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Nonlinear contractions and Caputo tempered implicit fractional differential equations in b-metric spaces with infinite delay

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Abstract. This paper deals with some existence and uniqueness results for a class of problems for nonlinear Caputo tempered implicit fractional differential equations in b-Metric spaces with initial condition and infinite delay. The results are based on the $\omega - \psi$-Geraghty type contraction, the F-contraction and the fixed point theory. Furthermore, an two illustrations are presented to demonstrate the plausibility of our results.

1. Introduction

In recent years, fractional calculus has shown to be a very useful tool for tackling the complexity structures seen in several disciplines of research. It is concerned with the extension of integer order differentiation and integration of a function to non-integer order, and its theory and application are substantial. We refer the reader to the monographs [1–3, 36, 39] and the papers [24, 25, 27, 31, 33, 37]. Many papers and monographs have lately been published in which the authors studied a wide range of results for various forms of fractional differential equations, inclusions with different types of conditions. One may see the papers [1, 23, 32], and the references therein.

In [12, 13], Czerwik introduced the notion of b-metric. Following these early studies, the existence fixed point for various classes of operators in the context of b-metric spaces has been intensively researched; see [4–6, 11, 16, 17, 21, 30] for more details on the concept of b-metric and contractions.

Wardowski [38] has asserted a novel inequality using auxiliary functions to ensure the existence and uniqueness of a particular mapping in the setting of standard metric space. This inequality is referred to as F-contraction. For more details on the $\omega - \psi$-Geraghty type contraction and the F-contraction, we refer the reader to the recent papers [8, 9, 18, 20, 22].

Tempered fractional calculus can be considered as the extension of fractional calculus. Buschman's earlier work [10] was the first to disclose the definitions of fractional integration with weak singular and

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exponential kernels. See the papers [7, 15, 26, 28, 29, 34, 35] and references therein for more details and results on the tempered fractional calculus.

In [21], the authors considered the following conformable impulsive problem:

$$\begin{cases} \mathcal{T}^{\vartheta}_{\zeta_\jmath}\chi(\zeta) = \aleph\left(\zeta, \chi_\zeta, \mathcal{T}^{\vartheta}_{\jmath}\chi(\zeta)\right), \quad \zeta \in \Omega_\jmath; \jmath = 0, 1, \ldots, \beta, \\ \Delta\chi|_{\zeta=\zeta_\jmath} = \Upsilon_\jmath(\chi_{\zeta_\jmath^-}), \quad \jmath = 1, 2, \ldots, \beta, \\ \chi(\zeta) = \mu(\zeta), \quad \zeta \in (-\infty, \varkappa], \end{cases}$$

where $0 \leq \varkappa = \zeta_0 < \zeta_1 < \cdots < \zeta_\beta < \zeta_{\beta+1} = \bar{\varkappa} < \infty$, $\mathcal{T}^{\vartheta}_{\zeta_\jmath}\chi(\zeta)$ is the conformable fractional derivative of order $0 < \vartheta < 1$, $\aleph : \Omega \times Q \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, $\Omega := [\varkappa, \bar{\varkappa}]$, $\Omega_0 := [\varkappa, \zeta_1]$, $\Omega_\jmath := (\zeta_\jmath, \zeta_{\jmath+1}]; \jmath = 1, 2, \ldots, \beta$, $\mu : (-\infty, \varkappa] \to \mathbb{R}$ and $\Upsilon_\jmath : Q \to \mathbb{R}$ are given continuous functions, and Q is called a phase space.

Taking inspiration of the previous mentioned publications, in this paper, we study the existence and uniqueness of solutions for the implicit problem with nonlinear fractional differential equation involving the Caputo tempered fractional derivative:

$$\left({}^{C}_{0}\mathfrak{D}^{\vartheta,\ell}_{\zeta}\chi\right)(\zeta) = \aleph\left(\zeta, \chi_\zeta, \left({}^{C}_{0}\mathfrak{D}^{\vartheta,\ell}_{\zeta}\chi\right)(\zeta)\right); \ \zeta \in \Omega := [0, \kappa], \tag{1}$$

$$\chi(\zeta) = \mu(\zeta), \ \zeta \in (-\infty, 0], \tag{2}$$

where $0 < \vartheta < 1$, $\ell \geq 0$, ${}^{C}_{0}\mathfrak{D}^{\vartheta,\ell}_{\zeta}$ is the Caputo tempered fractional derivative, $\aleph : \Omega \times Q \times \mathbb{R} \to \mathbb{R}$ and $\mu : (-\infty, 0] \to \mathbb{R}$ are given functions where $\mu(0) = \chi_0$. For any $\zeta \in \Omega$, we define $\chi_\zeta \in Q$ by

$$\chi_\zeta(\sigma) = \chi(\zeta + \sigma); \ \text{for } \sigma \in (-\infty, 0].$$

To the best of our knowledge, there are no existing publications in the literature that address the implicit Caputo tempered fractional problems with infinite delay. The limited number of published works on tempered fractional calculus highlights the need for further exploration and development. Therefore, our objective is to advance the field by exploring various problems with novel conditions that have not been previously studied. Furthermore, our study intends to incorporate different techniques such as the *F*-contraction method, which distinguishes it from prior research such as [21].

The study of implicit differential equations using the Caputo tempered fractional derivative in *b*-metric spaces is initiated in this paper. It is organized as follows: Section 2 introduces some preliminaries, definitions, lemmas and auxiliary results about the tempered fractional derivative. In section 3, we give some existence and uniqueness results for the problem (1)-(2) that are based on the $\omega - \psi$-Geraghty type contraction, *F*-contraction and the fixed point theory. Finally we present an example to show the validity of our results.

