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# Existence of global solutions and blow-up results for a class of $p(x)-$Laplacian Heat equations with logarithmic nonlinearity

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**Abstract.** This paper's main objective is to examine an initial boundary value problem of a quasilinear parabolic equation of non-standard growth and logarithmic nonlinearity by utilizing the logarithmic Sobolev inequality and potential well method. Results of global existence, estimates of polynomial decay, and blowing up of weak solutions have been obtained under certain conditions that will be stated later. Our results extend those of a recent paper that appeared in the literature.

## 1. Introduction

In the present paper, we deal with the global existence and blowing-up of weak solutions for the following initial boundary value problem with logarithmic nonlinearity

$$\begin{cases} u_t - \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = |u|^{p(x)-2} u \log|u|, & x \in \Omega, \quad t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. The function $p(.)$ is continuous on $\overline{\Omega}$ into $\mathbb{R}_+$ such that

$$2 < p_- \leq p(x) \leq p_+ < p^*(x), \tag{1.2}$$

with

$$p_- = ess \inf_{x\in\Omega} p(x), \quad p_+ = ess \sup_{x\in\Omega} p(x) \quad \text{and} \quad p^*(x) = \begin{cases} \frac{np(x)}{n-p(x)}, & \text{if} \quad p_+ < n, \\ +\infty, & \text{if} \quad p_+ \geq n, \end{cases}$$

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and $u_0 \in W_0^{1,p(x)}(\Omega) \setminus \{0\}$.

According to recent research, studying problems with nonstandard $p(x)-$growth can be an effective tool for modeling a wide range of events in many scientific and technological fields. Their applications in elastic mechanics, fluid dynamics, or calculus of variations, (see, for example, [1, 2, 36, 37]), have stimulated the study of problems with variable exponents. Variable exponent spaces are regarded as a good tool for handling problems of this type. The variable exponent Lebesgue and Sobolev spaces have been developed as a result of numerous theoretical studies (see [13, 19–22, 25, 32]). As a result, this topic is becoming increasingly important and well-known; for further details on variable exponent spaces, see [10, 12, 33, 39].

On the level of classic Lebesgue and Sobolev spaces, we may find quite many global existence or nonglobality, blow-up, a long time behavior of weak solution results on differential equations in the literature; see for instance [8, 11, 16, 18, 24, 29, 38].

When $p(x) \equiv 2$, the problem (1.1) has been considered in [9]; the authors investigated global existence and blow-up in infinite time of the solutions. If $p(x) \equiv p$ a constant exponent, Cong Nhan Le and Xuan Truong Le established in [28] the global existence and blow-up results. They showed that when $p > 2$, solutions blow up in finite time and obtained sufficient conditions on the existence of global weak solutions.

On the other hand, works on differential equations at the level of variable exponent Lebesgue and Sobolev spaces are fairly fewer.

Hua Wang and Yijun He in [40] were interested in the case where

$$\begin{cases} u_t = \Delta u + |u|^{p(x)}, & x \in \Omega, \quad t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, \quad t \geq 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases}$$

with $u_0(x) \geq 0$. They demonstrated that under condition $1 < p_- \leq p_+ \leq \frac{n+2}{n-2}$ and certain initial data, the solution blows up in finite time for a positive initial energy.

M. Kbiri Alaoui et al. [3] considered the problem

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}\left(|\nabla u|^{m(x)-2} \nabla u\right) = |u|^{p(x)-2} u + f, & \text{in } Q = \Omega \times (0,T), \\ u(x,t) = 0, & \text{on } \partial Q = \partial\Omega \times [0,T), \\ u(x,0) = u_0(x), & \text{in } \Omega, \end{cases}$$

They proved that any solution with a nontrivial initial datum blows up in finite time whenever

$$\int_\Omega u_0^2 dx > 0, \quad f \equiv 0 \quad \text{and} \quad \int_\Omega \left(\frac{1}{p(x)} |u_0|^{p(x)} - \frac{1}{m(x)} |\nabla u_0|^{m(x)}\right) dx \geq 0.$$

Boudjriou in [7] studied the problem

$$\begin{cases} u_t - \operatorname{div}\left(|\nabla u|^{p(x)-2} \nabla u\right) = |u|^{q(x)-2} u \log|u|, & x \in \Omega, \quad t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases}$$

Under suitable conditions, the author discusses the global existence and finite time blow-up of solutions by using the potential well method via the Pohozaev manifold and the concavity method.

In light of the extensive literature on polynomial nonlinear terms, physicists and mathematicians have shown a keen interest in logarithmic nonlinearity. Both the relativistic wave equation for spinless particles and the non-relativistic wave equation describing spinning particles traversing in an external electromagnetic field were also studied by introducing the logarithmic nonlinearity (see [4]). The global-in-time well posedness of the solution to the problem of the evolution equation with such logarithmic type nonlinearity also draws a lot of attention. This type of nonlinearity is also encountered in many branches of physics, including incoherent white light solitons in logarithmically saturable noninstantaneous nonlinear media [8], inflationary cosmology [17], nuclear physics [25], optics [26], and geophysics [30].

It should be noted that the presence of logarithmic nonlinearity causes a few difficulties in the deployment of the potential well method. So as to deal with this situation, we need the logarithmic Sobolev inequality, presented in [15, 28].

**Lemma 1.1.** *Let $q > 1, \mu > 0$, and $u \in W^{1,q}(\mathbb{R}^n) \setminus \{0\}$. Then we have*

$$q \int_{\mathbb{R}^n} |u(x)|^q \log\left(\frac{|u(x)|}{\|u(x)\|_{L^q(\mathbb{R}^n)}}\right) dx + \frac{n}{q} \log\left(\frac{q\mu e}{n\mathcal{L}_q}\right) \int_{\mathbb{R}^n} |u(x)|^q \, dx \le \mu \int_{\mathbb{R}^n} |\nabla u(x)|^q \, dx,$$

*where*

$$\mathcal{L}_q = \frac{q}{n}\left(\frac{q-1}{e}\right)^{q-1} \pi^{-\frac{q}{2}} \left[\frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(n\frac{q-1}{q}+1\right)}\right]^{\frac{q}{n}}.$$

In this paper, we consider the problem (1.1) with the presence of a nonlinear diffusion term of variable exponent $\Delta_{p(x)} = \operatorname{div}\left(|\nabla u|^{p(x)-2} \nabla u\right)$ and logarithmic nonlinearity $|u|^{p(x)-2} u \log|u|$. This extends the problem in [28] from the level of classical Lebesgue and Sobolev spaces to the level of variable exponent Lebesgue and Sobolev spaces. Our goal is to establish a global existence, a long time decay and blow up in finite time of solutions of the problem (1.1) within the framework of the variable exponent Lebesgue and Sobolev spaces by means of the potential well method (see [34]) via the Nehari manifold, and the concavity method (see [29]) in order to obtain global existence and blow up of weak solutions to (1.1). In our work, we discuss the following details:

-The solution of problem (1.1) exists locally and globally in time if it holds the condition $p_+ < p_-^*$, but here we have discussed two cases

Case 1: If $2 < p_- \le p_+ < \left(1 + \frac{2}{n}\right) p_-$,
Case 2: If $p_+ < \left(1 + \frac{2}{n}\right) p_-$ does note holds, i.e., $\left(\left(1 + \frac{2}{n}\right) p_- \le p_+ < p_-^*\right)$.

We point out that the case 1 with $p_- = p$ and $p_+ = q$ has been discussed in [23] where the authors obtained results of decay and finite time blow-up of solutions for the Pseudo-parabolic $p(x)$−Laplacian equation with logarithmic nonlinearity.

$$u_t - \Delta u_t - \operatorname{div}\left(|\nabla u|^{p(x)-2} \nabla u\right) = |u|^{q(x)-2} u \log|u|, \quad x \in \Omega, \quad t > 0. \tag{1.3}$$

But case 2 is not studied in the previous cited papers in the literature.

We montion also that an initial boundary value problem like (1.3) has been considered in [5, 14], (see also [41])

An other two cases that we have discussed for prouving the coercivity of the energy functional defined in (2.4) (see the proof of Lemma 2.6 below) that are:

Case 3: If $2 < p_- \le p_+ < \left(1 + \frac{p_-}{n}\right) p_-$,
Case 4: If $p_+ < \left(1 + \frac{p_-}{n}\right) p_-$ does note holds, i.e., $\left(\left(1 + \frac{p_-}{n}\right) p_- \le p_+ < p_-^*\right)$.

Also, these cases are not considered in the literature.

We also note that the inequality (2.6), which we have used throughout the paper and without which the potential well method does not work, plays an important role in this inequality, as does the generic constant $\gamma$, which we defined for the first time in ($a$).

There is no result for the logarithmic Sobolev inequality in variable exponent Sobolev spaces to our knowledge, but this is the first result in the literature that allows the treatment of non-standard growth parabolic equations by using the classical logarithmic Sobolev inequality, which is a fundamental inequality, to get the results in [28], for dealing with the logarithmic nonlinear term (see Lemma 2.5)

**Remark 1.1.** *If $u \in W_0^{1,p(x)}(\Omega) \setminus \{0\}$, then, by defining $u(x) = 0$ for $x \in \mathbb{R}^n \setminus \Omega$ it holds*

$$p_- \int_\Omega |u|^{p_-} \log\left(\frac{|u|}{\|u\|_{p_-}}\right) dx + \frac{n}{p_-} \log\left(\frac{p_- \mu e}{n \mathcal{L}_{p_-}}\right) \int_\Omega |u|^{p_-}\, dx \le \mu \int_\Omega |\nabla u|^{p_-}\, dx, \tag{1.4}$$

$$p_+ \int_\Omega |u|^{p_+} \log\left(\frac{|u|}{\|u\|_{p_+}}\right) dx + \frac{n}{p_+} \log\left(\frac{p_+ \mu e}{n \mathcal{L}_{p_+}}\right) \int_\Omega |u|^{p_+}\, dx \le \mu \int_\Omega |\nabla u|^{p_+}\, dx, \tag{1.5}$$

*for any real number $\mu > 0$, where*

$$\mathcal{L}_{p_-} = \frac{p_-}{n}\left(\frac{p_- - 1}{e}\right)^{p_- - 1} \pi^{-\frac{p_-}{2}} \left[\frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(n\frac{p_- - 1}{p_-}+1\right)}\right]^{\frac{p_-}{n}}, \quad \mathcal{L}_{p_+} = \frac{p_+}{n}\left(\frac{p_+ - 1}{e}\right)^{p_+ - 1} \pi^{-\frac{p_+}{2}} \left[\frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(n\frac{p_+ - 1}{p_+}+1\right)}\right]^{\frac{p_+}{n}}.$$

## 2. Preliminaries

### 2.1. *Functional framwork*

We give some results about the Lebesgue and Sobolev spaces with variable exponents, which are well-known in [12, 25, 32]. The following notations will be used in the sequel:

$$\|v\|_{p(.)} = \|v\|_{L^{p(.)}(\Omega)}, \quad \|v\|_{p_-} = \|v\|_{L^{p_-}(\Omega)}, \quad \|v\|_{p_+} = \|v\|_{L^{p_+}(\Omega)}.$$

Let $\mathcal{P}(\Omega)$ be the set of all Lebesgue measurable functions $p(.) : \Omega \to [1, \infty]$, where $\Omega$ is a bounded domain in $\mathbb{R}^n$.

A function $p(.)$ is said to satisfy the log-Hölder continuous condition in $\Omega$ if

$$\forall x, y \in \Omega \text{ with } |x - y| \le \frac{1}{2}, \quad |p(x) - p(y)| \le \frac{C_0}{-\log(|x - y|)}, \tag{2.1}$$

where $C_0 > 0$ is a constant.

We say that $p(.)$ satisfy the log-Hölder decay condition in $\Omega$ if

$$\forall x \in \Omega, \quad |p(x) - p_\infty| \le \frac{C_\infty}{\log(e + |x|)}, \tag{2.2}$$

where $p_\infty = \lim_{|x|\to\infty} p(x)$ and $C_\infty > 0$ are constants.

By $\mathcal{P}^{\log}(\Omega)$ we denote the class of variable exponents:

$$\mathcal{P}^{\log}(\Omega) = \{p(.) \in \mathcal{P}(\Omega) : 1/p(.) \text{ is globally log-Hölder continuous}\}.$$

Note that $r(.) \in \mathcal{P}(\Omega)$ is globally log-Hölder continuous in $\Omega$, if $r(.)$ satisfies both (2.1)-(2.2) conditions.

**Proposition 2.1** (see [10]). *Given a domain $\Omega$*
*1) If $p(.)$ fulfills (2.1), then it is uniformly continuous and fulfills (2.2) on every bounded subset. $E \subset \Omega$.*
*2) If $p(.) \in \mathcal{P}(\Omega)$ and $p_+ < +\infty$, then $1/p(.)$ satisfies either conditions (2.1), (2.2) or both if and only if $p(.)$ is also.*

**Remark 2.1.** *From Proposition 2.1 we deduce that if $\Omega$ is bounded, $p(.) \in C(\overline{\Omega})$ and satisfies the conditions (1.2), (2.1) then $p(.), 1/p(.) \in \mathcal{P}^{\log}(\Omega)$*

Now we define the $p(.)$ modular of a measurable function $u : \Omega \to \mathbb{R}$ as follows:

$$\rho_{p(x)}(u) = \int_{\Omega\setminus\Omega_\infty} |u(x)|^{p(x)}\, dx + ess \sup_{x\in\Omega_\infty} |u(x)|,$$

where

$$\Omega_\infty = \{x \in \Omega : p(x) = \infty\}.$$

The generalized Lebesgue space $L^{p(.)}(\Omega)$ is the class of those measurable functions $u$ defined on $\Omega$ as follows

$$L^{p(x)}(\Omega) = \left\{ u : u \in \mathcal{P}(\Omega), \int_\Omega |u(x)|^{p(x)}\, dx < \infty \right\}$$

We define the Luxembourg norm of the space $L^{p(.)}(\Omega)$ by

$$\|u\|_{p(x)} = \inf\left\{\kappa > 0 : \rho_{p(x)}(u/\kappa) \le 1\right\}.$$

The space $L^{p(.)}(\Omega)$ equipped with this norm, is a Banach space.

