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Pointwise bi-slant submanifolds of a Kenmotsu manifold

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Abstract. In the present paper, point wise bi-slant submanifolds are studied. Some geometrically important results are obtained in the second section and these findings are used in the third section to investigate bi-slant warped product submanifolds of a Kenmotsu manifold. In the last section of the paper, an inequality for the squared norm of the second fundamental form of a sequential warped product submanifold of a Kenmotsu manifold is established.

Introduction

Pointwise slant distributions on submanifolds of an almost Hermitian manifolds were defined as a generalized version of slant distribution by B.Y.Chen and Garay [3] This further lead to the study of pointwise semi-slant and hemi-slant submanifolds (cf. [14]). The study was later extended to contact settings and pointwise semi-slant, pseudo-slant and more generally bi-slant submanifolds in various contact settings have also been studied (cf. [5], [8], [12] etc). With regards to Kenmotsu manifolds, it is known that these manifolds are themselves locally warped product of the type $I \times_f N$ where I is an interval and N, a Kaehler manifold, with warping function $f(t) = c\exp(t)$ where c is a non-zero constant [6]. With this observation, it is natural to ask whether the warping function of a warped product submanifolds of a Kenmotsu manifold is the restriction of f on the first factor of the warped product submanifold. In the study of semi-slant and pseudo slant warped product submanifold, it has been observed that the warping function of the warped product submanifolds are non-trivial along the integral curve of ξ whereas it vanishes on the orthogonal complement of the curve [9]. We have analyzed point wise bi-slant submanifolds and observed the similar phenomenon in this general case as well. This fact plays an important role in the study of bi-slant submanifolds of Kenmotsu manifolds as warped products.

The paper is divided into four sections. The first section is meant to recollect some basic results and formulas relevant to the study of pointwise bi-slant submanifolds and warped product submanifolds of a Kenmotsu manifold with slant factors.

Section 2 deals with pointwise bi-slant submanifolds of a Kenmotsu manifold. To initiate the study, first some useful formulas for further investigations of the submanifolds are obtained. These formulas helped in establishing integrability conditions for the two distributions on the submanifold. Moreover, necessary and sufficient conditions are worked out for the totally geodesicness and totally umbilicalness of the involutive distributions on the submanifold. This is relevant in view of the fact that the first factor of a warped product

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manifold M is totally geodesic whereas the second factor is spherical in M.

Section 3, is mainly devoted to obtain a characterization under which a pointwise bi-slant submanifold of a Kenmotsu manifold is a warped product submanifold. Preceeding the Theorem, several formulas are obtained and a few Lemmas are proved which paved way to establish the characterization Theorem.

Finally, in section 4, Sequential warped product submanifolds of a Kenmotsu manifolds are studied and an inequality for the squared norm of the second fundamental form of these submanifolds is obtained. Equality case is also discussed.

1. Preliminaries

Let $\bar{M}$ be a $(2m+1)$-dimensional almost contact manifold endowed with almost contact structures denoted respectively by the standard symbols ϕ, ξ and η,with ϕ being a 1:1-tensor field, ξ, a structure vector field and η its dual 1-form, connected with each other as:

$$\phi^2 = -I + \eta \otimes \xi \qquad \eta(\xi) = 1 \qquad \phi\xi = 0 \qquad \eta o \phi = 0. \tag{1}$$

An almost contact manifold $(\bar{M}, \phi, \xi, \eta)$ can be endowed with a Riemannian metric g compatible with almost contact structures in the sense that

$$g(\phi U, \phi V) = g(U, V) - \eta(U)\eta(V). \tag{2}$$

From (1) and (2), it follows that

$$g(\phi U, V) + g(U, \phi V) = 0 \tag{3}$$

for any $U, V \in \Gamma T(\bar{M})$, where $\Gamma T\bar{M}$ denotes the set of locally defined sections of the tangent bundle $T\bar{M}$ of $\bar{M}$. The Riemannian manifold $(\bar{M}, g)$ is called an almost contact metric manifold.

An almost contact metric manifold $\bar{M}$ is a Kenmotsu manifold [6], if the Levi-Civita connection $\bar{\nabla}$ on $\bar{M}$ satisfies

$$(\bar{\nabla}_U \phi)V = g(\phi U, V)\xi - \eta(V)\phi U, \tag{4}$$

where the covariant derivative of ϕ is defined by the formula:

$$(\bar{\nabla}_U \phi)V = \bar{\nabla}_U \phi V - \phi \bar{\nabla}_U V. \tag{5}$$

From (1), (4) and (5), it follows that

$$\bar{\nabla}_U \xi = U - \eta(U)\xi. \tag{6}$$

Let M be an n-dimensional immersed submanifold of an almost contact metric manifold $\bar{M}$. From now on, we will be denoting by U, V etc., the vector fields tangential to M and by E, vector fields normal to M. If ∇ and $\nabla^{\perp}$ denote the induced connections on the tangent and normal bundle of M respectively then Gauss and Weingarten formulas are given by

$$\bar{\nabla}_U V = \nabla_U V + h(U, V) \tag{7}$$

$$\bar{\nabla}_U E = -A_E U + \nabla^{\perp}{}_U E \tag{8}$$

where h, A_E are the second fundamental form and shape operator associated with E respectively and they are related as $g(h(U, V), E) = g(A_E U, V)$, where g denotes the Riemannian metric on $\bar{M}$ as well as the metric induced on M.