2. Preliminaries

First, we give the definitions and the notations that we will use throughout this paper. We denote by $C(\Omega, \mathbb{R})$ the Banach space of all continuous functions from Ω into $\mathbb{R}$ with the following norm

$$\|\aleph\|_\infty = \sup_{\zeta \in \Omega}\{|\aleph(\zeta)|\}.$$

As usual, $AC(\Omega)$ denotes the space of absolutely continuous functions from Ω into $\mathbb{R}$. For any $n \in \mathbb{N}$, we denote by $AC^n(\Omega)$ the space defined by

$$AC^n(\Omega) := \left\{\aleph : \Omega \to \mathbb{R} : \frac{d^n}{d\zeta^n}\aleph(\zeta) \in AC(\Omega)\right\}.$$

Consider the space $X^p_b(\kappa_1,\kappa_2)$, ($b\in\mathbb{R}$, $1\le p\le\infty$) of those complex-valued Lebesgue measurable functions f on $[\kappa_1,\kappa_2]$ for which $\|\aleph\|_{X^p_b}<\infty$, where the norm is defined by:

$$\|\aleph\|_{X^p_b}=\left(\int_{\kappa_1}^{\kappa_2}|\zeta^b\aleph(\zeta)|^p\frac{d\zeta}{\zeta}\right)^{\frac{1}{p}},\ \ (1\le p<\infty,b\in\mathbb{R}).$$

Definition 2.1 (The Riemann-Liouville tempered fractional integral [15, 34, 35]). *Suppose that the real function f is piecewise continuous on $[\kappa_1,\kappa_2]$ and $f\in X^p_b(\kappa_1,\kappa_2)$, $\ell>0$. Then, the Riemann-Liouville tempered fractional integral of order ϑ is defined by*

$$_{\kappa_1}I^{\vartheta,\ell}_{\zeta}\aleph(\zeta)=e^{-\ell\zeta}\ {}_{\kappa_1}I^{\vartheta}_{\zeta}\left(e^{\ell\zeta}\aleph(\zeta)\right)=\frac{1}{\Gamma(\vartheta)}\int_{\kappa_1}^{\zeta}\frac{e^{-\ell(\zeta-\tau)}\aleph(\tau)}{(\zeta-\tau)^{1-\vartheta}}d\tau, \tag{3}$$

where ${}_{\kappa_1}\mathcal{I}^{\vartheta}_{\zeta}$ denotes the Riemann-Liouville fractional integral [19], defined by

$$_{\kappa_1}I^{\vartheta}_{\zeta}\aleph(\zeta)=\frac{1}{\Gamma(\vartheta)}\int_{\kappa_1}^{\zeta}\frac{\aleph(\tau)}{(\zeta-\tau)^{1-\vartheta}}d\tau. \tag{4}$$

Obviously, the tempered fractional integral (3) reduces to the Riemann-Liouville fractional integral (4) if $\ell=0$.

Definition 2.2 (The Riemann-Liouville tempered fractional derivative [15, 34]). *For $n-1<\vartheta<n;\ n\in\mathbb{N}$, $\ell\ge0$. The Riemann-Liouville tempered fractional derivative is defined by*

$$_{\kappa_1}\mathfrak{D}^{\vartheta,\ell}_{\zeta}\aleph(\zeta)=e^{-\ell\zeta}{}_{\kappa_1}\mathfrak{D}^{\vartheta}_{\zeta}\left(e^{\ell\zeta}\aleph(\zeta)\right)=\frac{e^{-\ell\zeta}}{\Gamma(n-\vartheta)}\frac{d^n}{d\zeta^n}\int_{\kappa_1}^{\zeta}\frac{e^{\ell\tau}f(\tau)}{(\zeta-\tau)^{\vartheta-n+1}}d\zeta,$$

where ${}_{\kappa_1}\mathfrak{D}^{\vartheta}_{\zeta}$ denotes the Riemann-Liouville fractional derivative [19], given by

$$_{\kappa_1}\mathfrak{D}^{\vartheta}_{\zeta}\left(e^{\ell\zeta}\aleph(\zeta)\right)=\frac{d^n}{d\zeta^n}\left({}_{\kappa_1}\mathcal{I}^{n-\vartheta}_{\zeta}\left(e^{\ell\zeta}\aleph(\zeta)\right)\right)=\frac{1}{\Gamma(n-\vartheta)}\frac{d^n}{d\zeta^n}\int_{\kappa_1}^{\zeta}\frac{\left(e^{\ell\tau}\aleph(\tau)\right)}{(\zeta-\tau)^{\vartheta-n+1}}d\tau.$$

Definition 2.3 (The Caputo tempered fractional derivative [15, 35]). *For $n-1<\vartheta<n;\ n\in\mathbb{N}^+$, $\ell\ge0$. The Caputo tempered fractional derivative is defined as*

$$^{C}_{\kappa_1}\mathfrak{D}^{\vartheta,\ell}_{\zeta}\aleph(\zeta)=e^{-\ell\zeta}\ {}^{C}_{\kappa_1}\mathfrak{D}^{\vartheta}_{\zeta}\left(e^{\ell\zeta}\aleph(\zeta)\right)=\frac{e^{-\ell\zeta}}{\Gamma(n-\vartheta)}\int_{\kappa_1}^{\zeta}\frac{1}{(\zeta-\tau)^{\vartheta-n+1}}\frac{d^n\left(e^{\ell\tau}\aleph(\tau)\right)}{d\tau^n}d\tau,$$

where ${}^{C}_{\kappa_1}\mathfrak{D}^{\vartheta,\ell}_{\zeta}$ denotes the Caputo fractional derivative [19], given by

$$^{C}_{\kappa_1}\mathfrak{D}^{\vartheta}_{\zeta}\left(e^{\ell\zeta}\aleph(\zeta)\right)=\frac{1}{\Gamma(n-\vartheta)}\int_{\kappa_1}^{\zeta}\frac{1}{(\zeta-\tau)^{\vartheta-n+1}}\frac{d^n\left(e^{\ell\tau}\aleph(\tau)\right)}{d\tau^n}d\tau.$$

Lemma 2.4 ([15]). *For a constant C,*

$$_{\kappa_1}\mathfrak{D}^{\vartheta,\ell}_{\zeta}C=Ce^{-\ell\zeta}{}_{\kappa_1}\mathfrak{D}^{\vartheta}_{\zeta}e^{\ell\zeta},\quad {}^{C}_{\kappa_1}\mathfrak{D}^{\vartheta,\ell}_{\zeta}C=Ce^{-\ell\zeta}\,{}^{C}_{\kappa_1}\mathfrak{D}^{\vartheta}_{\zeta}e^{\ell\zeta}.$$

Obviously, ${}_{\kappa_1}\mathfrak{D}^{\vartheta,\ell}_{\zeta}(C)\neq{}^{C}_{\kappa_1}\mathfrak{D}^{\vartheta,\ell}_{\zeta}(C)$. And, ${}^{C}_{\kappa_1}\mathfrak{D}^{\vartheta,\ell}_{\zeta}(C)$ is no longer equal to zero, being different from ${}^{C}_{\kappa_1}\mathfrak{D}^{\vartheta}_{\zeta}(C)\equiv0$.