Now we present some results that concern variable-exponent Lebesgue spaces

**Proposition 2.2** (see [20, 21]). *Let $u \in L^{p(x)}(\Omega)$, $(u_n)_{n\in\mathbb{N}} \subset L^{p(x)}(\Omega)$, then*
*1)* $\|u\|_{p(x)} < 1\,(= 1; > 1) \Leftrightarrow \rho_{p(x)}(u) < 1\,(= 1; > 1)$,
*2) If* $\|u\|_{p(x)} > 1$, *then* $\|u\|^{p_-}_{p(x)} \le \rho_{p(x)}(u) \le \|u\|^{p_+}_{p(x)}$,
*3) If* $\|u\|_{p(x)} < 1$, *then* $\|u\|^{p_+}_{p(x)} \le \rho_{p(x)}(u) \le \|u\|^{p_-}_{p(x)}$,
*4)* $\|u_n\|_{p(x)} \underset{n\to+\infty}{\to} 0 \Leftrightarrow \rho_{p(x)}(u_n) \underset{n\to+\infty}{\to} 0$,
*5)* $\|u_n\|_{p(x)} \underset{n\to+\infty}{\to} +\infty \Leftrightarrow \rho_{p(x)}(u_n) \underset{n\to+\infty}{\to} +\infty$.

**Proposition 2.3** (see [20]). *Let $u$, $u_n \in L^{p(x)}(\Omega)$, $n = 1, 2, \ldots$ . Then the following statements are equivalent to each other :*
*1)* $\lim_{n\to\infty} \|u_n - u\|_{p(x)} = 0$,
*2)* $\lim_{n\to\infty} \rho_{p(x)}(u_n - u) = 0$,
*3) $u_n$ converges to $u$ in $\Omega$ in measure and* $\lim_{n\to\infty} \rho_{p(x)}(u_n) = \rho_{p(x)}(u)$.

**Proposition 2.4** (see [20]). *Assume that $\Omega$ has finite measure, $p_1(x), p_2(x) \in \mathcal{P}(\Omega)$. If $p_1(x) \le p_2(x)$ for almost all $x \in \Omega$ and $1 \le p_{i-} \le p_{i+} < +\infty$, $(i = 1, 2)$, then $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$ and the embedding is continuous.*

**Proposition 2.5** (generalized Hölder inequality, see [25, 33]). *Let $p(.), p'(.) \in \mathcal{P}(\Omega)$ such that $p_- > 1$ and*

$$\frac{1}{p(.)} + \frac{1}{p'(.)} = 1, \quad a.e.\ x \in \Omega.$$

*Then the inequality*

$$\int_\Omega |u(x)\, v(x)|\, dx \le \left(\frac{1}{p_-} + \frac{1}{p'_-}\right) \|u\|_{p(x)} \|v\|_{p'(x)},$$

*holds for every $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$.*

The interpolation inequality given in [39, Lemma 8.2 page 37] is also needed

**Lemma 2.1.** *If $1 \leq p_0 < p_\theta < p_1 \leq \infty$, then*

$$\|u\|_{p_\theta} \leq \|u\|_{p_0}^{1-\theta} \|u\|_{p_1}^{\theta}$$

*for all $u \in L^{p_0}(\Omega) \cap L^{p_1}(\Omega)$ with $\theta \in (0,1)$ defined by $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.*

Generally, variable-exponent Lebesgue spaces and classical Lebesgue spaces are similar in many properties. Moreover, several results that concern variable-exponent Lebesgue spaces have been obtained, see for instance [12, 25] for the following statements.

• The modular $\rho_{p(.)}$ and the norm $\|.\|_{p(x)}$ are lower semi-continuous with respect to (sequential) weak convergence and almost everywhere convergence.
• The space $L^{p(.)}(\Omega)$ is reflexive if and only if $1 < p_- < p_+ < \infty$.
• Continuous functions are dense if $p_+ < \infty$.

The space $W^{1,p(.)}(\Omega)$ is the variable-exponent Sobolev space defined by

$$W^{1,p(.)}(\Omega) = \left\{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\right\}$$

This space endowed with the norm

$$\|u\|_{1,p(x)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)},$$

is a Banach space.

We define $W_0^{1,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ with respect to the norm $\|u\|_{1,p(x)} = \|u\|_{W_0^{1,p(.)}(\Omega)} = \|\nabla u\|_{p(x)}$, since $p(.)$ satisfies (1.2) and (2.1). The space $W^{-1,p'(.)}(\Omega)$ is the dual space of $W_0^{1,p(x)}(\Omega)$ where $p'(x)$ is the conjugate exponent function of $p(x)$ such that $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

We denote $X_0 = W_0^{1,p(x)}(\Omega) \setminus \{0\}$, with the norm $\|u\|_{1,p(x)} = \|\nabla u\|_{p(x)}$.

**Proposition 2.6** (see [12, Theorem 8.1.13])**.** *Let $p \in \mathcal{P}(\mathbb{R}^n)$. The space $W_0^{1,p(.)}(\Omega)$ is a Banach space, which is separable if $p(.)$ is bounded, and reflexive and uniformaly convexe if $1 < p_- \leq p_+ < +\infty$.*

**Proposition 2.7.** *Assume that $1 \leq ess\inf_{x\in\Omega} p_i(x) \leq p_i(x) \leq ess\sup_{x\in\Omega} p_i(x) < +\infty$, $(i = 1, 2)$. If $p_1(x) \leq p_2(x)$, then $W^{1,p_2(x)}(\Omega) \hookrightarrow W^{1,p_1(x)}(\Omega)$.*

**Proposition 2.8** (see [32])**.** *If $\Omega$ is bounded, $p(x) \in C\left(\overline{\Omega}\right)$ such that $p_+ < n$ and $q(x)$ defined in $\Omega$ with $q_- \geq 1$ and*

$$\underset{x\in\Omega}{ess\inf}\left(p^*(x) - q(x)\right) > 0,$$

*then the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact.*

**Proposition 2.9** (see [10, Theorem 6.29] and [12, Theorem 8.3.1])**.**
*1) Given $\Omega$ and $p(.) \in \mathcal{P}(\Omega)$ such that $p_+ < n$ suppose that the maximal operator is bounded on $L^{(p^*(.)/n')'}(\Omega)$. Then $W_0^{1,p(.)}(\Omega) \subset L^{p^*(.)}(\Omega)$, and*

$$\|u\|_{p^*(.)} \leq C \|\nabla u\|_{p(.)}.$$

*2) Let $p \in \mathcal{P}^{\log}(\Omega)$ satisfy $1 \leq p_- \leq p_+ < n$. Then for every $u \in W_0^{1,p(.)}(\Omega)$, the inequality*

$$\|u\|_{p^*(.)} \leq c \|\nabla u\|_{p(.)}$$

*holds with a constant $c$ depending only on the dimension $n$, $c_{\log}(p)$, and $p_+$.*

**Lemma 2.2.** *Let $\varrho(.)$ be a continuous function from $\overline{\Omega}$ into $(0,+\infty)$ such that $0 < ess\inf_{x\in\Omega}\varrho(x) = \varrho_- \leq \varrho(x) \leq \varrho_+ = ess\sup_{x\in\Omega}\varrho(x) < +\infty$. Then the following inequalities hold*

$$\log s \leq \frac{e^{-1}}{\varrho(x)} s^{\varrho(x)}, \text{ for all } s \in [1,+\infty) \quad \text{and} \quad -\frac{e^{-1}}{\varrho(x)} \leq s^{\varrho(x)}\log s \leq 0, \text{ for all } s \in (0,1].$$

*Proof.* The proof follows directly by studying the variations of the function $\varphi(s) = \log s - \frac{e^{-1}}{\varrho(x)} s^{\varrho(x)}$, for $s \in [1,+\infty)$ and the function $\phi(s) = s^{\varrho(x)}\log s$, for $s \in (0,1]$. □

**Remark 2.2.** *It follows from Lemma 2.2 that*

$$s^{p(x)}\log s \leq \frac{e^{-1}}{\varrho(x)} s^{p(x)+\varrho(x)}, \quad \text{for all } s \in [1,+\infty).$$

**Lemma 2.3** (see [27]).
*(a) For any function $u \in W_0^{1,p}(\Omega)$, we have the inequality*

$$\|u\|_q \leq B_{q,p}\|\nabla u\|_p,$$

*for all $q \in [1,\infty)$ if $n \leq p$, and $1 \leq q \leq \frac{np}{n-p}$ if $n > p$. The best constant $B_{q,p}$ depends only on $\Omega, n, p$ and $q$. We will denote the constant $B_{p,p}$ by $B_p$.*
*(b) Let $2 \leq s \leq p < q < p^*$. For any $u \in W_0^{1,p}(\Omega)$ we have*

$$\|u\|_q \leq C\|\nabla u\|_p^{\alpha}\|u\|_s^{1-\alpha},$$

*where $C$ is a positive constant and*

$$\alpha = \left(\frac{1}{s} - \frac{1}{q}\right)\left(\frac{1}{n} - \frac{1}{p} + \frac{1}{s}\right)^{-1}.$$

Now we define what a weak solution of problem (1.1) means.

**Definition 2.1** (Weak solution). *A function $u \in L^\infty(0,T;X_0)$ with*

$$u_t \in L^{p'(x)}\left(0,T;W^{-1,p'(x)}(\Omega)\right) \cap L^2\left(0,T;L^2(\Omega)\right),$$

*is said to be a weak solution of problem (1.1) on $\Omega \times [0,T)$ if it satisfies the initial condition*

$$u(.,0) = u_0(.) \in X_0.$$

*and*

$$\int_\Omega u_t w dx + \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla w dx = \int_\Omega |u|^{p(x)-2} u \log|u| w dx, \tag{2.3}$$

*for all $w \in W_0^{1,p(x)}(\Omega)$, and for almost every $t \in (0,T)$.*

*2.2. Potential well*

Let us consider the functionals $J$ and $I$ defined on $X_0$ by

$$J(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)} dx - \int_\Omega \frac{1}{p(x)}|u|^{p(x)}\log|u| dx + \int_\Omega \frac{1}{p^2(x)}|u|^{p(x)} dx,$$

$$= J_1(u) - \int_\Omega \frac{1}{p(x)} |u|^{p(x)} \log|u|\, dx + \int_\Omega \frac{1}{p^2(x)} |u|^{p(x)}\, dx, \tag{2.4}$$

$$I(u) = \int_\Omega |\nabla u|^{p(x)}\, dx - \int_\Omega |u|^{p(x)} \log|u|\, dx = I_1(u) - \int_\Omega |u|^{p(x)} \log|u|\, dx. \tag{2.5}$$

The functionals $I$ and $J$ are defined as in [28] with some modifications, they are well-defined in $X_0$. Furthermore, the following proposition given in [22] characterizes the functionals $J_1$ and $I_1$ in the space $W_0^{1,p(x)}(\Omega)$.

**Proposition 2.10.** *Let $J_1(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx$ for $u \in W_0^{1,p(x)}(\Omega)$, then $J_1 \in C^1\left(W_0^{1,p(x)}(\Omega), \mathbb{R}\right)$ and the $p(x)-$Laplacian is the derivative operator of $J_1$. We define $J_1' : W_0^{1,p(x)}(\Omega) \to \left(W_0^{1,p(x)}(\Omega)\right)^*$, then*

$$\langle J_1'(u), v\rangle = \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v dx, \quad \forall u, v \in W_0^{1,p(x)}(\Omega),$$

*and $J_1'$ satisfies the following properties:*
*(i) $J_1'$ is a continuous, bounded, strictly monotone operator, and is a homeomorphism.*
*(ii) $J_1'$ is a mapping of type $(S_+)$, i.e., if $u_n \rightharpoonup u$ in $W_0^{1,p(x)}(\Omega)$ and $\overline{\lim_{n\to\infty}}\langle J_1'(u_n) - J_1'(u), u_n - u\rangle \le 0$, then $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$.*

**Remark 2.3.** *Note that in Proposition 2.10, $\langle J_1'(u), u\rangle = I_1(u)$, for all $u \in X_0$, and then $\langle J'(u), u\rangle = I(u)$, for all $u \in X_0$. Indeed*

$$\begin{aligned}\langle J'(u), u\rangle &= \langle J_1'(u), u\rangle - \int_\Omega |u|^{p(x)-2} u \log(|u|)\, u dx - \int_\Omega \frac{1}{p(x)} |u|^{p(x)}\, dx + \int_\Omega \frac{1}{p(x)} \left(|u|^{p(x)-2} u\right) u dx \\ &= I_1(u) - \int_\Omega |u|^{p(x)} \log|u|\, dx = I(u).\end{aligned}$$

**Remark 2.4.** *It is easy to show by lemma 2.2 that the functional $u \mapsto \int_\Omega |u|^{p(x)} \log|u|\, dx$ is continuous on $X_0$, and then by Proposition 2.10 and Remark 2.3 we deduce that the functionals $J$ and $I$ are continuous from $X_0$ into $\mathbb{R}$. Furthermore we have $J \in C^1(X_0, \mathbb{R})$.*

On the other hand, since $I(u)$ changes sign (see Lemma 2.5 below), so we denote by $\gamma_{I(u)} \equiv \gamma$ a generic constant, i.e. a constant changing value according to the sign of $I(u)$, such that

$$\gamma = \frac{1}{2}\left(\frac{1}{p_+} - \operatorname{sgn}(I(u))\frac{1}{p_-}\right) + \frac{1}{2}\left(\frac{1}{p_-} + \operatorname{sgn}(I(u))\frac{1}{p_+}\right) = \begin{cases} 1/p_-, & \text{if } I(u) \le 0 \\ 1/p_+, & \text{if } I(u) > 0 \end{cases} \tag{a}$$

then from (2.4)-(2.5) we have

$$J(u) \ge \gamma I(u) + \frac{1}{p_+^2} \int_\Omega |u|^{p(x)}\, dx. \tag{2.6}$$

Let $u \in X_0$ and consider the real function $j : \lambda \to J(\lambda u)$ for $\lambda > 0$, defined by

$$j(\lambda) = J(\lambda u) = \int_\Omega \frac{\lambda^{p(x)}}{p(x)} |\nabla u|^{p(x)}\, dx - \int_\Omega \frac{\lambda^{p(x)}}{p(x)} |u|^{p(x)} \log|u|\, dx - \log\lambda \int_\Omega \frac{\lambda^{p(x)}}{p(x)} |u|^{p(x)}\, dx + \int_\Omega \frac{\lambda^{p(x)}}{p^2(x)} |u|^{p(x)}\, dx. \tag{2.7}$$

In the following lemma we show that $j(\lambda)$ has a unique positive critical point $\lambda^* = \lambda^*(u)$ see [16, 28, 34, 35].

