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On n-fractional polynomial P-functions

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Abstract. In this paper, we introduce and study the concept of n-fractional polynomial P-functions and establish Hermite-Hadamard's inequalities for this type of functions. In addition, we obtain some new Hermite-Hadamard type inequalities for functions whose first derivative in absolute value is n-fractional polynomial P-functions by using Hölder and power-mean integral inequalities. We also extend our initial results to functions of several variables. Next, we point out some applications of our results to give estimates for the approximation error of the integral the function in the trapezoidal formula and for some inequalities related to special means of real numbers.

1. Preliminaries

Let $\Psi : I \to \mathbb{R}$ be a convex function. Then the following inequalities hold

$$\Psi\left(\frac{r+s}{2}\right) \leq \frac{1}{s-r}\int_r^s \Psi(u)du \leq \frac{\Psi(r)+\Psi(s)}{2} \tag{1}$$

for all $r, s \in I$ with $r < s$. Both inequalities hold in the reversed direction if the function Ψ is concave. This double inequality is well known as the Hermite-Hadamard inequality [6]. Note that some of the classical inequalities for means can be derived from Hermite-Hadamard integral inequalities for appropriate particular selections of the mapping Ψ.

In [5], Dragomir et al. gave the following definition and related Hermite-Hadamard integral inequalities as follow:

Definition 1.1. *A nonnegative function $\Psi : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be P-function if the inequality*

$$\Psi(\theta r + (1-\theta)s) \leq \Psi(r) + \Psi(s)$$

holds for all $r, s \in I$ and $\theta \in (0,1)$.

Theorem 1.2. *Let $\Psi \in P(I)$, $r, s \in I$ with $r < s$ and $\Psi \in L[r,s]$. Then*

$$\Psi\left(\frac{r+s}{2}\right) \leq \frac{2}{s-r}\int_r^s \Psi(u)du \leq 2\left[\Psi(r)+\Psi(s)\right]. \tag{2}$$

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Definition 1.3. *[14] Let $h : J \to \mathbb{R}$ be a non-negative function, $h \neq 0$. We say that $\Psi : I \to \mathbb{R}$ is an h-convex function, or that Ψ belongs to the class $SX(h, I)$, if Ψ is non-negative and for all $u, v \in I$, $\theta \in (0, 1)$ we have*

$$\Psi(\theta r + (1-\theta)s) \leq h(\theta)\Psi(r) + h(1-\theta)\Psi(s).$$

If this inequality is reversed, then Ψ is said to be h-concave, i.e. $\Psi \in SV(h, I)$. It is clear that, if we choose $h(\theta) = \theta$ and $h(\theta) = 1$, then the h-convexity reduces to convexity and definition of P-function, respectively.

Readers can look at [1, 14] for studies on h-convexity.

In [7], İşcan gave the following definition and related Hermite-Hadamard integral inequalities as follow:

Definition 1.4. *Let $n \in \mathbb{N}$. A non-negative function $f : I \subset \mathbb{R} \to \mathbb{R}$ is called n-fractional polynomial convex function if the inequality*

$$f(ta + (1-t)b) \leq \frac{1}{n}\sum_{i=1}^{n} t^{1/i} f(a) + \frac{1}{n}\sum_{i=1}^{n} (1-t)^{1/i} f(b) \tag{3}$$

holds for every $a, b \in I$ and $t \in [0, 1]$.

Remark 1.5. *We especially note that; if we take $n = 1$ in the inequality (3), then the 1-polynomial convexity reduces to the clasical convexity.*

With the help of the following remark, we can give all non-negative convex functions as an example of an n-fractional polynomial convex functions.

Remark 1.6. *Every nonnegative convex function is also a n-fractional polynomial convex function. Indeed, since*

$$t \leq t^{1/2} \leq t^{1/3} \leq \ldots \leq t^{1/n}$$

for all $t \in [0, 1]$ and $n \in \mathbb{N}$. We can write

$$t \leq \frac{1}{n}\sum_{i=1}^{n} t^{1/i} \text{ and } 1-t \leq \frac{1}{n}\sum_{i=1}^{n} (1-t)^{1/i}$$

for all $t \in [0, 1]$ and $n \in \mathbb{N}$. and thus

$$f(ta + (1-t)b) \leq tf(a) + (1-t)f(b) \leq \frac{1}{n}\sum_{i=1}^{n} t^{1/i} f(a) + \frac{1}{n}\sum_{i=1}^{n} (1-t)^{1/i} f(b).$$