A submanifold M of $\bar{M}$ is said to be a totally geodesic submanifold if $h \equiv 0$. On the other hand, M is a totally umbilical submanifold, if the second fundamental form h is along the mean curvature vector H. More precisely, $h(U,V) = g(U,V)H$, where $H = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i)$ and $\{e_i\}_{1\leq i\leq n}$ is an orthonormal frame of vector fields on M and $\{e_i\}_{n+1\leq i\leq 2m+1-n}$ is a local orthonormal frame of normal vector fields. The squared norm of second fundamental form is defined as:

$$||h||^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)) \tag{9}$$

As,

$$T_p(\bar{M}) = T_p(M) \oplus T_p^{\perp}(M), \;\; p \in M,$$

we can decompose ϕU for any $U \in T_p(M)$ into tangential and normal parts associated with the above direct sum decomposition. More specifically, we write

$$\phi U = TU + FU \tag{10}$$

where T is an endomorphism on $T_p(M)$ and F is a normal valued linear map on $T_p(M)$. The tensor fields on M determined by the endomorphism T and the normal valued 1-form F are denoted by the same symbols. Similarly, for $E \in \Gamma T^{\perp}(M)$, we decompose ϕE into tangential and normal parts as:

$$\phi E = tE + fE, \tag{11}$$

with t and f being the tangential and normal valued 1-1 tensor fields on the normal bundle $T^{\perp}(M)$ of M. The covariant derivative of the tensor fields $T,\ F$ are defined as

$$(\bar{\nabla}_U T)V = \nabla_U TV - T\nabla_U V \tag{12}$$

$$(\bar{\nabla}_U F)V = \nabla_U^{\perp} FV - F\nabla_U V. \tag{13}$$

Similarly, one may obtain the formulas for the covariant derivatives of t and f.
On a submanifold of Kenmotsu manifold, from equation (4), (5), (7),(8), (10), (11), (12) and (13), we obtain

$$(\bar{\nabla}_U T)V = A_{FV}U + th(U,V) - g(U,TV)\xi - \eta(V)TU \tag{14}$$

and

$$(\bar{\nabla}_U F)V = fh(U,V) - h(U,TV) - \eta(V)FU. \tag{15}$$

Moreover, from equations (6) and (7) we get

$$\nabla_U \xi = U - \eta(U)\xi \tag{16}$$

and

$$h(U,\xi) = 0. \tag{17}$$

Let D be a differentiable distribution on the submanifold M of $\bar{M}$. Then for each $p \in M$,

$$T_p(M) =< \xi_p > \oplus D_p \oplus D'_p,$$

where D' is the distribution complementary to $D\oplus < \xi >$. The distribution D is called a pointwise slant distribution on M if for each $p \in M$ and non-zero tangent vector $U \in D_p$, the angle $\theta = \theta(U)$ between ϕU and D_p is constant. In this case, we call θ, a slant function of the distribution D. If the slant function is independent of choice of $p \in M$, then the distribution D is called a slant distribution with constant function

θ. In particular, if for each $p \in M$, $T_p(M) = D \oplus <\xi>$, with D being a point wise slant distribution on M, then M is called a point wise slant submanifold of $\bar{M}$. In other words, a submanifold M of an almost contact metric manifold $\bar{M}$ such that ξ is tangential to M is pointwise slant if and only if the distribution $D = TM \cap <\xi>^{\perp}$ is a pointwise slant distribution with slant function $\theta \in C^{\infty}(M)$. In particular, if $\theta = 0$ on a point wise slant submanifold M, then M is an invariant submanifold whereas if $\theta = \pi/2$ then M is ϕ-anti invariant submanifold.

Park [12] obtained following charachterization for pointwise slant submanifold.

Lemma 1.1. *Let M be a submanifold of almost contact metric manifold $\bar{M}$. Then M is pointwise slant submanifold if and only if*

$$T^2 = \cos^2\theta(-I + \eta \otimes \xi) \tag{18}$$

for some function $\theta : M \to \mathbb{R}$.

As an immediate consequence of the above formula, one can easily deduce that

$$g(TU, TV) = \cos^2\theta\{g(U, V) - \eta(U)\eta(V)\}, \tag{19}$$

$$g(FU, FV) = \sin^2\theta(g(U, V) - \eta(U)\eta(V)), \tag{20}$$

$$tFU = \sin^2\theta(-U + \eta(U)\xi), \tag{21}$$

$$fFU = -FTU \tag{22}$$

for any $U,\ V \in \Gamma TM$.

If TM admits two orthogonal complementry pointwise slant distribution $D^{\theta_1},\ D^{\theta_2}$ (with slant functions $\theta_1,\ \theta_2$ respectively), then M is said to be a pointwise bi-slant submanifold. Thus, the tangent bundle of a pointwise bi-slant submanifold admits the following direct sum decomposition

$$TM = D^{\theta_1} \oplus D^{\theta_2}.$$

In particular, if θ_1 is constant and $\theta_2 = 0$, then M is a semi-slant submanifold. On the other hand if θ_1 is constant and $\theta_2 = \pi/2$, then M is a pseudo-slant submanifold. However, if both the slant functions θ_1, θ_2 are constants then M is called simply a bi-slant submanifold.

2. Pointwise bi-slant submanifold

Througout this section we consider a pointwise bi-slant submanifold M of Kenmotsu manifold with structure vector field ξ tangential to M. In this case, the tangent and normal bundles of M are decomposed as:

$$TM = D^{\theta_1} \oplus D^{\theta_2} \oplus <\xi> \qquad \text{and} \qquad T^{\perp}M = FD^{\theta_1} \oplus FD^{\theta_2} \oplus \mu,$$

where, D^{θ_1} and D^{θ_2} are orthogonal point wise slant distributions on M with slant angles θ_1, θ_2 respectively and μ is ϕ-invariant normal subbundle of M.

Note. Throughout this section, we assume that the slant angles θ_1, θ_2 of the two distributions are non-trivial and different from each other. Further, U_1, V_1 denote vector fields in $D^{\theta_1} \oplus <\xi>$ and U_2, V_2 vector fields in D^{θ_2}.

We first obtain some prepatory formulas:

Lemma 2.1. *Let $\bar{M}$ be a Kenmotsu manifold and M be a pointwise bi-slant submanifold of $\bar{M}$ with slant functions $\theta_1,\ \theta_2$ $(\theta_1 \neq \theta_2)$ such that structure vector field ξ is tangential to M. Then the following formulas hold*

(i) $(\bar{\nabla}_{U_1}\phi)V_1 = g(TU_1, V_1)\xi - \eta(V_1)\phi U_1$

(ii) $(\bar{\nabla}_{U_2}\phi)V_2 = g(TU_2, V_2)\xi$
(iii) $(\bar{\nabla}_{U_2}\phi)U_1 = -\eta(U_1)\phi U_2$
(iv) $(\bar{\nabla}_{U_1}\phi)U_2 = 0$

The proof follows on taking account of formula (4) and the facts that for each $U_i \in D^{\theta_i}$, $TU_i \in D^{\theta_i}$ $(1 \le i \le 2)$ and the structure vector field ξ is orthogonal to D^{θ_2}.