Lemma 2.5 ([15, 35]). *Let $f(\zeta)\in AC^n[\kappa_1,\kappa_2]$, $\ell\ge0$ and $n-1<\vartheta<n$. Then the Caputo tempered fractional derivative and the Riemann-Liouville tempered fractional integral have the following composite properties:*

$$_{\kappa_1}I^{\vartheta,\ell}_{\zeta}\left[{}^{C}_{\kappa_1}\mathfrak{D}^{\vartheta,\ell}_{\zeta}f(\zeta)\right]=f(\zeta)-\sum_{\jmath=0}^{n-1}e^{-\ell\zeta}\frac{(\zeta-\kappa_1)^{\jmath}}{\jmath!}\left[\frac{d^{\jmath}\left(e^{\ell\zeta}f(\zeta)\right)}{d\zeta^{\jmath}}\bigg|_{\zeta=\kappa_1}\right],$$

and

$$^{C}_{\kappa_1}\mathfrak{D}^{\vartheta,\ell}_{\zeta}\left[{}_{a}I^{\vartheta,\ell}_{\zeta}f(\zeta)\right]=f(\zeta),\ for\ \vartheta\in(0,1).$$

Lemma 2.6. *Let $\bar{\aleph} \in L^1(\Omega)$ and $0 < \vartheta \leq 1$. Then the initial value problem*

$$\begin{cases} \left({}^{C}_{0}\mathfrak{D}^{\vartheta,\ell}_{\zeta}\chi\right)(\zeta) = \bar{\aleph}(\zeta);\ \zeta \in \Omega := [0, \kappa], \\ \chi(0) = \chi_0, \end{cases} \tag{5}$$

has a unique solution defined by

$$\chi(\zeta) = \chi_0 e^{-\ell\zeta} + \frac{1}{\Gamma(\vartheta)} \int_0^{\zeta} e^{-\ell(\zeta-\tau)}(\zeta - \tau)^{(\vartheta-1)}\bar{\aleph}(\tau)d\tau. \tag{6}$$

Proof. Applying the Riemann-Liouville tempered fractional integral of order ϑ to

$$\left({}^{C}_{0}\mathfrak{D}^{\vartheta,\ell}_{\zeta}\chi\right)(\zeta) = \bar{\aleph}(\zeta),$$

and by employing Lemma 2.5 and if $\zeta \in \Omega$, we obtain

$$\chi(\zeta) - \chi(0)e^{-\ell\zeta} = \frac{1}{\Gamma(\vartheta)} \int_0^{\zeta} e^{-\ell(\zeta-\tau)}(\zeta - \tau)^{(\vartheta-1)}\bar{\aleph}(\tau)d\tau.$$

From the initial conditions, we get

$$\chi(\zeta) = \chi_0 e^{-\ell\zeta} + \frac{1}{\Gamma(\vartheta)} \int_0^{\zeta} e^{-\ell(\zeta-\tau)}(\zeta - \tau)^{(\vartheta-1)}\bar{\aleph}(\tau)d\tau.$$

Conversely, by Lemma 2.4 and Lemma 2.5, we deduce that if χ verifies equation (6), then it satisfied the problem (5). □

As a consequence of Lemma 2.6, we give the following result.

Lemma 2.7. *A function χ is a solution of problem* (1)-(2) *if and only if χ satisfies the following:*

$$\chi(\zeta) = \begin{cases} \chi_0 e^{-\ell\zeta} + \dfrac{1}{\Gamma(\vartheta)} \displaystyle\int_0^{\zeta} e^{-\ell(\zeta-\tau)}(\zeta - \tau)^{(\vartheta-1)}\bar{\aleph}(\tau)d\tau, \quad \zeta \in \Omega, \\ \\ \mu(\zeta), \quad \zeta \in (-\infty, 0], \end{cases} \tag{7}$$

where $\bar{\aleph} \in C(\Omega)$ such that $\bar{\aleph}(\zeta) = \aleph(\zeta, \chi_\zeta, \bar{\aleph}(\zeta))$.

Definition 2.8 ([4]). *Let $\mathfrak{L}$ be a set and $\varepsilon \geq 1$. A distance function $\varpi : \mathfrak{L} \times \mathfrak{L} \to (0, \infty)$ is a b-metric if the following requirements hold for all $\chi_1, \chi_2, \chi_3 \in \mathfrak{L}$:*

(1) $\varpi(\chi_1, \chi_2) = 0$ *if and only if* $\chi_1 = \chi_2$,

(2) $\varpi(\chi_1, \chi_2) = \varpi(\chi_2, \chi_1)$,

(3) $\varpi(\chi_1, \chi_2) \leq \varepsilon[\varpi(\chi_1, \chi_3) + \varpi(\chi_3, \chi_2)]$.

Then, $(\mathfrak{L}, \varpi, \varepsilon)$ is called a b-metric space with parameter ε.

Let Λ be the set of all increasing and continuous function $\psi : (0, \infty) \to (0, \infty)$ satisfying: $\psi(\varepsilon\chi) \leq \varepsilon\psi(\chi) \leq \varepsilon\chi$, for $\varepsilon > 1$ and $\psi(0) = 0$. We denote by Θ the family of all nondecreasing functions $\eta : (0, \infty) \to [0, \frac{1}{\varepsilon^2})$ for some $\varepsilon \geq 1$.

Definition 2.9 ([4]). *Let $(\mathfrak{L}, \varpi, \varepsilon)$ be a b-metric space, $\mathfrak{S} : \mathfrak{L} \to \mathfrak{L}$ is said to be a generalized ω-ψ-Geraghty mapping whenever there exists $\omega : \mathfrak{L} \times \mathfrak{L} \to (0, \infty)$ such that*

$$\omega(\chi_1, \chi_2)\psi(\varepsilon^3 d(\mathfrak{S}(\chi_1), \mathfrak{S}(\chi_2)) \leq \eta(\psi(\varpi(\chi_1, \chi_2))\psi(\varpi(\chi_1, \chi_2))),$$

for $\chi_1, \chi_2 \in \mathfrak{L}$, where $\eta \in \Theta$.