Theorem 1.7 ([7]). *Let $n \in \mathbb{N}$ and $\Psi : [r, s] \to \mathbb{R}$ be a n-fractional polynomial convex function. If $r < s$ and $\Psi \in L[r, s]$, then the following Hermite-Hadamard type inequalities hold:*

$$\frac{n}{2\sum_{i=1}^{n}\left(\frac{1}{2}\right)^{1/i}}\Psi\left(\frac{r+s}{2}\right) \leq \frac{1}{s-r}\int_{r}^{s}\Psi(u)du \leq \left[\frac{\Psi(r)+\Psi(s)}{n}\right]\sum_{i=1}^{n}\frac{i}{i+1}. \tag{4}$$

The main purpose of this paper is to introduce the concept of n-fractional polynomial P-function which is connected with the concepts of P-function and n-fractional polynomial convex function and establish some new Hermite-Hadamard type inequality for this class of functions. In recent years many authors have studied error estimations of Hermite-Hadamard type inequalities; for refinements, counterparts, generalizations, for some related papers see [2–5, 8–13].

2. The definition of *n*-fractional polynomial *P*-function

In this section, we introduce a new concept, which is called n-fractional polynomial P-function and we give by setting some algebraic properties for the n-fractional polynomial P-function, as follows:

Definition 2.1. *Let $n \in \mathbb{N}$. A non-negative function $\Psi : I \subset \mathbb{R} \to \mathbb{R}$ is called n-fractional polynomial P-function if for every $r, s \in I$ and $\theta \in [0,1]$,*

$$\Psi\left(\theta r + (1-\theta)s\right) \leq \left(\frac{1}{n}\left[\sum_{i=1}^{n}\theta^{1/i} + \sum_{i=1}^{n}(1-\theta)^{1/i}\right]\right)\left[\Psi(r) + \Psi(s)\right]. \tag{5}$$

We will denote by $nFPP\,(I)$ the class of all n-fractional polynomial P-functions on interval I.

We note that, every n-fractional polynomial P-function is a h-convex function with the function $h(\theta) = \frac{1}{n}\left[\sum_{i=1}^{n}\theta^{1/i} + \sum_{i=1}^{n}(1-\theta)^{1/i}\right]$. Therefore, if $\Psi, \Phi \in nFPP\,(I)$, then

i.) $\Psi + \Phi \in nFPP\,(I)$ *and for $c \in \mathbb{R}$ $(c \geq 0)$ $c\Psi \in nFPP\,(I)$ (see [14], Proposition 9).*

ii.) If Ψ and g be a similarly ordered functions on I , then $\Psi.\Phi \in nFPP\,(I)$ *.(see [14], Proposition 10).*

Also, if $\Psi : I \to J$ is a convex and $\Phi \in nFPP\,(J)$ and nondecreasing, then $\Phi \circ \Psi \in nFPP\,(I)$ *(see [14], Theorem 15).*

Proposition 2.2. *Every n-fractional polynomial convex function is also an n-fractional polynomial P-function.*

Proof. Let $\Psi : I \subset \mathbb{R} \to \mathbb{R}$ be an arbitrary n-fractional polynomial functions. Then Ψ is nonnegative and the following inequality holds

$$\Psi\left(\theta r + (1-\theta)s\right) \leq \frac{1}{n}\sum_{i=1}^{n}\theta^{1/i}\Psi(r) + \frac{1}{n}\sum_{i=1}^{n}(1-\theta)^{1/i}\Psi(s)$$

for every $r, s \in I$ and $\theta \in [0,1]$. By $\Psi(r) \leq \Psi(r) + \Psi(s)$ and $\Psi(s) \leq \Psi(r) + \Psi(s)$, we obtain desired result. □

Proposition 2.3. *Every P-function is also a n-fractional polynomial P-function.*

Proof. The proof is clear from the following inequalities

$$\theta \leq \frac{1}{n}\sum_{i=1}^{n}\theta^{1/i} \text{ and } 1-\theta \leq \frac{1}{n}\sum_{i=1}^{n}(1-\theta)^{1/i}$$

for all $\theta \in [0,1]$. In this case, we can write

$$1 \leq \frac{1}{n}\left[\sum_{i=1}^{n}\theta^{1/i} + \sum_{i=1}^{n}(1-\theta)^{1/i}\right].$$

Therefore, the desired result is obtained. □

We can give the following corollary for every nonnegative convex function is also a P-function.

