smooth, for $1 \le m \le i$, for $p \le 0$, then*

$$Q_{n-i,mp}(K_1, \dots, K_i)^m \ge \prod_{j=1}^{m} Q_{n-i,mp}(\underbrace{K_j, \dots, K_j}_{m}, K_{m+1}, \dots, K_i), \tag{24}$$

with equality if and only if $K_1, \dots, K_m$ are homothetic.

Proof. By (22) and Hölder's inequality for $p \neq 0$, we have

$$\begin{aligned}
&Q_{n-i,mp}(K_1, \dots, K_i)^m \\
&= \left(\frac{\omega_n}{\omega_i}\right)^m \left(\int_{G_{n,i}} V_\xi(K_1|\xi, \dots, K_i|\xi)^{mp} \mathrm{d}\mu_i(\xi)\right)^{1/p} \\
&\ge \left(\frac{\omega_n}{\omega_i}\right)^m \left(\int_{G_{n,i}} \prod_{j=1}^{m} V_\xi(\underbrace{K_j|\xi, \dots, K_j|\xi}_{m}, K_{m+1}|\xi, \dots, K_i|\xi)^p \mathrm{d}\mu_i(\xi)\right)^{1/p} \\
&\ge \left(\frac{\omega_n}{\omega_i}\right)^m \prod_{j=1}^{m} \left(\int_{G_{n,i}} V_\xi(\underbrace{K_j|\xi, \dots, K_j|\xi}_{m}, K_{m+1}|\xi, \dots, K_i|\xi)^{p\cdot m} \mathrm{d}\mu_i(\xi)\right)^{1/pm} \\
&= \prod_{j=1}^{m} Q_{n-i,mp}(\underbrace{K_j, \dots, K_j}_{m}, K_{m+1}, \dots, K_i).
\end{aligned}$$

Assume the equality holds in (24), then equalities in the third line and the fourth line both hold. The first equality condition can deduced from (23), that is, $K_1, \dots, K_m$ are homothetic. The equality condition of Hölder's inequality implies that $V_\xi(K_k|\xi, m; K_{m+1}|\xi, \dots, K_i|\xi)$ and $V_\xi(K_l|\xi, m; K_{m+1}|\xi, \dots, K_i|\xi)$ are proportional for $1 \le k, l \le m$. Therefore, they are also homothetic. On the other hand, if $K_1, \dots, K_m$ are homothetic, it is obvious that the equality holds in (24).

Similarly, the case of $p = 0$ can be deduced by applying the properties of logarithmic function, exponential function and (6), we obtain

$$
\begin{aligned}
&Q_{n-i,0}(K_1,\ldots,K_i)^m\\
&\quad=\left(\frac{\omega_n}{\omega_i}\right)^m \exp\left(\int_{G_{n,i}} \log V_\xi(K_1|\xi,\ldots,K_i|\xi)^m \mathrm{d}\mu_i(\xi)\right)\\
&\quad\geq\left(\frac{\omega_n}{\omega_i}\right)^m \exp\left(\int_{G_{n,i}} \log \prod_{j=1}^m V_\xi(\underbrace{K_j|\xi,\ldots,K_j|\xi}_{m},K_{m+1}|\xi,\ldots,K_i|\xi)\mathrm{d}\mu_i(\xi)\right)\\
&\quad=\left(\frac{\omega_n}{\omega_i}\right)^m \exp\left(\sum_{j=1}^m\int_{G_{n,i}} \log V_\xi(\underbrace{K_j|\xi,\ldots,K_j|\xi}_{m},K_{m+1}|\xi,\ldots,K_i|\xi)\mathrm{d}\mu_i(\xi)\right)\\
&\quad=\left(\frac{\omega_n}{\omega_i}\right)^m \prod_{j=1}^m\exp\left(\int_{G_{n,i}} \log V_\xi(\underbrace{K_j|\xi,\ldots,K_j|\xi}_{m},K_{m+1}|\xi,\ldots,K_i|\xi)\mathrm{d}\mu_i(\xi)\right)\\
&\quad=\prod_{j=1}^m Q_{n-i,0}(\underbrace{K_j,\ldots,K_j}_{m},K_{m+1},\ldots,K_i). \qquad (25)
\end{aligned}
$$