Definition 2.10 ([4]). *Let $\mathfrak{L}$ be a non empty set, $\mathfrak{S} : \mathfrak{L} \to \mathfrak{L}$ and $\omega : \mathfrak{L} \times \mathfrak{L} \to (0, \infty)$ be given mappings. The operator $\mathfrak{S}$ is orbital ω-admissible if for $\chi \in \mathfrak{L}$, we have*

$$\omega(\chi, \mathfrak{S}(\chi)) \geq 1 \Rightarrow \omega(\mathfrak{S}(\chi), \mathfrak{S}^2(\chi)) \geq 1.$$

Definition 2.11 ([8]). *A mapping $\Psi : \mathfrak{L} \to \mathfrak{L}$ is said to be a generalized nonlinear $F-$contraction if there exist the functions $F : (0, \infty) \to \mathbb{R}$ and $\wp : (0, \infty) \to (0, \infty)$ such that for all $\chi, \mathfrak{I} \in \mathfrak{L}$ such that $\Psi\chi \neq \Psi\mathfrak{I}$,*

$$\wp(\varpi(\chi, \mathfrak{I})) + F(\overline{\omega}\varpi(\Psi\chi, \Psi\mathfrak{I})) \leq F(A^{\epsilon\varpi}(\chi, \mathfrak{I})), \tag{8}$$

where $\overline{\omega} > 1$, and

$$A^{\epsilon\varpi}(\chi, \mathfrak{I}) = \max\left\{\varpi(\chi, \mathfrak{I}), \varpi(\chi, \Psi\chi), \varpi(\mathfrak{I}, \Psi\mathfrak{I}), \frac{\beta}{2\epsilon}\left[\varpi(\mathfrak{I}, \Psi\chi) + \varpi(\chi, \Psi\mathfrak{I})\right]\right\}, \beta \in [0, 1].$$

Theorem 2.12 ([4]). *Let $(\mathfrak{L}, \varpi)$ be a complete b-metric space and $\Psi : \mathfrak{L} \to \mathfrak{L}$ be a generalized ω-ψ-Geraghty mapping where*

(a) *Ψ is ω-admissible;*

(b) *there exists $\chi_0 \in \mathfrak{L}$ where $\omega(\chi_0, \Psi(\chi_0)) \geq 1$;*

(c) *If $(\chi_n)_{n\in N} \subset \mathfrak{L}$ with $\chi_n \to \chi$ and $\omega(\chi_n, \chi_{n+1}) \geq 1$, then $\omega(\chi_n, \chi) \geq 1$.*

Then Ψ has a fixed point. Moreover, if

(d) *for all fixed points χ, χ' of Ψ, either*

$$\omega(\chi, \chi') \geq 1 \text{ or } \omega(\chi', \chi) \geq 1,$$

then Ψ has a unique fixed point.

Theorem 2.13 ([8]). *Let $(\mathfrak{L}, \varpi, \epsilon)$ be a complete b-metric space. A generalized nonlinear F-contraction Ψ has a fixed point if the following statements are true:*

(1) *F is strictly increasing, that is, if $a < b$, then $F(a) < F(b)$, for all $a, b \in (0, \infty)$;*

(2) *$\beta < 1$;*

(3) *$\frac{\varepsilon}{\vartheta} < 1$;*

(4) *$\liminf_{\chi \to \zeta^+} \wp(\chi) > 0$, for any $\zeta \geq 0$.*

3. Main Results

In this section, we establish some existence results for problem (1)-(2).

Let $(C(\Omega), \varpi, 2)$ be the complete b-metric space with $\varepsilon = 2$, such that $\varpi : C(\Omega) \times C(\Omega) \to (0, \infty)$, is given by:

$$\varpi(\chi, \mathfrak{I}) = \|(\chi - \mathfrak{I})^2\|_\infty := \sup_{\zeta\in\Omega} |\chi(\zeta) - \mathfrak{I}(\zeta)|^2.$$

Let the space $(Q, \|\cdot\|_Q)$ is a seminormed linear space of functions mapping $(-\infty, 0]$ into $\mathbb{R}$, and verifying the following axioms which were derived from Hale and Kato's originals [14]:

(Ax_1) If $\chi : (-\infty, 0] \to \mathbb{R}$, and $\chi_0 \in \mathcal{Q}$, then there exist constants $\xi_1, \xi_2, \xi_3 > 0$, such that for each $\zeta \in \Omega$; we have:

(*i*) χ_ζ *is in* $\mathcal{Q}$,

(*ii*) $\|\chi_\zeta\|_{\mathcal{Q}} \leq \xi_1 \|\chi_1\|_{\mathcal{Q}} + \xi_2 \sup_{\zeta \in [0,\zeta]} |\chi(\sigma)|$,

(*iii*) $|\chi(\zeta)| \leq \xi_3 \|\chi_\zeta\|_{\mathcal{Q}}$.

(Ax_2) For the function $\chi(\cdot)$ *in* (Ax_1), y_ζ is a $\mathcal{Q}-$ valued continuous function on Ω.

(Ax_3) The space $\mathcal{Q}$ is complete.

Consider the space

$$\Theta = \{\chi : (-\infty, \kappa] \to \mathbb{R},\ \chi|_{(-\infty,0]} \in \mathcal{Q},\ \chi|_\Omega \in C([0,\kappa], \mathbb{R})\}.$$

The hypotheses:

(H_1) There exist continuous functions $\overline{p} : \Omega \to (0, \infty)$ and $\overline{q} : \Omega \to (0, 1)$ such that for each $\chi, \chi_1 \in \mathcal{Q}$, $\mathfrak{I}, \mathfrak{I}_1 \in \mathbb{R}$ and $\zeta \in \Omega$

$$|\aleph(\zeta, \chi, \mathfrak{I}) - \aleph(\zeta, \chi_1, \mathfrak{I}_1)| \leq \overline{p}(\zeta)\|\chi - \chi_1\|_{\mathcal{Q}} + \overline{q}(\zeta)|\mathfrak{I} - \mathfrak{I}_1|$$

with

$$\left\| \frac{1}{\Gamma(\vartheta)} \int_0^\zeta e^{-\ell(\zeta-\tau)} (\zeta - \tau)^{(\vartheta-1)} \frac{\overline{p}(\tau)}{1 - \overline{q}*} d\tau \right\|_\infty^2 \leq \psi(\|(\chi - \chi_1)^2\|_\infty).$$

(H_2) There exist $\psi \in \Lambda$ and $\overline{\ell}_0 \in C(\Omega)$ and a function $\xi : \mathbb{R}^2 \to \mathbb{R}$, such that

$$\xi\left(\overline{\ell}_0(\zeta), \frac{1}{\Gamma(\vartheta)} \int_0^\zeta e^{-\ell(\zeta-\tau)} (\zeta - \tau)^{(\vartheta-1)} \bar{\aleph}(\tau) d\tau\right) \geq 0$$

where $\bar{\aleph} \in C(\Omega)$ such that $\bar{\aleph}(\zeta) = \aleph(\zeta, \overline{\ell}_0(\zeta), \bar{\aleph}(\zeta))$.