Corollary 2.4. *Every nonnegative convex function is also an n-fractional polynomial P-function.*

Theorem 2.5. *If $\Psi : [r,s] \subset \mathbb{R} \to \mathbb{R}$ is an n-fractional polynomial P-function, then Ψ is bounded on $[r,s]$.*

Proof. Let $M = \max\{\Psi(r), \Psi(s)\}$. For any $x \in [r,s]$, there exists an $\theta \in [0,1]$ such that $x = \theta r + (1-\theta)s$. Since Ψ is a n-fractional polynomial P-function on $[r,s]$, we have

$$\Psi(x) \leq \frac{1}{n}\left[\sum_{i=1}^{n}\theta^{1/i} + \sum_{i=1}^{n}(1-\theta)^{1/i}\right]\left[\Psi(r) + \Psi(s)\right] \leq 2M(\theta^{1/n} + (1-\theta)^{1/n}).$$

This shows that Ψ is bounded from above. For any $x \in [r,s]$, there exists an $\theta \in [0,1]$ such that either $x = \frac{r+s}{2} + \theta$ or $x = \frac{r+s}{2} - \theta$. Since it will lose nothing generality we can assume $x = \frac{r+s}{2} + \theta$. Thus we can write

$$\begin{aligned}
\Psi\left(\frac{r+s}{2}\right) &= \Psi\left(\frac{1}{2}\left[\frac{r+s}{2}+\theta\right]+\frac{1}{2}\left[\frac{r+s}{2}-\theta\right]\right) \\
&\leq \frac{2}{n}\sum_{i=1}^{n}\left(\frac{1}{2}\right)^{1/i}\left[\Psi(x)+\Psi\left(\frac{r+s}{2}-\theta\right)\right] \\
&\leq 2^{1-1/n}\left[\Psi(x)+\Psi\left(\frac{r+s}{2}-\theta\right)\right]
\end{aligned}$$

and from here we have

$$\begin{aligned}
\Psi(x) &\geq \frac{1}{2^{1-1/n}}\Psi\left(\frac{r+s}{2}\right)-\Psi\left(\frac{r+s}{2}-\theta\right) \\
&\geq \frac{1}{2^{1-1/n}}\Psi\left(\frac{r+s}{2}\right)-2M(\theta^{1/n}+(1-\theta)^{1/n}) = m.
\end{aligned}$$

This completes the proof. □

Theorem 2.6. *Let $s > r$ and $\Psi_\alpha : [r,s] \to \mathbb{R}$ be an arbitrary family of n-fractional polynomial P-function and let $\Psi(x) = \sup_\alpha \Psi_\alpha(x)$. If $J = \{u \in [r,s] : \Psi(u) < \infty\}$ is nonempty, then J is an interval and Ψ is an n-fractional polynomial P-function on J.*

Proof. Let $\theta \in [0,1]$ and $r,s \in J$ be arbitrary. Then

$$\begin{aligned}
& \Psi(\theta r+(1-\theta)s) \\
= & \sup_\alpha \Psi_\alpha(\theta r+(1-\theta)s) \\
\leq & \sup_\alpha\left\{\frac{1}{n}\left[\sum_{i=1}^{n}\theta^{1/i}+\sum_{i=1}^{n}(1-\theta)^{1/i}\right][\Psi_\alpha(r)+\Psi_\alpha(s)]\right\} \\
\leq & \frac{1}{n}\left[\sum_{i=1}^{n}\theta^{1/i}+\sum_{i=1}^{n}(1-\theta)^{1/i}\right]\left[\sup_\alpha\Psi_\alpha(r)+\sup_\alpha\Psi_\alpha(s)\right] \\
= & \frac{1}{n}\left[\sum_{i=1}^{n}\theta^{1/i}+\sum_{i=1}^{n}(1-\theta)^{1/i}\right][\Psi(r)+\Psi(s)] < \infty.
\end{aligned}$$

This shows simultaneously that J is an interval, since it contains every point between any two of its points, and that Ψ is an n-fractional polynomial P-function on J. This completes the proof of theorem. □

3. Hermite-Hadamard's inequality for *n*-fractional polynomial *P*-functions

The goal of this paper is to establish some inequalities of Hermite-Hadamard type for n-fractional polynomial P-functions. In this section, we will denote by $L[r,s]$ the space of (Lebesgue) integrable functions on $[r,s]$.

Theorem 3.1. *Let $\Psi : [r,s] \to \mathbb{R}$ be an n-fractional polynomial P-function. If $r < s$ and $\Psi \in L[r,s]$, then the following Hermite-Hadamard type inequalities hold:*

$$\frac{n}{4\sum_{i=1}^{n}\left(\frac{1}{2}\right)^{1/i}}\Psi\left(\frac{r+s}{2}\right) \leq \frac{1}{s-r}\int_r^s \Psi(u)du \leq \left[\frac{\Psi(r)+\Psi(s)}{n}\right]\sum_{i=1}^{n}\frac{2i}{i+1}. \tag{6}$$

Proof. Since Ψ is an n-fractional polynomial P-function, we get

$$\begin{aligned} & \Psi\left(\frac{r+s}{2}\right) \\ = & \Psi\left(\frac{1}{2}\left[\theta r+(1-\theta) s\right]+\frac{1}{2}\left[\theta s+(1-\theta) r\right]\right) \\ \leq & \frac{2}{n} \sum_{i=1}^{n}\left(\frac{1}{2}\right)^{1/i}\left[\Psi\left(\theta r+(1-\theta) s\right)+\Psi\left(\theta s+(1-\theta) r\right)\right]. \end{aligned}$$