As an immediate consequence of the formulas obtained in Lemma 2.1 and the formulas (19) and (20), we obtain:

Corollary 2.2. *Let $\bar{M}$ be a Kenmotsu manifold and M be a pointwise bi-slant submanifold of $\bar{M}$ with slant functions θ_1, θ_2 $(\theta_1 \neq \theta_2)$ such that structure vector field ξ is tangential to M. Then the following formulas hold*

(i) $(\bar{\nabla}_{U_2}\phi)TV_2 = \cos^2\theta_2\, g(U_2, V_2)\xi$
(ii) $(\bar{\nabla}_{U_1}\phi)TV_1 = \cos^2\theta_1\, g(U_1, V_1)\xi$
(iii) $(\bar{\nabla}_{U_2}\phi)TV_1 = 0 = (\bar{\nabla}_{V_1}\phi)TU_2$
(iv) $(\bar{\nabla}_{U_2}\phi)FV_2 = \sin^2\theta_2\, g(U_2, V_2)\xi$
(v) $(\bar{\nabla}_{U_1}\phi)FV_1 = \sin^2\theta_1 g(U_1, V_1)\xi$
(vi) $(\bar{\nabla}_{U_2}\phi)FU_1 = 0 = (\bar{\nabla}_{U_1}\phi)FU_2$.

Corollary 2.3. *Let M be pointwise bi-slant submanifold of a Kenmotsu manifold such that the structure vector field ξ is tangential to the submanifold M. Then*

(i) $g(\bar{\nabla}_{U_1}TV_1, \phi U_2) = \cos^2\theta_1 g(\nabla_{U_1}V_1, U_2) + g(A_{FTV_1}U_1, U_2)$
(ii) $g(\bar{\nabla}_{U_1}TU_2, \phi V_1) = \cos^2\theta_2 g(\nabla_{U_1}U_2, V_1) + g(A_{FTU_2}U_1, V_1)$
(iii) $g(\bar{\nabla}_{U_2}TV_2, \phi U_1) = \cos^2\theta_2\{g(\nabla_{U_2}V_2, U_1) + \eta(U_1)g(U_2, V_2)\} + g(A_{FTV_2}U_1, U_2)$

Proof. Making use of formulas (3),(5), (8),(10), (18) and Corollary 2.2 we obtain (i), (ii) and (iii). □

Lemma 2.4. *Let M be pointwise bi-slant submanifold of Kenmotsu manifold tangential to the structure vector field ξ. Then*

(i) $g(\bar{\nabla}_{U_1}FV_1, \phi U_2) = \sin^2\theta_2 g(\nabla_{U_1}V_1, U_2) + g(A_{FTU_2}V_1 - A_{FV_1}TU_2 - A_{FU_2}TV_1, U_1)$
(ii) $g(\bar{\nabla}_{U_2}FV_2, \phi U_1) = \sin^2\theta_1\{g(\nabla_{U_2}V_2, U_1) + \eta(U_1)g(U_2, V_2)\} + g(A_{FTU_1}V_2 - A_{FV_2}TU_1 - A_{FU_1}TV_2, U_2)$

Proof. From Corollary 2.2, while using (3) we obtain

$$g(\bar{\nabla}_{U_1}FU_2, \phi V_1) = -g(\bar{\nabla}_{U_1}\phi FU_2, V_1).$$

The above equation on making use of (11), (21) and the fact that D^{θ_1} and D^{θ_2} are mutually orthogonal, reduces to

$$g(\bar{\nabla}_{U_1}FU_2, \phi V_1) = \sin^2\theta_2 g(\nabla_{U_1}U_2, V_1) - g(A_{FTU_2}U_1, V_1). \tag{23}$$

Now, taking account of (10) and (8), we may write

$$\begin{aligned} g(\bar{\nabla}_{U_1}FV_1, \phi U_2) &= g(\bar{\nabla}_{U_1}FV_1, TU_2) + g(\bar{\nabla}_{U_1}FV_1, FU_2)\\ &= -g(A_{FV_1}U_1, TU_2) - g(\bar{\nabla}_{U_1}FU_2, FV_1)\\ &= -g(A_{FV_1}U_1, TU_2) - g(\bar{\nabla}_{U_1}FU_2, \phi V_1 - TV_1)\\ &= -g(A_{FV_1}U_1, TU_2) - g(A_{FU_2}U_1, TV_1) - g(\bar{\nabla}_{U_1}FU_2, \phi V_1). \end{aligned}$$

The above equation, on substitution from (23), yields

$$g(\bar{\nabla}_{U_1}FV_1, \phi U_2) = \sin^2\theta_2 g(\nabla_{U_1}V_1, U_2) - g(A_{FV_1}U_1, TU_2) - g(A_{FU_2}U_1, TV_1) + g(A_{FTU_2}U_1, V_1)$$

This proves part (i) of the Lemma.