(H_3) For each $\zeta \in \Omega$, and $\chi, \mathfrak{I} \in C(\Omega)$, we have that

$$\xi(\chi(\zeta), \mathfrak{I}(\zeta)) \geq 0$$

implies

$$\xi\left(\frac{1}{\Gamma(\vartheta)} \int_0^\zeta e^{-\ell(\zeta-\tau)} (\zeta - \tau)^{(\vartheta-1)} \bar{\aleph}(\tau) d\tau, \frac{1}{\Gamma(\vartheta)} \int_0^\zeta e^{-\ell(\zeta-\tau)} (\zeta - \tau)^{(\vartheta-1)} \bar{\aleph}'(\tau) d\tau\right) \geq 0,$$

where $\bar{\aleph}, \bar{\aleph}' \in C(\Omega)$ such that

$$\bar{\aleph}(\zeta) = \aleph(\zeta, \chi_\zeta, \bar{\aleph}(\zeta))$$

and

$$\bar{\aleph}'(\zeta) = \aleph(\zeta, \mathfrak{I}_\zeta, \bar{\aleph}'(\zeta)).$$

(H_4) If $(\chi_n)_{n\in N} \subset C(\Omega)$ with $\chi_n \to \chi$ and $\xi(\chi_n(\zeta), \chi_{n+1}(\zeta)) \geq 1$, then

$$\xi(\chi_n(\zeta), \chi(\zeta)) \geq 1.$$

(H_5) For all fixed solutions χ, χ' of problem (1)-(2), either

$$\xi(\chi(\zeta), \chi'(\zeta)) \geq 0,$$

or

$$\xi(\chi'(\zeta), \chi(\zeta)) \geq 0.$$

First, we prove the existence and uniqueness results by utilizing the ω-ψ-Geraghty type contraction and the fixed point theory.

Theorem 3.1. *Assume that the hypotheses* (H_1)-(H_4) *hold. Then the problem (1)-(2) has at least one solution defined on* Ω. *Moreover, if* (H_5) *holds, then we get a unique solution.*

Proof. Consider the operator $\Psi:\ \Theta \to \Theta$ defined by:

$$(\Psi\chi)(\zeta) = \begin{cases} \mu(0)e^{-\ell\zeta} + \dfrac{1}{\Gamma(\vartheta)}\displaystyle\int_0^{\zeta} e^{-\ell(\zeta-\tau)}(\zeta-\tau)^{(\vartheta-1)}\bar{\aleph}(\tau)d\tau, & \zeta \in \Omega, \\ \mu(\zeta), & \zeta \in (-\infty, 0], \end{cases} \tag{9}$$

where $\bar{\aleph} \in C(\Omega)$ such that $\bar{\aleph}(\zeta) = \aleph(\zeta, \chi_\zeta, \bar{\aleph}(\zeta))$.
Let $w : (-\infty, \kappa] \to \mathbb{R}$ be a function given by

$$w(\zeta) = \begin{cases} \mu(\zeta); & \zeta \in (-\infty, 0], \\ \mu(0)e^{-\ell\zeta} & \zeta \in \Omega. \end{cases}$$

Then $w_0 = \mu$. For each $z \in C(\Omega)$, with $z(0) = 0$, we denote by $\bar{z}$ the function defined by

$$\bar{z} = \begin{cases} 0, & \zeta \in (-\infty, 0], \\ z(\zeta), & \zeta \in \Omega. \end{cases}$$

If $\chi(\cdot)$ satisfies the integral equation

$$\chi(\zeta) \quad = \mu(0)e^{-\ell\zeta} + \frac{1}{\Gamma(\vartheta)}\int_0^{\zeta} e^{-\ell(\zeta-\tau)}(\zeta-\tau)^{(\vartheta-1)}\bar{\aleph}(\tau)d\tau,$$

we can decompose $\chi(\cdot)$ as $\chi(\zeta) = \bar{z}(\zeta) + w(\zeta)$; for $\zeta \in \Omega$, which implies that $\chi_\zeta = \bar{z}_\zeta + w_\zeta$ for every $\zeta \in \Omega$, and the function $z(\cdot)$ satisfies

$$z(\zeta) = \frac{1}{\Gamma(\vartheta)}\int_0^{\zeta} e^{-\ell(\zeta-\tau)}(\zeta-\tau)^{(\vartheta-1)}\bar{\aleph}(\tau)d\tau,$$

where

$$\bar{\aleph}(\zeta) = \aleph(\zeta, \bar{z}_\zeta + w_\zeta, \bar{\aleph}(\zeta));\ \zeta \in \Omega.$$

Set

$$\mathcal{D}_0 = \{z \in C(\Omega);\ z_0 = 0\},$$

and let $\|\cdot\|_\kappa$ be the norm in $\mathcal{D}_0$ defined by

$$\|z\|_\kappa = \|z_0\|_Q + \sup_{\zeta\in\Omega}|z(\zeta)| = \sup_{\zeta\in\Omega}|z(\zeta)|;\ z \in \mathcal{D}_0,$$

where $\mathcal{D}_0$ is a Banach space with norm $\|\cdot\|_\kappa$. Define the operator $\mathcal{W} : \mathcal{D}_0 \to \mathcal{D}_0$ by

$$(\mathcal{W}z)(\zeta) = \frac{1}{\Gamma(\vartheta)}\int_0^{\zeta} e^{-\ell(\zeta-\tau)}(\zeta-\tau)^{(\vartheta-1)}\bar{\aleph}(\tau)d\tau, \tag{10}$$

where

$$\bar{\aleph}(\zeta) = \aleph(\zeta, \bar{z}_\zeta + w_\zeta, \bar{\aleph}(\zeta));\ \zeta \in \Omega.$$