The equality condition in (25) follows (22) immediately. □

Moreover, if $m = i$ in (24), then

Corollary 4.5. *Suppose $K_1,\ldots,K_i \in \mathcal{K}^n$, for $p \leq 0$, then*

$$Q_{n-i,ip}(K_1,\ldots,K_i)^i \geq Q_{n-i,ip}(K_1)\cdots Q_{n-i,ip}(K_i), \qquad (26)$$

with equality if and only if $K_1,\ldots,K_i$ are homothetic.

As a special case, let $K_1 = \cdots = K_j = K$ and $K_{j+1} = \cdots = K_m = L$ in Theorem 4.4, we can get

Corollary 4.6. *Suppose $K, L, K_{m+1},\ldots,K_i \in \mathcal{K}^n$, $K_{m+1},\ldots,K_i$ are smooth and $1 \leq j \leq m \leq i$, for $p \leq 0$, then*

$$
\begin{aligned}
&Q_{n-i,mp}(K,j;L,m-j;K_{m+1},\ldots,K_i)^m\\
&\qquad\geq Q_{n-i,mp}(K,m;K_{m+1},\ldots,K_i)^j Q_{n-i,mp}(L,m;K_{m+1},\ldots,K_i)^{m-j},
\end{aligned}
$$

with equality if and only if K, L are homothetic.
Moreover, if $m = i$, it follows that

$$Q_{n-i,ip}(K,j;L,i-j)^i \geq Q_{n-i,ip}(K)^j Q_{n-i,ip}(L)^{i-j},$$

with equality if and only if K, L are homothetic.

When $p = -n/i$, then (26) yields the following inequality.

Corollary 4.7. *Suppose $K_1,\ldots,K_i \in \mathcal{K}^n$, then*

$$\Phi_{n-i}(K_1,\ldots,K_i)^i \geq \Phi_{n-i}(K_1)\cdots\Phi_{n-i}(K_i),$$

with equality if and only if $K_1,\ldots,K_i$ are homothetic.

The inequality (26) implies the following inequality for $\Pi^*(K_1,\ldots,K_{n-1})$, which was proved by Lutwak [17].

Corollary 4.8. *Suppose $K_1, \dots, K_{n-1} \in \mathcal{K}^n$, then*

$$V(\Pi^*(K_1, \dots, K_{n-1}))^{n-1} \le V(\Pi^*(K_1)) \cdots V(\Pi^*(K_{n-1})),$$

with equality if and only if $K_1, \dots, K_{n-1}$ are homothetic.

Proof. The volume of $\Pi^*(K_1, \dots, K_{n-1})$ can be formulated as

$$V(\Pi^*(K_1, \dots, K_{n-1})) = \frac{1}{n} \int_{S^{n-1}} V_{n-1}(K_1|u^\perp, \dots, K_{n-1}|u^\perp)^{-n} du. \tag{27}$$

Then put $i = n-1, p = n/(1-n)$ in $Q_{n-i,ip}(K_1, \dots, K_{n-1})$, it follows that

$$\begin{aligned} Q_{1,-n}(K_1, \dots, K_{n-1}) &= \frac{\omega_n}{\omega_{n-1}} \left(\int_{G_{n,n-1}} V_\xi(K_1|\xi, \dots, K_{n-1}|\xi)^{-n} d\xi \right)^{-1/n} \\ &= \frac{\omega_n}{\omega_{n-1}} \left(\frac{1}{n\omega_n} \int_{S^{n-1}} V_{n-1}(K_1|u^\perp, \dots, K_{n-1}|u^\perp)^{-n} du \right)^{-1/n} \\ &= \frac{\omega_n}{\omega_{n-1}} \left(\frac{1}{\omega_n} V(\Pi^*(K_1, \dots, K_{n-1})) \right)^{-1/n}. \end{aligned}$$