By taking integral in the last inequality with respect to $\theta \in [0,1]$, we deduce that

$$\Psi\left(\frac{r+s}{2}\right) \leq \frac{4}{n} \sum_{i=1}^{n}\left(\frac{1}{2}\right)^{1/i} \frac{1}{s-r} \int_{r}^{s} \Psi(u) du$$

By using the property of the n-fractional polynomial P-function of Ψ, if the variable is changed as $u = \theta r+(1-\theta) s$, then

$$\begin{aligned} \frac{1}{s-r} \int_{r}^{s} \Psi(u) du & = \int_{0}^{1} \Psi\left(\theta r+(1-\theta) s\right) d\theta \\ & \leq \left[\frac{\Psi(r)+\Psi(s)}{n}\right] \int_{0}^{1}\left[\sum_{i=1}^{n} \theta^{1/i}+\sum_{i=1}^{n}(1-\theta)^{1/i}\right] d\theta \\ & = \left[\frac{\Psi(r)+\Psi(s)}{n}\right] \sum_{i=1}^{n} \frac{2i}{i+1}. \end{aligned}$$

This completes the proof of theorem. □

4. Some new inequalities for n-fractional polynomial P-functions

The main purpose of this section is to establish new estimates that refine Hermite-Hadamard inequality for functions whose first derivative in absolute value is n-fractional polynomial P-function. Dragomir and Agarwal [4] used the following lemma:

Lemma 4.1. *Let $f : I^{\circ} \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I°, $r, s \in I^{\circ}$ with $r < s$. If $f' \in L[r,s]$, then*

$$\frac{f(r)+f(s)}{2}-\frac{1}{s-r} \int_{r}^{s} f(x) dx=\frac{s-r}{2} \int_{0}^{1}(1-2t) f'\left(tr+(1-t)s\right) dt.$$

Theorem 4.2. *Let $f : I \rightarrow \mathbb{R}$ be a differentiable mapping on I°, $r, s \in I^{\circ}$ with $r < s$ and assume that $f' \in L[r,s]$. If $\left|f'\right|$ is n-fractional polynomial P-function on interval $[r,s]$, then the following inequality holds*

$$\left|\frac{f(r)+f(s)}{2}-\frac{1}{s-r} \int_{r}^{s} f(x) dx\right| \leq (s-r) \frac{1}{n} \sum_{i=1}^{n}\left[\frac{2i\left(1+2^{\frac{-1}{i}} i\right)}{2i^{2}+3i+1}\right] A\left(\left|f'(r)\right|,\left|f'(s)\right|\right), \tag{7}$$

where A is the arithmetic mean.

Proof. Using Lemma 4.1 and the inequality

$$\left|f'\left(tr+(1-t)s\right)\right| \leq \frac{1}{n}\left[\sum_{i=1}^{n} t^{1/i}+\sum_{i=1}^{n}(1-t)^{1/i}\right]\left[\left|f'(r)\right|+\left|f'(s)\right|\right],$$

we get

$$\begin{aligned}
&\left|\frac{f(r)+f(s)}{2}-\frac{1}{s-r}\int_r^s f(x)dx\right| \\
\leq\ & \frac{s-r}{2}\left[\left|f'(r)\right|+\left|f'(s)\right|\right]\frac{1}{n}\sum_{i=1}^{n}\int_0^1 |1-2t|\left(t^{1/i}+(1-t)^{1/i}\right)dt \\
=\ & (s-r)\,\frac{1}{n}\sum_{i=1}^{n}\left[\frac{2i\left(1+2^{\frac{-1}{i}}i\right)}{2i^2+3i+1}\right]A\left(\left|f'(r)\right|,\left|f'(s)\right|\right)
\end{aligned}$$

where

$$\frac{1}{n}\sum_{i=1}^{n}\int_0^1 |1-2t|\left(t^{1/i}+(1-t)^{1/i}\right)dt=\frac{1}{n}\sum_{i=1}^{n}\left(\frac{2i\left(1+2^{\frac{-1}{i}}i\right)}{2i^2+3i+1}\right)$$

This completes the proof of theorem. □

Theorem 4.3. *Let $f : I \to \mathbb{R}$ be a differentiable mapping on I°, $r, s \in I^\circ$ with $r < s$ and assume that $f' \in L[r,s]$. If $\left|f'\right|^q$, $q > 1$, is an n-fractional polynomial P-function on interval $[r,s]$, then the following inequality holds*

$$\begin{aligned}
&\left|\frac{f(r)+f(s)}{2}-\frac{1}{s-r}\int_r^s f(x)dx\right| \qquad (8)\\
\leq\ & \frac{s-r}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{n}\sum_{i=1}^{n}\frac{4i}{i+1}\right)^{\frac{1}{q}}A^{\frac{1}{q}}\left(\left|f'(r)\right|^q,\left|f'(s)\right|^q\right),
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$ and A is the arithmetic mean.