The function $\omega : C(\Omega) \times C(\Omega) \to (0, \infty)$ is given by:

$$\begin{cases} \omega(z, y) = 1; & if\ \xi(z(\zeta), y(\zeta)) \geq 0,\ \zeta \in \Omega, \\ \omega(z, y) = 0; & else. \end{cases}$$

First, we prove that $\mathcal{W}$ is a generalized ω-ψ-Geraghty operator:
Let $z, y \in \mathcal{D}_0$. Then, for each $\zeta \in \Omega$, we have

$$|(\mathcal{W}z)(\zeta) - (\mathcal{W}y)(\zeta)| \leq \frac{1}{\Gamma(\vartheta)} \int_0^{\zeta} e^{-\ell(\zeta-\tau)}(\zeta-\tau)^{(\vartheta-1)}|\bar{\aleph}(\sigma) - \bar{\aleph}'(\sigma)|d\tau,$$

where $\bar{\aleph},\ \bar{\aleph}' \in C(\Omega)$ such that

$$\bar{\aleph}(\zeta) = \aleph(\zeta, \bar{z}_\zeta + w_\zeta, \bar{\aleph}(\zeta)) \textit{ and } \bar{\aleph}'(\zeta) = \aleph(\zeta, \overline{y}_\zeta + w_\zeta, \bar{\aleph}'(\zeta)).$$

From (H_1) we have

$$\|\bar{\aleph} - \bar{\aleph}'\|_\infty \leq \frac{\overline{p}(\zeta)}{1-\overline{q}*}\|(z-y)^2\|_\infty^{\frac{1}{2}},$$

where $\overline{q}* = \sup_{\zeta\in\Omega}|\overline{q}(\zeta)|$. Next, we have

$$|(\mathcal{W}z)(\zeta) - (\mathcal{W}y)(\zeta)| \leq \frac{1}{\Gamma(\vartheta)} \int_0^{\zeta} e^{-\ell(\zeta-\tau)}(\zeta-\tau)^{(\vartheta-1)}\frac{\overline{p}(\tau)}{1-\overline{q}*}\|(z-y)^2\|_\infty^{\frac{1}{2}}d\tau.$$

Thus,

$$\begin{aligned}\omega(z,y)|(\mathcal{W}z)(\zeta) - (\mathcal{W}y)(\zeta)|^2 &\leq \|(z-y)^2\|_\infty \omega(z,y)\left\|\frac{1}{\Gamma(\vartheta)} \int_0^{\zeta} e^{-\ell(\zeta-\tau)}(\zeta-\tau)^{(\vartheta-1)}\frac{\overline{p}(\tau)}{1-\overline{q}*}d\tau\right\|_\infty^2 \\ &\leq \|(z-y)^2\|_\infty \psi(\|(z-y)^2\|_\infty).\end{aligned}$$

Hence,

$$\omega(z,y)\psi(2^3 d(\mathcal{W}(z), \mathcal{W}(y)) \leq \eta(\psi(\varpi(z,y))\psi(\varpi(z,y)),$$

where $\eta \in \Theta$, $\psi \in \Lambda$, with $\eta(\zeta) = \frac{1}{8}\zeta$, and $\psi(\zeta) = \zeta$. So, $\mathcal{W}$ is generalized ω-ψ-Geraghty operator.

Let $z, y \in C(\Omega)$ such that

$$\omega(z,y) \geq 1.$$

Thus, for each $\zeta \in \Omega$, we have

$$\xi(\varkappa_{2\zeta}, \varkappa'_{2\zeta}) \geq 0.$$

This implies from (H_3) that

$$\xi(\mathcal{W}z(\zeta), \mathcal{W}y(\zeta)) \geq 0,$$

which gives

$$\omega(\mathcal{W}(z), \mathcal{W}(y)) \geq 1.$$

Hence, $\mathcal{W}$ is a ω-admissible.
Now, by (H_2), there exists $\overline{\ell}_0 \in C(\Omega)$ such that

$$\omega(\overline{\ell}_0, \mathcal{W}(\overline{\ell}_0)) \geq 1.$$

Thus, by (H_4), if $\left(\overline{\ell}_n\right)_{n\in N} \subset \mathfrak{L}$ with $\overline{\ell}_n \to \overline{\ell}$ and $\omega(\overline{\ell}_n, \overline{\ell}_{n+1}) \geq 1$, then

$$\omega(\overline{\ell}_n, \overline{\ell}) \geq 1.$$

From an application of Theorem 2.12, we conclude that $\mathcal{W}$ has a fixed point. Consequently, Ψ has a fixed point which is the solution of problem.
Moreover, (H_5) implies that if z and y are fixed points of $\mathcal{W}$, then either

$$\xi(z,y) \geq 0 \ \textit{ or } \ \xi(y,z) \geq 0.$$

This implies that either

$$\omega(z,y) \geq 1 \ \ or \ \ \omega(y,z) \geq 1.$$

Hence, problem (1) has a unique solution.
□

Now, we prove an existence and uniqueness result by using the F-contraction fixed point theorem.

Theorem 3.2. *Assume there exist constants $\lambda, \widehat{\lambda} > 0$, where $\overline{\lambda} = \lambda(1-\widehat{\lambda}) > \sqrt{2}$ such that for each $\chi_\zeta, \widehat{\chi_\zeta} \in Q$, $\mathfrak{I}, \widehat{\mathfrak{I}} \in \mathbb{R}$ and $\zeta \in \Omega$*

$$|\aleph(\zeta, \chi_\zeta, \mathfrak{I}) - \aleph(\zeta, \widehat{\chi_\zeta}, \widehat{\mathfrak{I}})| \leq \frac{\Gamma(\vartheta)}{2\lambda\kappa^{(\vartheta-1)}\left[1+\sup_{\zeta\in\Omega}|\chi(\zeta)| + \sup_{\zeta\in\Omega}|\widehat{\chi(\zeta)}|\right]}|\chi(\zeta) - \widehat{\chi}(\zeta)| + \widehat{\lambda}|\mathfrak{I} - \widehat{\mathfrak{I}}|. \tag{11}$$

Then the problem (1)-(2) has a unique solution defined on Ω.

