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# Semi $P$-geometric-arithmetically functions and some new related inequalities

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**Abstract.** In this manuscript, the authors introduce the concept of the semi P-geometric–arithmetically functions (semi P-GA functions) and give their some algebraic properties. Then, they get Hermite-Hadamard's integral inequalities for semi P-GA-functions (geometric-arithmetically convex). In addition, the authors obtain new inequalities by using Hölder and Hölder-İşcan integral inequalities with the help of an identity. Then, the aouthors compare the results obtained with both Hölder, Hölder-İşcan integral inequalities and prove that the Hölder-İşcan integral inequality gives a better approximation than the Hölder integral inequality. Also, some applications to special means of real numbers are also given.

## 1. Preliminaries and fundamentals

Let real function $f$ be defined on some nonempty interval $I$ of real line $\mathbb{R}$. The function $f$ is said to be convex on $I$ if the following inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

is valid for all $x, y \in I$ and $t \in [0,1]$. If this inequality reverses, then $f$ is said to be concave on interval $I \neq \emptyset$. Convexity theory in connection with integral inequalities is an interesting and important field of research. Many inequalities are direct consequences of the applications of convex functions. Mathematical inequalities and convexity theory play a key role in understanding a range of problems in various fields of mathematics and the other branches of sciences such as economics and engineering.

One of the most famous inequality for convex functions is so called Hermite-Hadamard's inequality as follows:

**Theorem 1.1 (Hermite-Hadamard integral inequality).** *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following double integral inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

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for all $a, b \in I$ with $a < b$. Both inequalities hold in the reversed direction if the function $f$ is concave. This double integral inequality is well known as the Hermite-Hadamard integral inequality [5]. Some refinements of the Hermite-Hadamard integral inequality for convex functions have been obtained [3, 12].

In [2, 4], Dragomir et al. gave the following definition and related Hermite-Hadamard integral inequalities as follow:

**Definition 1.2.** *A nonnegative function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be P-function if the inequality*

$$f(tx + (1-t)y) \le f(x) + f(y)$$

*holds for all $x, y \in I$ and $t \in (0, 1)$.*

**Theorem 1.3.** *Let $f$ be a P-function on interval $I$ (or $f \in P(I)$), $a, b \in I$ with $a < b$ and $f \in L[a, b]$. Then*

$$f\left(\frac{a+b}{2}\right) \le \frac{2}{b-a}\int_a^b f(x)\,dx \le 2\left[f(a) + f(b)\right]. \tag{2}$$

**Definition 1.4 ([11]).** *A function $f : I \subseteq (0, \infty) \to \mathbb{R}$ is said to be GA-convex (geometric-arithmetically convex) if*

$$f\left(x^t y^{1-t}\right) \le t f(x) + (1-t) f(y)$$

*for all $x, y \in I$ and $t \in [0, 1]$, where $x^t y^{1-t}$ and $t f(x) + (1-t) f(y)$ are respectively called the weighted geometric mean of two positive numbers $x$ and $y$ and the weighted arithmetic mean of $f(x)$ and $f(y)$.*

**Definition 1.5 ([8]).** *A function $f : I \subseteq (0, \infty) \to \mathbb{R}$ is said to be P-GA-function (P-geometric-arithmetic function) on $I$ if*

$$f\left(x^t y^{1-t}\right) \le f(x) + f(y)$$

*for any $x, y \in I$ and $t \in [0, 1]$.*

**Theorem 1.6 (Hölder-İşcan integral inequality [9]).** *Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $f$ and $g$ are real functions defined on interval $[a, b]$ and if $\left|f\right|^p$, $\left|g\right|^q$ are integrable functions on $[a, b]$ then*

$$\begin{aligned} \int_a^b \left|f(x)g(x)\right| dx \;\; \le \;\; & \frac{1}{b-a}\left\{\left(\int_a^b (b-x)\left|f(x)\right|^p dx\right)^{\frac{1}{p}} \left(\int_a^b (b-x)\left|g(x)\right|^q dx\right)^{\frac{1}{q}}\right. \\ & \left. + \left(\int_a^b (x-a)\left|f(x)\right|^p dx\right)^{\frac{1}{p}} \left(\int_a^b (x-a)\left|g(x)\right|^q dx\right)^{\frac{1}{q}}\right\} \end{aligned} \tag{3}$$

This article is organized as follows. In chapter 2, we introduce a new concept, which is called semi $P$-GA function, and we give by setting some algebraic properties of semi $P$-GA function. In chapter 3, we obtain the Hermite-Hadamard integral inequality for the semi $P$-GA function. In chapter 4, by using an identity, we obtain some refinements of the Hermite-Hadamard integral inequality for functions whose first derivative in absolute value, raised to a certain power which is greater than one, respectively at least one, is semi $P$-GA functions. Then, we compare the results obtained with both Hölder, Hölder-İşcan integral inequalities and we prove that the Hölder-İşcan integral inequality gives a better approximation than the Hölder integral inequality. In chapter 5, we give some applications to special means of real numbers.

## 2. The definition of semi $P$-geometric-arithmetically function

In recent years, many function classes such as P-function, geometric convex, geometric P-function, quasi-geometric convex, GA-s-convex function in the first sense, GA-s-convex function in the second sense, geometric trigonometrically convex, GG-convex, GH-convex, geometric P-function, etc. have been studied by many authors, and integral inequalities belonging to these function classes have been studied in the literature (see [1, 4, 6–8, 10, 11]). There are many articles and books on this subject.

In this section, a new function class, semi-P-GA function definition will be given, and the relations of this function class with the above-mentioned function classes will also be given.