Proof. Using Lemma 4.1, Hölder's integral inequality and the following inequality

$$\left|f'(tr+(1-t)s)\right|\leq\frac{1}{n}\left[\sum_{i=1}^{n}t^{1/i}+\sum_{i=1}^{n}(1-t)^{1/i}\right]\left[\left|f'(r)\right|+\left|f'(s)\right|\right]$$

which is the n-fractional polynomial P-function of $\left|f'\right|^q$, we get

$$\begin{aligned}
&\left|\frac{f(r)+f(s)}{2}-\frac{1}{s-r}\int_r^s f(x)dx\right| \\
\leq\ & \frac{s-r}{2}\left(\int_0^1 |1-2t|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'(tr+(1-t)s)\right|^q dt\right)^{\frac{1}{q}} \\
\leq\ & \frac{s-r}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\left[\left|f'(r)\right|^q+\left|f'(s)\right|^q\right]\frac{1}{n}\sum_{i=1}^{n}\int_0^1\left(t^{1/i}+(1-t)^{1/i}\right)dt\right)^{\frac{1}{q}} \\
=\ & \frac{s-r}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{n}\sum_{i=1}^{n}\frac{4i}{i+1}\right)^{\frac{1}{q}}A^{\frac{1}{q}}\left(\left|f'(r)\right|^q,\left|f'(s)\right|^q\right).
\end{aligned}$$

This completes the proof of theorem. □

Theorem 4.4. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $r, s \in I^\circ$ with $r < s$ and assume that $f' \in L[r,s]$. If $\left|f'\right|^q$, $q \geq 1$, is an n-fractional polynomial P-function on the interval $[r,s]$, then the following inequality holds*

$$\begin{aligned} &\left|\frac{f(r)+f(s)}{2} - \frac{1}{s-r}\int_r^s f(x)dx\right| \\ \leq\ & \frac{s-r}{2^{2-\frac{2}{q}}}\left(\frac{1}{n}\sum_{i=1}^{n}\left[\frac{2i\left(1+2^{\frac{-1}{i}}i\right)}{2i^2+3i+1}\right]\right)^{\frac{1}{q}} A^{\frac{1}{q}}\left(\left|f'(r)\right|^q, \left|f'(s)\right|^q\right). \end{aligned} \tag{9}$$

Proof. From Lemma 4.1, well known power-mean integral inequality and the property of n-fractional polynomial P-function of $\left|f'\right|^q$, we obtain

$$\begin{aligned} &\left|\frac{f(r)+f(s)}{2} - \frac{1}{s-r}\int_r^s f(x)dx\right| \\ \leq\ & \frac{s-r}{2}\left(\int_0^1 |1-2t|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |1-2t|\left|f'(tr+(1-t)s)\right|^q dt\right)^{\frac{1}{q}} \\ \leq\ & \frac{s-r}{2^{2-\frac{1}{q}}}\left(\left[\left|f'(r)\right|^q + \left|f'(s)\right|^q\right]\int_0^1 |1-2t|\,\frac{1}{n}\sum_{i=1}^{n}\int_0^1\left(t^{1/i}+(1-t)^{1/i}\right)dt\right)^{\frac{1}{q}} \\ =\ & \frac{s-r}{2^{2-\frac{2}{q}}}\left(\frac{1}{n}\sum_{i=1}^{n}\left[\frac{2i\left(1+2^{\frac{-1}{i}}i\right)}{2i^2+3i+1}\right]\right)^{\frac{1}{q}} A^{\frac{1}{q}}\left(\left|f'(r)\right|^q, \left|f'(s)\right|^q\right). \end{aligned}$$

This completes the proof of theorem. □

Corollary 4.5. *Under the assumption of Theorem 4.4, If we take $q = 1$ in the inequality (9), then we get the following inequality:*

$$\left|\frac{f(r)+f(s)}{2} - \frac{1}{s-r}\int_r^s f(x)dx\right| \leq (s-r)\frac{1}{n}\sum_{i=1}^{n}\left[\frac{2i\left(1+2^{\frac{-1}{i}}i\right)}{2i^2+3i+1}\right] A\left(\left|f'(r)\right|, \left|f'(s)\right|\right).$$

This inequality coincides with the inequality (7).

5. An extention of Theorem 4.2

In this section we will denote by K an open and convex set of $\mathbb{R}^n$ $(n \geq 1)$.