To prove part (ii), working on the same lines, we obtain by making use of (3),(8),(11), (21), (22) and Corollary 2.2 that

$$g(\bar{\nabla}_{U_2}FU_1, \phi V_2) = -\sin^2\theta_1 g(\nabla_{U_2}V_2, U_1) + \sin^2\theta_1\eta(U_1)g(U_2, V_2) - g(A_{FTU_1}U_2, V_2). \tag{24}$$

On the other hand, using formulas (8),(11) and orthogonality of $D^{\theta_1}, D^{\theta_2}$, we obtain

$$g(\bar{\nabla}_{U_2}FV_2, \phi U_1) = -g(\bar{\nabla}_{U_2}FU_1, \phi V_2) + g(A_{FU_1}U_2, TV_2) - g(A_{FV_2}U_2, TU_1).$$

On substituting the value of $g(\bar{\nabla}_{U_2}FU_1, \phi V_2)$ from (24) into the above equation, we obtain part (ii). That proves the Lemma completely. □

With regards to the integrability of the two complementry distributions on the submanifold M, we have:

Theorem 2.5. *Let M be a point wise bi-slant submanifold of a Kenmotsu manifold with orthogonal complemetry slant distributions D^{θ_1} and D^{θ_2} ($\theta_1 \neq \theta_2$) and ξ be tangent to M. Then*

(i) *the distribution $D^{\theta_1}\oplus <\xi>$is parallel if and only if*

$$A_{FU_1}U_2 - A_{FU_2}U_1 \in D^{\theta_2}. \tag{25}$$

(ii) *the distribution D^{θ_2} is involutive and its leaves are totally umbilical in M with mean curvature vector along ξ, if and only if*

$$A_{FU_1}U_2 - A_{FU_2}U_1 \in D^{\theta_1}\oplus <\xi> . \tag{26}$$

Proof. We first consider $g(\nabla_{U_1}V_1, U_2)$. Making use of the tensorial equations of Kenmotsu manifolds and Lemma 2.1, we find that

$$g(\nabla_{U_1}V_1, U_2) = g(\phi\bar{\nabla}_{U_1}V_1, \phi U_2)$$

On applying the relevant formulas obtained in Corollary 2.3 and Lemma 2.4, the right hand side of the above equation reduces to

$$\cos^2\theta_1 g(\nabla_{U_1}V_1, U_2) + g(A_{FTV_1}U_1, U_2) + \sin^2\theta_2 g(\nabla_{U_1}V_1, U_2) + g(A_{FTU_2}V_1 - A_{FV_1}TU_2 - A_{FU_2}TV_1, U_1)$$

That gives,

$$(\sin^2\theta_1 - \sin^2\theta_2)g(\nabla_{U_1}V_1, U_2) = g(A_{FTU_2}V_1 - A_{FV_1}TU_2 - A_{FU_2}TV_1 + A_{FTV_1}U_2, U_1) \tag{27}$$

Thus, (i) follows from (27).
Next, consider $g(\nabla_{U_2}V_2, U_1)$. With the similar arguments as made above, we may obtain

$$\begin{aligned} g(\nabla_{U_2}V_2, U_1) =& g(\bar{\nabla}_{U_2}\phi V_2, \phi U_1) - \eta(U_1)g(U_2, V_2) \\ =& g(\bar{\nabla}_{U_2}PV_2, \phi U_1) + g(\bar{\nabla}_{U_2}FV_2, \phi U_1) - \eta(U_1)g(U_2, V_2) \end{aligned}$$

Again using relevant formulas of Corollary 2.3 and Lemma 2.4, the right hand side of the above equation reduces to

$$\cos^2\theta_2\{g(\nabla_{U_2}V_2, U_1) + \eta(U_1)g(U_2, V_2)\} + \sin^2\theta_1\{g(\nabla_{U_2}V_2, U_1)\eta(U_1)g(U_2, V_2)\}$$

$$-\eta(U_1)g(U_2, V_2) + g(A_{FTV_2}U_1 - A_{FU_1}TV_2 + A_{FTU_1}V_2 - A_{FV_2}TU_1, U_2).$$

That gives

$$(\sin^2\theta_2 - \sin^2\theta_1)g(\nabla_{U_2}V_2, U_1) + (\sin^2\theta_2 - \sin^2\theta_1)\eta(U_1)g(U_2, V_2) = g(A_{FTV_2}U_1 - A_{FU_1}TV_2 + A_{FTU_1}V_2 - A_{FV_2}TU_1, U_2).$$

or,

$$(\sin^2\theta_2 - \sin^2\theta_1)g(\nabla_{U_2}V_2 + g(U_2, V_2)\xi, U_1) = g(A_{FTV_2}U_1 - A_{FU_1}TV_2 + A_{FTU_1}V_2 - A_{FV_2}TU_1, U_2). \quad (28)$$

Now, suppose that (26) holds on a pointwise bi-slant submanifold of a Kenmotsu manifold. In this case, formula (28) reduces to

$$(\sin^2\theta_2 - \sin^2\theta_1)g(\nabla_{U_2}V_2 + g(U_2, V_2)\xi, U_1) = 0, \quad (29)$$

from which it follows that

$$g([U_2, V_2], U_1) = 0.$$

That is, the distribution D^{θ_2} is involutive. If h_2 denotes the second fundamental form of the immersion of a leaf N_2 of D^{θ_2} in M, then we have from (29) that

$$h_2(U_2, V_2) = -g(U_2, V_2)\xi.$$

Thus, in this case, N_2 is totally umbilical in M with mean curvature along the structure vector field ξ.

Conversely, if D^{θ_2} is involutive and its leaves are totally umbilical in M with mean curvature vector ξ, then by (28) we deduce (26). □

3. Warped product submanifolds of a Kenmotsu manifold

Bishop and O'Neill [1] , while constructing examples of manifolds of negative sectional curvatures, discovered the notion of warped product manifolds. Later, it was found that the warped product metric is a natural metric in various models of space-time manifolds. Thats how, warped product manifolds found applications in relativity. These manifolds are defined as:

Let (N_1, g_1) and (N_2, g_2) be Riemannian manifolds and λ be a positive differentiable function on N_1. Then the warped product manifold M denoted as $N_1 \times_\lambda N_2$ is the product manifold $N_1 \times N_2$ endowed with the Riemannian metric

$$g = \pi_1^* g_1 + \lambda^2 \pi_2^* g_2.$$

It is easy to observe that warped products are a generalization of Riemannian product of two manifolds.