When $K_1 = \cdots = K_{n-1} = K \in \mathcal{K}^n$, we have

$$Q_{1,-n}(K) = \frac{\omega_n}{\omega_{n-1}} \left(\frac{1}{\omega_n} V(\Pi^*(K)) \right)^{-1/n}.$$

Therefore, the desired result follows (26) immediately. □

The following theorem is the isoperimetric inequality for L^p-moment mixed quermassintegrals.

Theorem 4.9. *Suppose $K_1, \dots, K_i \in \mathcal{K}^n$ and $p \ge -n$, then*

$$Q_{n-i,p}(K_1, \dots, K_i) \ge Q_{n-i,p}(B_{K_1}, \dots, B_{K_i}), \tag{28}$$

with equality if and only if $K_1, \dots, K_i$ are balls.

Proof. From Proposition 3.4, Corollary 4.7, (4) and (11), for $p \ge -n$, we have

$$\begin{aligned} Q_{n-i,p}(K_1, \dots, K_i) &\ge Q_{n-i,-n}(K_1, \dots, K_i) \\ &= \Phi_{n-i}(K_1, \dots, K_i) \\ &\ge \Phi_{n-i}(K_1)^{1/i} \cdots \Phi_{n-i}(K_i)^{1/i} \\ &\ge \Phi_{n-i}(B_{K_1})^{1/i} \cdots \Phi_{n-i}(B_{K_i})^{1/i} \\ &= Q_{n-i,p}(B_{K_1}, \dots, B_{K_i}). \end{aligned}$$

If the equality holds in (28), then we have equalities in all inequalities above. Equality in the fourth line implies that $K_1, \dots, K_i$ are ellipsoids. Equality in the third line implies that $K_1, \dots, K_i$ are homothetic, then $K_1, \dots, K_i$ are homothetic ellipsoids. Let E is an ellipsoid such that $\lambda_1 K_1 + x_1 = \cdots = \lambda_i K_i + x_i = E$, then $Q_{n-i,p}(K_1, \dots, K_i) = \lambda_1^{-1} \cdots \lambda_i^{-1} Q_{n-i,p}(E)$ and $Q_{n-i,-n}(K_1, \dots, K_i) = \lambda_1^{-1} \cdots \lambda_i^{-1} Q_{n-i,-n}(E)$. Therefore, equality in the first line implies that $Q_{n-i,p}(E) = Q_{n-i,-n}(E)$, then it follows Jensen's inequality that $V_i(E|\xi)$ is a constant for all $\xi \in G_{n,i}$, therefore, E must be a ball. □

5. L^p-moment mixed quermassintegrals inequalities of polar bodies

Lemma 5.1. *Suppose $K_1 \in \mathcal{K}_o^n$ and K_1 is smooth, $K_2, \ldots, K_i \in \mathcal{K}^n$ and $\xi \in G_{n,i}$ with $0 \le i \le n$, then*

$$V_\xi(K_1|\xi, K_2|\xi, \ldots, K_i|\xi)V_\xi(K_1^*|\xi, K_2|\xi, \ldots, K_i|\xi) \ge V_\xi(B|\xi, K_2|\xi, \ldots, K_i|\xi)^2, \tag{29}$$

with equality if and only if K_1 is a ball.

Proof. From the Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned}
&V_\xi(K_1|\xi, K_2|\xi, \ldots, K_i|\xi)V_\xi(K_1^*|\xi, K_2|\xi, \ldots, K_i|\xi)\\
&\quad = c_{n,i}^2 V(K_1, K_2, \ldots, K_i; B \cap \xi^\perp, n-i)V(K_1^*, K_2, \ldots, K_i; B \cap \xi^\perp, n-i)\\
&\quad \ge c_{n,i}^2 V(B, K_2, \ldots, K_i; B \cap \xi^\perp, n-i)^2\\
&\quad = V_\xi(B|\xi, K_2|\xi, \ldots, K_i|\xi)^2.
\end{aligned}$$

the equality condition can be obtain from Lemma 2.2. □

Letting $K_2 = \cdots = K_i = B$ in Lemma 5.1 gives the following inequality.

