



Bivariate Lupaş-Durrmeyer type operators involving Pólya distribution

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Abstract. In this paper, we consider a bivariate extension of blending type approximation by Lupaş-Durrmeyer type operators involving Pólya Distribution. We illustrate the convergence rate of these type operators using Peetre's K -functional, modulus of smoothness and for functions in a Lipschitz type space.

1. Introduction and preliminaries

Most likely used Bernstein polynomial of r^{th} degree for the continuous function $\gamma(x)$ and the interval $I = [0, 1]$ is defined as

$$B_r(\gamma; x) = \sum_{k=0}^r P_{r,k}(x) \gamma\left(\frac{k}{r}\right),$$

where $P_{r,k}(x) = \binom{r}{k} x^k (1-x)^{r-k}$, $x \in [0, 1]$.

Gupta and Rassias [15] introduced Lupaş-Durrmeyer operators for any $\gamma \in C(I)$ is given by

$$D_r^{\left[\frac{1}{r}\right]}(\gamma; x) = (r+1) \sum_{k=0}^r b_{r,k}^{\left[\frac{1}{r}\right]}(x) \int_0^1 b_{r,k}(t) \gamma(t) dt, \quad t \in [0, 1]. \quad (1)$$

Agrawal et al. [6] constructed the bivariate modification of the linear positive operator given in (1) based on Pólya distribution and discussed the degree of approximation and the rate of convergence for these operators. Kajla and Acar [18] proposed the blending type approximation by generalised Bernstein-Durrmeyer type operator. Kajla et al. [19] constructed a Durrmeyer type generalization of the Lupaş and

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Lupaş operators [20] involving $\tau(x)$ ($0 < \tau(x) \leq 1$) as

$$D_{r,\tau,\rho} \left[\begin{matrix} 1 \\ r \end{matrix} \right] (\gamma; x) = \sum_{k=0}^r b_{r,k} \left[\begin{matrix} 1 \\ r \end{matrix} \right] (x) \int_0^1 \left(\frac{t^{k\rho+\tau(x)-1} (1-t)^{(r-k)\rho+\tau(x)-1}}{B(k\rho+\tau(x), (r-k)\rho+\tau(x))} \right) \gamma(t) dt, \tag{2}$$

where $B(l, m) = \int_0^1 t^{l-1} (1-t)^{m-1} dt$ ($l, m > 0$ and $\rho > 0$).

Acar and Kajla [2] proposed bivariate extension of Bernstein type operators and discussed the degree of approximation for these operator. Many researchers also defined different type of generalization of these operators and discussed their approximation behaviour which we refer in (cf. [1, 3–5, 7–9, 11–14, 16, 17, 21–25, 27–30]).

Let fixed a continuous function which is strictly positive $\tau : C(I) \rightarrow C(I)$. We propose a bivariate generalization of the operator defined in (2) involving $\tau(x)$ ($0 < \tau(x) \leq 1$) and $\gamma : C(I^2) \rightarrow C(I^2)$, $I^2 = [0, 1] \times [0, 1]$

$$D_{r_1, r_2, \tau, \rho_1, \rho_2} \left[\begin{matrix} \frac{1}{r_1}, \frac{1}{r_2} \end{matrix} \right] (\gamma; x_1, x_2) = \sum_{k_1=0}^{r_1} \sum_{k_2=0}^{r_2} b_{r_1, r_2, k_1, k_2} \left[\begin{matrix} \frac{1}{r_1}, \frac{1}{r_2} \end{matrix} \right] (x_1, x_2) \int_0^1 \int_0^1 \left(\frac{t^{k_1\rho_1+\tau(x_1)-1} (1-t)^{(r_1-k_1)\rho_1+\tau(x_1)-1}}{B(k_1\rho_1+\tau(x_1), (r_1-k_1)\rho_1+\tau(x_1))} \right) \times \left(\frac{s^{k_2\rho_2+\tau(x_2)-1} (1-s)^{(r_2-k_2)\rho_2+\tau(x_2)-1}}{B(k_2\rho_2+\tau(x_2), (r_2-k_2)\rho_2+\tau(x_2))} \right) \gamma(t, s) dt ds.$$

$$B(l, m) = \int_0^1 t^{l-1} (1-t)^{m-1} dt \quad (l, m > 0 \text{ and } \rho_1, \rho_2 > 0).$$

Lemma 1.1. Let the bivariate test function be defined by $e_{mn}(x_1, x_2) = x_1^m x_2^n$, $(m, n) \in \mathbb{N}_0 \times \mathbb{N}_0$, with $m + n \leq 4$. Then, we have

- (i) $D_{r_1, r_2, \tau, \rho_1, \rho_2} \left[\begin{matrix} \frac{1}{r_1}, \frac{1}{r_2} \end{matrix} \right] (e_{00}; x_1, x_2) = 1;$
- (ii) $D_{r_1, r_2, \tau, \rho_1, \rho_2} \left[\begin{matrix} \frac{1}{r_1}, \frac{1}{r_2} \end{matrix} \right] (e_{10}; x_1, x_2) = \frac{r_1 x_1 \rho_1 + \tau(x_1)}{r_1 \rho_1 + 2\tau(x_1)};$
- (iii) $D_{r_1, r_2, \tau, \rho_1, \rho_2} \left[\begin{matrix} \frac{1}{r_1}, \frac{1}{r_2} \end{matrix} \right] (e_{01}; x_1, x_2) = \frac{r_2 x_2 \rho_2 + \tau(x_2)}{r_2 \rho_2 + 2\tau(x_2)};$
- (iv) $D_{r_1, r_2, \tau, \rho_1, \rho_2} \left[\begin{matrix} \frac{1}{r_1}, \frac{1}{r_2} \end{matrix} \right] (e_{11}; x_1, x_2) = \left(\frac{r_1 x_1 \rho_1 + \tau(x_1)}{r_1 \rho_1 + 2\tau(x_1)} \right) \times \left(\frac{r_2 x_2 \rho_2 + \tau(x_2)}{r_2 \rho_2 + 2\tau(x_2)} \right);$
- (v) $D_{r_1, r_2, \tau, \rho_1, \rho_2} \left[\begin{matrix} \frac{1}{r_1}, \frac{1}{r_2} \end{matrix} \right] (e_{20}; x_1, x_2) = \frac{x_1^2 \rho_1^2 r_1^2 (r_1 - 1)}{(r_1 + 1)(r_1 \rho_1 + 2\tau(x_1))(r_1 \rho_1 + 2\tau(x_1) + 1)} + \frac{x_1 r_1 \rho_1 (r_1 + 1)(1 + 2\tau(x_1) + 2r_1 \rho_1)}{(r_1 + 1)(r_1 \rho_1 + 2\tau(x_1))(r_1 \rho_1 + 2\tau(x_1) + 1)} + \frac{\tau(x_1)(1 + \tau(x_1))}{(r_1 + 1)(r_1 \rho_1 + 2\tau(x_1))(r_1 \rho_1 + 2\tau(x_1) + 1)};$
- (vi) $D_{r_1, r_2, \tau, \rho_1, \rho_2} \left[\begin{matrix} \frac{1}{r_1}, \frac{1}{r_2} \end{matrix} \right] (e_{02}; x_1, x_2) = \frac{x_2^2 \rho_2^2 r_2^2 (r_2 - 1)}{(r_2 + 1)(r_2 \rho_2 + 2\tau(x_2))(r_2 \rho_2 + 2\tau(x_2) + 1)} + \frac{x_1 r_2 \rho_2 (r_2 + 1)(1 + 2\tau(x_2) + 2r_2 \rho_2)}{(r_2 + 1)(r_2 \rho_2 + 2\tau(x_2))(r_2 \rho_2 + 2\tau(x_2) + 1)} + \frac{\tau(x_2)(1 + \tau(x_2))}{(r_2 + 1)(r_2 \rho_2 + 2\tau(x_2))(r_2 \rho_2 + 2\tau(x_2) + 1)};$