Twisted products $N_1 \times_\lambda N_2$ are natural generalizations of warped products. In this case, the function λ may depend on both factors. When λ depends only on N_1, then the twisted product reduces to a warped product. Ponge and Reckziegel [13] obtained the following:

Theorem 3.1. *Let g be a Riemannian metric on $M = N_1 \times N_2$ and assume that canonical foliations L_{N_1} and L_{N_2} intersect perpendicularly every where. Then g is the metric tensor of the twisted product $N_1 \times_\lambda N_2$ if and only if L_{N_1} is totally geodesic foliation and L_{N_2} is a totally umbilical foliation.*

Let ∇ be the Levi-Civita connection on M associated with twisted product metric on M, and ∇^1, ∇^2 denote the Levi-Civita connections of the metric tensor g_1, g_2 on N_1 and N_2 respectively. Further, let $\mathcal{L}(N_1)$ and $\mathcal{L}(N_2)$ denote the set of lifts of vector fields on N_1 and N_2 respectively. We use the same notations for a vector field and for its lift. We have [11] .

Proposition 3.2. *If $U_1, V_1 \in \mathcal{L}(N_1)$ and $U_2, V_2 \in \mathcal{L}(N_2)$, then*

$$\nabla_{U_1} V_1 = \nabla^1_{U_1} V_1, \tag{30}$$

$$\nabla_{U_1} U_2 = \nabla_{U_2} U_1 = (U_1 \ln \lambda) U_2, \tag{31}$$

$$\nabla_{U_2} V_2 = \nabla^2_{U_2} V_2 + (U_2 \ln \lambda) V_2 + (V_2 \ln \lambda) U_2 - g(U_2, V_2) \nabla \ln \lambda, \tag{32}$$

where $\nabla \ln \lambda$ is the gradient of $\ln \lambda$ defined as:

$$g(\nabla \lambda, U) = U(\lambda),$$

for any U tangent to M. As a result of the above formula, we have

$$\|\nabla \lambda\|^2 = \sum_{i=1}^{n} (e_i(\lambda))^2$$

It follows from (30) and (32) respectively that N_1 is totally geodesic and N_2 is totally umbilical in M. Moreover, if h_2 denotes the second fundamental form of the immersion of N_2 in M, then from (32), $h_2(U_2, V_2) = -g(U_2, V_2) \nabla \ln \lambda|_{\mathcal{L}(N_1)}$. That is, the mean curvature vector of N_2 in M is $-\nabla \ln \lambda|_{\mathcal{L}(N_1)}$.

We have the following characterization theorem for warped products [4]

Theorem 3.3. *Let (M,g) be a connected Riemannian manifold equipped with orthogonal complementary involutive distributions D_1 and D_2. Further, let the leaves of D_1 be totally geodesic and the leaves of D_2 be extrinsic spheres in M. Then $(M, g) = N_1 \times_\lambda N_2$, where N_1, N_2 denote the leaves of the distributions D_1, D_2 respectively and $\lambda : N_1 \to (0, \infty)$ is a smooth function such that $\nabla \ln \lambda$ is the mean curvature of N_2 in M. Further, if M is simply connected and complete, then (M, g) is a globally warped product manifold.*
Where by extrinsic sphere we mean a totally umbilic submanifold such that the mean curvature vector is parallel in the normal bundle.

Further generalizing the notion of warped product (resp. twisted product) manifolds, B.Unal[16] introduced the notion of doubly warped products (resp. doubly twisted product). In particular, a doubly warped product manifold M of N_1 and N_2, denoted as ${}_{\lambda_2}N_1 \times_{\lambda_1} N_2$ is a Riemannian manifold with Riemannian metric tensor g defined as: $g = \lambda_2^2 g_1 + \lambda_1^2 g_2$ where $\lambda_1 : N_1 \to \mathbb{R}^+$ and $\lambda_2 : N_2 \to \mathbb{R}^+$ are smooth maps.

B.Y.Chen [2] initiated the study of warped products with extrinsic geometric point of view by considering CR submanifolds of Kaehler manifolds as warped products. Later, I.Mihai [10] extended the study to contact settings. Our aim, in this section is to study pointwise bi-slant submanifolds of Kenmotsu manifolds immersed as warped products.

It is known that non trivial doubly warped products in a Kenmotsu manifold are non existent (whether ξ is tangential or normal to the submanifold) [8]. Even non-trivial (single) warped products normal to the structural vector field ξ in a Kenmotsu manifold are non-existent [8] However, non-trivial (single) warped products tangential to structural vector fields ξ do exist in a Kenmotsu manifold. In this case, ξ remains tangential to the first factor of the warped product submanifolds (cf.[9]).

Let (N_1, g_1) and (N_2, g_2) be Riemannian manifolds and $M = N_1 \times_\lambda N_2$ be a warped product manifold isometrically immersed into a Kenmotsu manifold $\bar{M}$. Throughout, we will be denoting by U_1, V_1 etc., vector fields tangential to the first factor N_1 of M and by U_2, V_2 etc., vector fields tangential to N_2. Then on making use of the facts that TN_1 and TN_2 are mutually orthogonal, $TU_1,\ TV_1 \in \Gamma TN_1$; $TU_2,\ TV_2 \in \Gamma TN_2$ and ξ is tangential to N_1, formulae (14) and (15) yield

$$(\bar{\nabla}_{U_1} T) U_2 = A_{FU_2} U_1 + th(U_1, U_2) \tag{33}$$

$$(\bar{\nabla}_{U_2}T)U_1 = A_{FU_1}U_2 + th(U_1, U_2) - \eta(U_1)TU_2 \tag{34}$$

$$(\bar{\nabla}_{U_1}F)U_2 = nh(U_1, U_2) - h(U_1, TU_2) \tag{35}$$

$$(\bar{\nabla}_{U_2}F)U_1 = nh(U_1, U_2) - h(TU_1, U_2) - \eta(U_1)FU_2 \tag{36}$$

By virtue of (31), $(\bar{\nabla}_{U_1}T)U_2 = 0$ and $(\bar{\nabla}_{U_2}T)U_1 = (TU_1 \ln\lambda)U_2 - (U_1 \ln\lambda)TU_2$ and thus, equations (33) and (34) reduce respectively to

$$A_{FU_2}U_1 + th(U_1, U_2) = 0 \tag{37}$$

and

$$(\eta(U_1) - U_1 \ln\lambda)TU_2 = A_{FU_1}U_2 + th(U_1, U_2) - (TU_1 \ln\lambda)U_2. \tag{38}$$