**Definition 2.1.** *A nonnegative function $f : I \subseteq (0,\infty) \to \mathbb{R}$ is called semi P-GA function if for every $x, y \in I$ and $t \in [0,1]$,*

$$f\left(x^t y^{1-t}\right) \leq (tx + (1-t)y)\left[f(x) + f(y)\right]. \tag{4}$$

We will denote by $SPGA(I)$ the class of all semi $P$-GA functions on interval $I$.

We note that if the function $f : I \subseteq (0,\infty) \to \mathbb{R}$ is semi $P$-GA then

$$f(x) = f\left(x^t x^{1-t}\right) \leq (tx + (1-t)x)\left[f(x) + f(x)\right] = 2xf(x)$$

for all $x \in I$, i.e. $(2x-1)f(x) \geq 0$ for all $x \in I$. In this case, we can say that either "$x \geq 1/2$ and $f(x) \geq 0$" or "$x \leq 1/2$ and $f(x) \leq 0$". Therefore, it must be $x \geq 1/2$.

**Example 2.2.** *The function $f : [1/2,\infty) \to \mathbb{R}$, $f(x) = x$ is a semi P-GA function.*

**Example 2.3.** *The function $f : [1,\infty) \to \mathbb{R}$, $f(x) = x^r, r \in \mathbb{R}$, is a semi P-GA function.*

**Example 2.4.** *The function $f : [1,\infty) \to \mathbb{R}$, $f(x) = \ln x$ is a semi P-GA function.*

**Example 2.5.** *For every $c \in \mathbb{R}\,(c \geq 0)$, the function $f : \left[\frac{1}{2},\infty\right) \subset \mathbb{R} \to \mathbb{R}$, $f(x) = c$ is a semi P-GA function.*

**Remark 2.6.** *If $f : [1,\infty) \to [0,\infty)$ is a GA-function, then $f$ is also semi P-GA function. Since, $t \leq ta+(1-t)b$, $1-t \leq ta+(1-t)b$ for every $a, b \in [1,\infty)$ and $t \in [0,1]$, we can write*

$$f\left(a^t b^{1-t}\right) \leq tf(a) + (1-t)f(b) \leq (ta + (1-t)b)\left[f(a) + f(b)\right].$$

**Remark 2.7.** *If $f : [1,\infty) \to [0,\infty)$ is a P-GA function, then $f$ is also a semi P-GA function. Since, $1 \leq ta + (1-t)b$ for every $a, b \in [1,\infty)$ and $t \in [0,1]$, we can write*

$$f\left(a^t b^{1-t}\right) \leq f(a) + f(b) \leq (ta + (1-t)b)\, f(a) + (ta + (1-t)b)\, f(b) \leq (ta + (1-t)b)\left[f(a) + f(b)\right].$$

**Remark 2.8.** *If $f : [1,\infty) \to [0,\infty)$ is a quasi geometrically convex function, then $f$ is also a semi P-GA function. Since,*

$$f(a) \leq (ta + (1-t)b)\, f(a), \quad f(b) \leq (ta + (1-t)b)\, f(b)$$

*for $a, b \in [1,\infty)$ and $t \in [0,1]$, we can write*

$$f\left(a^t b^{1-t}\right) \leq \max\{f(a), f(b)\} \leq \max\{(ta + (1-t)b)\, f(a), (ta + (1-t)b)\, f(b)\} \leq (ta + (1-t)b)\left[f(a) + f(b)\right].$$

**Remark 2.9.** *Let $f : [1,\infty) \to [0,\infty)$ be a nonnegative and $s \in (0,1]$. If $f$ is a GA s-convex function in the first sense, then $f$ is also a semi P-GA function. Since, $t^s \leq 1 \leq ta + (1-t)b$ and $1 - t^s \leq 1 \leq ta + (1-t)b$ for every $a, b \in [1,\infty)$ and $t \in [0,1]$, we can write*

$$f\left(a^t b^{1-t}\right) \leq t^s f(a) + (1-t^s)f(b) \leq (ta + (1-t)b)\, f(a) + (ta + (1-t)b)\, f(b) = (ta + (1-t)b)\left[f(a) + f(b)\right].$$

**Remark 2.10.** *Let $f:[1,\infty)\to[0,\infty)$ be a nonnegative and $s\in(0,1]$. If $f$ is a GA s-convex function in the second sense, then $f$ is also a semi P-GA function. Since, $t^s\leq 1\leq ta+(1-t)b$ and $(1-t)^s\leq 1\leq ta+(1-t)b$ for every $a,b\in[1,\infty)$ and $t\in[0,1]$, we can write*

$$f\left(a^tb^{1-t}\right) \leq t^sf(a)+(1-t)^sf(b)\leq(ta+(1-t)b)f(a)+(ta+(1-t)b)f(b)=(ta+(1-t)b)\left[f(a)+f(b)\right].$$

**Remark 2.11.** *If $f:[1,\infty)\to[0,\infty)$ is a geometric trigonometrically convex function, then $f$ is also a semi P-GA function. Since, $\sin\frac{\pi t}{2}\leq 1,\ \cos\frac{\pi t}{2}\leq 1$ and $1\leq ta+(1-t)b$ for every $a,b\in[1,\infty)$ and $t\in[0,1]$, we can write*

$$\begin{aligned} f\left(a^tb^{1-t}\right)\leq\sin\frac{\pi t}{2}f(a)+\cos\frac{\pi t}{2}f(b)\leq f(a)+f(b) &\leq (ta+(1-t)b)f(a)+(ta+(1-t)b)f(b)\\ &= (ta+(1-t)b)\left[f(a)+f(b)\right].\end{aligned}$$