We say that a function $f : K \to \mathbb{R}$ is n-fractional polynomial P-function on K if

$$f(tx+(1-t)y) \leq \frac{1}{n}\left[\sum_{i=1}^{n} t^{1/i} + \sum_{i=1}^{n}(1-t)^{1/i}\right][f(x)+f(y)]$$

for all $x, y \in K$ and $t \in [0,1]$.

Lemma 5.1. *Let $f : K \to \mathbb{R}$ be a function. Then f is n-fractional polynomial P-function on K if and only if for all $x, y \in K$ the function $\Phi : [0,1] \to \mathbb{R}$, $\Phi(t) = f(tx+(1-t)y)$ is n-fractional polynomial P-function on $[0,1]$.*

Proof. "$\Longleftarrow$"Let $x, y \in K$ be fixed. Assume that $\Phi : [0,1] \to \mathbb{R}$, $\Phi(t) = f(tx+(1-t)y)$ is n-fractional polynomial P-function on $[0,1]$.

Let $t \in [0,1]$ be arbitrary, but fixed. Clearly, $t = (1-t).0 + t.1$ and thus,

$$\begin{aligned} f(tx+(1-t)y) &= \Phi(t) = \Phi(t.1+(1-t).0) \\ &\leq \frac{1}{n}\left[\sum_{i=1}^{n} t^{1/i} + \sum_{i=1}^{n} (1-t)^{1/i}\right][\Phi(0)+\Phi(1)] \\ &= \frac{1}{n}\left[\sum_{i=1}^{n} t^{1/i} + \sum_{i=1}^{n} (1-t)^{1/i}\right][f(x)+f(y)]. \end{aligned}$$

It follows that f is n-fractional polynomial P-function on K.

$''\Longrightarrow''$ Assume that f is n-fractional polynomial P-function on K. Let $x, y \in K$ be fixed and define $\Phi : [0,1] \to \mathbb{R}$, $\Phi(t) = f(tx+(1-t)y)$. We must show that Φ is n-fractional polynomial P-function on $[0,1]$.

Let $u_1, u_2 \in [0,1]$ and $t \in [0,1]$. Then

$$\begin{aligned} \Phi(tu_1+(1-t)u_2) &= f((tu_1+(1-t)u_2)x + (1-tu_1-(1-t)u_2)y) \\ &= f(t(u_1x+(1-u_1)y + (1-t)(u_2x+(1-u_2)y) \\ &\leq \frac{1}{n}\left[\sum_{i=1}^{n} t^{1/i} + \sum_{i=1}^{n} (1-t)^{1/i}\right][f(u_1x+(1-u_1)y) + f(u_2x+(1-u_2)y)] \\ &= \frac{1}{n}\left[\sum_{i=1}^{n} t^{1/i} + \sum_{i=1}^{n} (1-t)^{1/i}\right][\Phi(u_1)+\Phi(u_2)]. \end{aligned}$$

We deduce that Φ is n-fractional polynomial P-function on $[0,1]$.

The proof of Lemma 5.1 is complete. □

Using the above lemma we will prove an extension of Theorem 4.2 to functions of several variables.

Proposition 5.2. *Assume $f : K \subseteq \mathbb{R}^n \to \mathbb{R}^+$ is a n-fractional polynomial P-function on K. Then for any $x, y \in K$ and any $u, v \in (0,1)$ with $u < v$ the following inequality holds*

$$\begin{aligned} &\left|\frac{1}{2}\int_0^u f(sx+(1-s)y)\,ds + \frac{1}{2}\int_0^v f(sx+(1-s)y)\,ds \right. \\ &\left. -\frac{1}{v-u}\int_u^v \left(\int_0^\theta f(sx+(1-s)y)\,ds\right)d\theta\right| \\ \leq\ &(v-u)\frac{1}{n}\sum_{i=1}^{n}\left[\frac{2i\left(1+2^{\frac{-1}{i}}i\right)}{2i^2+3i+1}\right]A(f(ux+(1-u)y), f(vx+(1-v)y)). \end{aligned} \tag{10}$$

Proof. We fix $x, y \in K$ and $u, v \in (0,1)$ with $u < v$. Since f is n-fractional polynomial P-function, by Lemma 5.1 it follows that the function

$$\Phi : [0,1] \to \mathbb{R}, \Phi(t) = f(tx+(1-t)y),$$

is n-fractional polynomial P-function on $[0,1]$.

Define $\Psi : [0,1] \to \mathbb{R}$,

$$\Psi(t) = \int_0^t \Phi(s)ds = \int_0^t f(sx+(1-s)y)\,ds.$$

Obviously, $\Psi'(t) = \Phi(t)$ for all $t \in (0,1)$.

Since $f(K) \subseteq \mathbb{R}^+$ it results that $\Phi \geq 0$ on $[0,1]$ and thus, $\Psi' \geq 0$ on $(0,1)$.