Corollary 5.2. *Suppose $K \in \mathcal{K}_o^n$ and K is smooth, for $\xi \in G_{n,i}$ with $0 \le i \le n$, then*

$$V_\xi(K|\xi, B|\xi, \ldots, B|\xi)V_\xi(K^*|\xi, B|\xi, \ldots, B|\xi) \ge \omega_i^2, \tag{30}$$

with equality if and only if K is a ball.

Theorem 5.3. *Suppose $K_1 \in \mathcal{K}_o^n$ and K_1 is smooth, $K_2, \ldots, K_i \in \mathcal{K}^n$. For $0 \le i \le n$ and $p \ge 0$, then*

$$Q_{n-i,p}(K_1, K_2, \ldots, K_i)Q_{n-i,p}(K_1^*, K_2, \ldots, K_i) \ge Q_{n-i,p}(B, K_2, \ldots, K_i)^2, \tag{31}$$

with equality if and only if K_1 is a ball.

Proof. For $p > 0$, from the definition of $Q_{n-i,p}$ together with Cauchy-Schwarz inequality, and Lemma 5.1, we have

$$\begin{aligned}
&Q_{n-i,p}(K_1, K_2, \ldots, K_i)Q_{n-i,p}(K_1^*, K_2, \ldots, K_i)\\
&= \left(\frac{\omega_n}{\omega_i}\right)^2 \left(\int_{G_{n,i}} V_\xi(K_1|\xi, K_2|\xi, \ldots, K_i|\xi)^p \mathrm{d}\mu_i(\xi)\right)^{1/p} \left(\int_{G_{n,i}} V_\xi(K_1^*|\xi, K_2|\xi, \ldots, K_i|\xi)^p \mathrm{d}\mu_i(\xi)\right)^{1/p}\\
&\ge \left(\frac{\omega_n}{\omega_i}\right)^2 \left(\int_{G_{n,i}} V_\xi(K_1|\xi, K_2|\xi, \ldots, K_i|\xi)^{p/2} V_\xi(K_1^*|\xi, K_2|\xi, \ldots, K_i|\xi)^{p/2} \mathrm{d}\mu_i(\xi)\right)^{2/p}\\
&\ge \left(\frac{\omega_n}{\omega_i}\right)^2 \left(\int_{G_{n,i}} V_\xi(B|\xi, K_2|\xi, \ldots, K_i|\xi)^p \mathrm{d}\mu_i(\xi)\right)^{2/p}\\
&= Q_{n-i,p}(B, K_2, \ldots, K_i)^2.
\end{aligned}$$

Assume the equality holds in (31), then we get equality in all inequality above. The equality in third line yields that $V_\xi(K_1|\xi, K_2|\xi, \ldots, K_i|\xi)$ and $V_\xi(K_1^*|\xi, K_2|\xi, \ldots, K_i|\xi)$ are proportional. And the equality in fourth line holds when K_1 is a ball follows Lemma 2.2. Therefore, the equality condition in (31) is that K_1 is a ball.

The case of $p = 0$ can be obtained by the properties of logarithmic function, exponential function and Lemma 2.2. □

The following result can be obtained from Theorem 31 as a special case by letting $K_2 = \cdots = K_i = B$.

Corollary 5.4. *Suppose $K \in \mathcal{K}_o^n$ and K is smooth, for $0 \le i \le n$ and $p \ge 0$, then*

$$Q_{n-i,p}(K, B, \ldots, B)Q_{n-i,p}(K^*, B, \ldots, B) \ge \omega_n^2,$$

with equality if and only if K is a ball.

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