**Remark 2.12.** *If $f:[1,\infty)\to(0,\infty)$ is a GG convex function, then $f$ is also a semi P-GA function. Since, $t\leq ta+(1-t)b,\ 1-t\leq ta+(1-t)b$ for $a,b\in[1,\infty)$ and $t\in[0,1]$, we can write*

$$f\left(a^tb^{1-t}\right)\leq[f(a)]^t[f(b)]^{1-t}\leq tf(a)+(1-t)f(b)\leq(ta+(1-t)b)\left[f(a)+f(b)\right].$$

**Remark 2.13.** *If $f:[1,\infty)\to(0,\infty)$ is a geometric harmonic (GH) convex function, then $f$ is also a semi P-GA function. Since, $t\leq ta+(1-t)b,\ 1-t\leq ta+(1-t)b$ for $a,b\in[1,\infty)$ and $t\in[0,1]$, we can write*

$$f\left(a^tb^{1-t}\right)\leq\frac{f(a)f(b)}{tf(b)+(1-t)f(a)}\leq tf(a)+(1-t)f(b)\leq(ta+(1-t)b)\left[f(a)+f(b)\right].$$

**Proposition 2.14.** *Let $f:I\subseteq[1,\infty)\to\mathbb{R}$. If the function $f$ is a P-function and nondecreasing, then $f$ is a semi P-GA-function on interval $I$.*

*Proof.* This follows from

$$f\left(a^tb^{1-t}\right)\leq f(ta+(1-t)b)\leq f(a)+f(b)\leq(ta+(1-t)b)\left[f(a)+f(b)\right]$$

for all $a,b\in I$ and $t\in[0,1]$. □

**Proposition 2.15.** *Let $f:I\subseteq[1,\infty)\to\mathbb{R}$. If the function $f$ is a P-GA function and nonincreasing, then $f$ is a semi P-GA function on interval $I$.*

*Proof.* The conclusion follows from

$$f(ta+(1-t)b)\leq f\left(a^tb^{1-t}\right)\leq f(a)+f(b)\leq(ta+(1-t)b)\left[f(a)+f(b)\right]$$

for all $a,b\in I$ and $t\in[0,1]$, respectively. □

**Proposition 2.16.** *If the function $f:I\subset[e,\infty)\to\mathbb{R}$ is P-GA function on interval $I$ then $f\circ\exp:\ln I\to\mathbb{R}$ is semi P- function on the interval $\ln I=\{\ln x: x\in I\}$.*

*Proof. Let $f:I\subset[e,\infty)\to\mathbb{R}$ is a P-GA function. Then, we write*

$$\begin{aligned}(f\circ\exp)(t\ln a+(1-t)\ln b) &= f\left(a^tb^{1-t}\right)\\ &\leq f(a)+f(b)=(f\circ\exp)(\ln a)+(f\circ\exp)(\ln b)\\ &\leq (t\ln a+(1-t)\ln b)(f\circ\exp)(\ln a)+(f\circ\exp)(\ln b).\end{aligned}$$

*Hence, the function $f\circ\exp$ is a semi P- function on the interval $\ln I$.* □

**Theorem 2.17.** *Let $f,g:I\subset\mathbb{R}\to\mathbb{R}$. If the functions $f$ and $g$ are semi P-GA functions, then*
*(i) $f+g$ is a semi P-GA function,*
*(ii) For $c\in\mathbb{R}$ ($c\geq 0$) $cf$ is a semi P-GA function.*

*Proof.* (i) Let $f, g$ be semi $P$-GA functions, then

$$\begin{aligned}(f+g)\left(a^t b^{1-t}\right) &= f\left(a^t b^{1-t}\right)+g\left(a^t b^{1-t}\right)\\ &\leq (ta+(1-t)b)\left[f(a)+f(b)\right]+(ta+(1-t)b)\left[g(a)+g(b)\right]\\ &= (ta+(1-t)b)\left[f(a)+g(a)\right]+(ta+(1-t)b)\left[f(b)+g(b)\right]\\ &= (ta+(1-t)b)(f+g)(a)+(ta+(1-t)b)(f+g)(b),\end{aligned}$$

for every $a, b \in I$ and $t \in [0,1]$.

(ii) Let $f$ be semi $P$-GA function and $c \in \mathbb{R}$ *($c \geq 0$), then*

$$\begin{aligned}(cf)\left(a^t b^{1-t}\right) &\leq c\,(ta+(1-t)b)\left[f(a)+f(b)\right]\\ &= (ta+(1-t)b)\left[cf(a)+cf(b)\right]\\ &= (ta+(1-t)b)\left[(cf)(a)+(cf)(b)\right],\end{aligned}$$

for every $a, b \in I$ and $t \in [0,1]$. □

**Theorem 2.18.** *Let $f_\alpha : I \subset (0,\infty) \to \mathbb{R}$ be an arbitrary family of semi P-GA functions and let $f(x) = \sup_\alpha f_\alpha(x)$. If $J = \{u \in I : f(u) < \infty\}$ is nonempty, then $J$ is an interval and the function $f$ is a semi P-GA function on intervalJ.*

*Proof.* Let $t \in [0,1]$ and $a, b \in J$ be arbitrary. Then

$$\begin{aligned}f\left(a^t b^{1-t}\right) &= \sup_\alpha f_\alpha\left(a^t b^{1-t}\right)\\ &\leq \sup_\alpha\left[(ta+(1-t)b)\,f_\alpha(a)+(ta+(1-t)b)\,f_\alpha(b)\right]\\ &\leq (ta+(1-t)b)\sup_\alpha f_\alpha(a)+(ta+(1-t)b)\sup_\alpha f_\alpha(b)\\ &= (ta+(1-t)b)\,f(a)+(ta+(1-t)b)\,f(b)\\ &= (ta+(1-t)b)\left[f(a)+f(b)\right]<\infty.\end{aligned}$$

This shows simultaneously that $J$ is an interval, since it contains every point between any two of its points, and that $f$ is a semi $P$-GA function on $J$. This completes the proof of theorem. □

## 3. Hermite-Hadamard integral inequality for semi *P*-GA functions

The goal of this paper is to establish some inequalities of Hermite-Hadamard type integral inequality for semi $P$-GA functions. In this section, we will denote by $L[a,b]$ the space of (Lebesgue) integrable functions on interval $[a,b]$.