**Lemma 2.4.** *Let $u \in X_0$. Then it holds*
*(1) $\lim_{\lambda\to 0^+} j(\lambda) = 0$ and $\lim_{\lambda\to+\infty} j(\lambda) = -\infty$;*
*(2) there is a unique $\lambda^* = \lambda^*(u) > 0$ such that $j'(\lambda^*) = 0$;*
*(3) $j(\lambda)$ is increasing on $(0, \lambda^*)$, decreasing on $(\lambda^*, +\infty)$ and attains the maximum at $\lambda^*$;*
*(4) $I(\lambda u) > 0$ for $0 < \lambda < \lambda^*$, $I(\lambda u) < 0$ for $\lambda^* < \lambda < +\infty$ and $I(\lambda^* u) = 0$.*

*Proof.* Let $u \in X_0$, by (2.7) it is easy to show that (1) holds since $\int_\Omega \frac{\lambda^{p(x)}}{p(x)} |u|^{p(x)} dx \neq 0, (\lambda \neq 0)$. By simple calculation, we get

$$\frac{d}{d\lambda} j(\lambda) = \int_\Omega \lambda^{p(x)-1} |\nabla u|^{p(x)} dx - \int_\Omega \lambda^{p(x)-1} |u|^{p(x)} \log|u| dx - \log\lambda \int_\Omega \lambda^{p(x)-1} |u|^{p(x)} dx, \tag{2.8}$$

to show (2) and (3), it suffices to take

$$\lambda^* = \lambda^*(u) = \exp\left(\frac{\int_\Omega (\lambda^*)^{p(x)-1} \left(|\nabla u|^{p(x)} - |u|^{p(x)} \log|u|\right) dx}{\int_\Omega (\lambda^*)^{p(x)-1} |u|^{p(x)} dx}\right)$$

implicitly. The last property (4) follows from the relationship.

$$\begin{aligned} I(\lambda u) &= \lambda(\int_\Omega \lambda^{p(x)-1} |\nabla u|^{p(x)} dx - \int_\Omega \lambda^{p(x)-1} |u|^{p(x)} \log|u| dx - \log\lambda \int_\Omega \lambda^{p(x)-1} |u|^{p(x)} dx) \\ &= \lambda j'(\lambda). \end{aligned}$$

Thus the lemma is proved. □

**Lemma 2.5.** *Let $u_0 \in X_0$. Suppose that*

$$\min_{\overline{\Omega}} \int_\Omega |u_0|^{p(x)} \log(|u_0|) dx \geq 0, \tag{2.9}$$

*then there exists a positive reel number $R$ such that the following statements hold*
*(1) if $0 < \max\left\{\|u\|_{p(x)}, \|u\|_{p(x)}^{p_+/p_-}\right\} < R$ then $I(u) > 0$.*
*(2) if $I(u) < 0$ then $\min\left\{\|u\|_{p(x)}, \|u\|_{p(x)}^{p_+/p_-}\right\} > R$.*
*(3) if $I(u) = 0$ then $\min\left\{\|u\|_{p(x)}, \|u\|_{p(x)}^{p_+/p_-}\right\} \geq R$.*

*Proof.* Divided $\Omega$ into $l$ subsets in the following way $\overline{\Omega} = \cup_{i=1}^{l} \overline{\Omega_i}$ such that in every subset $\Omega_i$ we have $p_{i-} \leq p(x) \leq p_{i+}$, for all $1 \leq i \leq l$, where $p_{i-} = p_-(\Omega_i)$ and $p_{i+} = p_+(\Omega_i)$. Here we have used the assumption (1.2) and the continuity of the function $p(x)$. Now for $l$ large enough we suppose that

$$\max_{\overline{\Omega}_i} \int_{\Omega_i} |u|^{p(x)} dx \leq 1, \quad i = 1, 2, ..., l, \tag{2.10}$$

and by assumption (2.9) that

$$\int_{\Omega_i} |u|^{p_{i-}} \log(|u|) dx \geq 0, \quad i = 1, 2, ..., l. \tag{2.11}$$

On one hand it follows by Proposition 2.2 (with $\Omega$ is replaced by $\Omega_i$) that

$$\|u\|_{p(x),\Omega_i}^{p_{i+}} \leq \int_{\Omega_i} |u|^{p(x)}\, dx \leq \|u\|_{p(x),\Omega_i}^{p_{i-}}. \tag{2.12}$$

On the other hand by (2.10) we may write

$$\min\left\{\|u\|_{p_{i-},\Omega_i}^{p_{i-}}, \|u\|_{p_{i+},\Omega_i}^{p_{i+}}\right\} \leq \int_{\Omega_i} |u|^{p(x)}\, dx \leq \max\left\{\|u\|_{p_{i-},\Omega_i}^{p_{i-}}, \|u\|_{p_{i+},\Omega_i}^{p_{i+}}\right\}. \tag{2.13}$$

According to (2.5), (2.12) and (1.4)-(1.5) (with $p_-$, $p_+$ and $\Omega$ are replaced by $p_{i-}$, $p_{i+}$ and $\Omega_i$ respectively), we get

$$\begin{aligned}
\int_{\Omega_i} |\nabla u|^{p(x)}\, dx - \int_{\Omega_i} |u|^{p(x)} \log|u|\, dx \geq{}& \int_{\Omega_i} |\nabla u|^{p(x)}\, dx - \frac{\mu_i}{p_{i-}}\int_{\Omega_i} |\nabla u|^{p_{i-}}\, dx - \frac{\mu_i}{p_{i+}}\int_{\Omega_i} |\nabla u|^{p_{i+}}\, dx \\
&- \int_{\Omega_i} |u|^{p(x)} \left[\log\left(\frac{|u|}{\|u\|_{p(x),\Omega_i}}\right) + \frac{1}{p_{i+}} \log\left[\int_{\Omega_i} |u|^{p(x)}\, dx\right]\right] dx \\
&+ \int_{\Omega_i} |u|^{p_{i-}} \log\left(\frac{|u|}{\|u\|_{p_{i-},\Omega_i}}\right) dx + \frac{n}{p_{i-}^2} \log\left(\frac{p_{i-}\mu_i e}{n\mathcal{L}_{p_{i-}}}\right) \int_{\Omega_i} |u|^{p_{i-}}\, dx \\
&+ \int_{\Omega_i} |u|^{p_{i+}} \log\left(\frac{|u|}{\|u\|_{p_{i+},\Omega_i}}\right) dx + \frac{n}{p_{i+}^2} \log\left(\frac{p_{i+}\mu_i e}{n\mathcal{L}_{p_{i+}}}\right) \int_{\Omega_i} |u|^{p_{i+}}\, dx,
\end{aligned}$$

Choose $\mu_i = \int_{\Omega_i} |\nabla u|^{p(x)}\, dx \Big/ \left[p_{i-}^{-1} \int_{\Omega_i} |\nabla u|^{p_{i-}}\, dx + p_{i+}^{-1} \int_{\Omega_i} |\nabla u|^{p_{i+}}\, dx\right]$.

Since by (2.11), $\int_{\Omega_i} |u|^{p_{i-}} \log(|u|)\, dx \geq 0$ for all $i = 1, 2, ..., l$, we may write

$$\int_{\Omega_{i-}} \left(|u|^{p(x)} - |u|^{p_{i-}} - |u|^{p_{i+}}\right) \log(|u|)\, dx \leq -\int_{\Omega_{i-}} |u|^{p_{i-}} \log(|u|)\, dx \leq \int_{\Omega_{i+}} \left(|u|^{p_{i-}} + |u|^{p_{i+}} - |u|^{p(x)}\right) \log(|u|)\, dx$$

which means that

$$\int_{\Omega_i} |u|^{p_{i-}} \log(|u|)\, dx + \int_{\Omega_i} |u|^{p_{i+}} \log(|u|)\, dx - \int_{\Omega_i} |u|^{p(x)} \log(|u|)\, dx \geq 0. \tag{2.14}$$

It follows from (2.10), (2.12) and (2.13) that

$$\|u\|_{p_{i-},\Omega_i} \|u\|_{p_{i+},\Omega_i} \leq \min\left\{\|u\|_{p_{i-},\Omega_i}, \|u\|_{p_{i+},\Omega_i}\right\} \leq \|u\|_{p(x),\Omega_i}.$$

There exists $\delta \in (0, 1)$ such that by (2.13) we find

$$\begin{aligned}
&\log\left(\|u\|_{p_{i-},\Omega_i}\right) \|u\|_{p_{i-},\Omega_i} + \log\left(\|u\|_{p_{i+},\Omega_i}\right) \|u\|_{p_{i+},\Omega_i} \\
&\leq \left(\delta \max\left\{\|u\|_{p_{i-},\Omega_i}, \|u\|_{p_{i+},\Omega_i}\right\} + (1-\delta) \min\left\{\|u\|_{p_{i-},\Omega_i}, \|u\|_{p_{i+},\Omega_i}\right\}\right) \log\left(\|u\|_{p_{i-},\Omega_i} \|u\|_{p_{i+},\Omega_i}\right) \\
&\leq \log\left(\|u\|_{p(x),\Omega_i}\right) \int_{\Omega_i} |u|^{p(x)}\, dx.
\end{aligned} \tag{2.15}$$

So (2.14) and (2.15) give

$$0 \leq \int_{\Omega_i} |u|^{p_{i-}} \log\left(\frac{|u|}{\|u\|_{p_{i-},\Omega_i}}\right) dx + \int_{\Omega_i} |u|^{p_{i+}} \log\left(\frac{|u|}{\|u\|_{p_{i+},\Omega_i}}\right) dx - \int_{\Omega_i} |u|^{p(x)} \log\left(\frac{|u|}{\|u\|_{p(x),\Omega_i}}\right) dx$$

Therefore we obtain

$$\int_{\Omega_i} |\nabla u|^{p(x)}\, dx - \int_{\Omega_i} |u|^{p(x)} \log|u|\, dx \geq a\left[\log\left(\left(\frac{p_{i-}\mu_i e}{n\mathcal{L}_{p_{i-}}}\right)^{\frac{n}{p_{i-}^2}} \left(\frac{p_{i+}\mu_i e}{n\mathcal{L}_{p_{i+}}}\right)^{\frac{n}{p_{i+}^2}}\right) - \frac{1}{p_{i+}} \log\left(\int_{\Omega_i} |u|^{p(x)}\, dx\right)\right],$$

where $a = \min\left\{\|u\|_{p_{i-},\Omega_i}, \|u\|_{p_{i+},\Omega_i}\right\}$.

By applying the well known property of the logarithmic function $\sum_{i\geq 1} \log(\delta_i) \leq \log\left(\sum_{i\geq 1} \delta_i\right)$ for $\delta_i \leq 1$, we get

$$\begin{aligned} I(u) &= \sum_{i=1}^{l}\left(\int_{\Omega_i} |\nabla u|^{p(x)}\, dx - \int_{\Omega_i} |u|^{p(x)} \log|u|\, dx\right) \\ &\geq a\left[\log\prod_{i=1}^{l}\left(\left(\frac{p_{i-}\mu_i e}{n\mathcal{L}_{p_{i-}}}\right)^{\frac{n}{p_{i-}^2}} \left(\frac{p_{i+}\mu_i e}{n\mathcal{L}_{p_{i+}}}\right)^{\frac{n}{p_{i+}^2}}\right) - \sum_{i=1}^{l}\frac{1}{p_{i+}} \log\left(\int_{\Omega_i} |u|^{p(x)}\, dx\right)\right] \\ &\geq a\left[\log\prod_{i=1}^{l}\left(\left(\frac{p_{i-}\mu_i e}{n\mathcal{L}_{p_{i-}}}\right)^{\frac{n}{p_{i-}^2}} \left(\frac{p_{i+}\mu_i e}{n\mathcal{L}_{p_{i+}}}\right)^{\frac{n}{p_{i+}^2}}\right) - \frac{1}{p_-} \log\left(\int_{\Omega} |u|^{p(x)}\, dx\right)\right]. \end{aligned} \tag{2.16}$$

If $\|u\|_{p(x)} \leq 1$ then from (2.16) we have

$$I(u) \geq a\left[\log\prod_{i=1}^{l}\left(\left(\frac{p_{i-}\mu_i e}{n\mathcal{L}_{p_{i-}}}\right)^{\frac{n}{p_{i-}^2}} \left(\frac{p_{i+}\mu_i e}{n\mathcal{L}_{p_{i+}}}\right)^{\frac{n}{p_{i+}^2}}\right) - \log\left(\|u\|_{p(x)}\right)\right], \tag{2.17}$$

if $\|u\|_{p(x)} > 1$ then (2.16) gives

$$I(u) \geq a\left[\log\prod_{i=1}^{l}\left(\left(\frac{p_{i-}\mu_i e}{n\mathcal{L}_{p_{i-}}}\right)^{\frac{n}{p_{i-}^2}} \left(\frac{p_{i+}\mu_i e}{n\mathcal{L}_{p_{i+}}}\right)^{\frac{n}{p_{i+}^2}}\right) - \log\left(\|u\|_{p(x)}^{p_+/p_-}\right)\right]. \tag{2.18}$$

Seting

$$R = \prod_{i=1}^{l}\left(\left(\frac{p_{i-}\mu_i e}{n\mathcal{L}_{p_{i-}}}\right)^{\frac{n}{p_{i-}^2}} \left(\frac{p_{i+}\mu_i e}{n\mathcal{L}_{p_{i+}}}\right)^{\frac{n}{p_{i+}^2}}\right).$$

(1) From (2.17)-(2.18) we may deduce that for $0 < \max\left\{\|u\|_{p(x)}, \|u\|_{p(x)}^{p_+/p_-}\right\} < R$, then

$$I(u) > 0.$$

(2) suppose that $I(u) < 0$. then from (2.17)-(2.18) we find

$$\log\left(R/\|u\|_{p(x)}\right) < 0, \quad \text{and} \quad \log\left(R/\|u\|_{p(x)}^{p_+/p_-}\right) < 0,$$

this means that

$$R/\|u\|_{p(x)} < 1, \quad \text{and} \quad R/\|u\|_{p(x)}^{p_+/p_-} < 1$$

which implies

$$\min\left\{\|u\|_{p(x)}, \|u\|_{p(x)}^{p_+/p_-}\right\} > R.$$

To prove (3), we proceed in the same way as in the proof of (2), which completes the proof of the lemma. □

Let us denote by $\mathcal{N}$ the Nehari manifold

$$\mathcal{N} = \{u \in X_0 : I(u) = 0\}.$$

Clearly, $\mathcal{N}$ is not an empty set in accordance with Lemma 2.4 (Notice that $I(\lambda^* u) = 0$ then $\lambda^* u$ is in $\mathcal{N}$). Furthermore, we have the following result.