$$\begin{aligned}
 \text{(vii)} \quad D_{r_1, r_2, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r_1}, \frac{1}{r_2}\right]}(e_{30}; x_1, x_2) &= \frac{x_1^3 r_1^3 \rho_1^3 (r_1 - 1)(r_1 - 2)}{(r_1 + 1)(r_1 + 2)(r_2 \rho_1 + 2\tau(x_1))(r_1 \rho_1 + 2\tau(x_1) + 1)(r_1 \rho_1 + 2\tau(x_1) + 2)} \\
 &+ \frac{3x_1^2 r_1^2 \rho_1^2 (r_1 - 1)(r_1 + 2)(1 + \tau(x_1) + 2r_1 \rho_1)}{(r_1 + 1)(r_1 + 2)(r_1 \rho_1 + 2\tau(x_1))(r_1 \rho_1 + 2\tau(x_1) + 1)(r_1 \rho_1 + 2\tau(x_1) + 2)} \\
 &+ \frac{x_1(r_1(r_1 + 1)(r_1 + 2)(2 + 3\tau(x_1))(2 + \tau(x_2)))\rho_1 + 6r_1^2(r_1 + 2)(1 + \tau(x_1))\rho_1^2 + 6r_1^3 \rho_1^3}{(r_1 + 1)(r_1 + 2)(r_1 \rho_1 + 2\tau(x_1))(r_1 \rho_1 + 2\tau(x_1) + 1)(r_1 \rho_1 + 2\tau(x_1) + 2)} \\
 &+ \frac{\tau(x_1)(1 + \tau(x_1))(2 + \tau(x_1))}{(r_1 \rho_1 + 2\tau(x_1))(r_1 \rho_1 + 2\tau(x_1) + 1)(r_1 \rho_1 + 2\tau(x_1) + 2)}; \\
 \text{(viii)} \quad D_{r_1, r_2, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r_1}, \frac{1}{r_2}\right]}(e_{03}; x_1, x_2) &= \frac{x_2^3 r_2^3 \rho_2^3 (r_2 - 1)(r_2 - 2)}{(r_2 + 1)(r_2 + 2)(r_2 \rho_2 + 2\tau(x_2))(r_2 \rho_2 + 2\tau(x_2) + 1)(r_2 \rho_2 + 2\tau(x_2) + 2)} \\
 &+ \frac{3x_2^2 r_2^2 \rho_2^2 (r_2 - 1)(r_2 + 2)(1 + \tau(x_2) + 2r_2 \rho_2)}{(r_2 + 1)(r_2 + 2)(r_2 \rho_2 + 2\tau(x_2))(r_2 \rho_2 + 2\tau(x_2) + 1)(r_2 \rho_2 + 2\tau(x_2) + 2)} \\
 &+ \frac{x_2(r_2(r_2 + 1)(r_2 + 2)(2 + 3\tau(x_2))(2 + \tau(x_2)))\rho_2 + 6r_2^2(r_2 + 2)(1 + \tau(x_2))\rho_2^2 + 6r_2^3 \rho_2^3}{(r_2 + 1)(r_2 + 2)(r_2 \rho_2 + 2\tau(x_2))(r_2 \rho_2 + 2\tau(x_2) + 1)(r_2 \rho_2 + 2\tau(x_2) + 2)} \\
 &+ \frac{\tau(x_2)(1 + \tau(x_2))(2 + \tau(x_2))}{(r_2 \rho_2 + 2\tau(x_2))(r_2 \rho_2 + 2\tau(x_2) + 1)(r_2 \rho_2 + 2\tau(x_2) + 2)}; \\
 \text{(ix)} \quad D_{r_1, r_2, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r_1}, \frac{1}{r_2}\right]}(e_{40}; x_1, x_2) &= \frac{x_1^4 r_1^4 \rho_1^4 (r_1 - 1)(r_1 - 2)(r_1 - 3)}{(r_1 + 1)(r_1 + 2)(r_1 + 3)(r_1 \rho_1 + 2\tau(x_1))(r_1 \rho_1 + 2\tau(x_1) + 1)(r_1 \rho_1 + 2\tau(x_1) + 2)(r_1 \rho_1 + 2\tau(x_1) + 3)} \\
 &+ \frac{2x_1^3 r_1^3 \rho_1^3 (r_1 - 1)(r_1 - 2)((r_1 + 3)(3 + 2\tau(x_1)) + 6r_1 \rho_1)}{(r_1 + 1)(r_1 + 2)(r_1 + 3)(r_1 \rho_1 + 2\tau(x_1))(r_1 \rho_1 + 2\tau(x_1) + 1)(r_1 \rho_1 + 2\tau(x_1) + 2)(r_1 \rho_1 + 2\tau(x_1) + 3)} \\
 &+ \frac{x_1^2 r_1^2 \rho_1^2 ((r_1 - 1)(r_1 + 2)(r_1 + 3)(11 + 6\tau(x_1)(3 + \tau(x_1))) + 12r_1(r_1 - 1)(r_1 + 3)(3 + 2\tau(x_1))\rho_1 + 2r_1 \rho_1^2(1 - 19r_1 + 36r_1^2))}{(r_1 + 1)(r_1 + 2)(r_1 + 3)(r_1 \rho_1 + 2\tau(x_1))(r_1 \rho_1 + 2\tau(x_1) + 1)(r_1 \rho_1 + 2\tau(x_1) + 2)(r_1 \rho_1 + 2\tau(x_1) + 3)} \\
 &+ \frac{2r_1 x_1 \rho_1 \left((r_1 + 1)(r_1 + 2)(r_1 + 3)(3 + 2\tau(x_1))(1 + \tau(x_1)(3 + \tau(x_1))) + r_1(r_1 + 2)(r_1 + 3)(11 + 6\tau(x_1)(3 + \tau(x_1)))\rho_1 + 6r_1^2(r_1 + 3)(3 + 2\tau(x_1))\rho_1^2 - r_1^3 \rho_1^3 + 13r_1 \rho_1^3 \right)}{(r_1 + 1)(r_1 + 2)(r_1 + 3)(r_1 \rho_1 + 2\tau(x_1))(r_1 \rho_1 + 2\tau(x_1) + 1)(r_1 \rho_1 + 2\tau(x_1) + 2)(r_1 \rho_1 + 2\tau(x_1) + 3)} \\
 &+ \frac{\tau(x_1)(1 + \tau(x_1))(2 + \tau(x_1))(3 + \tau(x_1))}{(r_1 \rho_1 + 2\tau(x_1))(r_1 \rho_1 + 2\tau(x_1) + 1)(r_1 \rho_1 + 2\tau(x_1) + 2)(r_1 \rho_1 + 2\tau(x_1) + 3)}; \\
 \text{(x)} \quad D_{r_1, r_2, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r_1}, \frac{1}{r_2}\right]}(e_{04}; x_1, x_2) &= \frac{x_2^4 r_2^4 \rho_2^4 (r_2 - 1)(r_2 - 2)(r_2 - 3)}{(r_2 + 1)(r_2 + 2)(r_2 + 3)(r_2 \rho_2 + 2\tau(x_2))(r_2 \rho_2 + 2\tau(x_2) + 1)(r_2 \rho_2 + 2\tau(x_2) + 2)(r_2 \rho_2 + 2\tau(x_2) + 3)} \\
 &+ \frac{2x_2^3 r_2^3 \rho_2^3 (r_2 - 1)(r_2 - 2)((r_2 + 3)(3 + 2\tau(x_2)) + 6r_2 \rho_2)}{(r_2 + 1)(r_2 + 2)(r_2 + 3)(r_2 \rho_2 + 2\tau(x_2))(r_2 \rho_2 + 2\tau(x_2) + 1)(r_2 \rho_2 + 2\tau(x_2) + 2)(r_2 \rho_2 + 2\tau(x_2) + 3)} \\
 &+ \frac{x_2^2 r_2^2 \rho_2^2 ((r_2 - 1)(r_2 + 2)(r_2 + 3)(11 + 6\tau(x_2)(3 + \tau(x_2))) + 12r_2(r_2 - 1)(r_2 + 3)(3 + 2\tau(x_2))\rho_2 + 2r_2 \rho_2^2(1 - 19r_2 + 36r_2^2))}{(r_2 + 1)(r_2 + 2)(r_2 + 3)(r_2 \rho_2 + 2\tau(x_2))(r_2 \rho_2 + 2\tau(x_2) + 1)(r_2 \rho_2 + 2\tau(x_2) + 2)(r_2 \rho_2 + 2\tau(x_2) + 3)} \\
 &+ \frac{2r_2 x_2 \rho_2 \left((r_2 + 1)(r_2 + 2)(r_2 + 3)(3 + 2\tau(x_2))(1 + \tau(x_2)(3 + \tau(x_2))) + r_2(r_2 + 2)(r_2 + 3)(11 + 6\tau(x_2)(3 + \tau(x_2)))\rho_2 + 6r_2^2(r_2 + 3)(3 + 2\tau(x_2))\rho_2^2 - r_2^3 \rho_2^3 + 13r_2 \rho_2^3 \right)}{(r_2 + 1)(r_2 + 2)(r_2 + 3)(r_2 \rho_2 + 2\tau(x_2))(r_2 \rho_2 + 2\tau(x_2) + 1)(r_2 \rho_2 + 2\tau(x_2) + 2)(r_2 \rho_2 + 2\tau(x_2) + 3)} \\
 &+ \frac{\tau(x_2)(1 + \tau(x_2))(2 + \tau(x_2))(3 + \tau(x_2))}{(r_2 \rho_2 + 2\tau(x_2))(r_2 \rho_2 + 2\tau(x_2) + 1)(r_2 \rho_2 + 2\tau(x_2) + 2)(r_2 \rho_2 + 2\tau(x_2) + 3)}.
 \end{aligned}$$

Lemma 1.2. *By Direct computation, we have*

$$\text{(i)} \quad D_{r_1, r_2, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r_1}, \frac{1}{r_2}\right]}((t_1 - x_1); x_1, x_2) = \frac{\tau(x_1)(1 - 2x_1)}{r_1 \rho_1 + 2\tau(x_1)};$$