**Theorem 3.1.** *Let $f : I \subset [1/2,\infty) \to \mathbb{R}$ be a semi P-GA function. If $a < b$ and $f \in L[a,b]$, then the following Hermite-Hadamard type integral inequalities hold:*

$$\begin{aligned}f\left(\sqrt{ab}\right) &\leq \frac{1}{\ln b-\ln a}\int_a^b f(u)du+\frac{ab}{\ln b-\ln a}\int_a^b \frac{f(u)}{u^2}du, \\ &\leq A(a,b)L(a,b)2\left[f(a)+f(b)\right].\end{aligned} \tag{5}$$

*$A(a,b) = \frac{a+b}{2}$ is arithmetic mean and $L(a,b) = \frac{b-a}{\ln b-\ln a}$ is logarithmic mean.*

*Proof.* Since $f$ is a semi $P$-GA function on interval $[a,b]$, we have for all $x, y \in [a,b]$ (with $t = \frac{1}{2}$ in inequality (4))

$$f\left(\sqrt{xy}\right) \leq \frac{x+y}{2}\left[f(x)+f(y)\right].$$

By choosing $x = a^t b^{1-t}$ and $y = a^{1-t} b^t$, we have

$$f\left(\sqrt{ab}\right) \leq \frac{a^t b^{1-t} + a^{1-t} b^t}{2}\left[f\left(a^t b^{1-t}\right) + f\left(a^{1-t} b^t\right)\right].$$

Integrating the resulting inequality with respect to $t$ over $[0,1]$, we obtain

$$\begin{aligned}
f\left(\sqrt{ab}\right) &\leq \frac{1}{2}\left[\int_0^1 a^t b^{1-t} f\left(a^t b^{1-t}\right) dt + \int_0^1 a^t b^{1-t} f\left(a^{1-t} b^t\right) dt + \int_0^1 a^{1-t} b^t f\left(a^t b^{1-t}\right) dt + \int_0^1 a^{1-t} b^t f\left(a^{1-t} b^t\right) dt\right] \\
&= \frac{1}{2}\left[\frac{1}{\ln b - \ln a}\int_a^b f(u)du + \frac{ab}{\ln b - \ln a}\int_a^b \frac{f(u)}{u^2}du + \frac{ab}{\ln b - \ln a}\int_a^b \frac{f(u)}{u^2}du + \frac{1}{\ln b - \ln a}\int_a^b f(u)du\right] \\
&= \frac{1}{\ln b - \ln a}\int_a^b f(u)du + \frac{ab}{\ln b - \ln a}\int_a^b \frac{f(u)}{u^2}du,
\end{aligned}$$

and the first inequality is proved. For the proof of the second inequality in (5) we first note that if the function $f$ is a semi $P$-GA-function, then, for $t \in [0,1]$, it yields

$$\begin{aligned}
f\left(a^t b^{1-t}\right) &\leq (ta + (1-t)b)\left[f(a) + f(b)\right], \\
f\left(a^{1-t} b^t\right) &\leq (tb + (1-t)a)\left[f(a) + f(b)\right].
\end{aligned}$$

By adding side to side these inequalities and multiplying by $a^t b^{1-t}$, we have

$$a^t b^{1-t} f\left(a^t b^{1-t}\right) + a^t b^{1-t} f\left(a^{1-t} b^t\right) \leq a^t b^{1-t}(a+b)\left[f(a) + f(b)\right],$$

and, integrating the resulting inequality with respect to $t$ over $[0,1]$, we obtain

$$\begin{aligned}
\int_0^1 a^t b^{1-t} f\left(a^t b^{1-t}\right) dt + \int_0^1 a^t b^{1-t} f\left(a^{1-t} b^t\right) dt &\leq (a+b)\left[f(a) + f(b)\right]\int_0^1 a^t b^{1-t} dt \\
\frac{1}{\ln b - \ln a}\int_a^b f(u)du + \frac{ab}{\ln b - \ln a}\int_a^b \frac{f(u)}{u^2}du &\leq (a+b)\left[f(a) + f(b)\right]\frac{b-a}{\ln b - \ln a}.
\end{aligned}$$

This completes the proof of theorem. □

## 4. Some new integral inequalities for the semi *P*-GA functions

The main purpose of this section is to establish new estimates that refine Hermite-Hadamard integral inequality for functions whose first derivative in absolute value, raised to a certain power which is greater than one, respectively at least one, is semi $P$-GA function. In [13], Zhang et al. established some Hermite-Hadamard type inequalities for geometric aritmetically (GA) convex functions and applied these inequalities to construct several inequalities for special means and they used the following lemma to prove their results. We will use the following Lemma:

**Lemma 4.1 ([13]).** *Let $f : I \subseteq \mathbb{R}_+ = (0,\infty) \to \mathbb{R}$ be differentiable mapping on $I^\circ$, $a, b \in I^\circ$ with $a < b$. If $f' \in L[a,b]$, then*

$$bf(b) - af(a) - \int_a^b f(x)dx = (\ln b - \ln a)\int_0^1 b^{2t} a^{2(1-t)} f'\left(b^t a^{1-t}\right) dt.$$