From (37), we have

$$g(h(U_1, V_2), FU_2) = g(h(U_1, U_2), FV_2). \tag{39}$$

Using (37) in (38), we obtain

$$A_{FU_2}U_1 - A_{FU_1}U_2 = (U_1 \ln\lambda - \eta(U_1))TU_2 - (TU_1 \ln\lambda)U_2. \tag{40}$$

Taking product with V_2 in the above equation gives

$$g(h(U_1, V_2), FU_2) - g(h(U_2, V_2), FU_1) = (U_1 \ln\lambda - \eta(U_1))g(TU_2, V_2) - (TU_1 \ln\lambda)g(U_2, V_2).$$

That is, we have

$$g(h(U_1, V_2), FU_2) - g(h(U_2, V_2), FU_1) + (TU_1 \ln\lambda)g(U_2, V_2) = (U_1 \ln\lambda - \eta(U_1))g(TU_2, V_2). \tag{41}$$

All terms in the left hand side of of (41) are symmetric in U_2, V_2 (the first one by virtue of (39) and the others due to symmetry of h and g) whereas the right hand side is skew symmetric in U_2, V_2. Therefore, both sides must seperately be zero. That is,

$$(TU_1 \ln\lambda)g(U_2, V_2) = g(h(U_2, V_2), FU_1) - g(h(U_1, V_2), FU_2) \tag{42}$$

and

$$(U_1 \ln\lambda - \eta(U_1))g(TU_2, V_2) = 0. \tag{43}$$

(43) holds if either N_2 is ϕ-anti invariant or

$$U_1 \ln\lambda = \eta(U_1). \tag{44}$$

When N_2 is not ϕ-anti invariant, then by (44)

$$TU_1 \ln\lambda = 0. \tag{45}$$

The above findings lead to the following

Theorem 3.4. *Let $\bar{M}$ be a Kenmotsu manifold and $M = N_1 \times_\lambda N_2$ be a warped product manifold isometrically immersed in to $\bar{M}$ such that the structure vector field $\xi = \frac{\partial}{\partial t}$ is tangential to N_1. If the second factor N_2 of M is not ϕ-anti invariant submanifold of $\bar{M}$, then λ is constant on N_1, whereas along the integral curve of ξ, $\lambda(t) = e^t$.*

Further,when N_2 is not ϕ-anti invariant, equation (40) reduces to

$$A_{FU_1}U_2 = A_{FU_2}U_1 \tag{46}$$

consequently, we have

$$g(h(U_2,V_2),FU_1) = g(h(U_1,V_2),FU_2)$$

$$g(h(U_2,V_1),FU_1) = g(h(U_1,V_1),FU_2).$$

Case 1. In particular, when N_2 is ϕ-invariant, formula (37) and (46) imply that

$$th(U_1,U_2) = 0 = A_{FU_1}U_2. \tag{47}$$

In this case it follows from(35) and the above equation that

$$h(U_1,\phi U_2) = \phi h(U_1,U_2), \quad h(U_1,U_2), h(U_2,V_2) \in \mu. \tag{48}$$

Case 2. When N_2 is ϕ-anti invariant, from formula (40), we have

$$A_{FU_1}U_2 - A_{FU_2}U_1 = (TU_1 \ln\lambda)U_2. \tag{49}$$

Taking inner product with V_2 in the above equation gives

$$g(h(U_2,V_2),FU_1) - g(h(U_1,V_2),FU_2) = (TU_1 \ln\lambda)g(U_2,V_2). \tag{50}$$

Moreover, using (12), (31) in equation (35) and (36) yields

$$\nabla^{\perp}_{U_1}FU_2 = (U_1 \ln\lambda)FU_2 + nh(U_1,U_2) \tag{51}$$

and

$$\nabla^{\perp}_{U_2}FU_1 = (U_1 \ln\lambda - \eta(U_1))FU_2 + nh(U_1,U_2) - h(TU_1,U_2). \tag{52}$$

Lemma 3.5. *Let $M = N_1 \times_\lambda N_2$ be a warped product submanifold of a Kenmotsu manifold $\bar{M}$ such that structural vector field ξ is tangential to M. Then following is equivalent*

(i) $\eta(U_1) = U_1 \ln\lambda$
(ii) $H = \nabla \ln\lambda = \xi$.

Proof. By Theorem 3.3, N_2 is totally umbilical in M. If H is the mean curvature of N_2 in M, then

$$h_2(U_2,V_2) = g(U_2,V_2)H,$$

where h_2 denotes the second fundamental form of the immersion of N_2 in to M. On the other hand, by (32)

$$h_2(U_2,V_2) = -g(U_2,V_2)\nabla \ln\lambda$$

From the above two equations, we find that

$$H = -\nabla \ln\lambda$$

That means, for any $U_1 \in \Gamma TN_1$,

$$g(H,U_1) = -g(\nabla \ln\lambda, U_1) = -U_1 \ln\lambda$$

that is,

$$g(H,U_1) = -U_1 \ln\lambda \tag{53}$$

(53) proves the assertion completely. □

Remark 3.6. *Let $M = N_1 \times N_\perp$ arbitrary warped product submanifold of Kenmotsu manifold $\bar{M}$ such that, ξ is tangential to N_1, then mean curvature vector H of $N_\perp$ in M is not equal to ξ.*

It follows from Theorem 3.4 and Lemma3.5.

Theorem 2.5, Lemma 3.5 and Theorem 3.3 lead us to the following characterization Theorem

Theorem 3.7. *Let M be an immersed submanifold of a Kenmotsu manifold $\bar{M}$ such that $\xi \in TM$. Assume that M is pointwise bi-slant submanifold with two complementary pointwise slant distribution D^{θ_1} and D^{θ_2}, where θ_1, θ_2 are two slant functions on M such that $\theta_1 \neq \theta_2$, $\theta_2 \neq \pi/2$. Then M is locally a warped product submanifold $N_1 \times_\lambda N_2$ where N_1, N_2 are the leaves of the distribution $D^{\theta_1} \oplus \xi$ and D^{θ_2} if and only if the shape operator of M satisfies*

$$A_{FU_1}U_2 = A_{FU_2}U_1, \tag{54}$$

for $U_1 \in D^{\theta_1} \oplus <\xi>$ and $U_2 \in D^{\theta_2}$. In this case, the warping function λ satisfies $\lambda(t) = \exp(t)$ and $\xi = \frac{\partial}{\partial t}$.