$$\begin{aligned}
 \text{(ii)} \quad & D_{r_1, r_2, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r_1}, \frac{1}{r_2}\right]}((t_2 - x_2); x_1, x_2) = \frac{\tau(x_2)(1 - 2x_2)}{r_2\rho_2 + 2\tau(x_2)}; \\
 \text{(iii)} \quad & D_{r_1, r_2, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r_1}, \frac{1}{r_2}\right]}((t_1 - x_1)^2; x_1, x_2) = \frac{x_1(1 - x_1)(r_1\rho_1(1 + r_1 + 2r_1\rho_1) - 4(r_1 + 1)\tau^2(x_1) - 2(r_1 + 1)\tau(x_1))}{(r_1 + 1)(r_1\rho_1 + 2\tau(x_1))(r_1\rho_1 + 2\tau(x_1) + 1)}; \\
 \text{(iv)} \quad & D_{r_1, r_2, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r_1}, \frac{1}{r_2}\right]}((t_2 - x_2)^2; x_1, x_2) = \frac{x_2(1 - x_2)(r_2\rho_2(1 + r_2 + 2r_2\rho_2) - 4(r_2 + 1)\tau^2(x_2) - 2(r_2 + 1)\tau(x_2))}{(r_2 + 1)(r_2\rho_2 + 2\tau(x_2))(r_2\rho_2 + 2\tau(x_2) + 1)}; \\
 \text{(v)} \quad & D_{r_1, r_2, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r_1}, \frac{1}{r_2}\right]}((t_1 - x_1)^4; x_1, x_2) = \frac{6(r_1+1)(r_1+2)(r_1+3)\tau(x_1)+11(r_1+1)(r_1+2)(r_1+3)\tau^2(x_1)+6(r_1+1)(r_1+2)(r_1+3)\tau^3(x_1)}{(r_1+1)(r_1+2)(r_1+3)(2\tau(x_1)+r_2\rho_2)(1+2\tau(x_1)+r_2\rho_1)(2+2\tau(x_1)+r_1\rho_1)(3+2\tau(x_1)+r_1\rho_1)} \\
 & + \frac{(r_1 + 1)(r_1 + 2)(r_1 + 3)(2\tau(x_1) + r_2\rho_1)(1 + 2\tau(x_1) + r_1\rho_1)(2 + 2\tau(x_1) + r_1\rho_1)(3 + 2\tau(x_1) + r_1\rho_1)}{(r_1 + 1)(r_1 + 2)(r_1 + 3)(2\tau(x_1) + r_1\rho_1)(1 + 2\tau(x_1) + r_1\rho_1)(2 + 2\tau(x_1) + r_1\rho_1)(3 + 2\tau(x_1) + r_1\rho_1)} \\
 & + \frac{x_1(-24(r_1 + 1)(r_1 + 2)(r_1 + 3)\tau(x_1) - 52(r_1 + 1)(r_1 + 2)(r_1 + 3)\tau^2(x_1) - 36(r_1 + 1)(r_1 + 2)(r_1 + 3)\tau^3(x_1))}{(r_1 + 1)(r_1 + 2)(r_1 + 3)(2\tau(x_1) + r_1\rho_1)(1 + 2\tau(x_1) + r_1\rho_1)(2 + 2\tau(x_1) + r_1\rho_1)(3 + 2\tau(x_1) + r_1\rho_1)} \\
 & + \frac{x_1(-8(r_1 + 1)(r_1 + 2)(r_1 + 3)\tau(x_1)^4 + 6r_1(r_1 + 1)(r_1 + 2)(r_1 + 3)\rho_1 + 14r_1(r_1 + 1)(r_1 + 2)(r_1 + 3)\tau(x_1)\rho_2)}{(r_1 + 1)(r_1 + 2)(r_1 + 3)(2\tau(x_1) + r_2\rho_1)(1 + 2\tau(x_1) + r_2\rho_1)(2 + 2\tau(x_1) + r_1\rho_1)(3 + 2\tau(x_1) + r_1\rho_1)} \\
 & + \frac{x_1(6r_1(r_1 + 1)(r_1 + 2)(r_1 + 3)\tau^2(x_1)\rho_1 + 22r_1^2(r_1 + 1)(r_1 + 2)(r_1 + 3)\rho_1^2 + 36r_1^2(r_1 + 1)(r_1 + 2)(r_1 + 3)\tau(x_1)\rho_1^2)}{(r_1 + 1)(r_1 + 2)(r_1 + 3)(2\tau(x_1) + r_1\rho_1)(1 + 2\tau(x_1) + r_1\rho_1)(2 + 2\tau(x_1) + r_1\rho_1)(3 + 2\tau(x_1) + r_1\rho_1)} \\
 & + \frac{x_1(12r_1^2(r_1 + 1)(r_1 + 2)(r_1 + 3)\tau^2(x_1)\rho_1^2 + 36r_1^3(r_1 + 3)\rho_1^3 + 24r_1^3(3 + r_1)\tau(x_1)\rho_1^3 - 2r_1^3\rho_1^4 + 26r_1^4\rho_1^4)}{(r_1 + 1)(r_1 + 2)(r_1 + 3)(2\tau(x_1) + r_1\rho_1)(1 + 2\tau(x_1) + r_1\rho_1)(2 + 2\tau(x_1) + r_1\rho_1)(3 + 2\tau(x_1) + r_1\rho_1)} \\
 & + \left(x_1^3 \left(-24(r_1 + 1)(r_1 + 2)(r_1 + 3)\tau(x_1) - 88(r_1 + 1)(r_1 + 2)(r_1 + 3)\tau^2(x_1) - 96(r_1 + 1)(r_1 + 2)(r_1 + 3)\tau^3(x_1) \right. \right. \\
 & \quad - 32(r_1 + 1)(r_1 + 2)(r_1 + 3)\tau^4(x_1) + 36r_1(1 + r_1)(2 + r_1)(3 + r_1)\rho_1 + 88r_1(1 + r_1)(2 + r_1)(3 + r_1)\tau(x_1)\rho_1 \\
 & \quad + 48r_1(1 + r_1)(2 + r_1)(3 + r_1)\tau^2(x_1)\rho_1 - 6(-23 + r_1)r_1^2(2 + r_1)(3 + r_1)\rho^2 + 240r_1^2(2 + r_1)(3 + r_1) \\
 & \quad \left. \tau(x_1)\rho_1^2 + 96r_1^2(2 + r_1)(3 + r_1)\tau^2(x_1)\rho_1^2 + 120r_1^3(3 + r_1)\rho_1^3 - 12(-10 + r_1)r_1^3(3 + r_1)\rho_1^3 - 12r_1^4(3 + r_1)\rho_1^3 \right. \\
 & \quad \left. + 192r_1^3(3 + r_1)\tau\rho^3 - 24(-7 + r_1)r_1^4\rho_1^4\right) / ((r_1 + 1)(r_1 + 2)(r_1 + 3)(2\tau(x_1) + r_2\rho_2)(1 + 2\tau(x_1) + r_2\rho_1)(2 + 2\tau(x_1) + \\
 & r_1\rho_1)(3 + 2\tau(x_1) + r_1\rho_1)) + \left(x_1^4 \left(12(1 + r_1)(2 + r_1)(3 + r_1)\tau(x_1) + 44(1 + r_1)(2 + r_1)(3 + r_1)\tau^2(x_1) \right. \right. \\
 & \quad + 48(1 + r_1)(2 + r_1)(3 + r_1)\tau^3(x_1) + 16(1 + r_1)(2 + r_1)(3 + r_1)\tau^4(x_1) - 18r_1(1 + r_1)(2 + r_1)(3 + r_1)\rho_1 \\
 & \quad - 44r_1(1 + r_1)(2 + r_1)(3 + r_1)\tau(x_1)\rho_1 - 24r_1(1 + r_1)(2 + r_1)(3 + r_1)\tau^2(x_1)\rho_1 + 3(-23 + r_1)r_1^2(2 + r_1) \\
 & \quad \left. (3 + r_1)\rho_1^2 - 120r_1^2(2 + r_1)(3 + r_1)\tau(x_1)\rho_1^2 - 48r_1^2(2 + r_1)(3 + r_1)\tau^2(x_1)\rho_1^2 + 12(r_1 - 10)r_1^3(3 + r_1)\rho_1^3 \right. \\
 & \quad \left. - 96r_1^3(3 + r_1)\tau(x_1)\rho_1^3 + 12(-7 + r_1)r_1^4\rho_1^4\right) / ((1 + r_1)(2 + r_1)(3 + r_1)(2\tau(x_1) + r_1\rho_1)(1 + 2\tau(x_1) + r_1\rho_1)(2 + 2\tau(x_1) + \\
 & r_1\rho_1)(3 + 2\tau(x_1) + r_1\rho_1)) + \left(x_1^2 \left(36(1 + r_1)(2 + r_1)(3 + r_1)\tau(x_1) + 96(1 + r_1)(2 + r_1)(3 + r_1)\tau^2(x_1) \right. \right. \\
 & \quad + 84(1 + r_1)(2 + r_1)(3 + r_1)\tau^3(x_1) + 24(1 + r_1)(2 + r_1)(3 + r_1)\tau^4(x_1) - 24r_1(1 + r_1)(2 + r_1)(3 + r_1)\rho_1 \\
 & \quad - 58r_1(1 + r_1)(2 + r_1)(3 + r_1)\tau(x_1)\rho_1 - 30r_1(1 + r_1)(2 + r_1)(3 + r_1)\tau^2(x_1)\rho_1 - 22r_1^2(2 + r_1)(3 + r_1)\rho_1^2 \\
 & \quad \left. + 3(-23 + r_1)r_1^2(2 + r_1)(3 + r_1)\rho_1^2 - 156r_1^2(2 + r_1)(3 + r_1)\tau(x_1)\rho_1^2 - 60r_1^2(2 + r_1)(3 + r_1)\tau^2(x_1)\rho_1^2 \right. \\
 & \quad \left. - 156r_1^3(3 + r_1)\rho_1^3 + 12r_1^4(3 + r_1)\rho_1^3 - 120r_1^3(3 + r_1)\tau(x_1)\rho_1^3 + 2r_1^3\rho_1^4 - 26r_1^4\rho_1^4 + 12(-7 + r_1)r_1^4\rho_1^4\right) / ((1 + r_1)(2 + \\
 & r_1)(3 + r_1)(2\tau(x_1) + r_1\rho_1)(1 + 2\tau(x_1) + r_1\rho_1)(2 + 2\tau(x_1) + r_1\rho_1)(3 + 2\tau(x_1) + r_1\rho_1)) \\
 \text{(vi)} \quad & D_{r_1, r_2, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r_1}, \frac{1}{r_2}\right]}((t_2 - x_2)^4; x_1, x_2) = \frac{6(r_2+1)(r_2+2)(r_2+3)\tau(x_2)+11(r_2+1)(r_2+2)(r_2+3)\tau^2(x_2)+6(r_2+1)(r_2+2)(r_2+3)\tau^3(x_2)}{(r_2+1)(r_2+2)(r_2+3)(2\tau(x_2)+r_2\rho_2)(1+2\tau(x_2)+r_2\rho_1)(2+2\tau(x_2)+r_2\rho_2)(3+2\tau(x_2)+r_2\rho_2)} \\
 & + \frac{(r_2 + 1)(r_2 + 2)(r_2 + 3)(2\tau(x_2) + r_2\rho_2)(1 + 2\tau(x_2) + r_2\rho_2)(2 + 2\tau(x_2) + r_2\rho_2)(3 + 2\tau(x_2) + r_2\rho_2)}{(r_2 + 1)(r_2 + 2)(r_2 + 3)(2\tau(x_2) + r_2\rho_2)(1 + 2\tau(x_2) + r_2\rho_2)(2 + 2\tau(x_2) + r_2\rho_2)(3 + 2\tau(x_2) + r_2\rho_2)} \\
 & + \frac{x_2(-24(r_2 + 1)(r_2 + 2)(r_2 + 3)\tau(x_2) - 52(r_2 + 1)(r_2 + 2)(r_2 + 3)\tau^2(x_2) - 36(r_2 + 1)(r_2 + 2)(r_2 + 3)\tau^3(x_2))}{(r_2 + 1)(r_2 + 2)(r_2 + 3)(2\tau(x_2) + r_2\rho_2)(1 + 2\tau(x_2) + r_2\rho_2)(2 + 2\tau(x_2) + r_2\rho_2)(3 + 2\tau(x_2) + r_2\rho_2)} \\
 & + \frac{x_2(-8(r_2 + 1)(r_2 + 2)(r_2 + 3)\tau(x_2)^4 + 6r_2(r_2 + 1)(r_2 + 2)(r_2 + 3)\rho_2 + 14r_2(r_2 + 1)(r_2 + 2)(r_2 + 3)\tau(x_2)\rho_2)}{(r_2 + 1)(r_2 + 2)(r_2 + 3)(2\tau(x_2) + r_2\rho_2)(1 + 2\tau(x_2) + r_2\rho_2)(2 + 2\tau(x_2) + r_2\rho_2)(3 + 2\tau(x_2) + r_2\rho_2)}
 \end{aligned}$$

$$\frac{x_2(6r_2(r_2 + 1)(r_2 + 2)(r_2 + 3)\tau^2(x_2)\rho_2 + 22r_2^2(r_2 + 1)(r_2 + 2)(r_2 + 3)\rho_2^2 + 36r_2^2(r_2 + 1)(r_2 + 2)(r_2 + 3)\tau(x_2)\rho_2^3)}{(r_2 + 1)(r_2 + 2)(r_2 + 3)(2\tau(x_2) + r_2\rho_2)(1 + 2\tau(x_2) + r_2\rho_2)(2 + 2\tau(x_2) + r_2\rho_2)(3 + 2\tau(x_2) + r_2\rho_2)}$$

$$\frac{x_2(12r_2^2(r_2 + 1)(r_2 + 2)(r_2 + 3)\tau^2(x_2)\rho_2^2 + 36r_2^3(r_2 + 3)\rho_2^3 + 24r_2^3(3 + r_2)\tau(x_2)\rho_2^3 - 2r_2^3\rho_2^4 + 26r_2^4\rho_2^4)}{(r_2 + 1)(r_2 + 2)(r_2 + 3)(2\tau(x_2) + r_2\rho_2)(1 + 2\tau(x_2) + r_2\rho_2)(2 + 2\tau(x_2) + r_2\rho_2)(3 + 2\tau(x_2) + r_2\rho_2)}$$

$$+ \left(x_2^3 \left(-24(r_2 + 1)(r_2 + 2)(r_2 + 3)\tau(x_2) - 88(r_2 + 1)(r_2 + 2)(r_2 + 3)\tau^2(x_2) - 96(r_2 + 1)(r_2 + 2)(r_2 + 3)\tau^3(x_2) \right. \right.$$

$$\left. - 32(r_2 + 1)(r_2 + 2)(r_2 + 3)\tau^4(x_2) + 36r(1 + r_2)(2 + r_2)(3 + r_2)\rho_2 + 88r_2(1 + r_2)(2 + r_2)(3 + r_2)\tau(x_2)\rho_2 \right.$$

$$\left. + 48r_2(1 + r_2)(2 + r_2)(3 + r_2)\tau^2(x_2)\rho_2 - 6(-23 + r_2)r_2^2(2 + r_2)(3 + r_2)\rho_2^2 + 240r_2^2(2 + r_2)(3 + r_2) \right.$$

$$\left. \tau(x_2)\rho_2^2 + 96r_2^2(2 + r_2)(3 + r_2)\tau^2(x_2)\rho_2^2 + 120r_2^3(3 + r_2)\rho_2^3 - 12(-10 + r_2)r_2^3(3 + r_2)\rho_2^3 - 12r_2^4(3 + r_2)\rho_2^3 \right.$$

$$\left. + 192r_2^3(3 + r_2)\tau(x_2)\rho_2^3 - 24(-7 + r_2)r_2^4\rho_2^4 \right) / ((r_2 + 1)(r_2 + 2)(r_2 + 3)(2\tau(x_2) + r_2\rho_2)(1 + 2\tau(x_2) + r_2\rho_2)(2 +$$

$$2\tau(x_2) + r_2\rho_2)(3 + 2\tau(x_2) + r_2\rho_2)) + \left(x_2^4 \left(12(1 + r_2)(2 + r_2)(3 + r_2)\tau(x_2) + 44(1 + r_2)(2 + r_2)(3 + r_2)\tau^2(x_2) \right. \right.$$

$$\left. + 48(1 + r_2)(2 + r_2)(3 + r_2)\tau^3(x_2) + 16(1 + r_2)(2 + r_2)(3 + r_2)\tau^4(x_2) - 18r_2(1 + r_2)(2 + r_2)(3 + r_2)\rho_2 - 44r_2 \right.$$

$$\left. (1 + r_2)(2 + r_2)(3 + r_2)\tau(x_1)\rho_2 - 24r_2(1 + r_2)(2 + r_2)(3 + r_2)\tau^2(x_2)\rho_2 + 3(-23 + r_2)r_2^2(2 + r_2)(3 + r_2)\rho_2^2 \right.$$