**Theorem 4.2.** *Let $f : I \subseteq [1/2,\infty) \to \mathbb{R}$ be a differentiable function on $I^\circ$, $a, b \in I^\circ$ with $a < b$ and assume that $f' \in L[a,b]$. If $\left|f'\right|$ is a semi P-GA function on interval $[a,b]$, then the following inequality holds*

$$\left|bf(b) - af(a) - \int_a^b f(x)dx\right| \leq (b-a)L^{-1}(a,b)A\left(\left|f'(a)\right|, \left|f'(b)\right|\right)\left[L\left(a^2,b^2\right)(2a - L(a,b)) + b^2 L(a,b)\right], \quad (6)$$

*$A(a,b) = \frac{a+b}{2}$ is arithmetic mean and $L(a,b) = \frac{b-a}{\ln b - \ln a}$ is logarithmic mean.*

*Proof.* Using Lemma 4.1 and the inequality

$$\left|f'\left(a^{1-t}b^{t}\right)\right| \le (tb+(1-t)a)\left[\left|f'(a)\right|+\left|f'(b)\right|\right],$$

we get

$$\begin{aligned}
&\left|bf(b)-af(a)-\int_a^b f(x)dx\right| \\
\le\ & (\ln b-\ln a)\int_0^1 b^{2t}a^{2(1-t)}\left|f'\left(b^{t}a^{1-t}\right)\right|dt \\
\le\ & (\ln b-\ln a)\int_0^1 b^{2t}a^{2(1-t)}\left((tb+(1-t)a)\left[\left|f'(a)\right|+\left|f'(b)\right|\right]\right)dt \\
\le\ & (\ln b-\ln a)\left[\left|f'(a)\right|+\left|f'(b)\right|\right]\int_0^1 b^{2t}a^{2(1-t)}\left(tb+(1-t)a\right)dt \\
=\ & (\ln b-\ln a)\left[\left|f'(a)\right|+\left|f'(b)\right|\right]\left[(b-a)\int_0^1 tb^{2t}a^{2(1-t)}dt+a\int_0^1 b^{2t}a^{2(1-t)}dt\right] \\
=\ & (\ln b-\ln a)\left[\left|f'(a)\right|+\left|f'(b)\right|\right]\left[(b-a)\frac{b^2\left(2\ln b-2\ln a-1\right)+a^2}{4\left(\ln b-\ln a\right)^2}+a\frac{b^2-a^2}{2\left(\ln b-\ln a\right)}\right] \\
=\ & (\ln b-\ln a)\left[\frac{\left|f'(a)\right|+\left|f'(b)\right|}{2}\right]\left[L\left(a^2,b^2\right)(2a-L(a,b))+b^2L(a,b)\right] \\
=\ & (b-a)L^{-1}(a,b)A\left(\left|f'(a)\right|,\left|f'(b)\right|\right)\left[L\left(a^2,b^2\right)(2a-L(a,b))+b^2L(a,b)\right],
\end{aligned}$$

where

$$\begin{aligned}
\int_0^1 tb^{2t}a^{2(1-t)}dt &= \frac{b^2\left(2\ln b-2\ln a-1\right)+a^2}{4\left(\ln b-\ln a\right)^2}, \\
\int_0^1 b^{2t}a^{2(1-t)}dt &= \frac{b^2-a^2}{2\left(\ln b-\ln a\right)}.
\end{aligned}$$

This completes the proof of theorem. □

**Theorem 4.3.** *Let $f: I\subseteq[1/2,\infty)\to\mathbb{R}$ be a differentiable function on $I^\circ$, $a,b\in I^\circ$ with $a<b$, $q>1$, $\frac{1}{p}+\frac{1}{q}=1$ and assume that $f'\in L[a,b]$. If $\left|f'\right|^q$ is a semi P-GA function on interval $[a,b]$, then the following inequality holds*

$$\left|bf(b)-af(a)-\int_a^b f(x)dx\right| \le 2^{\frac{1}{q}}\left(b-a\right)L^{-1}(a,b)A^{\frac{1}{q}}(a,b)A^{\frac{1}{q}}\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right)L^{\frac{1}{p}}\left(a^{2p},b^{2p}\right) \tag{7}$$

*Proof.* Using Lemma 4.1, Hölder's integral inequality and the following inequality

$$\left|f'\left(a^{1-t}b^{t}\right)\right|^q \le (tb+(1-t)a)\left[\left|f'(a)\right|^q+\left|f'(b)\right|^q\right]$$

which is the semi $P$-GA function of the function $\left|f'\right|^q$, we have

$$\begin{aligned}
&\left|bf(b)-af(a)-\int_a^b f(x)dx\right| \\
=\;& (\ln b-\ln a)\int_0^1 b^{2t}a^{2(1-t)}\left|f'\left(b^ta^{1-t}\right)\right|dt \\
\leq\;& (\ln b-\ln a)\left(\int_0^1 \left|b^{2t}a^{2(1-t)}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'\left(b^ta^{1-t}\right)\right|^q dt\right)^{\frac{1}{q}} \\
\leq\;& (\ln b-\ln a)\left(\int_0^1 b^{2pt}a^{2p(1-t)}dt\right)^{\frac{1}{p}}\left(\int_0^1 (tb+(1-t)a)\left[\left|f'(a)\right|^q+\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}} \\
=\;& (\ln b-\ln a)\left[\left|f'(a)\right|^q+\left|f'(b)\right|^q\right]^{\frac{1}{q}}\left(\int_0^1 b^{2pt}a^{2p(1-t)}dt\right)^{\frac{1}{p}}\left(\int_0^1 (tb+(1-t)a)\,dt\right)^{\frac{1}{q}} \\
=\;& 2^{\frac{1}{q}}(b-a)\left(\frac{\ln b-\ln a}{b-a}\right)A^{\frac{1}{q}}\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right)L^{\frac{1}{p}}\left(a^{2p},b^{2p}\right)\left(\frac{a+b}{2}\right)^{\frac{1}{q}} \\
=\;& 2^{\frac{1}{q}}(b-a)L^{-1}(a,b)A^{\frac{1}{q}}(a,b)A^{\frac{1}{q}}\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right)L^{\frac{1}{p}}\left(a^{2p},b^{2p}\right),
\end{aligned}$$