**Lemma 2.6.**
*1) Assume that $p_+ + \varrho_+ < p_-^*$, then the functional $J$ is coercive on $\mathcal{N}$,*
*2) The functionals $J$ and $I$ are weakly lower semicontinuous.*

*Proof.* 1) Remember that $J$ is coercive on $\mathcal{N}$ if $\lim_{u\in\mathcal{N}, \|u\|_{W_0^{1,p(x)}}\to\infty} J(u) = \infty$. For this we assume that

$$\|u\|_{p(x)}, \|\nabla u\|_{p(x)} > 1. \tag{2.19}$$

Now suppose that $u \in \mathcal{N}$, from formula (2.6) we get

$$J(u) \geq \frac{1}{p_+^2}\int_\Omega |u|^{p(x)}\, dx. \tag{2.20}$$

According to Remark 2.2, we may write

$$\begin{aligned}\int_\Omega |u(x)|^{p(x)} \log|u(x)|\, dx &\leq \int_{\Omega_-} |u(x)|^{p_+} \log|u(x)|\, dx + \int_{\Omega_+} |u(x)|^{p_+} \log|u(x)|\, dx \\ &\leq \int_{\Omega_+} |u(x)|^{p_+} \log|u(x)|\, dx \\ &\leq \frac{1}{\varrho_-}\int_\Omega |u(x)|^{p_+ + \varrho_+}\, dx.\end{aligned}$$

Therefore we apply the Lemma 2.3 by using the assumption $p_- < p_+ + \varrho_+ < p_-^*$, we obtain

$$\int_\Omega |u(x)|^{p(x)} \log|u(x)|\, dx \leq C\|u\|_{p_-}^{(1-\alpha)(p_+ + \varrho_+)} \|\nabla u\|_{p_-}^{\alpha(p_+ + \varrho_+)},$$

where

$$p_-^* = \frac{np_-}{n - p_-} \quad \text{and} \quad \alpha = n\left(\frac{1}{p_-} - \frac{1}{p_+ + \varrho_+}\right) = \frac{n(p_+ + \varrho_+ - p_-)}{p_-(p_+ + \varrho_+)}.$$

By Proposition 2.4 and Proposition 2.7, the continuous embeddings $L^{p(x)} \hookrightarrow L^{p_-}(\Omega)$, $W_0^{1,p(x)}(\Omega) \hookrightarrow W_0^{1,p_-}(\Omega)$ ensure that

$$\int_\Omega |u(x)|^{p(x)} \log |u(x)|\, dx \leq C \|u\|_{p(x)}^{(1-\alpha)(p_+ + \varrho_+)} \|\nabla u\|_{p(x)}^{\alpha(p_+ + \varrho_+)}. \tag{2.21}$$

To complete the proof of 1) we have two cases for $p_+$.

**Case 1 :** if $p_+ < p_- + \frac{p_-^2}{n}$, in this case choose $0 < \varrho_+ < \left(p_- + \frac{p_-^2}{n}\right) - p_+$, then $p_- > \alpha(p_+ + \varrho_+)$, by Young's inequality, we get

$$\int_\Omega |u(x)|^{p(x)} \log |u(x)|\, dx \leq C_\varepsilon \left(\|u\|_{p(x)}^{p_-}\right)^\beta + \varepsilon \|\nabla u\|_{p(x)}^{p_-},$$

where $\varepsilon > 0$ and $\beta = \frac{(1-\alpha)(p_+ + \varrho)}{p_- - \alpha(p_+ + \varrho)} > 1$. Since $u \in \mathcal{N}$, so if $\|\nabla u\|_{p(x)} > 1$, we find from Proposition 2.2 (with $u$ is replaced by $\nabla u$) that

$$\|\nabla u\|_{p(x)}^{p_-} \leq \int_\Omega |\nabla u|^{p(x)}\, dx = \int_\Omega |u|^{p(x)} \log |u|\, dx \leq C_\varepsilon \left(\|u\|_{p(x)}^{p_-}\right)^\beta + \varepsilon \|\nabla u\|_{p(x)}^{p_-}.$$

Taking $\varepsilon < 1$ so from (2.20) and Proposition 2.2 by using the assumption $\|u\|_{p(x)} > 1$, yield

$$J(u) \geq c_\varepsilon \left(\|\nabla u\|_{p(x)}^{p_-}\right)^{\frac{1}{\beta}}, \tag{2.22}$$

Hence, $J$ is coercive on $\mathcal{N}$.

**Case 2 :** if $p_+ < p_- + \frac{p_-^2}{n}$ does not hold. So divided $\Omega$ into $l$ subsets in the following way $\overline{\Omega} = \cup_{j=1}^{l} \overline{\Omega}_j$ and in every subset $\Omega_j$ we have $p_{j+} < p_{j-} + \frac{p_{j-}^2}{n}$, $(j = 1, 2, ..., l)$ and $\varrho_{j+} < p_{j-}^* - p_{j+}$, $(j = 1, 2, ..., l)$. Here we have used the assumption (1.2) and the continuity of the functions $p(x)$ and $\varrho(x)$. For any $u \in \mathcal{N}$, we shall show that (2.22) holds, for this we assume that $\|u\|_{p(x),\Omega_j} \leq 1$ and $\|\nabla u\|_{p(x),\Omega_j} \leq 1$, for all $j = 1, 2, ..., l$. Therefore, choose $\varrho_{j+} < \left(p_{j-} + \frac{p_{j-}^2}{n}\right) - p_{j+}$. Then $p_{j-} > \alpha_j \left(p_{j+} + \varrho_{j+}\right)$, $(j = 1, 2, ..., l)$. As above we have

$$\begin{aligned}\int_{\Omega_j} |u(x)|^{p(x)} \log |u(x)|\, dx &\leq C_{j\varepsilon} \|u\|_{p(x),\Omega_j}^{\beta_j p_{j-}} + \varepsilon \|\nabla u\|_{p(x),\Omega_j}^{p_{j-}} \\ &\leq C_{j\varepsilon} \left(\int_{\Omega_j} |u|^{p(x)}\, dx\right)^{\beta_j p_{j-}/p_{j+}} + \varepsilon \left(\int_{\Omega_j} |\nabla u|^{p(x)}\, dx\right)^{p_{j-}/p_{j+}}\end{aligned}$$

for every $j = 1, 2, ..., l$, where $\beta_j = \frac{(1-\alpha_j)(p_{j+} + \varrho_{j+})}{p_{j-} - \alpha_j(p_{j+} + \varrho_{j+})} > 1$, here we have used Proposition 2.2. Because of

$$\begin{aligned}\int_\Omega |u(x)|^{p(x)} \log |u(x)|\, dx &\leq \sum_{j=1}^{l} \int_{\Omega_j} |u(x)|^{p(x)} \log |u(x)|\, dx \\ &\leq \sum_{j=1}^{l} \left[ C_{j\varepsilon} \left(\int_{\Omega_j} |u|^{p(x)}\, dx\right)^{\beta_j p_{j-}/p_{j+}} + \varepsilon \left(\int_{\Omega_j} |\nabla u|^{p(x)}\, dx\right)^{p_{j-}/p_{j+}} \right].\end{aligned}$$

Set $\beta = \min_{1\leq j\leq l}\beta_j > 1$, since $\|u\|_{p(x),\Omega_j}, \|\nabla u\|_{p(x),\Omega_j} \leq 1$ and $1 < p_- \leq p_{j-} \leq p_{j+} \leq p_+, \forall j = 1, 2, ..., l$, then

$$\begin{aligned}
\int_\Omega |u(x)|^{p(x)} \log|u(x)|\,dx &\leq C_\varepsilon \sum_{j=1}^{l}\left(\int_{\Omega_j} |u|^{p(x)}\,dx\right)^{\beta p_-/p_+} + \varepsilon \sum_{j=1}^{l}\left(\int_{\Omega_j} |\nabla u|^{p(x)}\,dx\right)^{p_-/p_+} \\
&\leq C_\varepsilon \left(\sum_{j=1}^{l}\int_{\Omega_j} |u|^{p(x)}\,dx\right)^{\beta p_-/p_+} + \varepsilon \left(\sum_{j=1}^{l}\int_{\Omega_j} |\nabla u|^{p(x)}\,dx\right)^{p_-/p_+} \\
&= C_\varepsilon \left(\left(\int_{\Omega} |u|^{p(x)}\,dx\right)^{p_-/p_+}\right)^{\beta} + \varepsilon \left(\int_{\Omega} |\nabla u|^{p(x)}\,dx\right)^{p_-/p_+}.
\end{aligned}$$

If (2.19) holds, Proposition 2.2 gives us

$$\int_\Omega |u(x)|^{p(x)} \log|u(x)|\,dx \leq C_\varepsilon \left(\|u\|_{p(x),\Omega}^{p_-}\right)^{\beta} + \varepsilon \|\nabla u\|_{p(x),\Omega}^{p_-}.$$

Similarly we get (2.22) as above, Thus, $J$ is coercive on $\mathcal{N}$.

2) Consider in $X_0$ the sequence $(u_n)_{n\in\mathbb{N}}$ such that $u_n \rightharpoonup u$ weakly in $X_0$, so Proposition 2.3, Proposition 2.6 and Proposition 2.7 ensure that $(u_n)_{n\in\mathbb{N}}$ is bounded in $X_0$, there exists a subsequence of $(u_n)_{n\in\mathbb{N}}$ still denoted by $(u_n)_{n\in\mathbb{N}}$ such that $u_n \to u$ a.e in $\Omega$. So by Lebesgue dominated convergence Theorem, in view of (1.2), Lemma 2.1 and Remark 2.2, we get

$$\lim_{n\to\infty}\int_\Omega \frac{1}{p(x)}|u_n|^{p(x)}\log|u_n|\,dx = \int_\Omega \frac{1}{p(x)}|u|^{p(x)}\log|u|\,dx, \tag{2.23}$$

and

$$\lim_{n\to\infty}\int_\Omega |u_n|^{p(x)}\log|u_n|\,dx = \int_\Omega |u|^{p(x)}\log|u|\,dx. \tag{2.24}$$

Moreover from Fatou Lemma we have

$$\int_\Omega \liminf_{n\to\infty}\frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx \leq \liminf_{n\to\infty}\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx, \tag{2.25}$$

and

$$\int_\Omega \liminf_{n\to\infty}\frac{1}{p^2(x)}|u_n|^{p(x)}\,dx \leq \liminf_{n\to\infty}\int_\Omega \frac{1}{p^2(x)}|u_n|^{p(x)}\,dx, \tag{2.26}$$

which means by (2.23), (2.25) and (2.26) that

$$\begin{aligned}
J(u) &\leq \liminf_{n\to\infty}\left(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx - \int_\Omega \frac{1}{p(x)}|u_n|^{p(x)}\log|u_n|\,dx + \int_\Omega \frac{1}{p^2(x)}|u_n|^{p(x)}\,dx\right) \\
&= \liminf_{n\to\infty} J(u_n).
\end{aligned}$$

For the funtional $I$ we use (2.24) and the weak lower semicontinuity of the modular $\rho_{p(.)}$ (see [12, Theorem 3.2.9 p. 77]) that is

$$\begin{aligned} I(u) &= \int_\Omega |\nabla u|^{p(x)}\,dx - \int_\Omega |u|^{p(x)} \log|u|\,dx \\ &\leq \liminf_{n\to\infty}\left(\int_\Omega |\nabla u_n|^{p(x)}\,dx - \int_\Omega |u_n|^{p(x)} \log|u_n|\,dx\right) \\ &= \liminf_{n\to\infty} I(u_n). \end{aligned}$$

Thus 2). This completes the proof. □

Now, define

$$d = \inf_{u\in\mathcal{N}} J(u), \tag{2.27}$$

and prove in the following lemma that there exists for $J$ defined on $X_0$ a nontrivial critical minimizing point $u \in \mathcal{N}$, as a solution to the stationary problem (2.27) associated to (1.1).