$$\left. - 120r_2^2(2 + r_2)(3 + r_2)\tau\rho^2 - 48r_2^2(2 + r_2)(3 + r_2)\tau^2(x_2)\rho_2^2 + 12(-10 + r_2)r_2^3(3 + r_2)\rho_2^3 - 96r_2^3(3 + r_2)\tau(x_2)\rho_2^3 \right.$$

$$\left. + 12(-7 + r_2)r_2^4\rho_2^4 \right) / ((1 + r_2)(2 + r_2)(3 + r_2)(2\tau(x_2) + r_2\rho_2)(1 + 2\tau(x_2) + r_2\rho_2)(2 + 2\tau(x_2) + r_2\rho_2)(3 + 2\tau(x_2) +$$

$$r_2\rho_2)) + \left(x_2^5 \left(36(1 + r_2)(2 + r_2)(3 + r_2)\tau(x_2) + 96(1 + r_2)(2 + r_2)(3 + r_2)\tau^2(x_2) + 84(1 + r_2)(2 + r_2) \right. \right.$$

$$\left. (3 + r_2)\tau^3(x_2) + 24(1 + r_2)(2 + r_2)(3 + r_2)\tau^4(x_2) - 24r_2(1 + r_2)(2 + r_2)(3 + r_2)\rho_2 - 58r_2(1 + r_2)(2 + r_2) \right.$$

$$\left. (3 + r_2)\tau(x_2)\rho_2 - 30r_2(1 + r_2)(2 + r_2)(3 + r_2)\tau^2(x_2)\rho_2 - 22r_2^2(2 + r_2)(3 + r_2)\rho_2^2 + 3(-23 + r_2)r_2^2 \right.$$

$$\left. (2 + r_2)(3 + r_2)\rho_2^2 - 156r_2^2(2 + r_2)(3 + r_2)\tau(x_2)\rho_2^2 - 60r_2^2(2 + r_2)(3 + r_2)\tau^2(x_2)\rho_2^2 - 156r_2^3(3 + r_2)\rho_2^3 \right.$$

$$\left. + 12r_2^4(3 + r_2)\rho_2^3 - 120r_2^3(3 + r_2)\tau(x_2)\rho_2^3 + 2r_2^3\rho_2^4 - 26r_2^4\rho_2^4 + 12(-7 + r_2)r_2^4\rho_2^4 \right) / ((1 + r_2)(2 + r_2)(3 + r_2)(2\tau(x_2) +$$

$$r_2\rho_2)(1 + 2\tau(x_2) + r_2\rho_2)(2 + 2\tau(x_2) + r_2\rho_2)(3 + 2\tau(x_2) + r_2\rho_2)).$$

Lemma 1.3. From Lemma 1.2, we have

$$\lim_{r_1 \rightarrow \infty} r_1 D_{r_1, r_2, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r_1}, \frac{1}{r_2} \right]} ((t_1 - x_1)^2; x_1, x_2) = \frac{1}{\rho_1} (1 + 2\rho_1)x_1(1 - x_1);$$

$$\lim_{r_2 \rightarrow \infty} r_2 D_{r_1, r_2, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r_1}, \frac{1}{r_2} \right]} ((t_2 - x_2)^2; x_1, x_2) = \frac{1}{\rho_2} (1 + 2\rho_2)x_2(1 - x_2);$$

$$\lim_{r_1 \rightarrow \infty} r_1^2 D_{r_1, r_2, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r_1}, \frac{1}{r_2} \right]} ((t_1 - x_1)^4; x_1, x_2) = \left(\frac{3x_1^2(1 - x_1)^2(1 + 2\rho_1)^2}{\rho_1^2} \right);$$

$$\lim_{r_2 \rightarrow \infty} r_2^2 D_{r_1, r_2, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r_1}, \frac{1}{r_2} \right]} ((t_2 - x_2)^4; x_1, x_2) = \left(\frac{3x_2^2(1 - x_2)^2(1 + 2\rho_2)^2}{\rho_2^2} \right).$$

Lemma 1.4. We have

$$D_{r_1, r_2, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r_1}, \frac{1}{r_2} \right]} ((t_1 - x_1)^2; x_1, x_2) = \frac{A_1}{r_1} \left\{ x_1(1 - x_1) + \frac{1}{r_1} \right\} = \Phi_{r_1, \rho_1}^2(x_1);$$

$$D_{r_1, r_2, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r_1}, \frac{1}{r_2} \right]} ((t_2 - x_2)^2; x_1, x_2) = \frac{A_2}{r_2} \left\{ x_2(1 - x_2) + \frac{1}{r_2} \right\} = \Psi_{r_2, \rho_2}^2(x_2),$$

where $A_i = A_i(\rho_i)$, $i = 1, 2$, is a positive constant.

Theorem 1.5. Suppose $\gamma \in C(I^2)$. Then,

$$\lim_{r_1, r_2 \rightarrow \infty} \|D_{r_1, r_2, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r_1}, \frac{1}{r_2}\right]}(\gamma) - \gamma\| = 0.$$

Proof. From the Lemma 1.1, we have

$$\lim_{r_1, r_2 \rightarrow \infty} D_{r_1, r_2, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r_1}, \frac{1}{r_2}\right]}(e_{mn}; x_1, x_2) = e_{mn}, (m, n) \in \{(0, 0), (1, 0), (0, 1)\}$$

and

$$\lim_{r_1, r_2 \rightarrow \infty} D_{r_1, r_2, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r_1}, \frac{1}{r_2}\right]}(e_{02} + e_{20}; x_1, x_2) = e_{02} + e_{20}$$

uniformly on I^2 . Hence, the result follows in view of the theorem given by Volkov [31] \square

Let $C^n(I^2), n \in \mathbb{N}$ denote the space of n – times partially differentiable continuous function in I^2 .

2. Voronovskaja Type Theorem

In the next theorem, we establish a Voronovskaja type asymptotic result.

Theorem 2.1. If $\gamma \in C^2(I^2)$, then

$$\begin{aligned} \lim_{r \rightarrow \infty} r \left(D_{r, r, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r}, \frac{1}{r}\right]}(\gamma; x_1, x_2) - \gamma(x_1, x_2) \right) &= \frac{\tau(x_1)(1 - 2x_1)}{\rho_1} \gamma'_{x_1}(x_1, x_2) + \frac{\tau(x_2)(1 - 2x_2)}{\rho_2} \gamma'_{x_2}(x_1, x_2) \\ &+ \frac{1}{2\rho_1} (1 + 2\rho_1)x_1(1 - x_1)\gamma''_{x_1x_1}(x_1, x_2) + \frac{1}{2\rho_2} (1 + 2\rho_2)x_2(1 - x_2)\gamma''_{x_2x_2}(x_1, x_2), \end{aligned}$$

uniformly in $(x_1, x_2) \in I^2$.

Proof. Using Taylor’s series expansion, we get

$$\begin{aligned} \gamma(u, v) &= \gamma(x_1, x_2) + \gamma'_{x_1}(x_1, x_2)(u - x_1) + \gamma'_{x_2}(x_1, x_2)(v - x_2) + \frac{1}{2} \{ \gamma''_{x_1x_1}(x_1, x_2)(u - x_1)^2 + 2\gamma''_{x_1x_2}(x_1, x_2)(u - x_1)(v - x_2) \\ &+ \gamma''_{x_2x_2}(x_1, x_2)(v - x_2)^2 \} + \beta(u, v; x_1, x_2) \sqrt{(u - x_1)^4 + (v - x_2)^4}, \end{aligned}$$

where $\beta(u, v; x_1, x_2) \in C(I^2)$ and $\beta(u, v; x_1, x_2) \rightarrow 0$ as $(u, v) \rightarrow (x_1, x_2)$.

Applying positivity and linearity of $D_{r, r, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r}, \frac{1}{r}\right]}(\gamma; x_1, x_2)$ on the above Taylor’s expansion, we have

$$\begin{aligned} D_{r, r, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r}, \frac{1}{r}\right]}(\gamma; x_1, x_2) &= \gamma(x_1, x_2) + \gamma'_{x_1}(x_1, x_2) D_{r, r, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r}, \frac{1}{r}\right]}((u - x_1); x_1, x_2) + \gamma'_{x_2}(x_1, x_2) D_{r, r, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r}, \frac{1}{r}\right]}((v - x_2); x_1, x_2) \\ &+ \frac{1}{2} (\gamma''_{x_1x_1}(x_1, x_2) D_{r, r, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r}, \frac{1}{r}\right]}((u - x_1)^2; x_1, x_2) + \gamma''_{x_2x_2}(x_1, x_2) D_{r, r, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r}, \frac{1}{r}\right]}((v - x_2)^2; x_1, x_2) \\ &+ 2\gamma''_{x_1x_2}(x_1, x_2) D_{r, r, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r}, \frac{1}{r}\right]}((u - x_1); x_1, x_2) D_{r, r, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r}, \frac{1}{r}\right]}((v - x_2); x_1, x_2) \end{aligned}$$

$$+D_{r,r,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r},\frac{1}{r}\right]} \left(\beta(u, v; x_1, x_2) \sqrt{(u - x_1)^4 + (v - x_2)^4}; x_1, x_2 \right)$$

From Lemma 1.2, we have

$$\lim_{r \rightarrow \infty} r \left(D_{r,r,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r},\frac{1}{r}\right]} (\gamma; x_1, x_2) - \gamma(x_1, x_2) \right) = \frac{\tau(x_1)(1 - 2x_1)}{\rho_1} \gamma'_{x_1}(x_1, x_2) + \frac{\tau(x_2)(1 - 2x_2)}{\rho_2} \gamma'_{x_2}(x_1, x_2) \tag{3}$$

$$+ \frac{1}{2\rho_1} (1 + 2\rho_1)x_1(1 - x_1) \gamma''_{x_1x_1}(x_1, x_2) + \frac{1}{2\rho_2} (1 + 2\rho_2)x_2(1 - x_2) \gamma''_{x_2x_2}(x_1, x_2) \\ + \lim_{r \rightarrow \infty} r D_{r,r,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r},\frac{1}{r}\right]} \left(\beta(u, v; x_1, x_2) \sqrt{(u - x_1)^4 + (v - x_2)^4}; x_1, x_2 \right), \tag{4}$$

uniformly in $(x_1, x_2) \in I^2$. By using Cauchy-Schwarz inequality, we obtain

$$r D_{r,r,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r},\frac{1}{r}\right]} \left(\beta(u, v; x_1, x_2) \sqrt{(u - x_1)^4 + (v - x_2)^4}; x_1, x_2 \right) \leq \left(D_{r,r,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r},\frac{1}{r}\right]} (\beta^2(u, v; x_1, x_2); x_1, x_2) \right)^{\frac{1}{2}} \\ \left(r^2 D_{r,r,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r},\frac{1}{r}\right]} ((u - x_1)^4; x_1, x_2) + r^2 D_{r,r,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r},\frac{1}{r}\right]} ((v - x_2)^4; x_1, x_2) \right)^{\frac{1}{2}}$$

and $\beta^2(u, v; x_1, x_2) \rightarrow 0$, and $(u, v) \rightarrow (x_1, x_2)$, using Theorem 1.5,

$$\lim_{r \rightarrow \infty} D_{r,r,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r},\frac{1}{r}\right]} (\beta^2(u, v; x_1, x_2); x_1, x_2) = 0,$$

uniformly in $(x_1, x_2) \in I^2$. By using Lemma 1.3, we may write

$$(r^2 D_{r,r,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r},\frac{1}{r}\right]} ((u - x_1)^4; x_1, x_2) + r^2 D_{r,r,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r},\frac{1}{r}\right]} ((v - x_2)^4; x_1, x_2)) = O(1),$$

uniformly in $(x_1, x_2) \in I^2$, as $r \rightarrow \infty$. Hence,

$$\lim_{r \rightarrow \infty} r D_{r,r,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r},\frac{1}{r}\right]} \left(\beta(u, v; x_1, x_2) \sqrt{(u - x_1)^4 + (v - x_2)^4}; x_1, x_2 \right) = 0,$$

uniformly in $(x_1, x_2) \in I^2$. Therefore, by (6) we obtain the desired result. \square

3. Grüss Voronovskaja type theorem

The following theorem shows the non-multiplicative behaviour of operator $D_{r,r,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r},\frac{1}{r}\right]}$.