where $\int_0^1 b^{2pt}a^{2p(1-t)}dt = L\left(a^{2p},b^{2p}\right)$, $\int_0^1 (tb+(1-t)a)\,dt = A(a,b)$. This completes the proof of theorem. □

**Theorem 4.4.** *Let $f: I\subseteq[1/2,\infty)\to\mathbb{R}$ be a differentiable function on $I^\circ$, $a,b\in I^\circ$ with $a<b$, $q\geq 1$ and assume that $f'\in L[a,b]$. If $\left|f'\right|^q$ is a semi P-GA function on the interval $[a,b]$, then the following inequality holds*

$$\begin{aligned}
&\left|bf(b)-af(a)-\int_a^b f(x)dx\right| \\
\leq\;& (b-a)L^{-1}(a,b)L^{1-\frac{1}{q}}\left(a^2,b^2\right)A^{\frac{1}{q}}\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right)\left[L\left(a^2,b^2\right)(2a-L(a,b))+b^2L(a,b)\right]^{\frac{1}{q}}. \qquad (8)
\end{aligned}$$

*Proof.* Assume first that $q>1$. From Lemma 4.1, Hölder integral inequality and the property of the semi $P$-GA function of the function $\left|f'\right|^q$, we get

$$\begin{aligned}
&\left|bf(b)-af(a)-\int_a^b f(x)dx\right| \\
\leq\;& (\ln b-\ln a)\left(\int_0^1 b^{2t}a^{2(1-t)}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 b^{2t}a^{2(1-t)}\left|f'b^ta^{1-t}\right|^q dt\right)^{\frac{1}{q}} \\
\leq\;& (\ln b-\ln a)\left(\int_0^1 b^{2t}a^{2(1-t)}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 b^{2t}a^{2(1-t)}(tb+(1-t)a)\left[\left|f'(a)\right|^q+\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}} \\
=\;& (\ln b-\ln a)\left(\frac{b^2-a^2}{\ln b^2-\ln a^2}\right)^{1-\frac{1}{q}}\left(\left[\left|f'(a)\right|^q+\left|f'(b)\right|^q\right]\left[(b-a)\frac{b^2(2\ln b-2\ln a-1)+a^2}{4(\ln b-\ln a)^2}+a\frac{b^2-a^2}{2(\ln b-\ln a)}\right]\right)^{\frac{1}{q}} \\
=\;& (\ln b-\ln a)\left(\frac{b^2-a^2}{\ln b^2-\ln a^2}\right)^{1-\frac{1}{q}}\left(\frac{\left[\left|f'(a)\right|^q+\left|f'(b)\right|^q\right]}{2}L\left(a^2,b^2\right)(2a-L(a,b))+b^2L(a,b)\right)^{\frac{1}{q}} \\
=\;& (b-a)L^{-1}(a,b)L^{1-\frac{1}{q}}\left(a^2,b^2\right)A^{\frac{1}{q}}\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right)\left[L\left(a^2,b^2\right)(2a-L(a,b))+b^2L(a,b)\right]^{\frac{1}{q}},
\end{aligned}$$

where $\int_0^1 b^{2t}a^{2(1-t)}dt = L\left(a^2,b^2\right)$. For $q=1$ we use the estimates from the proof of Theorem 4.2, which also follow step by step the above estimates. This completes the proof of theorem. □

**Corollary 4.5.** *Under the assumption of Theorem 4.4 with $q = 1$, we get the conclusion of Theorem 4.2.*

$$\left| bf(b) - af(a) - \int_a^b f(x)dx \right| \leq (b-a)L^{-1}(a,b)A\left(\left|f'(a)\right|, \left|f'(b)\right|\right)\left[L\left(a^2,b^2\right)(2a - L(a,b)) + b^2L(a,b)\right].$$

Now, we will prove the Theorem 4.3 by using Hölder-İşcan integral inequality. Then we will show that the result we have obtained in this theorem gives a better approach than that obtained in the Theorem 4.3.

**Theorem 4.6.** *Let $f : I \subseteq [1/2, \infty) \to \mathbb{R}$ be a differentiable function on $I^\circ$, $a, b \in I^\circ$ with $a < b$, $q > 1, \frac{1}{p} + \frac{1}{q} = 1$ and assume that $f' \in L[a,b]$. If $\left|f'\right|^q$ is a semi P-GA function on interval $[a,b]$, then the following inequality holds*

$$\begin{aligned} &\left| bf(b) - af(a) - \int_a^b f(x)dx \right| \leq \left(\frac{1}{2p}\right)^{\frac{1}{p}} \left(\frac{b-a}{L(a,b)}\right)^{\frac{1}{q}} A^{\frac{1}{q}}\left(\left|f'(a)\right|^q, \left|f'(b)\right|^q\right) \\ &\times \left\{ \left[L\left(a^{2p},b^{2p}\right) - a^{2p}\right]^{\frac{1}{p}} \left(\frac{b+2a}{3}\right)^{\frac{1}{q}} + \left[b^{2p} - L\left(a^{2p},b^{2p}\right)\right]^{\frac{1}{p}} \left(\frac{2b+a}{3}\right)^{\frac{1}{q}} \right\}. \end{aligned} \tag{9}$$