**Lemma 2.7.** *Let $u \in X_0$, and $p(.) \in \mathcal{P}^{\log}(\Omega)$, then the following assertions hold*
*(1) $d = \inf_{u\in X_0}\sup_{\lambda>0} J(\lambda u)$.*
*(2) There exists a positive lower bound for $d$, that is*

$$d \geq \frac{R^{p_-}}{p_+^2} = M, \tag{2.28}$$

*(3) The problem (2.27) has a positive extremal solution $u \in \mathcal{N}$, In other words, it means $J(u) = d$.*

*Proof.* Let $u \in X_0$, then by lemma 2.4, Proposition 2.2 and (2.6) (with $u$ is replaced by $\lambda^* u$) we may write

$$\begin{aligned} \sup_{\lambda>0} J(\lambda u) &= J(\lambda^* u) \geq \gamma I(\lambda^* u) + \frac{1}{p_+^2}\int_\Omega |\lambda^* u|^{p(x)} \\ &\geq \frac{1}{p_+^2} \min\left\{\|\lambda^* u\|_{p(x)}^{p_-}, \|\lambda^* u\|_{p(x)}^{p_+}\right\}. \end{aligned} \tag{2.29}$$

Firstly, we prove (1). By the definition of $\mathcal{N}$, and lemma 2.4 we can deduce that $\lambda^* u \in \mathcal{N}$. Consequently,

$$J(\lambda^* u) \geq \inf_{u\in\mathcal{N}} J(u) = d. \tag{2.30}$$

So (2.29) together with (2.30) yield that

$$\inf_{u\in X_0}\sup_{\lambda>0} J(\lambda u) \geq d. \tag{2.31}$$

In addition, if $u \in \mathcal{N}$ then it follows from (2.8) that $\lambda^* = 1$ is the only critical point in $(0, \infty)$ of the mapping $j(\lambda)$. Therefore,

$$\sup_{\lambda>0} J(\lambda u) = J(u),$$

for each $u \in \mathcal{N}$. Hence

$$\inf_{u\in X_0}\sup_{\lambda>0} J(\lambda u) \leq \inf_{u\in\mathcal{N}}\sup_{\lambda>0} J(\lambda u) = \inf_{u\in\mathcal{N}} J(u) = d. \tag{2.32}$$

Thus, (2.31) and (2.32) lead to the desired result.
(2) In accordance with lemma 2.4, for each $u \in X_0$, then $I(\lambda^* u) = 0$. Which means by lemma 2.5 that

$$\min\left\{\|\lambda^* u\|_{p(x)}, \|\lambda^* u\|_{p(x)}^{p_+/p_-}\right\} \geq R. \tag{2.33}$$

So (2.29) and (2.33) give

$$\sup_{\lambda>0} J(\lambda u) \geq \frac{R^{p_-}}{p_+^2} = M.$$

Hence, (2.28) arises from assertion (1) and therefore the assertion (2) is proved.
(3) Let $\{u_m\}_m^\infty \subset \mathcal{N}$, be a minimizing sequence for $J$, and suppose that $\{u_m\}_m^\infty$ is of positive terms ($u_m > 0$) a.e. $\Omega$ for all $m \in \mathbb{N}$, such that

$$\lim_{m\to\infty} J(u_m) = d.$$

Moreover $\{|u_m|\}_m^\infty \subset \mathcal{N}$ is also a minimizing sequence for $J$ and $J(|u_m|) = J(u_m)$ because of $u_m > 0$. In addition, we have previously seen that $J$ is coercive on $\mathcal{N}$ which implies that $\{u_m\}_m^\infty$ is bounded in $W_0^{1,p(x)}(\Omega)$. Since $p(.) \in \mathcal{P}^{\log}(\Omega)$, then the compact embedding $W_0^{1,p(.)}(\Omega) \hookrightarrow\hookrightarrow L^{p(.)}(\Omega)$ (see [12, Theorem 8.4.2]), guarantees that there exists a function $u$ and a subsequence of $\{u_m\}_m^\infty$, anyway denoted by $\{u_m\}_m^\infty$, such that

$$\begin{aligned} u_m &\to u \quad \text{weakly in} \quad W_0^{1,p(x)}(\Omega), \\ u_m &\to u \quad \text{strongly in} \quad L^{p(x)}(\Omega), \\ u_m(x) &\to u(x) \quad \text{a.e. in} \quad \Omega. \end{aligned}$$

so we have $u \geq 0$ a.e in $\Omega$. Since $p(.) \in \mathcal{P}(\Omega)$, the weak lower semicontinuity of the functional $J$ (see Lemma 2.6), yield

$$J(u) \leq \liminf_{m\to\infty} J(u_m) = d.$$

since $u_m \in \mathcal{N}$ then $u_m \in X_0$ and $I(u_m) = 0$ which implies by Lemma 2.5 that

$$\|u_m\|_{p(x)} \geq R.$$

This mean that $\|u_m\|_{p(x)} \neq 0$ by strong convergence in $L^{p(x)}(\Omega)$, that is, $u \in X_0$. Furthermore, from weak lower semicontinuity of $I(u)$ (see Lemma 2.6), we find

$$I(u) \leq \liminf_{k\to\infty} I(u_k) = 0.$$

So, to complete the proof of (3), we must show that $I(u) = 0$. Indeed, suppose that $I(u) < 0$, then, by Lemma 2.4, there exists a positive constant $\lambda^*$,

$$\lambda^* = \lambda^*(u) = \exp\left(\frac{\int_\Omega (\lambda^*)^{p(x)-1}\left(|\nabla u|^{p(x)} - |u|^{p(x)}\log|u|\right)dx}{\int_\Omega (\lambda^*)^{p(x)-1}|u|^{p(x)}\,dx}\right) < 1$$

satisfying $I(\lambda^* u) = 0$. Therefore, we have

$$0 < d \leq J(\lambda^* u) \leq \int_\Omega \frac{1}{p^2(x)}|\lambda^* u|^{p(x)}\,dx \leq (\lambda^*)^{p_-}\liminf_{k\to\infty} J(u_k) = (\lambda^*)^{p_-} d < d.$$

Which is impossible, and the lemma is proved. □

Now we presnt the potential well sets that were introduced in [28] (see also [34])

$$\begin{array}{lll} \mathcal{W}_1 = \{u \in X_0 : J(u) < d\}, & \mathcal{W}_2 = \{u \in X_0 : J(u) = d\}, & \mathcal{W} = \mathcal{W}_1 \cup \mathcal{W}_2, \\ \mathcal{W}_1^+ = \{u \in \mathcal{W}_1 : I(u) > 0\}, & \mathcal{W}_2^+ = \{u \in \mathcal{W}_2 : I(u) > 0\}, & \mathcal{W}^+ = \mathcal{W}_1^+ \cup \mathcal{W}_2^+, \\ \mathcal{W}_1^- = \{u \in \mathcal{W}_1 : I(u) < 0\}, & \mathcal{W}_2^- = \{u \in \mathcal{W}_2 : I(u) < 0\}, & \mathcal{W}^- = \mathcal{W}_1^- \cup \mathcal{W}_2^-. \end{array}$$

It is clear that, $\mathcal{W}^+ \cap \mathcal{W}^- = \varnothing$ and $\mathcal{W}^+ \cup \mathcal{W}^- = \mathcal{W}$. We call $\mathcal{W}$ the potential well and $d$ the depth of the well. Note that $\mathcal{W}^+$ is the best part of the well, so we will prove that if the initial datum belongs to $\mathcal{W}^+$ then every weak solution of the problem (1.1) exists globally in time. On the other hand, a result of blow up for weak solutions may be obtained if the initial datum belongs to $\mathcal{W}^-$.

**Remark 2.5.** *According to (2.6), it is easy to show that*

$$\mathcal{W}_1^+ = \{u \in X_0 : \ 0 < J(u) < d, \ I(u) > 0\}.$$

## 3. Global existence and decay estimates

This section is firstly devoted to establishing the local existence of solutions of problem (1.1), and thus we prove the global existence of weak solutions of problem (1.1), taking into account that the initial datum belong to $\mathcal{W}^+$. Next, we shall show as in [28] that the decay of the norm $\|u(t)\|_2$ is polynomial rather than exponential as given in [9] with respect to the case $p(x) \equiv 2$. The proof of the last objective is based on the following lemma presented by Martinez [31].

**Lemma 3.1.** *Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a nonincreasing function and $\sigma$ is a nonegative constant such that*

$$\int_t^{+\infty} f^{1+\sigma}(s)\, ds \le \frac{1}{\omega} f^{\sigma}(0)\, f(t), \quad \forall t \ge 0.$$

*Then we have*
*(1) $f(t) \le f(0)\, e^{1-\omega t}$, for all $t \ge 0$, whenever $\sigma = 0$,*
*(2) $f(t) \le f(0) \left(\frac{1+\sigma}{1+\omega\sigma t}\right)^{\frac{1}{\sigma}}$, for all $t \ge 0$, whenever $\sigma > 0$.*

Now we start with the local existence of solutions to our problem.

**Theorem 3.1** (Local existence)**.** *Assume that $u_0 \in X_0, p \in C\left(\overline{\Omega}\right)$ satisfying the condition $2 < p_- \le p(x) \le p_+ < p_-^*$ and the log-Hölder continuous condition (2.1). Then the problem (1.1) has a weak local solution $u(x,t)$ on $\Omega \times (0, T_0)$, satisfying the energy inequality*

$$\int_0^t \|u_s(s)\|_2^2\, ds + J(u(t)) \le J(u_0), \quad t \in [0, T_0). \tag{3.1}$$

*where $T_0$ is a positive constant.*

*Proof.* By using The Faedo-Galerkin's methods: We consider in the space $W_0^{1,p(x)}(\Omega)$, the basis $\left\{w_j\right\}_{j=1}^{\infty}$ and let $V_m$ be the finite dimensional space defined by

$$V_m = \operatorname{span}\{w_1, w_2, ..., w_m\}.$$

Suppose that $u_{0m}$ an element of $V_m$ such that

$$u_{0m} = \sum_{j=1}^{m} a_{mj} w_j \to u_0 \quad \text{strongly in } W_0^{1,p(x)}(\Omega). \tag{3.2}$$

when $m \to +\infty$. Defining the approximate solution $u_m(x,t)$ of the problem (1.1) as follows

$$u_m(x,t) = \sum_{j=1}^{m} \alpha_{mj}(t)\, w_j(x),$$

with coefficients $\alpha_{mj}$ $(1 \le j \le m)$ satisfying the following ordinary differential equations

$$\int_\Omega u_{mt}(t)\, w_i dx + \int_\Omega |\nabla u_m(t)|^{p(x)-2} \nabla u_m(t) \nabla w_i dx = \int_\Omega |u_m(t)|^{p(x)-2} u_m(t) \log|u_m(t)|\, w_i dx, \tag{3.3}$$

$1 \le i \le m$, with the initial conditions

$$\alpha_{mj}(0) = a_{mj}, \quad 1 \le j \le m. \tag{3.4}$$

The Theorem of Peano guarantees the local existence of the solution of a system (3.3)-(3.4). Let us multiply the $i^{th}$ equation in (3.3) by $\alpha_{mi}(t)$ and take the sum over $i$ from 1 to $m$ we get

$$\frac{1}{2}\frac{d}{dt}\|u_m(t)\|_2^2 + \int_\Omega |\nabla u_m(t)|^{p(x)} dx = \int_\Omega |u_m(t)|^{p(x)} \log|u_m(t)|\, dx. \tag{3.5}$$

In view of Remark 2.2, we have

$$\begin{aligned}\int_\Omega |u_m(t)|^{p(x)} \log|u_m(t)|\, dx &\le \int_{\Omega_-} |u_m(t)|^{p_+} \log|u_m(t)|\, dx + \int_{\Omega_+} |u_m(t)|^{p_+} \log|u_m(t)|\, dx \\ &\le \frac{e^{-1}}{\varrho_-} \int_{\Omega_+} |u_m(t)|^{p_+ + \varrho_+} dx \\ &\le \frac{1}{\varrho_-} \int_\Omega |u_m(t)|^{p_+ + \varrho_+} dx, \end{aligned} \tag{3.6}$$

for some $\varrho_+$ chosen small enough such that $2 < p_- < p_+ + \varrho_+ \le p_-^* = \frac{np_-}{n-p_-}$ with $\Omega_- = \{x \in \Omega \ \ |u_m(t)| \le 1\}$ and $\Omega_+ = \{x \in \Omega \ \ |u_m(t)| > 1\}$. By Lemma 2.1 we get from (3.6),

$$\int_\Omega |u_m(t)|^{p(x)} \log|u_m(t)|\, dx \le C \|u_m(t)\|_{p_-^*}^{\theta(p_+ + \varrho_+)} \|u_m(t)\|_2^{(1-\theta)(p_+ + \varrho_+)}$$

with $0 < \theta < 1$ and $\frac{1}{p_+ + \varrho_+} = \frac{\theta(n-p_-)}{np_-} + \frac{1-\theta}{2}$. From the continuous embeddings $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{p^*(x)}(\Omega) \hookrightarrow L^{p_-^*}(\Omega)$, we have

$$\int_\Omega |u_m(t)|^{p(x)} \log|u_m(t)|\, dx \le C \|\nabla u_m(t)\|_{p(x)}^{\theta(p_+ + \varrho_+)} \|u_m(t)\|_2^{(1-\theta)(p_+ + \varrho_+)}. \tag{3.7}$$

We distinguishe two cases
**Case 1:** if $\|\nabla u_m\|_{p(x)} > 1$. Assume that $p_+ < (1 + \frac{2}{n})p_-$, it follows from (3.7) that

$$\int_\Omega |u_m(t)|^{p(x)} \log|u_m(t)|\, dx \le C \left( \int_\Omega |\nabla u_m(t)|^{p(x)} dx \right)^{\frac{\theta(p_+ + \varrho_+)}{p_-}} \|u_m(t)\|_2^{(1-\theta)(p_+ + \varrho_+)},$$

by Young's inequality we have

$$\int_{\Omega} |u_m(t)|^{p(x)} \log |u_m(t)|\, dx \leq \varepsilon \int_{\Omega} |\nabla u_m(t)|^{p(x)}\, dx + C_\varepsilon \left( \|u_m(t)\|_2^2 \right)^{\nu}, \tag{3.8}$$

where we have chosen $0 < \varrho_+ < (1 + \frac{2}{n})p_- - p_+$ so that $\theta(p_+ + \varrho_+) < p_-$, with

$$\theta = \left( \frac{1}{2} - \frac{1}{p_+ + \varrho_+} \right) \left( \frac{1}{n} - \frac{1}{p_-} + \frac{1}{2} \right)^{-1} \text{ and } \nu = \frac{p_-(1-\theta)(p_+ + \varrho_+)}{2\left[p_- - \theta(p_+ + \varrho_+)\right]} > 1.$$