Theorem 3.1. For $\gamma, \zeta \in C^2(I^2)$, there holds the following equality

$$\lim_{r \rightarrow \infty} r \left\{ D_{r,r,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r},\frac{1}{r}\right]} (\gamma\zeta; x_1, x_2) - D_{r,r,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r},\frac{1}{r}\right]} (\gamma; x_1, x_2) D_{r,r,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r},\frac{1}{r}\right]} (\zeta; x_1, x_2) \right\} = \frac{1}{\rho_1} (1 + 2\rho_1)x_1(1 - x_1) \gamma'_{x_1}(x_1, x_2) \zeta'_{x_1}(x_1, x_2) \\ + \frac{1}{\rho_2} (1 + \rho_2)x_2(1 - x_2) \gamma'_{x_2}(x_1, x_2) \zeta'_{x_2}(x_1, x_2),$$

uniformly in $(x_1, x_2) \in I^2$.

Proof.

$$\begin{aligned}
 & \lim_{r \rightarrow \infty} r \left\{ D_{r,r,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r}, \frac{1}{r} \right]} (\gamma \zeta; x_1, x_2) - D_{r,r,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r}, \frac{1}{r} \right]} (\gamma; x_1, x_2) D_{r,r,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r}, \frac{1}{r} \right]} (\zeta; x_1, x_2) \right\} \\
 &= r \left(D_{r,r,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r}, \frac{1}{r} \right]} (\gamma \zeta; x_1, x_2) - \gamma(x_1, x_2) \zeta(x_1, x_2) - (\gamma(x_1, x_2) \zeta'_{x_1}(x_1, x_2) + \zeta(x_1, x_2) \gamma'_{x_1}(x_1, x_2)) \right. \\
 & \quad \left. D_{r,r,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r}, \frac{1}{r} \right]} ((u - x_1); x_1, x_2) - (\gamma(x_1, x_2) \zeta'_{x_2}(x_1, x_2) + \zeta(x_1, x_2) \gamma'_{x_2}(x_1, x_2)) D_{r,r,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r}, \frac{1}{r} \right]} ((v - x_2); x_1, x_2) \right. \\
 & \quad \left. - \frac{1}{2} (\gamma(x_1, x_2) \zeta''_{x_1 x_1}(x_1, x_2) + 2\gamma'_{x_1}(x_1, x_2) \zeta'_{x_1}(x_1, x_2) + \zeta(x_1, x_2) \gamma''_{x_1 x_1}(x_1, x_2)) D_{r,r,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r}, \frac{1}{r} \right]} ((u - x_1)^2; x_1, x_2) \right. \\
 & \quad \left. - (\gamma(x_1, x_2) \zeta''_{x_1 x_2}(x_1, x_2) + \gamma'_{x_1}(x_1, x_2) \zeta'_{x_2}(x_1, x_2) + \gamma'_{x_2}(x_1, x_2) \zeta'_{x_1}(x_1, x_2)) \right. \\
 & \quad \left. + \zeta(x_1, x_2) \gamma''_{x_1 x_2}(x_1, x_2)) D_{r,r,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r}, \frac{1}{r} \right]} ((u - x_1)(v - x_2); x_1, x_2) \right. \\
 & \quad \left. - \frac{1}{2} (\gamma(x_1, x_2) \zeta''_{x_2 x_2}(x_1, x_2) + 2\gamma'_{x_2}(x_1, x_2) \zeta'_{x_2}(x_1, x_2) + \zeta(x_1, x_2) \gamma''_{x_2 x_2}(x_1, x_2)) D_{r,r,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r}, \frac{1}{r} \right]} ((v - x_2)^2; x_1, x_2) \right. \\
 & \quad \left. - \zeta(x_1, x_2) (D_{r,r,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r}, \frac{1}{r} \right]} (\gamma; x_1, x_2) - \gamma(x_1, x_2) - \gamma'_{x_1}(x_1, x_2) D_{r,r,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r}, \frac{1}{r} \right]} ((u - x_1); x_1, x_2)) \right. \\
 & \quad \left. - \gamma'_{x_2}(x_1, x_2) D_{r,r,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r}, \frac{1}{r} \right]} ((v - x_1); x_1, x_2) - \frac{1}{2} \gamma''_{x_1 x_1}(x_1, x_2) D_{r,r,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r}, \frac{1}{r} \right]} ((u - x_1)^2; x_1, x_2) \right. \\
 & \quad \left. - \gamma''_{x_1 x_2}(x_1, x_2) D_{r,r,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r}, \frac{1}{r} \right]} ((u - x_1)(v - x_2); x_1, x_2) \right. \\
 & \quad \left. - \frac{1}{2} \gamma''_{x_2 x_2}(x_1, x_2) D_{r,r,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r}, \frac{1}{r} \right]} ((v - x_2)^2; x_1, x_2) - \gamma(x_1, x_2) (D_{r,r,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r}, \frac{1}{r} \right]} (\zeta; x_1, x_2) - \zeta(x_1, x_2)) \right. \\
 & \quad \left. - \zeta'_{x_1}(x_1, x_2) D_{r,r,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r}, \frac{1}{r} \right]} ((u - x_1); x_1, x_2) - \zeta'_{x_2}(x_1, x_2) D_{r,r,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r}, \frac{1}{r} \right]} ((v - x_2); x_1, x_2) \right. \\
 & \quad \left. - \frac{1}{2} \zeta''_{x_1 x_1}(x_1, x_2) D_{r,r,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r}, \frac{1}{r} \right]} ((u - x_1)^2; x_1, x_2) \right. \\
 & \quad \left. - \zeta''_{x_1 x_2}(x_1, x_2) D_{r,r,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r}, \frac{1}{r} \right]} ((u - x_1)(v - x_2); x_1, x_2) - \frac{1}{2} \zeta''_{x_2 x_2}(x_1, x_2) D_{r,r,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r}, \frac{1}{r} \right]} ((v - x_2)^2; x_1, x_2) \right. \\
 & \quad \left. + \zeta'_{x_1}(x_1, x_2) D_{r,r,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r}, \frac{1}{r} \right]} ((u - x_1); x_1, x_2) (\gamma(x_1, x_2) - D_{r,r,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r}, \frac{1}{r} \right]} (\gamma; x_1, x_2)) + \zeta'_{x_2}(x_1, x_2) D_{r,r,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r}, \frac{1}{r} \right]} ((v - x_2); x_1, x_2) \right. \\
 & \quad \left. (\gamma(x_1, x_2) - D_{r,r,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r}, \frac{1}{r} \right]} (\gamma; x_1, x_2)) + \zeta''_{x_1 x_1} \frac{D_{r,r,\tau,\rho_1}^{\left[\frac{1}{r} \right]} ((u - x_1)^2; x_1, x_2)}{2} (\gamma(x_1, x_2) - D_{r,r,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r}, \frac{1}{r} \right]} (\gamma; x_1, x_2)) \right)
 \end{aligned}$$

$$\begin{aligned}
 & + \zeta''_{x_2 x_2}(x_1, x_2) \frac{D_{r, \tau, \rho_2}^{\left[\frac{1}{r}\right]}((v - x_2)^2; x_1, x_2)}{2} (\gamma(x_1, x_2) - D_{r, \tau, \rho_2}^{\left[\frac{1}{r}, \frac{1}{r}\right]}(\gamma; x_1, x_2)) \\
 & + \zeta''_{x_1 x_2}(x_1, x_2) D_{r, \tau, \rho_1}^{\left[\frac{1}{r}\right]}((u - x_1); x_1, x_2) D_{r, \tau, \rho_2}^{\left[\frac{1}{r}\right]}((v - x_2); x_1, x_2) \\
 & \left. \left((\gamma(x_1, x_2) - D_{r, \tau, \rho_2}^{\left[\frac{1}{r}, \frac{1}{r}\right]}(\gamma; x_1, x_2)) + \gamma'_{x_1}(x_1, x_2) \zeta'_{x_1}(x_1, x_2) D_{r, \tau, \rho_1}^{\left[\frac{1}{r}\right]}((u - x_1)^2; x_1, x_2) + \gamma'_{x_1}(x_1, x_2) \zeta'_{x_2}(x_1, x_2) \right. \right. \\
 & \left. \left. D_{r, \tau, \rho_1}^{\left[\frac{1}{r}\right]}((u - x_1); x_1, x_2) D_{r, \tau, \rho_2}^{\left[\frac{1}{r}\right]}((v - x_2); x_1, x_2) + \gamma'_{x_2}(x_1, x_2) \zeta'_{x_2}(x_1, x_2) D_{r, \tau, \rho_1}^{\left[\frac{1}{r}\right]}((v - x_2)^2; x_1, x_2) \right) \right),
 \end{aligned}$$

from Theorem (2.1), for each $\gamma \in C^2(I^2)$ it follows that

$$\begin{aligned}
 & \lim_{r \rightarrow \infty} r \left\{ D_{r, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r}, \frac{1}{r}\right]}(\gamma; x_1, x_2) - \gamma(x_1, x_2) - \gamma'_{x_1}(x_1, x_2) D_{r, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r}, \frac{1}{r}\right]}((u - x_1); x_1, x_2) \right. \\
 & + \gamma'_{x_2}(x_1, x_2) D_{r, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r}, \frac{1}{r}\right]}((v - x_2); x_1, x_2) - \frac{1}{2} (\gamma''_{x_1 x_1}(x_1, x_2) D_{r, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r}, \frac{1}{r}\right]}((u - x_1)^2; x_1, x_2) \\
 & \left. + \gamma''_{x_2 x_2}(x_1, x_2) D_{r, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r}, \frac{1}{r}\right]}((v - x_2)^2; x_1, x_2) - 2\gamma''_{x_1 x_2}(x_1, x_2) D_{r, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r}, \frac{1}{r}\right]}((u - x_1); x_1, x_2) D_{r, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r}, \frac{1}{r}\right]}((v - x_2); x_1, x_2) \right\} \\
 & = \lim_{r \rightarrow \infty} r \left\{ D_{r, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r}, \frac{1}{r}\right]}(\beta(u, v; x_1, x_2) \sqrt{(u - x_1)^4 + (v - x_2)^4}; x_1, x_2) \right\} = 0,
 \end{aligned}$$

and from Theorem (1.5), $D_{r, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r}, \frac{1}{r}\right]}(\gamma; x_1, x_2) \rightarrow \gamma(x_1, x_2)$, as $r \rightarrow \infty$, uniformly in $(x_1, x_2) \in I^2$, hence applying Lemma 1.3, we get the required result. \square

4. Degree of approximation

Next, we shall prove a qualitative result by using the Lipschitz class. For $0 < \alpha_1, \alpha_2 \leq 1$, $Lip_k^{(\alpha_1, \alpha_2)}$ is the Lipschitz class for the bivariate case as follows:

$$|\gamma(t_1, t_2) - \gamma(x_1, x_2)| \leq K |t_1 - x_1|^{\alpha_1} |t_2 - x_2|^{\alpha_2}$$

where K is some positive constant.