*Proof.* Using Lemma 4.1, Hölder-İşcan integral inequality and the following inequality

$$\left|f'\left(b^t a^{1-t}\right)\right|^q \leq (tb + (1-t)a)\left[\left|f'(a)\right|^q + \left|f'(b)\right|^q\right]$$

which is the semi $P$-GA function of $\left|f'\right|^q$, we obtain

$$\begin{aligned} &\left| bf(b) - af(a) - \int_a^b f(x)dx \right| \\ \leq\ & (\ln b - \ln a)\left(\int_0^1 (1-t)b^{2pt}a^{2p(1-t)}dt\right)^{\frac{1}{p}} \left(\int_0^1 (1-t)\left|f'\left(b^t a^{1-t}\right)\right|^q dt\right)^{\frac{1}{q}} \\ &+ (\ln b - \ln a)\left(\int_0^1 t b^{2pt}a^{2p(1-t)}dt\right)^{\frac{1}{p}} \left(\int_0^1 t\left|f'\left(b^t a^{1-t}\right)\right|^q dt\right)^{\frac{1}{q}} \\ \leq\ & (\ln b - \ln a)\left(\int_0^1 (1-t)b^{2pt}a^{2p(1-t)}dt\right)^{\frac{1}{p}} \left(\left[\left|f'(a)\right|^q + \left|f'(b)\right|^q\right] \int_0^1 (1-t)(tb + (1-t)a)\,dt\right)^{\frac{1}{q}} \\ &+ (\ln b - \ln a)\left(\int_0^1 t b^{2pt}a^{2p(1-t)}dt\right)^{\frac{1}{p}} \left(\left[\left|f'(a)\right|^q + \left|f'(b)\right|^q\right] \int_0^1 t(tb + (1-t)a)\,dt\right)^{\frac{1}{q}} \\ =\ & (\ln b - \ln a)\left(\frac{L\left(a^{2p},b^{2p}\right) - a^{2p}}{2p(\ln p - \ln a)}\right)^{\frac{1}{p}} \left(\left[\left|f'(a)\right|^q + \left|f'(b)\right|^q\right]\frac{b+2a}{6}\right)^{\frac{1}{q}} \\ &+ (\ln b - \ln a)\left(\frac{b^{2p} - L\left(a^{2p},b^{2p}\right)}{2p(\ln p - \ln a)}\right)^{\frac{1}{p}} \left(\left[\left|f'(a)\right|^q + \left|f'(b)\right|^q\right]\frac{2b+a}{6}\right)^{\frac{1}{q}} \\ =\ & \left(\frac{1}{2p}\right)^{\frac{1}{p}} \left(\frac{b-a}{L(a,b)}\right)^{\frac{1}{q}} \left[L\left(a^{2p},b^{2p}\right) - a^{2p}\right]^{\frac{1}{p}} A^{\frac{1}{q}}\left(\left|f'(a)\right|^q, \left|f'(b)\right|^q\right)\left(\frac{b+2a}{3}\right)^{\frac{1}{q}} \\ &+ \left(\frac{1}{2p}\right)^{\frac{1}{p}} \left(\frac{b-a}{L(a,b)}\right)^{\frac{1}{q}} \left[b^{2p} - L\left(a^{2p},b^{2p}\right)\right]^{\frac{1}{p}} A^{\frac{1}{q}}\left(\left|f'(a)\right|^q, \left|f'(b)\right|^q\right)\left(\frac{2b+a}{3}\right)^{\frac{1}{q}} \\ =\ & \left(\frac{1}{2p}\right)^{\frac{1}{p}} \left(\frac{b-a}{L(a,b)}\right)^{\frac{1}{q}} A^{\frac{1}{q}}\left(\left|f'(a)\right|^q, \left|f'(b)\right|^q\right)\left\{\left[L\left(a^{2p},b^{2p}\right) - a^{2p}\right]^{\frac{1}{p}}\left(\frac{b+2a}{3}\right)^{\frac{1}{q}} + \left[b^{2p} - L\left(a^{2p},b^{2p}\right)\right]^{\frac{1}{p}}\left(\frac{2b+a}{3}\right)^{\frac{1}{q}}\right\} \end{aligned}$$

where

$$\begin{aligned} \int_0^1 (1-t)b^{2pt}a^{2p(1-t)}dt &= \frac{L\left(a^{2p},b^{2p}\right)-a^{2p}}{2p\left(\ln p-\ln a\right)}, \int_0^1 tb^{2pt}a^{2p(1-t)}dt = \frac{b^{2p}-L\left(a^{2p},b^{2p}\right)}{2p\left(\ln p-\ln a\right)}, \\ \int_0^1 (1-t)\left(tb+(1-t)a\right)dt &= \frac{b+2a}{6}, \int_0^1 t\left(tb+(1-t)a\right)dt = \frac{2b+a}{6}. \end{aligned}$$

This completes the proof of theorem. □

**Remark 4.7.** *The inequality (9) gives better results than the inequality (7). Let us show that*

$$\begin{aligned} &\left(\frac{1}{2p}\right)^{\frac{1}{p}}\left(\frac{b-a}{L(a,b)}\right)^{\frac{1}{q}} A^{\frac{1}{q}}\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right)\left\{\left[L\left(a^{2p},b^{2p}\right)-a^{2p}\right]^{\frac{1}{p}}\left(\frac{b+2a}{3}\right)^{\frac{1}{q}}+\left[b^{2p}-L\left(a^{2p},b^{2p}\right)\right]^{\frac{1}{p}}\left(\frac{2b+a}{3}\right)^{\frac{1}{q}}\right\} \\ \leq\ & 2^{\frac{1}{q}}\left(b-a\right)L^{-1}(a,b)A^{\frac{1}{q}}(a,b)A^{\frac{1}{q}}\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right)L^{\frac{1}{p}}\left(a^{2p},b^{2p}\right) \end{aligned}$$