Proof. Let $\mathcal{W} : \mathcal{D}_0 \to \mathcal{D}_0$ defined as in (10), For any $z,\ y \in \mathcal{D}_0$. For each $\zeta \in \Omega$ we have

$$|(\mathcal{W}z)(\zeta) - (\mathcal{W}y)(\zeta)|^2 \quad \leq \quad \left\{\frac{1}{\Gamma(\vartheta)}\int_0^\zeta e^{-\ell(\zeta-\tau)}(\zeta-\tau)^{(\vartheta-1)}|\bar{\aleph}(\tau) - \bar{\aleph}'(\tau)|d\tau\right\}^2,$$

where $\bar{\aleph},\ \bar{\aleph}' \in C(\Omega)$ such that

$$\bar{\aleph}(\zeta) = \aleph(\zeta, \bar{z}_\zeta + w_\zeta, \bar{\aleph}(\zeta)) \text{ and } \bar{\aleph}'(\zeta) = \aleph(\zeta, \overline{y}_\zeta + w_\zeta, \bar{\aleph}'(\zeta)).$$

Since, for each $\zeta \in \Omega$, we have

$$\|\bar{\aleph}(\zeta) - \bar{\aleph}'(\zeta)\| \leq \frac{\Gamma(\vartheta)}{2(1-\widehat{\lambda})\lambda\kappa^{(\vartheta-1)}\left[1+\sup_{\zeta\in\Omega}|z(\zeta)| + \sup_{\zeta\in\Omega}|y(\zeta)|\right]}|z(\zeta) - y(\zeta)|.$$

Then, we get

$$\begin{aligned}
|(\mathcal{W}z)(\zeta) - (\mathcal{W}y)(\zeta)|^2 &\leq \left\{\frac{1}{\overline{\lambda}\left[2+2\sup_{\zeta\in\Omega}|z(\zeta)| + 2\sup_{\zeta\in\Omega}|y(\zeta)|\right]}|z(\zeta)-y(\zeta)|\right\}^2 \\
&\leq \frac{1}{\overline{\lambda}^2\left[2+2\sup_{\zeta\in\Omega}|z(\zeta)| + 2\sup_{\zeta\in\Omega}|y(\zeta)|\right]^2}\left\{\sqrt{|z(\zeta)-y(\zeta)|^2}\right\}^2 \\
&\leq \frac{1}{\overline{\lambda}^2\left[2+2\sup_{\zeta\in\Omega}|z(\zeta)| + 2\sup_{\zeta\in\Omega}|y(\zeta)|\right]^2}\left\{\sqrt{\sup_{\zeta\in\Omega}|z(\zeta)-y(\zeta)|^2}\right\}^2 \\
&\leq \frac{1}{\overline{\lambda}^2\left[2+\sup_{\zeta\in\Omega}|z(\zeta)-y(\zeta)|^2\right]}\left\{\sqrt{\sup_{\zeta\in\Omega}|z(\zeta)-y(\zeta)|^2}\right\}^2.
\end{aligned}$$

Consequently, we get

$$\overline{\lambda}^2\varpi(\mathcal{W}z, \mathcal{W}y) \leq \frac{\varpi(z,y)}{2+\varpi(z,y)}.$$

Now, applying natural logarithm on the previous inequality, we obtain

$$\ln(2+\varpi(z,y)) + \ln(\overline{\lambda}^2\varpi(\mathcal{W}z,\mathcal{W}y)) \leq \ln(\varpi(z,y)) \leq \ln(A^{\epsilon\varpi}(z,y)),$$

where

$$A^{\epsilon\varpi}(z,y) = \max\left\{\varpi(z,y), \varpi(z,\mathcal{W}z), \varpi(y,\mathcal{W}y), \frac{\beta}{2\epsilon}\left[\varpi(y,\mathcal{W}z) + \varpi(z,\mathcal{W}y)\right]\right\}, \beta < \frac{1}{2}.$$

If we choose $F(\zeta) = \ln(\zeta)$ and $\wp(\zeta) = \ln(2+\zeta)$ we see that all the conditions of Theorem 2.13 are satisfied, so that $\mathcal{W}$ has a unique fixed point. Consequently, Ψ has a unique fixed point which is the solution of problem.

□

4. Some Examples

Example 4.1. *Consider the following problem which is an example of problem* (1)-(2)*:*

$$\begin{cases} \left({}_0^C\mathfrak{D}_\zeta^{\frac{1}{2},\ell}\chi\right)(\zeta) = \dfrac{\arctan(\|\chi_\zeta\|_Q)}{380(1+\|\chi_\zeta\|_Q)} + \dfrac{1}{380\left(1+\left|\left({}_0^C\mathfrak{D}_\zeta^{\frac{1}{2},\ell}\chi\right)(\zeta)\right|\right)};\ \zeta \in \Omega := [0,1], \\ \chi(\zeta) = \zeta + 1;\ \zeta \in (-\infty, 0]. \end{cases} \tag{12}$$

Let γ be a positive real constant and

$$B_\gamma = \{\chi \in C((-\infty,1],\mathbb{R},) : \lim_{\tau\to-\infty} e^{\gamma\tau}\chi(\tau) \text{ exists in } \mathbb{R}\}. \tag{13}$$

The norm of B_γ is given by

$$\|\chi\|_\gamma = \sup_{\tau\in(-\infty,1]} e^{\gamma\tau}|\chi(\tau)|.$$

Let $\chi : (-\infty,0] \to \mathbb{R}$ be such that $\chi_0 \in B_\gamma$. Then

$$\begin{aligned} \lim_{\tau\to-\infty} e^{\gamma\tau}\chi_\zeta(\tau) &= \lim_{\tau\to-\infty} e^{\gamma\tau}\chi(\zeta+\tau-1) = \lim_{\tau\to-\infty} e^{\gamma(\tau-\zeta+1)}\chi(\tau) \\ &= e^{\gamma(-\zeta+1)} \lim_{\tau\to-\infty} e^{\gamma(\tau)}\chi_1(\tau) < \infty. \end{aligned}$$