If $p_+ < (1 + \frac{2}{n})p_-$ does not hold. here we use the same as in the proof of Lemma 2.6. By dividing $\Omega$ into $l$ subsets such that $\overline{\Omega} = \cup_{j=1}^{l} \overline{\Omega}_j$ with $p_{j+} < (1 + \frac{2}{n})p_{j-}$, $(j = 1, 2, ..., l)$ and $\varrho_{j+} < p_{j-}^{*} - p_{j+}$, $(j = 1, 2, ..., l)$. For $l$ large enough, assume that $\|\nabla u_m(t)\|_{p(x),\Omega_j} \leq 1$, for all $j = 1, 2, ..., l$. So, choose $\varrho_{j+} < (1 + \frac{2}{n})p_{j-} - p_{j+}$. Then $p_{j-} > \theta_j \left(p_{j+} + \varrho_{j+}\right)$, $(j = 1, 2, ..., l)$. By Proposition 2.2 we find

$$\int_{\Omega_j} |u_m(t)|^{p(x)} \log |u_m(t)|\, dx \leq C_j \left( \int_{\Omega_j} |\nabla u_m(t)|^{p(x)}\, dx \right)^{\frac{\theta_j\left(p_{j+}+\varrho_{j+}\right)}{p_{j+}}} \|u_m(t)\|_{2,\Omega_j}^{(1-\theta_j)\left(p_{j+}+\varrho_{j+}\right)}$$

Young's inequality gives

$$\int_{\Omega_j} |u_m(t)|^{p(x)} \log |u_m(t)|\, dx \leq \varepsilon \left( \int_{\Omega_j} |\nabla u_m(t)|^{p(x)}\, dx \right)^{p_{j-}/p_{j+}} + C_{j\varepsilon} \left( \|u_m(t)\|_{2,\Omega_j}^2 \right)^{\nu_j}$$

where

$$\theta_j = \left( \frac{1}{2} - \frac{1}{p_{j+} + \varrho_{j+}} \right) \left( \frac{1}{n} - \frac{1}{p_{j-}} + \frac{1}{2} \right)^{-1} \text{ and } \nu_j = \frac{p_{j-}\left(1-\theta_j\right)\left(p_{j+} + \varrho_{j+}\right)}{2\left[p_{j-} - \theta_j\left(p_{j+} + \varrho_{j+}\right)\right]} > 1.$$

We can see that

$$\begin{aligned}
\int_{\Omega} |u(x)|^{p(x)} \log |u(x)|\, dx &\leq \sum_{j=1}^{l} \int_{\Omega_j} |u(x)|^{p(x)} \log |u(x)|\, dx \\
&\leq \sum_{j=1}^{l} C_{j\varepsilon} \left( \int_{\Omega_j} |u_m(t)|^2\, dx \right)^{\nu_j} + \varepsilon \sum_{j=1}^{l} \left( \int_{\Omega_j} |\nabla u_m(t)|^{p(x)}\, dx \right)^{p_{j-}/p_{j+}}.
\end{aligned}$$

Set $\nu = \min_{1 \leq j \leq l} \nu_j > 1$, since $\|\nabla u_m(t)\|_{p(x),\Omega_j} \leq 1$ and $1 < p_- \leq p_{j-} \leq p_{j+} \leq p_+$, $\forall j = 1, 2, ..., l$, then

$$\begin{aligned}
\int_{\Omega} |u(x)|^{p(x)} \log |u(x)|\, dx &\leq C_\varepsilon \sum_{j=1}^{l} \left( \int_{\Omega_j} |u_m(t)|^2\, dx \right)^{\nu} + \varepsilon \sum_{j=1}^{l} \left( \int_{\Omega_j} |\nabla u_m(t)|^{p(x)}\, dx \right)^{p_-/p_+} \\
&\leq C_\varepsilon \left( \sum_{j=1}^{l} \int_{\Omega_j} |u_m(t)|^2\, dx \right)^{\nu} + \varepsilon \left( \sum_{j=1}^{l} \int_{\Omega_j} |\nabla u_m(t)|^{p(x)}\, dx \right)^{p_-/p_+}
\end{aligned}$$

$$= C_\varepsilon \left(\|u_m(t)\|_{2,\Omega}^2\right)^\nu + \varepsilon \int_\Omega |\nabla u_m(t)|^{p(x)}\, dx,$$

and (3.8) follows again
**Case 2:** if $\|\nabla u_m\|_{p(x)} \leq 1$, so (3.7) becomes

$$\int_\Omega |u_m(t)|^{p(x)} \log|u_m(t)|\, dx \leq C\left(\|u_m(t)\|_2^2\right)^{\frac{(1-\theta)(p_+ + \varrho_+)}{2}}$$

Since $\frac{p_-}{p_- - \theta(p_+ + \varrho_+)} > 1$ then $\frac{2\nu}{(1-\theta)(p_+ + \varrho_+)} > 1$. Young's inequality gives

$$\int_\Omega |u_m(t)|^{p(x)} \log|u_m(t)|\, dx \leq C + \left(\|u_m(t)\|_2^2\right)^\nu. \tag{3.9}$$

We combine (3.8) and (3.9) we find

$$\int_\Omega |u_m(t)|^{p(x)} \log|u_m(t)|\, dx \leq \varepsilon \int_\Omega |\nabla u_m(t)|^{p(x)}\, dx + c_\varepsilon \left(\|u_m(t)\|_2^2\right)^\nu. \tag{3.10}$$

This combined with (3.5) yields

$$\frac{1}{2}\frac{d}{dt}\|u_m(t)\|_2^2 + (1-\varepsilon)\int_\Omega |\nabla u_m(t)|^{p(x)}\, dx \leq c_\varepsilon \left(\|u_m(t)\|_2^2\right)^\nu.$$

Choose $0 < \varepsilon < 1$, then for all $t \in [0, T_0]$ where $T_0$ is a positive constant, we find

$$\frac{1}{2}\frac{d}{dt}\|u_m(t)\|_2^2 \leq c_\varepsilon \left(\|u_m(t)\|_2^2\right)^\nu.$$

setting $\eta = \max_{m\in\mathbb{N}} \|u_{0m}\|_2^2$, $r(s) = s^\nu$, $g(s) = 2c_\varepsilon$ and choose $T_0$ such that there exists a positive constant $C_0$ large enough satisfying $T_0 \leq \frac{\eta^{1-\nu} - C_0^{1-\nu}}{2c_\varepsilon(\nu-1)}$, so that Bihari's integral inequality yields

$$\|u_m(t)\|_2^2 \leq C_0, \quad \forall t \in [0, T_0]. \tag{3.11}$$

Let us multiply again the two sides of (3.3) by $\alpha'_{mi}(t)$, and take the sum over $i = 1, 2...m$, and then integrate with respect to time on $[0, t]$. We obtain

$$\int_0^t \|u_{ms}(s)\|_2^2\, ds + J(u_m(t)) = J(u_m(0)). \tag{3.12}$$

Then (3.2) means that there is a positive constant $C_1$ such that

$$J(u_m(0)) \leq C_1, \quad \text{for all } m. \tag{3.13}$$

On the other hand, (3.10) and (3.11) with the help of (2.6) (where $u$ is replaced by $u_m(t)$), derive that

$$J(u_m(t)) \geq \gamma(1-\varepsilon)\int_\Omega |\nabla u_m(t)|^{p(x)}\, dx + \frac{1}{p_+^2}\int_\Omega |u_m(t)|^{p(x)}\, dx - \gamma c_\varepsilon (C_0)^\nu, \tag{3.14}$$

where $C_0$ is a positive constant depending on $T_0$.
So (3.12) and (3.14) give

$$\int_0^t \|u_{ms}(s)\|_2^2\, ds + \frac{1-\varepsilon}{p_+}\int_\Omega |\nabla u_m(t)|^{p(x)}\, dx \le C_1 + \frac{c_\varepsilon (C_0)^\nu}{p_-}$$

This means, for $\varepsilon < 1$ that

$$\int_\Omega |\nabla u_m(t)|^{p(x)}\, dx \le C, \tag{3.15}$$

and

$$\|u_{mt}\|_{L^2(0,T_0;L^2(\Omega))} \le C. \tag{3.16}$$

If we combine a priori estimates (3.15), (3.16) we conclude that there exists a function $u$ and a subsequence of $\{u_m\}_{m=1}^\infty$ again denoted by $\{u_m\}_{m=1}^\infty$ such that

$$u_m \to u \quad \text{weakly}^* \text{ in } L^\infty\left(0,T_0; W_0^{1,p(x)}(\Omega)\right), \tag{3.17}$$

$$u_{mt} \to u_t \quad \text{weakly in } L^2\left(0,T_0; L^2(\Omega)\right), \tag{3.18}$$

$$|\nabla u_m|^{p(x)-2}\nabla u_m \to \chi \quad \text{weakly}^* \text{ in } L^\infty\left(0,T_0; W_0^{-1,p'(x)}(\Omega)\right). \tag{3.19}$$

Due to the compact embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{r(x)}(\Omega)$ given in Proposition 2.8 and by the compactness theorem of Aubin-Lions-Simon, it follows from (3.17) and (3.18) that

$$u_m \to u \text{ strongly in } C\left([0,T_0]; L^{r(x)}(\Omega)\right),$$

for all function $r(.)$ such that $2 \le r(x) \ll p^*(x) = \frac{np(x)}{n-p(x)}$. Obviously, this means by the continuity of the function $u_m \mapsto |u_m|^{p(x)-2} u_m \log|u_m|$ that

$$|u_m|^{p(x)-2} u_m \log|u_m| \to |u|^{p(x)-2} u \log|u| \quad \text{a.e. } (x,t) \in \Omega\times(0,T_0). \tag{3.20}$$

On the other hand, a simple calculation, gives

$$\begin{aligned}\int_\Omega \left|\psi_m(x,t)\right|^{p'(x)} dx &\le \int_{\Omega_-} \left|\psi_m(x,t)\right|^{p'_+} dx + \int_{\Omega_+} \left|\psi_m(x,t)\right|^{p'_+} dx \\ &\le \left(\frac{e^{-1}}{p_- - 1}\right)^{p'_+} |\Omega| + \left(\frac{p_+}{q_-}\right)^{p'_+} \int_{\Omega_+} |u_m(t)|^{q(x)}\, dx\end{aligned}$$

where $q(.) : \Omega \to \mathbb{R}$ is a measurable function satisfies $p_+ \le q_- \le q(x) \le q_+ < p^*(x)$, $\psi_m(x,t) = |u_m(x,t)|^{p(x)-1}\log|u_m(x,t)|$ and

$$\Omega_- = \{x\in\Omega : |u_m(x,t)| \le 1\}, \quad \Omega_+ = \{x\in\Omega : |u_m(x,t)| > 1\},$$

by Proposition 2.2 we get

$$\int_\Omega \left|\psi_m(x,t)\right|^{p'(x)} dx \le \left(\frac{e^{-1}}{p_- - 1}\right)^{p'_+} |\Omega| + \left(\frac{p_+}{q_-}\right)^{p'_+} \max\left\{\|u_m(t)\|_{q(x)}^{q_-}, \|u_m(t)\|_{q(x)}^{q_+}\right\}$$

Using the embeddings $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{p^*(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ (see Proposition 2.4 and Proposition 2.9) we find

$$\int_\Omega \left|\psi_m(x,t)\right|^{p'(x)} dx \le \left(\frac{1}{p_- - 1}\right)^{p'_+} |\Omega| + \left(\frac{p_+}{q_-}\right)^{p'_+} \max\left\{S^{q_-} \|\nabla u_m(t)\|_{p(x)}^{q_-}, S^{q_+} \|\nabla u_m(t)\|_{p(x)}^{q_+}\right\} \le C_0,$$

since by (3.15) where $S$ is the best constant of the Sobolev embedding. Then, using Lions lemma (see [6, Lemma 1.3 p. 12]), we deduce from (3.18) and (3.19) that

$$|u_m|^{p(x)-2} u_m \log|u_m| \to |u|^{p(x)-2} u \log|u| \quad \text{weakly}^* \text{ in } L^\infty\left(0,T_0;L^{p'(x)}(\Omega)\right). \tag{3.21}$$

Taking, in (3.3) and (3.4) the limit as $m \to +\infty$, and then by using (3.17)-(3.19) and (3.21), it is readily shown that $u$ satisfies the initial condition $u(0) = u_0$ and

$$\int_\Omega u_t(t)\, w dx + \int_\Omega \chi(t) \nabla w dx = \int_\Omega |u(t)|^{p(x)-2} u(t) \log|u(t)|\, w dx, \tag{3.22}$$

for all $w \in W_0^{1,p(x)}(\Omega)$ and for almost every $t \in [0,T_0]$. Finally, by means of well-known arguments from the theory of monotone operators in variable exponent spaces (see [13]) and Minty's trick we obtain

$$\chi = |\nabla u|^{p(x)-2} \nabla u$$

Indeed, we show that

$$\limsup_{m\to\infty} \int_0^{T_0}\int_\Omega |\nabla u_m(t)|^{p(x)} dxdt \le \int_0^{T_0}\int_\Omega \chi(t) \nabla u(t)\, dxdt.$$

So on one hand because of $u_m$ is a test function we have from (3.3)

$$\int_0^{T_0}\int_\Omega |\nabla u_m(t)|^{p(x)} dxdt = -\int_0^{T_0}\int_\Omega u_{mt}(t)\, u_m(t)\, dxdt + \int_0^{T_0}\int_\Omega |u_m(t)|^{p(x)} \log|u_m(t)|\, dxdt$$

an integration by parts gives

$$\int_0^{T_0}\int_\Omega |\nabla u_m(t)|^{p(x)} dxdt = -\frac{1}{2}\|u_m(T_0)\|_2^2 + \frac{1}{2}\|u_m(0)\|_2^2 + \int_0^{T_0}\int_\Omega |u_m(t)|^{p(x)} \log|u_m(t)|\, dxdt$$

Taking $\limsup_{m\to\infty}$ on both sides and using the fact that $V_m$ is dense in $W_0^{1,p(x)}(\Omega)$ and the lower semicontinuity of the norm as well as (3.17)-(3.19) and (3.21) yields

$$\begin{aligned}\limsup_{m\to\infty} \int_0^{T_0}\int_\Omega |\nabla u_m(t)|^{p(x)} dxdt &\le -\frac{1}{2}\|u(T_0)\|_2^2 + \frac{1}{2}\|u(0)\|_2^2 + \int_0^{T_0}\int_\Omega |u(t)|^{p(x)} \log|u(t)|\, dxdt \\ &= -\int_0^{T_0}\int_\Omega u_t(t)\, u(t)\, dxdt + \int_0^{T_0}\int_\Omega |u(t)|^{p(x)} \log|u(t)|\, dxdt \\ &= \int_0^{T_0}\int_\Omega \chi(t) \nabla u(t)\, dxdt\end{aligned}$$

since by (3.22) (with $w$ is replaced by $u$). Thus, the function $u$ is a desirable solution of the problem (1.1).