Proof. Assume that $N_1 \times_\lambda N_2$ is a warped product submanifold of $\bar{M}$ such that N_2 is not anti-invariant submanifold. Then the condition (54) directly follows from (46).
Conversely, suppose that (54) holds on a pointwise bi slant submanifold M of a Kenmotsu manifold $\bar{M}$, then by Theorem 2.5, the distributions $D^{\theta_1} \oplus <\xi>$ and D^{θ_2} are involutive and their leaves are respectively totally geodesic and totally umbilical in M. Moreover, as $\nabla_{U_2}\xi = U_2$, we have $nor(\nabla_{U_2}\xi) = 0$. That means the mean curvature H is parallel showing that the leaves of D^{θ_2} are spherical in M. Hence, by Theorem 3.3, M is a warped product manifold with warping function λ satisfying $\lambda(t) = \exp(t)$. $\square$

As an immediate consequence of the above Theorem, we have

Corollary 3.8. *Let M be pointwise semi-slant submanifold of Kenmotsu manifold $\bar{M}$ with $\xi \in TM$. Then for any $U_1 \in D^\theta \oplus <\xi>$ and $X \in D^T$, M is locally warped product submanifold of the type $N_\theta \times_\lambda N_T$, where N_θ is integral pointwise slant submanifold of distribution $D^\theta \oplus <\xi>$ having slant function θ and N_T is invariant submanifold if and only if the shape operator satisfies*

$$A_{FU}X \equiv 0.$$

4. Sequential warped product submanifold

As a next step forward, we consider sequential warped products (i.e., a manifold M of the type $(N_1 \times_\lambda N_2) \times_\mu N_3$). We can view M as a single warped product manifold $N \times_\mu N_3$ with N itself a warped product manifold $N_1 \times_\lambda N_2$ with factors N_1, N_2 as submanifolds of $\bar{M}$, where $\lambda : N_1 \to \mathbb{R}^+$ and $\mu : N_1 \times N_2 \to \mathbb{R}^+$ are C^∞- functions. If μ depends only on N_1, then sequential warped product reduces to bi-warped product.

Let $M = (N_1 \times_\lambda N_2) \times_\mu N_3$ be sequential warped product submanifold of Kenmotsu $\bar{M}$ such that ξ is tangential to N_1. Throughout, we assume that the factors of M are ϕ-invariant, anti invariant and slant submanifolds of $\bar{M}$. It is known that if D^θ is a slant distribution on a submanifold M of a Kenmotsu manifold such that $D^\theta \oplus <\xi>$ is involutive, then the leaves of $D^\theta \oplus <\xi>$ are considered as slant submanifolds of M, denoted as N_θ. Moreover, in the case, when N_3 is non-anti invariant, the warping function μ is constant on N_2, as a result the sequential warped product will reduce to simply a bi-warped product. Thus, for a non-trivial sequential warped product submanifold, we assume that N_3 is $\phi-$ anti invariant submanifold of $\bar{M}$. More specifically, we consider sequential warped product submanifolds of the type $(N_\theta \times_\lambda N_T) \times_\mu N_\perp$, where N_T and $N_\perp$ are ϕ-invariant and anti-invariant submanifolfds of $\bar{M}$.
Further, let the $dim(N_\theta) = 2p$, $dim(N_T) = 2q$, and $dim(N_\perp) = r$.

Since the shape operator and and the second fundamental form of the immersion of M into $\bar{M}$ play important role in the study of extrinsic geometry of the submanifold, we begin the section by looking into properties of the the second fundamental form h of M into $\bar{M}$.

Note. Throughout we will be denoting by X, Y etc, vector fields tangential to N_T and by Z, W etc, vector fields tangential to $N_\perp$, and by U, V etc, vector fields tangential to N_θ.

By (44), we have $U \ln \lambda = \eta(U)$, in particular $TU \ln \lambda = 0$. Taking account of these observation in equation (40), we get

$$A_{FU}X = 0$$

That gives the following relations of second fundamental form

$$g(h(X, Y), FU) = 0, \tag{55}$$

$$g(h(X, U), FV) = 0 \tag{56}$$

and

$$g(h(X, Z), FU) = 0. \tag{57}$$

From (40), we have

$$A_{FU}Z - A_{FZ}U = (TU \ln \mu)Z, \quad A_{FZ}X = -(TX \ln \mu)Z$$

These relations yield the following equations:

$$g(h(X, Y), FZ) = g(h(X, U), FZ) = g(h(Z, X), FU) = 0, \tag{58}$$

$$g(h(X, W), FZ) = -(TX \ln \mu)g(Z, W) \tag{59}$$

$$g(h(Z, V), FU) = g(h(U, V), FZ), \tag{60}$$

$$g(h(Z, W), FU) = g(h(U, W), FZ) + (TU \ln \mu)g(Z, W). \tag{61}$$

Let $\{U_1, U_2, ...U_{2p}, \xi\}, \{X_1, X_2, ..., X_{2q}\}$ and $\{Z_1, Z_2, ...Z_r\}$be orthonormal frames of vector fields on N_θ, N_T and $N_\perp$ respectively.
Throughout this section, we use the following convention for the indices.

$$1 \leq i, j, k \leq 2p; \quad 1 \leq l, m, n \leq 2q; \quad 1 \leq a, b, c \leq r$$

Theorem 4.1. *Let $M = (N_\theta \times_\lambda N_T) \times_\mu N_\perp$ be a sequential warped product submanifold of a Kenmotsu manifold $\bar{M}$, where N_θ and N_T are slant and ϕ-invariant submanifolds of $\bar{M}$ whereas $N_\perp$ is a totally umbilical , ϕ-anti invariant submanifold of $\bar{M}$. Then the squared norm of the second fundamental form satisfies*

$$||h||^2 \geq r^2\{\sin^2 \theta ||H_\theta||^2 + ||H_\perp||^2 - \sin 2\theta ||H_\theta|| ||\nabla \ln \mu_\theta||\}.$$

The equality in the above inequality holds iff N_θ is totally geodesic in $\bar{M}$.