Theorem 4.1. Let $\gamma \in Lip_K^{(\alpha_1, \alpha_2)}$. Then, for each $(x_1, x_2) \in I^2$ we have

$$|D_{r_1, r_2, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r_1}, \frac{1}{r_2}\right]}(\gamma; x_1, x_2) - \gamma(x_1, x_2)| \leq K \Phi_{r_1, \rho_1}^{\alpha_1}(x_1) \Psi_{r_2, \rho_2}^{\alpha_2}(x_2)$$

where $\Phi_{r_1, \rho_1}(x_1)$ and $\Psi_{r_2, \rho_2}(x_2)$ are defined as in Lemma 1.4.

Proof. Using the operator $D_{r_1, r_2, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r_1}, \frac{1}{r_2}\right]}(\cdot; x_1, x_2)$, we may write

$$\begin{aligned} |D_{r_1, r_2, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r_1}, \frac{1}{r_2}\right]}(\gamma; x_1, x_2) - \gamma(x_1, x_2)| &\leq D_{r_1, r_2, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r_1}, \frac{1}{r_2}\right]}(|(\gamma(t_1, t_2) - \gamma(x_1, x_2))|; x_1, x_2) \\ &\leq KD_{r_1, r_2, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r_1}, \frac{1}{r_2}\right]}(|t_1 - x_1|^{\alpha_1} |t_2 - x_2|^{\alpha_2}; x_1, x_2) \\ &= KD_{r_1, \tau, \rho_1}^{\left[\frac{1}{r_1}\right]}(|t_1 - x_1|^{\alpha_1}; x_1, x_2) D_{r_2, \tau, \rho_2}^{\left[\frac{1}{r_2}\right]}(|t_2 - x_2|^{\alpha_2}; x_1, x_2) \end{aligned}$$

By the Hölder’s inequality with $v_1 = \frac{2}{\alpha_1}, u_1 = \frac{2}{2 - \alpha_1}$ and $v_2 = \frac{2}{\alpha_2}, u_2 = \frac{2}{2 - \alpha_2}$ and using Lemma 1.4, we get

$$\begin{aligned} |D_{r_1, r_2, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r_1}, \frac{1}{r_2}\right]}(\gamma; x_1, x_2) - \gamma(x_1, x_2)| &\leq K \left(D_{r, \tau, \rho_1}^{\left[\frac{1}{r}\right]}(|t_1 - x_1|^2; x_1, x_2) \right)^{\frac{\alpha_1}{2}} \left(D_{r, \tau, \rho_1}^{\left[\frac{1}{r}\right]}(e_{00}; x_1, x_2) \right)^{\frac{2 - \alpha_1}{2}} \\ &\quad \times \left(D_{r, \tau, \rho_1}^{\left[\frac{1}{r}\right]}(|t_2 - x_2|^2; x_1, x_2) \right)^{\frac{\alpha_2}{2}} \left(D_{r, \tau, \rho_1}^{\left[\frac{1}{r}\right]}(e_{00}; x_1, x_2) \right)^{\frac{2 - \alpha_2}{2}} \\ &\leq \Phi_{r, \rho_1}^{\alpha_1}(x_1) \Psi_{r, \rho_2}^{\alpha_2}(x_2). \end{aligned}$$

□

In these theorem, we determine the convergence rate of $D_{r_1, r_2, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r_1}, \frac{1}{r_2}\right]}(\gamma)$ to γ for $\gamma \in C^1(I^2)$.

Theorem 4.2. Let $\gamma \in C^1(I^2)$ and $(x_1, x_2) \in I^2$. Then, we get

$$|D_{r_1, r_2, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r_1}, \frac{1}{r_2}\right]}(\gamma; x_1, x_2) - \gamma(x_1, x_2)| \leq \|\gamma'_{x_1}\|_{C(I^2)} \Phi_{r_1, \rho_1}(x_1) + \|\gamma'_{x_2}\|_{C(I^2)} \Psi_{r_2, \rho_2}(x_2)$$

where $\Phi_{r_1, \rho_1}(x_1)$ and $\Psi_{r_2, \rho_2}(x_2)$ are defined as in Lemma 1.4.

Proof. For any $(x_1, x_2) \in I^2$, we may write

$$\gamma(t_1, t_2) - \gamma(x_1, x_2) = \int_{x_1}^{t_1} \gamma'_w(w, t_2) dw + \int_{x_2}^{t_2} \gamma'_h(t_1, h) dh \tag{5}$$

operating $D_{r_1, r_2, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r_1}, \frac{1}{r_2}\right]}(\cdot; x_1, x_2)$ on both sides in equation (5), we get

$$|D_{r_1, r_2, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r_1}, \frac{1}{r_2}\right]}(\gamma; x_1, x_2) - \gamma(x_1, x_2)| \leq D_{r_1, r_2, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r_1}, \frac{1}{r_2}\right]} \left(\int_{x_1}^{t_1} \gamma'_w(w, t_2) dw; x_1, x_2 \right) + D_{r_1, r_2, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r_1}, \frac{1}{r_2}\right]} \left(\int_{x_2}^{t_2} \gamma'_h(t_1, h) dh; x_1, x_2 \right)$$

Hence using the inequalities

$\left| \int_{x_1}^{t_1} \gamma'_w(w, t_2) dw \right| \leq \|\gamma'_{x_1}\|_{C(I^2)} |t_1 - x_1|$ and $\left| \int_{x_2}^{t_2} \gamma'_h(t_1, h) dh \right| \leq \|\gamma'_{x_2}\|_{C(I^2)} |t_2 - x_2|$, we have

$$|D_{r_1, r_2, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r_1}, \frac{1}{r_2}\right]}(\gamma; x_1, x_2) - \gamma(x_1, x_2)| \leq \|\gamma'_{x_1}\|_{C(I^2)} D_{r_1, r_2, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r_1}, \frac{1}{r_2}\right]}(|t_1 - x_1|; x_1, x_2) + \|\gamma'_{x_2}\|_{C(I^2)} D_{r_1, r_2, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r_1}, \frac{1}{r_2}\right]}(|t_2 - x_2|; x_2).$$

By Lemma 1.2 and Cauchy-Schwarz inequality, we have

$$\begin{aligned} \left| D_{r_1, r_2, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r_1}, \frac{1}{r_2} \right]}(\gamma, x_1, x_2) - \gamma(x_1, x_2) \right| &\leq \|\gamma'_{x_1}\|_{C(I^2)} \left(D_{r_1, \tau, \rho_1}^{\left[\frac{1}{r_1} \right]}(|t_1 - x_1|^2; x_1, \right) \frac{1}{2} \left(D_{r_1, \tau, \rho_1}^{\left[\frac{1}{r_1} \right]}(e_{00}; x_1, x_2) \right)^{\frac{1}{2}} \\ &\quad + \|\gamma'_{x_2}\|_{C(I^2)} \left(D_{r_2, \tau, \rho_2}^{\left[\frac{1}{r_2} \right]}(|t_2 - x_2|^2; x_1, x_2) \right)^{\frac{1}{2}} \left(D_{r_2, \tau, \rho_2}^{\left[\frac{1}{r_2} \right]}(e_{00}; x_1, x_2) \right)^{\frac{1}{2}} \\ &\leq \|\gamma'_{x_1}\|_{C(I^2)} \Phi_{r_1, \rho_1}(x_1) + \|\gamma'_{x_2}\|_{C(I^2)} \Psi_{r_2, \rho_2}(x_2). \end{aligned}$$

Hence the proof is completed. \square

For $\gamma \in C(I^2)$ and any $\varsigma_1, \varsigma_2 > 0$, for the bivariate case the complete modulus of continuity of γ is defined as

$$\bar{\omega}(\gamma; \varsigma_1, \varsigma_2) = \sup\{|\gamma(t_1, t_2) - \gamma(x_1, x_2)| : (t_1, t_2), (x_1, x_2) \in I^2 \text{ and } |t_1 - x_1| \leq \varsigma_1, |t_2 - x_2| \leq \varsigma_2\}.$$

For $\gamma \in C(I^2)$ and any $\zeta > 0$, the partial moduli of continuity of γ with respect to x_1 and x_2 is given by

$$\bar{\omega}^1(\gamma, \zeta) = \sup\{|\gamma(x_{11}, x_2) - \gamma(x_{12}, x_2)| : y \in I \text{ and } |x_{11} - x_{12}| \leq \zeta\}$$

$$\bar{\omega}^2(\gamma, \zeta) = \sup\{|\gamma(x_1, x_{21}) - \gamma(x_1, x_{22})| : x \in I \text{ and } |x_{21} - x_{22}| \leq \zeta\}.$$

Evidently, these moduli of continuity satisfy the properties of the usual modulus of continuity.

Theorem 4.3. For $\gamma \in C(I^2)$ and $(x_1, x_2) \in I^2$, we have the following inequality :

$$\left| D_{r_1, r_2, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r_1}, \frac{1}{r_2} \right]}(\gamma, x_1, x_2) - \gamma(x_1, x_2) \right| \leq 4\bar{\omega}(\gamma; \Phi_{r_1, \rho_1}(x_1), \Psi_{r_2, \rho_2}(x_2))$$

where $\Phi_{r_1, \rho_1}(x_1)$ and $\Psi_{r_2, \rho_2}(x_2)$ are defined as in Lemma 1.4.

Proof. By using the theorem (4.1) of [26] and Lemma 1.4, for any $\delta_1, \delta_2 > 0$, we get

$$\begin{aligned} \left| D_{r_1, r_2, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r_1}, \frac{1}{r_2} \right]}(\gamma, x_1, x_2) - \gamma(x_1, x_2) \right| &\leq \bar{\omega}(\gamma; \delta_1, \delta_2) \left(1 + \delta_1^{-1} \sqrt{D_{r, \tau, \rho_1}^{\frac{1}{r}}((e_{10} - x_1)^2; x_1, x_2)} \right. \\ &\quad \left. + \delta_2^{-1} \sqrt{D_{r, \tau, \rho_2}^{\frac{1}{r}}((e_{01} - x_2)^2; x_1, x_2)} \right. \\ &\quad \left. + \delta_1^{-1} \delta_2^{-1} \sqrt{D_{r, \tau, \rho_1}^{\frac{1}{r}}((e_{10} - x_1)^2; x_1, x_2)} \sqrt{D_{r, \tau, \rho_2}^{\frac{1}{r}}((e_{01} - x_2)^2; x_1, x_2)} \right). \end{aligned}$$

Taking the value of $\delta_1 = \Phi_{r_1, \rho_1}(x_1)$ and $\delta_2 = \Psi_{r_2, \rho_2}(x_2)$, we get the desired result.

In the next theorem, we estimate the degree of approximation of γ by $D_{r_1, r_2, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r_1}, \frac{1}{r_2} \right]}(\gamma)$ in terms of partial moduli of continuity. \square

Theorem 4.4. For $\gamma \in C(I^2)$ and $(x_1, x_2) \in I^2$, we have the following inequality

$$\left| D_{r_1, r_2, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r_1}, \frac{1}{r_2} \right]}(\gamma; x_1, x_2) - \gamma(x_1, x_2) \right| \leq 2(\omega(\gamma; \Phi_{r_1, \rho_1}(x_1)) + \omega^2(\gamma; \Psi_{r_2, \rho_2}(x_2)))$$

where $\Phi_{r_1, \rho_1}(x_1)$ and $\Psi_{r_2, \rho_2}(x_2)$ are defined as in Lemma 1.4.