Applying Theorem 4.2 to the function Ψ we obtain

$$\left|\frac{\Psi(u)+\Psi(v)}{2}-\frac{1}{v-u}\int_u^v \Psi(\theta)d\theta\right| \leq (v-u)\frac{1}{n}\sum_{i=1}^{n}\left[\frac{2i\left(1+2^{\frac{-1}{i}}i\right)}{2i^2+3i+1}\right]A\left(\Psi'(u),\Psi'(v)\right),$$

and we deduce that relation (10) holds true. □

Remark 5.3. *We point out that a similar result as those of Proposition 5.2 can be stated by using Theorem 4.3 and Theorem 4.4 .*

6. Applications to the trapezoidal formula

Assume $\wp$ is a division of the interval $[r,s]$ such that

$$\wp: \quad r = x_0 < x_1 < \ldots < x_{n-1} < x_n = s.$$

For a given function $f:[r,s]\to\mathbb{R}$ we consider the trapezoidal formula

$$T(f,\wp)=\sum_{i=0}^{n-1}\frac{f(x_i)+f(x_{i+1})}{2}(x_{i+1}-x_i).$$

It is well known that if f is twice differentiable on (r,s) and $M=\sup_{x\in(r,s)}\left|f''(x)\right|<\infty$ then

$$\int_r^s f(x)dx = T(f,\wp)+E(f,\wp),$$

where $E(f,\wp)$ is the approximation error of the integral $\int_r^s f(x)dx$ by the trapezoidal formula and satisfies,

$$\left|E(f,\wp)\right| \leq \frac{M}{12}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^3. \tag{11}$$

Clearly, if the function f is not twice differentiable or the second derivative is not bounded on (r,s), then (11) does not hold true. In that context, the following results are important in order to obtain some estimates of $E(f,\wp)$.

Proposition 6.1. *Assume $r,s\in\mathbb{R}$ with $r<s$ and $f:[r,s]\to\mathbb{R}$ is a differentiable function on (r,s). If $\left|f'\right|$ is n-fractional polynomial P-function on $[r,s]$ then for each division $\wp$ of the interval $[r,s]$ we have,*

$$\left|E(f,\wp)\right| \leq \frac{2}{n^2}\left(\sum_{i=1}^{n}\left[\frac{2i\left(1+2^{\frac{-1}{i}}i\right)}{2i^2+3i+1}\right]\right)\left(\sum_{i=1}^{n}\left(\frac{1}{2}\right)^{\frac{1}{i}}\right)A\left(\left|f'(r)\right|,\left|f'(s)\right|\right)\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2. \tag{12}$$

Proof. We apply Theorem 4.2 on the sub-intervals $[x_i,x_{i+1}]$, $i=0,1,\ldots,n-1$ given by the division $\wp$. Adding from $i=0$ to $i=n-1$ we deduce

$$\left|T(f,\wp)-\int_r^s f(x)dx\right| \leq \frac{1}{n}\sum_{i=1}^{n}\left[\frac{2i\left(1+2^{\frac{-1}{i}}i\right)}{2i^2+3i+1}\right]\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2 A\left(\left|f'(x_i)\right|,\left|f'(x_{i+1})\right|\right). \tag{13}$$

On the other hand, for each $x_i\in[r,s]$ there exists $t_i\in[0,1]$ such that $x_i=t_ir+(1-t_i)s$. Since $\left|f'\right|$ is n-fractional polynomial P-function and $\frac{1}{n}\left[\sum_{i=1}^{n}t^{1/i}+\sum_{i=1}^{n}(1-t)^{1/i}\right]\leq\frac{2}{n}\left[\sum_{i=1}^{n}\left(\frac{1}{2}\right)^{\frac{1}{i}}\right]$for all $t\in[0,1]$, we deduce

$$\left|f'(x_i)\right| \leq \frac{1}{n}\left[\sum_{i=1}^{n}t^{1/i}+\sum_{i=1}^{n}(1-t)^{1/i}\right][f(r)+f(s)] \leq \frac{4}{n}\left(\sum_{i=1}^{n}\left(\frac{1}{2}\right)^{\frac{1}{i}}\right)A\left(\left|f'(r)\right|,\left|f'(s)\right|\right) \tag{14}$$

for each $i=0,1,\ldots,n-1$. Relations (13) and (14) imply that relation (12) holds true. Thus, Proposition 7.1 is completely proved. □

A similar method as that used in the proof of Proposition 6.1 but based on Theorem 4.3 and Theorem 4.4 shows that the following results are valid.