Hence $\chi_\zeta \in B_\gamma$. Finally we prove that

$$\|\chi_\zeta\|_\gamma \le \xi_1\|\chi_1\|_\gamma + \xi_2 \sup_{\sigma\in[0,\zeta]} |\chi(\sigma)|,$$

where $\xi_1 = \xi_2 = 1$ and $\xi_3 = 1$. We have

$$|\chi_\zeta(\tau)| = |\chi(\zeta+\tau)|.$$

If $\zeta + \tau \le 1$, we get

$$|\chi_\zeta(\xi)| \le \sup_{\sigma\in(-\infty,0]} |\chi(\sigma)|.$$

For $\zeta + \tau \ge 0$, then we have

$$|\chi_\zeta(\xi)| \le \sup_{\sigma\in[0,\zeta]} |\chi(\sigma)|.$$

Thus for all $\zeta + \tau \in \Omega$, we get

$$|\chi_\zeta(\xi)| \le \sup_{\sigma\in(-\infty,0]} |\chi(\sigma)| + \sup_{\sigma\in[0,\zeta]} |\chi(\sigma)|.$$

Then

$$\|\chi_\zeta\|_\gamma \le \|\chi_0\|_\gamma + \sup_{\sigma\in[0,\zeta]} |\chi(\sigma)|.$$

It is clear that $(B_\gamma, \|\cdot\|)$ is a Banach space. We can conclude that B_γ a phase space.

Set

$$\aleph(\zeta,\chi,\mathfrak{I})=\frac{\arctan(\|\chi\|_{Q})}{380(1+\|\chi\|_{Q})}+\frac{1}{380(1+|\mathfrak{I}|)},$$

where $\zeta\in\Omega,\ \chi\in Q,\mathfrak{I}\in\mathbb{R}$.

Let $(C(\Omega),\varpi,2)$ *be the complete b-metric space with* $\varepsilon=2$, *such that* $\varpi:C(\Omega)\times C(\Omega)\to(0,\infty)$, *is given by:*

$$\varpi(\chi,\mathfrak{I})=\|(\chi-\mathfrak{I})^{2}\|_{\infty}:=\sup_{\zeta\in\Omega}|\chi(\zeta)-\mathfrak{I}(\zeta)|^{2}.$$

For any $\chi,\mathfrak{I}\in Q,\ \bar{\chi},\bar{\mathfrak{I}}\in\mathbb{R}$ *and* $\zeta\in\Omega$, *we have*

$$|\aleph(\zeta,\chi,\bar{\chi})-\aleph(\zeta,\mathfrak{I},\bar{\mathfrak{I}})|\leq\frac{\|\chi-\mathfrak{I}\|_{Q}}{380}+\frac{|\bar{\chi}-\bar{\mathfrak{I}}|}{380}.$$

Thus, hypothesis (H_1) *is satisfied with*

$$p(\zeta)=q(\zeta)=\frac{1}{380}.$$

Define the functions $\eta(\zeta)=\frac{1}{8}\zeta$, $\phi(\zeta)=\zeta$, $\vartheta:C(\Omega)\times C(\Omega)\to\mathbb{R}_{+}^{*}$ *with*

$$\begin{cases}\vartheta(\chi,\mathfrak{I})=1;\ if\ \varpi(\chi(\zeta),\mathfrak{I}(\zeta))\geq 0,\ \zeta\in\Omega,\\ \vartheta(\chi,\mathfrak{I})=0;\ else,\end{cases}$$

and $\varpi:C(\Omega)\times C(\Omega)\to\mathbb{R}$ *with* $\underline{\varpi}(\chi,\mathfrak{I})=\|\chi-\mathfrak{I}\|_{\infty}$.
Hypothesis (H_2) *is satisfied with* $\overline{\ell}_0(\zeta)=\chi_0$. *Also,* (H_3) *holds from the definition of the function* ϖ.

Simple computations show that all conditions of Theorem 3.1 are satisfied. Hence, we get the existence and the uniqueness of solutions for problem (12).

Example 4.2. *Next, consider the following problem:*

$$\begin{cases}\left({}_{0}^{C}\mathfrak{D}_{\zeta}^{\frac{1}{2},\ell}\chi\right)(\zeta)=\dfrac{\Gamma(\frac{1}{2})}{4(1+\sup_{\zeta\in\Omega}|\chi(\zeta)|)}+\dfrac{1}{20\left(1+\left|\left({}_{0}^{C}\mathfrak{D}_{\zeta}^{\frac{1}{2},\ell}\chi\right)(\zeta)\right|\right)};\ \zeta\in\Omega,\\ \chi(\zeta)=2\zeta+4;\ \zeta\in(-\infty,0].\end{cases}\tag{14}$$

Set

$$\aleph(\zeta,\chi_{\zeta},\mathfrak{I})=\frac{\Gamma(\frac{1}{2})}{4(1+\sup_{\zeta\in\Omega}|\chi(\zeta)|)}+\frac{1}{20(1+|\mathfrak{I}|)},$$

where $\zeta\in\Omega,\ \chi\in C(\Omega),\mathfrak{I}\in\mathbb{R}$.

Let $(C(\Omega),\varpi,2)$ *be a complete b-metric space with* $\varepsilon=2$, *such that* $\varpi:C(\Omega)\times C(\Omega)\to(0,\infty)$, *is given by:*

$$\varpi(\chi,\mathfrak{I})=\|(\chi-\mathfrak{I})^{2}\|_{\infty}:=\sup_{\zeta\in\Omega}|\chi(\zeta)-\mathfrak{I}(\zeta)|^{2}.$$

For any $\chi,\mathfrak{I}\in C(\Omega),\ \bar{\chi},\bar{\mathfrak{I}}\in\mathbb{R}$ *and* $\zeta\in\Omega$, *we have*

$$|\aleph(\zeta,\chi_{\zeta},\bar{\chi})-\aleph(\zeta,\mathfrak{I}_{\zeta},\bar{\mathfrak{I}})|\leq\frac{\Gamma(\frac{1}{2})|\chi(\zeta)-\mathfrak{I}(\zeta)|}{4(1+\sup_{\zeta\in\Omega}|\chi(\zeta)|+\sup_{\zeta\in\Omega}|\mathfrak{I}(\zeta)|)}+\frac{1}{20}|\bar{\chi}-\bar{\mathfrak{I}}|.$$

Then, hypothesis (11) *is satisfied with*

$$\lambda=2,\ \widehat{\lambda}=\frac{1}{20}\ and\ \overline{\lambda}=\frac{19}{10}>\sqrt{2}.$$

Since all requirements of Theorem 2.13 are verified, we conclude the existence the uniqueness of solutions and for problem (14).

Declarations

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