Proof. Using Cauchy-Schwarz inequality, Lemma 1.4 and definition of the partial moduli of continuity, we have

$$\begin{aligned}
 \left| D_{r_1, r_2, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r_1}, \frac{1}{r_2} \right]} (\gamma, x_1, x_2) - \gamma(x_1, x_2) \right| &\leq D_{r_1, r_2, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r_1}, \frac{1}{r_2} \right]} (|\gamma(t_1, t_2)| - \gamma(x_1, x_2); x_1, x_2) \\
 &\leq D_{r_1, r_2, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r_1}, \frac{1}{r_2} \right]} (|\gamma(t_1, x_2) - \gamma(x_1, x_2)|; x_1, x_2) + D_{r_1, r_2, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r_1}, \frac{1}{r_2} \right]} (|\gamma(x_1, t_2) - \gamma(x_1, x_2)|; x_1, x_2) \\
 &\leq D_{r_1, r_2, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r_1}, \frac{1}{r_2} \right]} \bar{\omega}^1(\gamma; |t_1 - x_1|); x_1, x_2 + D_{r_1, r_2, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r_1}, \frac{1}{r_2} \right]} (\bar{\omega}^2(\gamma; |t_2 - x_2|); x_1, x_2) \\
 &\leq \bar{\omega}^1(\gamma; \Phi_{r_1, \rho_1}(x_1) \left[1 + \frac{1}{\Phi_{r_1, \rho_1}(x_1)} D_{r_1, \tau, \rho_1}^{\frac{1}{r_1}} (|t_1 - x_1|; x_1, x_2) \right] \\
 &\quad + \bar{\omega}^2(\gamma; \Psi_{r_2, \rho_2}(x_2) \left[1 + \frac{1}{\Phi_{r_2, \rho_2}(x_2)} D_{r_2, \tau, \rho_2}^{\frac{1}{r_2}} (|t_2 - x_2|; x_1, x_2) \right]) \\
 &\leq \bar{\omega}^1(\gamma; \Phi_{r_1, \rho_1}(x_1) \left[1 + \frac{1}{\Phi_{r_1, \rho_1}(x_1)} \left(D_{r_1, \tau, \rho_1}^{\frac{1}{r_1}} ((e_{10} - x_1)^2; x_1, x_2) \right)^{\frac{1}{2}} \right] \\
 &\quad + \bar{\omega}^2(\gamma; \Psi_{r_2, \rho_2}(x_2) \left[1 + \frac{1}{\Phi_{r_2, \rho_2}(x_2)} \left(D_{r_2, \tau, \rho_2}^{\frac{1}{r_2}} ((e_{01} - x_2)^2; x_1, x_2) \right)^{\frac{1}{2}} \right])
 \end{aligned}$$

from which the desired result is directly proof. \square

Let $\gamma^{i,j}(x_1, x_2) = \left(\frac{\partial^{i+j}}{\partial x_1^i \partial x_2^j} \right) \gamma(x_1, x_2)$ $i, j \in \mathbb{N}_0$ such that $0 \leq i + j \leq 2$ and $C^2(I^2)$ be endowed with the norm

$$\|\gamma\|_{C^2(I^2)} = \|\gamma\|_{C(I^2)} + \|\gamma\|_{C(I^2)}^1 + \|\gamma\|_{C(I^2)}^2,$$

where

$$\|\gamma\|_{C(I^2)}^1 = \sup_{(x_1, x_2) \in I^2} \{ |\gamma(x_1, x_2)|, |\gamma^{(1,0)}(x_1, x_2)|, |\gamma^{(0,1)}(x_1, x_2)| \}$$

and

$$\|\gamma\|_{C(I^2)}^2 = \sup \{ |\gamma(x, x_2)|, |\gamma^{(1,0)}(x_1, x_2)|, |\gamma^{(0,1)}(x_1, x_2)|, |\gamma^{(2,0)}(x_1, x_2)|, |\gamma^{(1,1)}(x_1, x_2)|, |\gamma^{(0,2)}(x_1, x_2)| \}.$$

Now, we proceed to determine the approximation order of the sequence $D_{r_1, r_2, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r_1}, \frac{1}{r_2} \right]} (\gamma, \cdot)$ to the function $\gamma \in C(I^2)$ in terms of the Peetre’s K-function given by

$$\kappa(\gamma; \delta) = \inf_{\mu \in C^2(I^2)} \{ \|\gamma - \mu\|_{C(I^2)} + \delta \|\mu\|_{C^2(I^2)}, \delta > 0 \}$$

It is known [27] that the inequality

$$\kappa(\gamma; \delta) \leq M_1 \{ \bar{\omega}_2(\gamma; \sqrt{\delta}) + \min \{ 1, \delta \} \|\gamma\|_{C(I^2)} \}$$

holds for each $\delta > 0$, where the constant M_1 is independent of γ, δ and $\bar{\omega}_2(\gamma; \sqrt{\delta})$ is defined as second order modulus of continuity for the bivariate extension.

Theorem 4.5. For any $(x_1, y) \in I^2$, we have

$$\left| D_{r_1, r_2, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r_1}, \frac{1}{r_2} \right]} (\gamma, x_1, x_2) - \gamma(x_1, x_2) \right| \leq M \left\{ \bar{\omega}_2 \left(\gamma; \sqrt{\xi_{r_1, r_2, \tau, \rho_1, \rho_2}(x_1, x_2)} \right) \right\}$$

$$\begin{aligned}
 & + \min \left\{ 1, \frac{\xi_{r_1, r_2, \tau, \rho_1, \rho_2}(x_1, x_2)}{4} \right\}_{\|\gamma\|_{C(I^2)}} \Bigg\} \\
 & + \bar{\omega} \left(\gamma; \left| \frac{\tau(x_1)(1 - 2x_1)}{(r_1\rho_1 + 2\tau(x_1))} \right|, \left| \frac{\tau(x_2)(1 - 2x_2)}{(r_2\rho_2 + 2\tau(x_2))} \right| \right),
 \end{aligned}$$

where

$$\xi_{r_1, r_2, \tau, \rho_1, \rho_2}(x_1, x_2) = \left\{ \Phi_{r_1, \rho_1}(x_1)\Psi_{r_2, \rho_2}(x_2) + \left| \frac{\tau(x_1)(1 - 2x_1)}{(r_1\rho_1 + 2\tau(x_1))} \right| \times \left| \frac{\tau(x_2)(1 - 2x_2)}{(r_2\rho_2 + 2\tau(x_2))} \right| \right\},$$

$\Phi_{r_1, \rho_1}(x_1)$ and $\Psi_{r_2, \rho_2}(x_2)$ are defined in Lemma 1.4.

Proof. The following auxiliary operator are defined as follows:

$$\begin{aligned}
 D_{r_1, r_2, \tau, \rho_1, \rho_2}^* \left[\frac{1}{r_1}, \frac{1}{r_2} \right] (\gamma; x_1, x_2) &= D_{r_1, r_2, \tau, \rho_1, \rho_2}^* \left[\frac{1}{r_1}, \frac{1}{r_2} \right] (\gamma; x_1, x_2) - \gamma \left(\frac{r_1 x_1 \rho_1 + 2\tau(x_1)x_1 + (1 - 2x_1)\tau(x_1)}{r_1 \rho_1 + 2\tau(x_1)} \right. \\
 &\quad \left. , \frac{r_2 x_2 \rho_2 + 2\tau(x_2)x_2 + (1 - 2x_2)\tau(x_2)}{r_2 \rho_2 + 2\tau(x_2)} \right) + \gamma(x_1, x_2).
 \end{aligned} \tag{6}$$

Then, by Lemma 1.1,

$$\begin{aligned}
 & D_{r_1, r_2, \tau, \rho_1, \rho_2}^* \left[\frac{1}{r_1}, \frac{1}{r_2} \right] (e_{00}; x_1, x_2) = 1 \\
 & D_{r_1, r_2, \tau, \rho_1, \rho_2}^* \left[\frac{1}{r_1}, \frac{1}{r_2} \right] (e_{10}; x_1, x_2) = x_1 \text{ and } D_{r_1, r_2, \tau, \rho_1, \rho_2}^* \left[\frac{1}{r_1}, \frac{1}{r_2} \right] (e_{01}; x_1, x_2) = x_2. \\
 \text{Hence } D_{r_1, r_2, \tau, \rho_1, \rho_2}^* \left[\frac{1}{r_1}, \frac{1}{r_2} \right] ((t_1 - x_1); x_1, x_2) &= 0, \quad D_{r_1, r_2, \tau, \rho_1, \rho_2}^* \left[\frac{1}{r_1}, \frac{1}{r_2} \right] ((t_2 - x_2); x_1, x_2) = 0.
 \end{aligned} \tag{7}$$

Let $\lambda \in C^2(I^2)$ and $(t_1, t_2) \in I^2$ be arbitrary. Then, by Taylor’s formula

$$\begin{aligned}
 \lambda(t_1, t_2) - \lambda(x_1, x_2) &= \lambda(t_1, x_2) + \lambda(x_1, x_2) + \lambda(t_1, t_2) - \tau(t_1, x_2) \\
 &= \frac{\partial \lambda(x_1, x_2)}{\partial x_1} (t_1 - x_1) + \int_{x_1}^{t_1} (t_1 - u) \frac{\partial^2 \lambda(u, x_2)}{\partial u^2} + \frac{\partial \lambda(x_1, x_2)}{\partial x_2} (t_2 - x_2) + \int_{x_2}^{t_2} (t_2 - v) \frac{\partial^2 \lambda(x_1, v)}{\partial v^2} dv \\
 &\quad + \int_{x_2}^{t_2} \int_{x_1}^{t_1} \frac{\partial^2 \lambda(t_1, t_2)}{\partial u \partial v} dudv.
 \end{aligned} \tag{8}$$

Applying the operator $D_{r_1, r_2, \tau, \rho_1, \rho_2}^* \left[\frac{1}{r_1}, \frac{1}{r_2} \right] (e_{00}; x_1, x_2)$ on both sides to the equation (8), from (6) we obtain

$$\begin{aligned}
 & D_{r_1, r_2, \tau, \rho_1, \rho_2}^* \left[\frac{1}{r_1}, \frac{1}{r_2} \right] (\lambda; x_1, x_2) - \lambda(x_1, x_2) \\
 &= D_{r_1, r_2, \tau, \rho_1, \rho_2}^* \left[\frac{1}{r_1}, \frac{1}{r_2} \right] \left(\int_{x_1}^{t_1} (t_1 - u) \frac{\partial^2 \lambda(u, x_2)}{\partial u^2} du; x_1, x_2 \right) \\
 &\quad + D_{r_1, r_2, \tau, \rho_1, \rho_2}^* \left[\frac{1}{r_1}, \frac{1}{r_2} \right] \left(\int_{x_1}^{t_1} (t_2 - v) \frac{\partial^2 \lambda(x_1, v)}{\partial v^2} dv; x_1, x_2 \right)
 \end{aligned}$$