Proposition 6.2. *Assume $r, s \in \mathbb{R}$ with $r < s$ and $f : [r, s] \to \mathbb{R}$ is a differentiable function on (r, s). If $\left|f'\right|^q$, $q > 1$, is a n-fractional polynomial P-function on interval $[r, s]$, then for each division $\wp$ of the interval $[r, s]$ we have,*

$$\left|E(f,\wp)\right| \leq \frac{1}{n}\left(\sum_{i=1}^{n}\left(\frac{1}{2}\right)^{\frac{1}{i}}\right)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{n}\sum_{i=1}^{n}\frac{4i}{i+1}\right)^{\frac{1}{q}} A^{\frac{1}{q}}\left(\left|f'(r)\right|^q, \left|f'(s)\right|^q\right)\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2,$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proposition 6.3. *Assume $r, s \in \mathbb{R}$ with $r < s$ and $f : [r, s] \to \mathbb{R}$ is a differentiable function on (r, s). If $\left|f'\right|^q$, $q > 1$, is a n-fractional polynomial P-function on interval $[r, s]$, then for each division $\wp$ of the interval $[r, s]$ we have,*

$$\left|E(f,\wp)\right| \leq \frac{\frac{1}{n}\left(\sum_{i=1}^{n}\left(\frac{1}{2}\right)^{\frac{1}{i}}\right)}{2^{1-\frac{2}{q}}}\left(\sum_{i=1}^{n}\left[\frac{2i\left(1+2^{\frac{-1}{i}}i\right)}{2i^2+3i+1}\right]\right)^{\frac{1}{q}} A^{\frac{1}{q}}\left(\left|f'(r)\right|^q, \left|f'(s)\right|^q\right)\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2.$$

7. Applications for special means

Throughout this section, for shortness, the following notations will be used for special means of two nonnegative numbers r, s with $s > r$:

1. The arithmetic mean

$$A := A(r,s) = \frac{r+s}{2}, \quad r,s \geq 0.$$

2. The geometric mean

$$G := G(r,s) = \sqrt{rs}, \quad r,s \geq 0.$$

3. The harmonic mean

$$H := H(r,s) = \frac{2rs}{r+s}, \quad r,s > 0.$$

4. The logarithmic mean

$$L := L(r,s) = \begin{cases} \frac{s-r}{\ln s - \ln r}, & r \neq s \\ r, & r = s \end{cases} ; \quad r,s > 0.$$

5. The p-logarithmic mean

$$L_p := L_p(r,s) = \begin{cases} \left(\frac{s^{p+1}-r^{p+1}}{(p+1)(s-r)}\right)^{\frac{1}{p}}, & r \neq s, p \in \mathbb{R}\setminus\{-1,0\} \\ r, & r = s \end{cases} ; \quad r,s > 0.$$

6.The identric mean

$$I := I(r,s) = \frac{1}{e}\left(\frac{s^s}{r^r}\right)^{\frac{1}{s-r}}, \quad r,s > 0.$$

These means are often used in numerical approximation and in other areas. However, the following simple relationships are known in the literature:

$$H \leq G \leq L \leq I \leq A.$$

It is also known that L_p is monotonically increasing over $p \in \mathbb{R}$, denoting $L_0 = I$ and $L_{-1} = L$.

Proposition 7.1. *Let $r, s \in [0, \infty)$ with $r < s$ and $n \in (-\infty, 0) \cup [1, \infty) \setminus \{-1\}$. Then, the following inequalities are obtained:*

$$\frac{n}{4\sum_{i=1}^{n}\left(\frac{1}{2}\right)^{1/i}} A^n(r,s) \leq L_n^n(r,s) \leq \frac{1}{n}\sum_{i=1}^{n}\frac{4i}{i+1}A(r^n, s^n).$$

Proof. The assertion follows from the inequalities (6) for the function

$$\Psi(x) = x^n, \quad x \in [0, \infty).$$

□

Proposition 7.2. *Let $r, s \in (0, \infty)$ with $r < s$. Then, the following inequalities are obtained:*

$$\frac{n}{4\sum_{i=1}^{n}\left(\frac{1}{2}\right)^{1/i}} A^{-1}(r,s) \leq L^{-1}(r,s) \leq \sum_{i=1}^{n}\frac{4i}{i+1}H^{-1}(r,s).$$

Proof. The assertion follows from the inequalities (6) for the function

$$\Psi(x) = x^{-1}, \quad x \in (0, \infty).$$

□

Proposition 7.3. *Let $r, s \in (0, 1]$ with $r < s$. Then, the following inequalities are obtained:*

$$\sum_{i=1}^{n}\frac{4i}{i+1}\ln G(r,s) \leq \ln I(r,s) \leq \frac{n}{4\sum_{i=1}^{n}\left(\frac{1}{2}\right)^{1/i}}\ln A(r,s).$$

Proof. The assertion follows from the inequalities (6) for the function

$$\Psi(x) = -\ln x, \quad x \in (0, 1].$$

□

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