Now we show that the solution $u$ satisfies the energy inequality (3.1). To this end, we consider the positive continuous function $\theta \in C([0, T_0])$. Then, from (3.12) we get

$$\int_0^{T_0} \theta(t)\,dt \int_0^t \|u_{ms}(s)\|_2^2\,ds + \int_0^{T_0} J(u_m(t))\,\theta(t)\,dt = \int_0^{T_0} J(u_m(0))\,\theta(t)\,dt. \tag{3.23}$$

The right-hand side of (3.23) converges to

$$\int_0^{T_0} J(u_0)\,\theta(t)\,dt$$

as $m \to +\infty$. The lower semi-continuity of the second term in left-hand side of (3.23) with respect to the weak topology of $W_0^{1,p(x)}(\Omega)$, means that

$$\int_0^{T_0} J(u(t))\,\theta(t)\,dt \le \liminf_{m\to+\infty} \int_0^{T_0} J(u_m(t))\,\theta(t)\,dt.$$

Therefore, we get

$$\int_0^{T_0} \theta(t)\,dt \int_0^t \|u_s(s)\|_2^2\,ds + \int_0^{T_0} J(u(t))\,\theta(t)\,dt \le \int_0^{T_0} J(u_0)\,\theta(t)\,dt.$$

$\theta$ is arbitrary, then, it holds the energy inequality

$$\int_0^t \|u_s(s)\|_2^2\,ds + J(u(t)) \le J(u_0), \quad t \in [0, T_0].$$

Which completes the proof. □

**Remark 3.1.** *Note that the continuous embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{p^*(x)}(\Omega)$ in Proposition 2.9 (see [10, 12]) can be also obtained from the embeddings $W_0^{1,p(x)}(\Omega) \hookrightarrow W^{1,p(x)}(\Omega)$ which is straightforward and the continuous one $W^{1,p(x)}(\Omega) \hookrightarrow L^{p^*(x)}(\Omega)$ given in [19, Theorem 1.1, Theorem 1.2].*

Now we present our main theorem of this section.

**Theorem 3.2.** *Assume that $u_0 \in \mathcal{W}^+$. Then the weak solution of the problem (1.1) is globally in time and satisfies:*

$$u(t) \in \overline{\mathcal{W}^+} \quad \text{for} \quad 0 \le t < +\infty,$$

*and the following energy estimate holds*

$$\int_0^t \|u_s(s)\|_2^2\,ds + J(u(t)) \le J(u_0), \quad a.e.\ t \ge 0. \tag{3.24}$$

*Furthermore, the decay of the solution is polynomial, as follows,*
*(i) if $J(u_0) < M$, then it holds the estimate*

$$\|u(t)\|_2 \le \|u_0\|_2 \left( \frac{p_-}{2\left(1 + \zeta(p_- - 2)\|u_0\|_2^{p_- - 2}\, t\right)} \right)^{1/(p_- - 2)}, \quad t \ge 0,$$

*where* $\zeta = \frac{a}{p_- R^{p_-}} \log \frac{M}{J(u_0)} > 0$;
*(ii) if* $J(u_0) = M$, *then there exists a time* $t_\varepsilon > 0$ *such that the norm* $\|u(t)\|_2$ *satisfies the estimate*

$$\|u(t)\|_2 \leq \|u(t_\varepsilon)\|_2 \left( \frac{p_-}{2\left(1 + \zeta_\varepsilon (p_- - 2) \|u(t_\varepsilon)\|_2^{p_- - 2} t\right)} \right)^{1/(p_- - 2)}, \quad t \geq t_\varepsilon,$$

*where* $\zeta_\varepsilon = \frac{a}{p_- R^{p_-}} \log \frac{M}{J(u(t_\varepsilon))} > 0$.

*Proof.* Two cases can be distinguished

**Case I: the initial datum** $u_0 \in \mathcal{W}_1^+$

**Global existence of the Weak Solutions**

As in the proof of Theorem 3.1, we consider the same sequences $\{w_j\}_{j=1}^{+\infty}$, $\{u_{0m}\}_{m=1}^{+\infty}$, and $\{u_m\}_{m=1}^{+\infty}$. Let us multiply the two sides of (3.3) by $\alpha'_{mi}(t)$, and taking the sum over $i \in \{1, 2, ..., m\}$, and then integrating with respect to time on $[0, t]$, we have

$$\int_0^t \|u_{ms}(s)\|_2^2 \, ds + J(u_m(t)) = J(u_m(0)), \quad 0 \leq t < T_m, \tag{3.25}$$

where $T_m$ is the maximal existence time of solution $u_m(x, t)$.
Since $J$ is continuous, then from (3.2), (3.4) and (3.25) we get

$$J(u_m(0)) \longrightarrow J(u_0) \quad \text{as } m \longrightarrow +\infty,$$

with $J(u_0) < d$ and

$$\int_0^t \|u_{ms}(s)\|_2^2 \, ds + J(u_m(t)) < d, \quad 0 \leq t < T_m, \tag{3.26}$$

for some $m$ large enough. We shall show that

$$u_m(t) \in \mathcal{W}_1^+, \quad \forall t \geq 0, \tag{3.27}$$

for some $m$ large enough. Indeed, suppose that (3.27) is not true and that $t_*$ be the smallest time such that $u_m(t_*) \notin \mathcal{W}_1^+$. Then, by the continuity of $u_m(t)$, we have $u_m(t_*) \in \partial \mathcal{W}_1^+$. Therefore

$$J(u_m(t_*)) = d, \tag{3.28}$$

or

$$I(u_m(t_*)) = 0. \tag{3.29}$$

However, it is obvious that (3.28) could not arise from (3.26) whereas if (3.29) is verified then, through the formula (2.27), we have

$$J(u_m(t_*)) \geq \inf_{u \in \mathcal{N}} J(u) = d,$$

this contradicts (3.26). then (3.27) holds.

On the other hand, because of $u_m(t) \in \mathcal{W}_1^+$ so that $I(u_m(t)) > 0$ and

$$J(u_m(t)) \geq \frac{1}{p_+} I(u_m(t)) + \frac{1}{p_+^2} \int_\Omega |u_m(t)|^{p(x)} dx, \quad \forall t \in [0, T_m), \tag{3.30}$$

then through (3.26) we derive,

$$\int_\Omega |u_m(t)|^{p(x)} dx < p_+^2 d, \quad \text{and} \quad \int_0^t \|u_{ms}(s)\|_2^2 ds < d, \tag{3.31}$$

for some $m$ large enough and $t \in [0, T_m)$. Moreover, from the fact that $I(u_m(t)) > 0$ there exists a constant $0 < \delta < 1$ such that

$$\int_\Omega |u_m(t)|^{p(x)} \log |u_m(t)| dx \leq \delta \int_\Omega |\nabla u_m(t)|^{p(x)} dx$$

this combined with (3.30), we may write

$$\begin{aligned}\int_\Omega |\nabla u_m(t)|^{p(x)} dx &\leq p_+ J(u_m(t)) + \int_\Omega |u_m(t)|^{p(x)} \log |u_{,m}(t)| dx - \frac{1}{p_+} \int_\Omega |u_m(t)|^{p(x)} dx \\ &\leq p_+ J(u_m(t)) + \delta \int_\Omega |\nabla u_m(t)|^{p(x)} dx - \frac{1}{p_+} \int_\Omega |u_m(t)|^{p(x)} dx,\end{aligned}$$

since $0 < \delta < 1$ we conclude from (3.26) and (3.31) that

$$\int_\Omega |\nabla u_m(t)|^{p(x)} dx \leq C_d, \tag{3.32}$$

for all $t \in [0, T_m)$. The above inequalities lead us to take $T_m = T$ for all $m$, with any $T > 0$.

According to (3.31) and (3.32), it can be seen by proceeding in the same way as in the proof of Theorem 3.1 that the problem (1.1) has a weak solution $u$ in the interval $[0, T]$, and furthermore (3.24) is fulfulled.

**Decay Estimates**

Case $J(u_0) < M$ : due to $u(t) \in \overline{\mathcal{W}_1^+}$, through (2.6) and the energy inequality we can conclude that

$$\int_\Omega |u_m(t)|^{p(x)} dx \leq p_+^2 J(u(t)) \leq p_+^2 J(u_0).$$

utilising (2.16), Proposition 2.2 and Lemma 2.5, because of $I(u(t)) \geq 0$ we have

$$\begin{aligned}I(u(t)) &\geq \frac{a}{R^{p_-}} \left( \log(R) - \frac{1}{p_-} \log\left( \int_\Omega |u|^{p(x)} dx \right) \right) \max\left\{ \|u\|_{p(x)}^{p_-}, \|u\|_{p(x)}^{p_+} \right\} \\ &\geq \frac{a}{R^{p_-}} \left( \log(R) - \frac{1}{p_-} \log\left( p_+^2 J(u_0) \right) \right) \int_\Omega |u|^{p(x)} dx \\ &\geq \zeta \|u(t)\|_2^{p_-},\end{aligned} \tag{3.33}$$

because of $p_- > 2$, where $\zeta = \frac{a}{p_- R^{p_-}} \log \frac{M}{J(u_0)} > 0$.

On the other hand, from the first equation of problem (1.1) we get

$$\int_t^T I(u(s))\,ds = -\int_t^T\int_\Omega u_s(s)\,u(s)\,dxds \le \frac{1}{2}\|u(t)\|_2^2, \tag{3.34}$$

for all $t \in [0,T]$.

We Combine (3.31) and (3.32), resulting in

$$\int_t^T \|u(t)\|_2^{p_-}\,ds \le \frac{1}{2\zeta}\|u(t)\|_2^2,\quad \forall t \in [0,T].$$

Tending $T$ to $+\infty$ then by Lemma 3.1 (with $f(t)$ replaced by $\|u(t)\|_2^2$, $\sigma = (p_- - 2)/2$ and $\omega = 2\zeta\|u_0\|_2^{p_- -2}$), we obtain the following decay estimate

$$\|u(t)\|_2 \le \|u_0\|_2\left(\frac{p_-}{2\left(1+\zeta(p_- -2)\|u_0\|_2^{p_- -2}\,t\right)}\right)^{1/(p_- -2)},\quad t\ge 0.$$

Case $J(u_0) = M$ : in view of $I(u(t)) > 0$ for all $t \ge 0$ we may write

$$\int_\Omega u_t(t)\,u(t)\,dx = -I(u(t)) < 0,\quad \forall t > 0.$$

And as $\|u_t(t)\|_2^2$ should be positive $\left(\|u_t(t)\|_2^2 > 0\right)$, for all $t > 0$. Since $t \to \int_0^t \|u_s(s)\|_2^2\,ds$ is a continuous function and by the help of the energy inequality (3.24), thus, for any number $\varepsilon > 0$ small enough, there exists $t_\varepsilon > 0$ so that

$$J(u(t_\varepsilon)) \le J(u_0) - \int_0^{t_\varepsilon}\|u_s(s)\|_2^2\,ds = M - \varepsilon.$$

Finally, we consider the initial time $t_\varepsilon$ and we proceed in the same way as in the case $0 < J(u_0) < M$ then, we find the estimate

$$\|u(t)\|_2 \le \|u(t_\varepsilon)\|_2\left(\frac{p_-}{2\left(1+\zeta_\varepsilon(p_- -2)\|u(t_\varepsilon)\|_2^{p_- -2}\,t\right)}\right)^{1/(p_- -2)},\quad t \ge t_\varepsilon,$$

where $\zeta_\varepsilon = \frac{a}{p_- R^{p_-}}\log\frac{M}{J(u(t_\varepsilon))} > 0$.