Proof. By definition,

$$
\begin{aligned}
||h||^2 = & \sum_{i,j,k=1}^{2p} g(h(U_i,U_j),FU_k)^2 + \sum_{a=1}^{r}\sum_{i,j=1}^{2p} g(h(U_i,U_j),FZ_a)^2 + \sum_{l=1}^{2q}\sum_{i,j=1}^{2p} g(h(U_i,X_l),FU_j)^2 \\
& + \sum_{a=1}^{r}\sum_{l=1}^{2q}\sum_{i=1}^{2p} g(h(U_i,X_l),FZ_a)^2 + \sum_{a=1}^{r}\sum_{i,j=1}^{2p} g(h(U_i,Z_a),FU_j)^2 + \sum_{a,b=1}^{r}\sum_{i,j=1}^{2p} g(h(U_i,Z_a),FZ_b)^2 \\
& + \sum_{l,m=1}^{2q}\sum_{i,=1}^{2p} g(h(X_l,X_m),FU_i)^2 + \sum_{l,m=1}^{2q}\sum_{a=1}^{r} g(h(X_l,X_m),FZ_a)^2 + \sum_{l=1}^{2q}\sum_{i,j=1}^{2p} g(h(X_l,U_i),FU_j)^2 \\
& + \sum_{a=1}^{r}\sum_{l=1}^{2q}\sum_{i=1}^{2p} g(h(X_l,U_i),FZ_a)^2 + \sum_{a=1}^{r}\sum_{l=1}^{2q}\sum_{i=1}^{2p} g(h(X_l,Z_a),FU_i)^2 + \sum_{a,b=1}^{r}\sum_{l=1}^{2q} g(h(X_l,Z_a),FZ_b)^2 \\
& + \sum_{a,b=1}^{r}\sum_{i=1}^{2p} g(h(Z_a,Z_b),FU_i)^2 + \sum_{a,b,c=1}^{r} g(h(Z_a,Z_b),FZ_c)^2 + \sum_{a=1}^{r}\sum_{i,j=1}^{2p} g(h(Z_a,U_i),FU_j)^2 \\
& + \sum_{a,b=1}^{r}\sum_{i,j=1}^{2p} g(h(Z_a,U_i),FZ_b)^2 + \sum_{l=1}^{2q}\sum_{a=1}^{r}\sum_{i=1}^{2p} g(h(Z_a,X_l),FU_i)^2 + \sum_{l=1}^{2q}\sum_{a,b=1}^{r} g(h(Z_a,X_l),FZ_b)^2
\end{aligned}
$$

On using formulae (55), (56), (58) and the symmetry of h, the above equality reduces to

$$
\begin{aligned}
||h||^2 = & \sum_{i,j,k=1}^{2p} g(h(U_i,U_j),FU_k)^2 + \sum_{a=1}^{r}\sum_{i,j=1}^{2p} g(h(U_i,U_j),FZ_a)^2 \\
& + 2\Big\{ \sum_{a=1}^{r}\sum_{i,j=1}^{2p} g(h(U_i,Z_a),FU_j)^2 + \sum_{a,b=1}^{r}\sum_{i,j=1}^{2p} g(h(U_i,Z_a),FZ_b)^2 + \sum_{a,b=1}^{r}\sum_{l=1}^{2q} g(h(X_l,Z_a),FZ_b)^2 \Big\} \\
& + \sum_{a,b=1}^{r}\sum_{i=1}^{2p} g(h(Z_a,Z_b),FU_i)^2 + \sum_{a,b,c=1}^{r} g(h(Z_a,Z_b),FZ_c)^2
\end{aligned}
$$

As M is a submanifold of a Kenmotsu manifold, $h(U,\xi) = 0$ for any $U \in TM$, and on further ignoring the terms involving $h(U_i,U_j)$ and making use of (60), we may write:

$$
\begin{aligned}
||h||^2 \geq & \, 2\sum_{a,b=1}^{r}\sum_{i,j=1}^{2p} g(h(U_i,Z_a),FZ_b)^2 + 2\sum_{a,b=1}^{r}\sum_{l=1}^{2q} g(h(X_l,Z_a),FZ_b)^2 + \sum_{a,b=1}^{r}\sum_{i=1}^{2p} g(h(Z_a,Z_b),FU_i)^2 \\
& + \sum_{a,b,c=1}^{r} g(h(Z_a,Z_b),FZ_c)^2.
\end{aligned} \tag{62}
$$

Further, if the submanifold $N_\perp$ is totally umbilical in $\bar{M}$ with mean curvature H, then on denoting the components of H in $F(TN_\theta)$ and $F(TN_\perp)$ by H_θ and $H_\perp$ respectively, we have

$$
\sum_{a,b=1}^{r}\sum_{i=1}^{2p} g(h(Z_a,Z_b),FU_i)^2 = r^2 \sin^2\theta ||H_\theta||^2
$$

and

$$\sum_{a,b,c=1}^{r} g(h(Z_a, Z_b), FZ_c)^2 = r^2\|H_\perp\|^2.$$

The other two terms in the inequality (62), on using (59) and (61) take the form:

$$\sum_{a,b=1}^{r}\sum_{i,j=1}^{2p} g(h(U_i, Z_a), FZ_b)^2 = r^2\{\sin\theta\|H_\theta\| - cos\theta\|\nabla \ln \mu_\theta\|\}^2$$

and

$$\sum_{a,b=1}^{r}\sum_{l=1}^{2q} g(h(X_l, Z_a), FZ_b)^2 = r^2 cos^2\theta\|\nabla \ln \mu_\theta\|^2.$$

Substituting from the above equations into the inequality (62), we obtain

$$\|h\|^2 \geq r^2\{\sin^2\theta\|H_\theta\|^2 + \|H_\perp\|^2 - \sin 2\theta\|H_\theta\|\|\nabla \ln \mu_\theta\|\}.$$

□

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