*Using the well known classic inequality* $x^{1/p}y^{1/q}+z^{1/p}w^{1/q}\leq(x+z)^{1/p}(y+w)^{1/q},\ x,y,z,w\in(0,\infty)$, *by sample calculation we get*

$$\begin{aligned} &\left(\frac{1}{2p}\right)^{\frac{1}{p}}\left(\frac{b-a}{L(a,b)}\right)^{\frac{1}{q}} A^{\frac{1}{q}}\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right)\left\{\left[L\left(a^{2p},b^{2p}\right)-a^{2p}\right]^{\frac{1}{p}}\left(\frac{b+2a}{3}\right)^{\frac{1}{q}}+\left[b^{2p}-L\left(a^{2p},b^{2p}\right)\right]^{\frac{1}{p}}\left(\frac{2b+a}{3}\right)^{\frac{1}{q}}\right\} \\ \leq\ & \left(\frac{1}{2p}\right)^{\frac{1}{p}}\left(\frac{b-a}{L(a,b)}\right)^{\frac{1}{q}} A^{\frac{1}{q}}\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right)\left[b^{2p}-a^{2p}\right]^{\frac{1}{p}}\left(a+b\right)^{\frac{1}{q}} \\ =\ & 2^{\frac{1}{q}}\left(b-a\right)L^{-1}(a,b)A^{\frac{1}{q}}(a,b)A^{\frac{1}{q}}\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right)L^{\frac{1}{p}}\left(a^{2p},b^{2p}\right) \end{aligned}$$

*which is the required.*

## 5. Applications for special means

Throughout this section, for shortness, the following notations will be used for special means of two nonnegative numbers $a,b$ with $b>a$:

1. The arithmetic mean

$$A:=A(a,b)=\frac{a+b}{2},\quad a,b\geq 0,$$

2. The geometric mean

$$G:=G(a,b)=\sqrt{ab},\quad a,b\geq 0$$

3. The harmonic mean

$$H:=H(a,b)=\frac{2ab}{a+b},\quad a,b>0,$$

4. The logarithmic mean

$$L:=L(a,b)=\begin{cases} \frac{b-a}{\ln b-\ln a}, & a\neq b \\ a, & a=b \end{cases};\ a,b>0$$

5. The $p$-logarithmic mean

$$L_p:=L_p(a,b)=\begin{cases} \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}, & a\neq b, p\in\mathbb{R}\setminus\{-1,0\} \\ a, & a=b \end{cases};\ a,b>0.$$

6.The identric mean

$$I := I(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, \quad a,b>0,$$

These means are often used in numerical approximation and in other areas. However, the following simple relationships are known in the literature:

$$H \leq G \leq L \leq I \leq A.$$

It is also known that $L_p$ is monotonically increasing over $p \in \mathbb{R}$, denoting $L_0 = I$ and $L_{-1} = L$.

**Proposition 5.1.** *Let $a,b \in [1,\infty)$ with $a<b$ and $n \in \mathbb{R}\setminus\{-1,0,1,2\}$. Then, the following inequalities are obtained:*

$$G^n(a,b) \leq L(a,b)\left[L_n^n(a,b) + L_{n-2}^{n-2}(a,b)G^2(a,b)\right] \leq 4A(a,b)L(a,b)A(a^n,b^n).$$

*Proof.* The assertion follows from the inequalities (5) for the function

$$f(x) = x^n, \quad x \in [1,\infty).$$

□

**Proposition 5.2.** *Let $a,b \in [1/2,\infty)$ with $a<b$ . Then, the following inequalities are obtained:*

$$1 \leq A(a,b)L(a,b) + L(a^{-1},b^{-1})G^2(a,b) \leq 4A(a,b)L(a,b).$$

*Proof.* The assertion follows from the inequalities (5) for the function

$$f(x) = 1, \quad x \in [1/2,\infty).$$

□

**Proposition 5.3.** *Let $a,b \in [1,\infty)$ with $a<b$. Then, the following inequalities are obtained:*

$$G(a,b) \leq A(a,b)L(a,b) + G^2(a,b) \leq 4A^2(a,b)L(a,b).$$

*Proof.* The assertion follows from the inequalities (5) for the function

$$f(x) = x, \quad x \in [1,\infty).$$

□

**Proposition 5.4.** *Let $a,b \in [1,\infty)$ with $a<b$. Then, the following inequalities are obtained:*

$$G^2(a,b) \leq 2L(a,b)\left[A(a^2,b^2) + G^2(a,b)\right] \leq 4A(a,b)L(a,b)A(a^2,b^2).$$

*Proof.* The assertion follows from the inequalities (5) for the function

$$f(x) = x^2, \quad x \in [1,\infty).$$

□

**Proposition 5.5.** *Let $a,b \in [1,\infty)$ with $a<b$. Then, the following inequalities are obtained:*

$$G^{-1}(a,b) \leq 1 + \frac{A(a,b)(b-a)}{G^2(a,b)} \leq 4A(a,b)L(a,b)H^{-1}(a,b).$$

*Proof.* The assertion follows from the inequalities (5) for the function

$$f(x) = x^{-1}, \quad x \in [1,\infty).$$

□

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