**Case II: the Initial Datum** $u_0 \in \mathcal{W}_2^+$

To prove that the solution of problem (1.1) is globally in time, we need to define the sequence

$$\{\gamma_m\}_{m=1}^{+\infty} \subset (0,1)$$

with $\lim_{m\to+\infty}\gamma_m = 1$. And consider the following problem

$$\begin{cases} u_t - \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = |u|^{p(x)-2}\,u\log|u|, & (x,t)\in\Omega\times\mathbb{R}^+,\\ u(x,t) = 0, & (x,t)\in\partial\Omega\times\mathbb{R}^+,\\ u(x,0) = u_{0m}(x), & x\in\Omega, \end{cases} \tag{3.35}$$

where $u_{0m} = \gamma_m u_0$. Owing to $I(u_0) \geq 0$ then from Lemma 2.4 we have

$$\lambda^* = \lambda^*(u_0) = \exp\left(\frac{\int_\Omega \lambda^{*p(x)-1}\left(|\nabla u_0|^{p(x)} - |u_0|^{p(x)} \log|u_0|\right)dx}{\int_\Omega \lambda^{*p(x)-1}|u_0|^{p(x)}\,dx}\right) \geq 1.$$

Therefore, we obtain

$$I(u_{0m}) = I(\gamma_m u_0) > 0 \quad \text{and} \quad J(u_{0m}) = J(\gamma_m u_0) < J(u_0) = d$$

it means by the definition of $\mathcal{W}_1^+$ that $u_{0m} \in \mathcal{W}_1^+$. Proceeding in the same way as in the previous subsection, leads to the problem (3.35) having a global weak solution $u_m$ as follows

$$u_m \in L^\infty\left(0, T; W_0^{1,p(x)}(\Omega)\right), \quad u_{mt} \in L^{p'(x)}\left(0, T; W_0^{-1,p'(x)}(\Omega)\right) \cap L^2\left(0, T; L^2(\Omega)\right)$$

and

$$\int_\Omega u_{mt}(t)\,w dx + \int_\Omega |\nabla u_m(t)|^{p(x)-2}\,\nabla u_m(t)\,\nabla w dx = \int_\Omega |u_m(t)|^{p(x)-2}\,u_m(t)\log|u_m(t)|\,w dx,$$

for all $w \in W_0^{1,p(x)}(\Omega) \cap L^2(\Omega)$ and for almost every $t > 0$ as well, we have

$$u_m(t) \in \overline{\mathcal{W}^+},$$

for all $t \in [0, +\infty)$, and

$$\int_0^t \|u_{ms}(s)\|_2^2\,ds + J(u_m(t)) \leq J(u_{0m}) < d, \quad t \in [0, +\infty).$$

The rest of the proof can be formulated in a similar way as earlier. □

## 4. Blow up of Weak Solutions

The aim of this section is to prove that blowing up in finite time results of weak solutions to problem (1.1) provided that the initial datum $u_0$ is in $\mathcal{W}^-$ and fulfills the condition $J(u_0) \leq M$. To this end, we need some lemmas, which are presented in [9, 28].

**Lemma 4.1.** *Let $\psi \in W_{loc}^{1,1}(\mathbb{R}^+)$ be a nonnegative function satisfying $\psi(0) > 0$ and the differential inequality*

$$\frac{d\psi}{dt}(t) \geq c\psi^\sigma(t), \quad \text{for a.e. } t \geq 0,$$

*where $\sigma > 1$ and $c$ is positive constant. Then we have*

$$\psi(t) \geq \left(\frac{1}{\psi^{1-\sigma}(0) - c(\sigma - 1)\,t}\right)^{1/(\sigma-1)}, \quad t \in [0, T_*),$$

*which implies that $\lim_{t\to T_*^-} \psi(t) = \infty$, with $T_* = \frac{\psi^{1-\sigma}(0)}{c(\sigma-1)}$.*

*Proof.* For the proof of this lemma see [18]. □

**Lemma 4.2.** *Let $\Phi$ be a positive, twice differentiable function satisfying the following conditions*

$$\Phi(\bar{t}) > 0, \quad and \quad \Phi'(\bar{t}) > 0,$$

*for some $\bar{t} \in [0, T)$, and the inequality*

$$\Phi(t)\Phi''(t) - \alpha\left(\Phi'(t)\right)^2 \geq 0, \quad \forall t \in [\bar{t}, T],$$

*where $\alpha > 1$. Then, we have*

$$\Phi(t) \geq \left(\frac{1}{\Phi^{1-\alpha}(\bar{t}) - \hat{\Phi}(t-\bar{t})}\right)^{1/(\alpha-1)}, \quad t \in \left[\bar{t}, T_*\right),$$

*with $\hat{\Phi}$ is a positive constant, and*

$$T_* = \bar{t} + \frac{\Phi(\bar{t})}{(\alpha-1)\,\Phi'(\bar{t})}.$$

*This implies*

$$\lim_{t \to T_*^-} \Phi(t) = +\infty.$$

The main result of this section is the following theorem

**Theorem 4.1.** *Let $u_0 \in \mathcal{W}^-$ with $J(u_0) \leq M$. Suppose that the local weak solution $u(x,t)$ of problem (1.1) which corresponds to $u_0$ and for which the energy inequality*

$$\int_0^t \|u_s(s)\|_2^2\, ds + J(u(t)) \leq J(u_0), \quad \forall t \in [0, T). \tag{4.1}$$

*holds. Then, it follow the assertions*
*(i) The solution $u(x,t)$ blows up at finite time when $J(u_0) \leq 0$ so we have*

$$\lim_{t \to T_*^-} \|u(t)\|_2^2 = +\infty, \quad where \quad T_* = \frac{p_+^2 l_{2,p(x)}^{p_-} \|u_0\|_2^{2-p_-}}{p_-(p_- - 2)}. \tag{4.2}$$

*In addition, the following estimate holds*

$$\|u\,(t)\|_2^2 \geq \left(\frac{1}{\|u_0\|_2^{2-p_-} - \left(p_-/p_+^2 l_{2,p(x)}^{p_-}\right)(p_- - 2)\,t}\right)^{2/(p_- - 2)}.$$

*(ii) The solution $u(x,t)$ blows up at finite time when $0 < J(u_0) \leq M$, which means that there exists a time $T_* > 0$ for which*

$$\lim_{t \to T_*^-} \|u(t)\|_2^2 = +\infty.$$

*Proof.* We start by proving that $u(t) \in \mathcal{W}_1^-$, for all $t \geq 0$, when $u_0 \in \mathcal{W}_1^-$. Indeed, reasoning by absurd, we suppose that there exists a time $t_0 \in (0, T)$ for which

$$u(.,t) \in \mathcal{W}_1^- \quad \text{for all } t \in [0, t_0), \quad \text{and} \quad u(.,t_0) \in \partial\mathcal{W}_1^-,$$

so

$$I(u(t_0)) = 0 \quad \text{or} \quad J(u(t_0)) = d.$$

Nevertheless, owing to (2.28) and the energy inequality (4.1) it is obvious that the identity $J(u(t_0)) = d$ can not arise. However, if $I(u(t_0)) = 0$, it follows that $\min\left\{\|u\|_{p(x)}, \|u\|_{p(x)}^{p_+/p_-}\right\} \geq R > 0$ in view of Lemma 2.5. We conclude that $u(t_0) \in \mathcal{N}$ so, through the formula (2.27), we find $J(u(t_0)) \geq d$ which contracts the inequality of energy (4.1).

After that, weak solutions to problem (1.1) blow up at finite time i.e., (4.2) holds. Thus we assume that

$$\|u(t)\|_{p(x)} > 1. \tag{4.3}$$

since by the embedding $L^{p(x)}(\Omega) \hookrightarrow L^2(\Omega), p(x) > 2.$

Define the functional

$$\Gamma(t) = \int_0^t \|u(s)\|_2^2 \, ds + (T - t)\|u_0\|_2^2 \quad t \in [0, T], \tag{4.4}$$

we have

$$\Gamma'(t) = \|u(t)\|_2^2 - \|u_0\|_2^2 = \int_0^t \frac{d}{ds}\left(\|u(s)\|_2^2\right) ds = 2\int_0^t \int_\Omega u_s(s)\, u(s)\, dx ds. \tag{4.5}$$

From (4.5) and by Setting $u = w$ in (2.3) we get

$$\Gamma''(t) = 2\int_\Omega u_t(t)\, u(t)\, dx = -2\int_\Omega |\nabla u(t)|^{p(x)} dx + 2\int_\Omega |u(t)|^{p(x)} \log|u(t)|\, dx = -2I(u(t)). \tag{4.6}$$

By Proposition 2.2, (2.6), (4.3), and the inequality of energy (4.1), the formula (4.6) becomes

$$\begin{aligned}\Gamma''(t) &\geq -\frac{2}{\gamma}J(u(t)) + \frac{2}{\gamma p_+^2}\int_\Omega |u(t)|^{p(x)} dx \\ &\geq \frac{2}{\gamma}\int_0^t \|u_s(s)\|_2^2 \, ds + \frac{2}{\gamma p_+^2}\|u(t)\|_{p(x)}^{p_-} - \frac{2}{\gamma}J(u_0).\end{aligned} \tag{4.7}$$

Three cases can be considered:
(i) Case $J(u_0) \leq 0$.
In the present case, we use the embedding $L^{p(x)}(\Omega) \hookrightarrow L^2(\Omega), p(x) > 2$. Then (4.3), (4.5) and (4.7) give us

$$\Gamma''(t) \geq \frac{2p_-}{p_+^2}\|u(t)\|_{p(x)}^{p_-} \geq \frac{2p_-}{p_+^2 l_{2,p(x)}^{p_-}}\left(\|u(t)\|_2^2\right)^{p_-/2} = \frac{2p_-}{p_+^2 l_{2,p(x)}^{p_-}}\left(\Gamma'(t) + \|u_0\|_2^2\right)^{p_-/2},$$

since by the definition of $\gamma$, where $l_{2,p(x)}^{p_-}$ is the embedding constant. Therefore we can apply Lemma 4.1 (with $\psi(t)$ replaced by $\Gamma'(t) + \|u_0\|_2^2$, $c = 2p_-/p_+^2 l_{2,p(x)}^{p_-}$ and $\sigma = p_-/2$) to obtain the estimate

$$\|u(t)\|_2^2 \geq \left(\frac{1}{\|u_0\|_2^{2-p_-} - \left(p_-/p_+^2 l_{2,p(x)}^{p_-}\right)(p_- - 2)\, t}\right)^{2/(p_- - 2)},$$

this demonstrates that

$$\lim_{t\to T_*^-} \|u(t)\|_2^2 = +\infty, \quad \text{where} \quad T_* = \frac{p_+^2 l_{2,p(x)}^{p_-} \|u_0\|_2^{2-p_-}}{p_-(p_- - 2)}.$$

(ii) Case $0 < J(u_0) < M$ :
Since $u(t) \in \mathcal{W}_1^-$, for all $t \in [0,T]$, we have $I(u(t)) < 0$ this involves by Lemma 2.5 and (4.3) that

$$\min\left\{\|u\|_{p(x)}^{p_-}, \|u\|_{p(x)}^{p_+}\right\} = \|u\|_{p(x)}^{p_-} \geq R^{p_-} \quad \text{for all } t \in [0,T],$$

Then, through (4.7) by using the fact that $M - J(u_0) > 0$, and the definition of $\gamma$ we have

$$\begin{aligned}\Gamma''(t) &\geq \frac{2}{\gamma}\int_0^t \|u_s(s)\|_2^2\,ds + \frac{2}{\gamma}(M - J(u_0)) \\ &\geq 2p_- \int_0^t \|u_s(s)\|_2^2\,ds + 2p_-(M - J(u_0)), \quad t \in [0,T]. \end{aligned} \tag{4.8}$$

So we get

$$\Gamma'(t) = \Gamma'(0) + \int_0^t \Gamma''(s)ds \geq 2p_-(M - J(u_0))\,t \geq 0, \quad t \in [0,T]. \tag{4.9}$$

Therefore, (4.5) and (4.9) give us by using Hölder inequality that

$$\frac{1}{4}(\Gamma'(t))^2 \leq \left(\int_0^t \int_\Omega u_s(s)\,u(s)\,dxds\right)^2 \leq \int_0^t \|u_s(s)\|_2^2\,ds \int_0^t \|u(s)\|_2^2\,ds, \tag{4.10}$$

for all $t \in [0,T]$. Combining (4.4), (4.8) and (4.10), we find

$$\begin{aligned}\Gamma(t)\Gamma''(t) &\geq 2p_- \int_0^t \|u_s(s)\|_2^2\,ds \int_0^t \|u(s)\|_2^2\,ds + 2p_-(M - J(u_0))\,\Gamma(t) \\ &\geq \frac{p_-}{2}(\Gamma'(t))^2 + 2p_-(M - J(u_0))\,\Gamma(t)\end{aligned}$$

for all $t \in [0,T]$. Consequently

$$\Gamma(t)\Gamma''(t) - \frac{p_-}{2}(\Gamma'(t))^2 \geq 2p_-(M - J(u_0))\,\Gamma(t) > 0. \quad \text{for all } t \in [0,T]$$

In accordance with Lemma 4.2, there exists $T_* > 0$ such that

$$\lim_{t\to T_*^-} \Gamma(t) = +\infty,$$

which means that $\lim_{t\to T_*^-} \int_0^t \|u(s)\|_2^2\,ds = +\infty$. And then, we obtain

$$\lim_{t\to T_*^-} \|u(t)\|_2^2 = +\infty.$$

(iii) Case $J(u_0) = M$ : Because $I(u(t)) < 0$ for all $t \geq 0$ this means that

$$\int_{\Omega} u_t(t)\, u(t)\, dt = -\int_{\Omega} |\nabla u(t)|^{p(x)}\, dx + \int_{\Omega} |u(t)|^{p(x)} \log|u(t)|\, dx = -I(u(t)) > 0, \quad \forall t > 0.$$

And as $\|u_t(t)\|_2^2$ should be positive, for all $t > 0$. Since $t \to \int_0^t \|u_s(s)\|_2^2\, ds$ is continuous function and by the help of the energy inequality (4.1), thus, for any number $\varepsilon > 0$ small enough, there exists $t_\varepsilon > 0$ such that

$$J(u(t_\varepsilon)) \leq J(u_0) - \int_0^{t_\varepsilon} \|u_s(s)\|_2^2\, ds = M - \varepsilon$$

Finally, we consider the initial time, $t_\varepsilon$ and we proceed in the same way as in the case $0 < J(u_0) < M$ then we obtain the finite blow up result

$$\lim_{t \to T_*^-} \|u(t)\|_2^2 = +\infty.$$

which complete the proof of the theorem. □

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