$$\begin{aligned}
 & + D_{r_1, r_2, \tau, \rho_1, \rho_2}^* \left[\frac{1}{r_1}, \frac{1}{r_2} \right] \left(\int_{x_2}^{t_2} \int_{x_1}^{t_1} (t_1 - u) \frac{\partial^2 \lambda(u, x_2)}{\partial u \partial v} dudv; x_1, x_2 \right) \\
 = & D_{r_1, r_2, \tau, \rho_1, \rho_2} \left[\frac{1}{r_1}, \frac{1}{r_2} \right] \left(\int_{x_1}^{t_1} (t_1 - u) \frac{\partial^2 \lambda(u, x_2)}{\partial u^2} du; x_1, x_2 \right) \\
 & - \int_{x_1} \frac{r_1 x_1 \rho_1 + 2\tau(x_1)x_1 + (1 - 2x_1)\tau(x_1)}{r_1 \rho_1 + 2\tau(x_1)} \left(\frac{r_1 x_1 \rho_1 + 2\tau(x_1)x_1 + (1 - 2x_1)\tau(x_1)}{r_1 \rho_1 + 2\tau(x_1)} - u \right) \frac{\partial^2 \lambda(u, x_2)}{\partial u^2} du \\
 & + D_{r_1, r_2, \tau, \rho_1, \rho_2} \left[\frac{1}{r_1}, \frac{1}{r_2} \right] \left(\int_{x_2}^{t_2} (t_2 - v) \frac{\partial^2 \lambda(x_1, v)}{\partial v^2} dv; x_1, x_2 \right) \\
 & - \int_{x_2} \frac{r_2 x_2 \rho_2 + 2\tau(x_2)x_2 + (1 - 2x_2)\tau(x_2)}{r_2 \rho_2 + 2\tau(x_2)} \left(\frac{r_2 x_2 \rho_2 + 2\tau(x_2)x_2 + (1 - 2x_2)\tau(x_2)}{r_2 \rho_2 + 2\tau(x_2)} - v \right) \frac{\partial^2 \lambda(x_1, v)}{\partial v^2} dv \\
 & + D_{r_1, r_2, \tau, \rho_1, \rho_2} \left[\frac{1}{r_1}, \frac{1}{r_2} \right] \left(\int_{x_2}^{t_2} \int_{x_1}^{t_1} \frac{\partial^2 \lambda(u, x_2)}{\partial u \partial v} dudv; x_1, x_2 \right) \\
 & - \int_{x_1} \frac{r_1 x_1 \rho_1 + 2\tau(x_1)x_1 + (1 - 2x_1)\tau(x_1)}{r_1 \rho_1 + 2\tau(x_1)} \int_{x_2} \frac{r_2 x_2 \rho_2 + 2\tau(x_2)x_2 + (1 - 2x_2)\tau(x_2)}{r_2 \rho_2 + 2\tau(x_2)} \frac{\partial^2 \lambda(t_1, t_2)}{\partial u \partial v} dudv.
 \end{aligned}$$

Hence Cauchy-Schwarz theorem and by Lemma 1.2 and, we have

$$\begin{aligned}
 & |D_{r_1, r_2, \tau, \rho_1, \rho_2}^* \left[\frac{1}{r_1}, \frac{1}{r_2} \right] (\lambda; x_1, x_2) - \lambda(x_1, x_2)| \\
 \leq & D_{r_1, r_2, \tau, \rho_1, \rho_2} \left[\frac{1}{r_1}, \frac{1}{r_2} \right] \left(\left| \int_{x_1}^{t_1} |(t_1 - u)| \left| \frac{\partial^2 \lambda(u, x_2)}{\partial u^2} \right| du \right|; x_1, x_2 \right) \\
 & + \left| \int_{x_1} \frac{r_1 x_1 \rho_1 + 2\tau(x_1)x_1 + (1 - 2x_1)\tau(x_1)}{r_1 \rho_1 + 2\tau(x_1)} \right| \\
 & \left| \left(\frac{r_1 x_1 \rho_1 + 2\tau(x_1)x_1 + (1 - 2x_1)\tau(x_1)}{r_1 \rho_1 + 2\tau(x_1)} - u \right) \left| \frac{\partial^2 \lambda(u, x_2)}{\partial u^2} \right| du \right| \\
 & + D_{r_1, r_2, \tau, \rho_1, \rho_2} \left[\frac{1}{r_1}, \frac{1}{r_2} \right] \left(\left| \int_{x_2}^{t_2} |(t_2 - v)| \left| \frac{\partial^2 \lambda(x_1, v)}{\partial v^2} \right| dv \right|; x_1, x_2 \right) \\
 & + \left| \int_{x_2} \frac{r_2 x_2 \rho_2 + 2\tau(x_2)x_2 + (1 - 2x_2)\tau(x_2)}{r_2 \rho_2 + 2\tau(x_2)} \right| \left| \left(\frac{r_2 x_2 \rho_2 + 2\tau(x_2)x_2 + (1 - 2x_2)\tau(x_2)}{r_2 \rho_2 + 2\tau(x_2)} - v \right) \left| \frac{\partial^2 \lambda(x_1, v)}{\partial v^2} \right| dv \right| \\
 & + D_{r_1, r_2, \tau, \rho_1, \rho_2} \left[\frac{1}{r_1}, \frac{1}{r_2} \right] \left(\left| \int_{x_2}^{t_2} \int_{x_1}^{t_1} \left| \frac{\partial^2 \lambda(u, x_2)}{\partial u \partial v} \right| dudv \right|; x_1, x_2 \right) \\
 & + \left| \int_{x_1} \frac{r_1 x_1 \rho_1 + 2\tau(x_1)x_1 + (1 - 2x_1)\tau(x_1)}{r_1 \rho_1 + 2\tau(x_1)} \int_{x_2} \frac{r_2 x_2 \rho_2 + 2\tau(x_2)x_2 + (1 - 2x_2)\tau(x_2)}{r_2 \rho_2 + 2\tau(x_2)} \left| \frac{\partial^2 \lambda(t_1, t_2)}{\partial u \partial v} \right| dudv \right|
 \end{aligned}$$

$$\begin{aligned}
 &\leq \left\{ x_1 D_{r_1, \tau, \rho_1}^{\left[\frac{1}{r_1} \right]} \left((t_1 - x_1)^2; x_1, x_2 \right) + \left(\frac{r_1 x_1 \rho_1 + 2\tau(x_1)x_1 + (1 - 2x_1)\tau(x_1)}{r_1 \rho_1 + 2\tau(x_1)} - x_1 \right)^2 \right\} \\
 &\quad \|\lambda\|_{C^2(I^2)} + \left\{ x_2 D_{r_2, \tau, \rho_2}^{\left[\frac{1}{r_2} \right]} \left((t_2 - x_2)^2; x_1, x_2 \right) + \left(\frac{r_2 x_2 \rho_2 + 2\tau(x_2)x_2 + (1 - 2x_2)\tau(x_2)}{r_2 \rho_2 + 2\tau(x_2)} - x_2 \right)^2 \right\} \|\lambda\|_{C^2(I^2)} \\
 &\quad + \left\{ x_1 D_{r_1, \tau, \rho_1}^{\left[\frac{1}{r_1} \right]} (|t_1 - x_1|; x_1, x_2) x_2 D_{r_2, \tau, \rho_2}^{\left[\frac{1}{r_2} \right]} (|t_2 - x_2|; x_1, x_2) \right. \\
 &\quad \left. + \left| \frac{r_1 x_1 \rho_1 + 2\tau(x_1)x_1 + (1 - 2x_1)\tau(x_1)}{r_1 \rho_1 + 2\tau(x_1)} \right| \left| \frac{r_2 x_2 \rho_2 + 2\tau(x_2)x_2 + (1 - 2x_2)\tau(x_2)}{r_2 \rho_2 + 2\tau(x_2)} \right| \right. \\
 &\quad \left. \|\tau\|_{C^2(I^2)} \right\} \\
 &= \xi_{r_1, r_2, \tau, \rho_1, \rho_2}(x_1, x_2) \|\lambda\|_{C^2(I^2)}.
 \end{aligned}$$

Also, from (7) and Lemma 1.1,

$$\begin{aligned}
 |D_{r_1, r_2, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r_1}, \frac{1}{r_2} \right]} (\gamma; x_1, x_2)| &\leq |D_{r_1, r_2, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r_1}, \frac{1}{r_2} \right]} (\gamma; x_1, x_2)| \\
 &\quad + \left| \gamma \left(\frac{r_1 x_1 \rho_1 + 2\tau(x_1)x_1 + (1 - 2x_1)\tau(x_1)}{r_1 \rho_1 + 2\tau(x_1)}, \frac{r_2 x_2 \rho_2 + 2\tau(x_2)x_2 + (1 - 2x_2)\tau(x_2)}{r_2 \rho_2 + 2\tau(x_2)} \right) \right| \\
 &\quad + |\gamma(x_1, x_2)| \\
 &\leq 3\|\gamma\|_{C(I^2)}.
 \end{aligned}$$

Therefore, for $\gamma \in C(I^2)$ and $\lambda \in C^2(I^2)$, using eqn (7), we get

$$\begin{aligned}
 &|D_{r_1, r_2, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r_1}, \frac{1}{r_2} \right]} (\lambda; x_1, x_2) - \lambda(x_1, x_2)| \\
 &\leq |D_{r_1, r_2, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r_1}, \frac{1}{r_2} \right]} (\lambda; x_1, x_2) - \lambda(x_1, x_2)| + \left| \gamma \left(\frac{r_1 x_1 \rho_1 + 2\tau(x_1)x_1 + (1 - 2x_1)\tau(x_1)}{r_1 \rho_1 + 2\tau(x_1)}, \right. \right. \\
 &\quad \left. \left. \frac{r_2 x_2 \rho_2 + 2\tau(x_2)x_2 + (1 - 2x_2)\tau(x_2)}{r_2 \rho_2 + 2\tau(x_2)} \right) - \gamma(x_1, x_2) \right| \\
 &\leq |D_{r_1, r_2, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r_1}, \frac{1}{r_2} \right]} (\gamma - \lambda; x_1, x_2)| + |D_{r_1, r_2, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r_1}, \frac{1}{r_2} \right]} (\lambda; x_1, x_2) - \lambda(x_1, x_2)| \\
 &\quad + |\lambda(x_1, x_2) - \gamma(x_1, x_2)| + \left| \gamma \left(\frac{r_1 x_1 \rho_1 + 2\tau(x_1)x_1 + (1 - 2x_1)\tau(x_1)}{r_1 \rho_1 + 2\tau(x_1)}, \right. \right. \\
 &\quad \left. \left. \frac{r_2 x_2 \rho_2 + 2\tau(x_2)x_2 + (1 - 2x_2)\tau(x_2)}{r_2 \rho_2 + 2\tau(x_2)} \right) - \gamma(x_1, x_2) \right| \\
 &\leq 4\|\gamma - \lambda\|_{C(I^2)} + \xi_{r_1, r_2, \tau, \rho_1, \rho_2} \|\lambda\|_{C^2(I^2)} + \bar{\omega} \left(\gamma; \left| \frac{(1 - 2x_1)\tau(x_1)}{r_1 \rho_1 + 2\tau(x_1)} \right|, \left| \frac{(1 - 2x_2)\tau(x_2)}{r_2 \rho_2 + 2\tau(x_2)} \right| \right).
 \end{aligned}$$

On the right-hand side we take infimum for each $\lambda \in C^2(I^2)$ and by equation (7), we have

$$\begin{aligned}
 |D_{r_1, r_2, \tau, \rho_1, \rho_2}^{\left[\frac{1}{r_1}, \frac{1}{r_2} \right]} (\lambda; x_1, x_2) - \lambda(x_1, x_2)| &\leq 4\kappa \left(\gamma; \frac{\xi_{r_1, r_2, \tau, \rho_1, \rho_2}}{4} \right) + \bar{\omega} \left(\gamma; \frac{(1 - 2x_1)\tau(x_1)}{r_1 \rho_1 + 2\tau(x_1)}, \frac{(1 - 2x_2)\tau(x_2)}{r_2 \rho_2 + 2\tau(x_2)} \right) \\
 &\leq M \left\{ \bar{\omega}_2 \left(\gamma; \frac{\sqrt{\xi_{r_1, r_2, \tau, \rho_1, \rho_2}}}{2} \right) + \min \left\{ 1, \frac{\xi_{r_1, r_2, \tau, \rho_1, \rho_2}}{4} \right\} \|\gamma\|_{C(I^2)} \right\}
 \end{aligned}$$

$$+\bar{\omega}\left(\gamma; \frac{(1-2x_1)\tau(x_1)}{r_1\rho_1+2\tau(x_1)}, \frac{(1-2x_2)\tau(x_2)}{r_2\rho_2+2\tau(x_2)}\right).$$

This completes the proof. \square

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