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Some results on existence and regularity for non-linear $p(x)$-parabolic equations with quadratic growth with respect to the gradient and general data

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Abstract. This work investigates the regularity of solutions to a nonlinear parabolic equation with perturbations and general measure data. Our approach involves a combination of convergence and compactness techniques in variable exponent Sobolev spaces.

1. Introduction

In this manuscript, we show a certain regularity of solutions for nonlinear $p(x)-$parabolic problems including a low order term with natural growth. More precisely, we are interested in the following problem

$$(\mathcal{P})\begin{cases} \dfrac{\partial b(u)}{\partial t} - div\Big[\phi(t,x,u)(1+|u|)^{s(x)}|\nabla u|^{p(x)-2}\nabla u\Big] + \zeta(x,t)(1+|u|)^{q(x)-1}u|\nabla u|^{p(x)} = \mu & \text{in } Q_T,\\ u(t,x) = 0 & \text{on } (0,T)\times\partial\Omega,\\ b(u)(0,x) = b(u_0)(x) & \text{in } \Omega, \end{cases}$$

where Ω is a bounded domain, with a smooth boundary $\partial\Omega$ and $Q_T := (0,T)\times\Omega,\ \Omega\subset\mathbb{R}^N (N\geq 2), T>0$. The vector filed $\phi(t,x,u)$ verified certain appropriate hypotheses, μ is a bounded Radon measure on Q_T, the initial data $u_0\in L^1(\Omega)$ and $\zeta(x,t)$ is a measurable positive function.

The notion of existence and regularity results was introduced by Boccardo and al [20] when the right hand side is in $W^{-1,p'}(\Omega)$. The following quasi-linear elliptic problem

$$\begin{cases} -div((\phi(x)+|u|^q)\nabla u) = f - \zeta(x)|u|^{p-1}u|\nabla u|^2 \quad in\ \Omega,\\ u=0 \quad on \quad \partial\Omega, \end{cases} \tag{1}$$

with f is non-negative, $f\in L^1(\Omega), a\leq\phi(x)\leq b$ and $p,\ q\leq 2q$, it has been examined in [19] (see also [18]). Moreover, similar results have also been shown taking into account the parameters $p,\ q$ and the summability of the data f. The authors have enriched the work of [1, 19] by establishing the existence of solutions of the problem (1) by taking $p,\ q$ as real without any condition.

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The key point of the existence result in [36] is to show that $|u|^q|\nabla u| \in L^1(\Omega)$ for any $q > 0$. L. Aharouch and colleagues in [6] established the existence of weak solutions for degenerate parabolic equations when $f \in L^{p'}(0, T, W^{-1,p'}(\Omega, W^*))$ and $\phi(x, t, u, \nabla u)$ is strictly monotone. The authors proposed in [3, 4, 48] a novel method using diffuse measures as data and perturbation terms, which avoids the need to apply the specific structure of the measure decomposition and makes it more versatile for a wider range of problems. This theory has applications in various disciplines of PDE analysis, including specialised electro-rheological fluid models and image processing (see e.g. [11, 30, 50] and reference therein). In addition, the generalised variational capacity, a Choquet capacity with respect to space, is widely used in nonlinear theory.

The authors of [3, 37] investigated the connection between these chosen capacities and diffuse measures. Given the use of this capacity in geometric function theory and stochastic processes, such as its behaviour under various forms of symmetrization and other geometric transformations, Harjulehto et al. [31] created a relative capacity, studied its properties and compared it with the Sobolev capacity. In the case where $\zeta(x) = u$, there are several publications dealing with different aspects of this topic, such as (1). In addition, as far as we know, there are some extended results in the framework of generalised Lebesgue spaces.

This paper improves and generalises previously published results and addresses more challenging problems, such as nonlinear parabolic problems with variable exponent $(\mathcal{P})$. The method used to prove the main results is a combination of convergence results in appropriate spaces and compactness estimates via some approximation problems.

The main contribution of this study is to extend the results for problems with measures to the case with variable exponent. To obtain global estimates from a priori estimates, additional assumptions on the exponent $s(x)$ are required. When dealing with a potentially perturbed term with natural growth, more general strategies such as those described in [2, 43] are applied. To achieve strong convergence of approximate solutions, which is essential for generalised estimates of "near/far", certain types of test functions are used instead of modifying the unknown variable, similar to the strategies used in [39, 41] without a low-order term and [38] with an absorption term.

The structure of this paper is as follows. In Section 2, we provide some preliminary remarks, including important properties and results on Lebesgue-Sobolev spaces with variable exponents, the generalised parabolic capacity $p(\cdot)-$, and measure decompositions that will be used throughout the proof. These results and decompositions will be discussed in more detail later. In addition, the basic assumptions that must be made about $\phi,\ \mu,\ c,\ u_0$ are presented. To provide our general definition when the operator is modified, a new primary result is proved in Section 3.

2. Preliminary results and notations

In our analysis of the problem $(\mathcal{P})$ we will use definitions and fundamental properties of generalised Lebesgue-Sobolev spaces $L^{p(x)}(\Omega)$, $W_0^{1,p(x)}(\Omega)$, as well as the theory of parabolic capacities. We will only give brief summaries of the necessary results here, and refer the reader to the references [27, 28] for more information.

2.1. *Sobolev spaces with variable exponents*

We define a real-valued continuous function p to be log-Hölder continuous in a bounded open subset Ω of $\mathbb{R}^N$ (with $N \geq 2$) if

$$|p(x) - p(y)| \leq \frac{C}{|\log|x - y||} \text{ for all } x,\ y \in \overline{\Omega} \text{ such that } |x - y| < \frac{1}{2},$$

where C is a constant. We designate by

$$C_+(\overline{\Omega}) = \Big\{ \text{log-Hölder continuous function } p : \overline{\Omega} \to \mathbb{R} \text{ with } 1 < p^- \leq p(x) \leq p^+ < N \Big\},$$

where

$$p^- = \min\Big\{p(x) : x \in \overline{\Omega}\Big\} \text{ and } p^+ = \max\Big\{p(x) : x \in \overline{\Omega}\Big\}.$$

Therefore, the variable exponent Lebesgue space $L^{p(x)}(Q_T)$ is introduced as follows

$$L^{p(x)}(\Omega) = \Big\{u : \Omega \to \mathbb{R} \text{ is measurable such that } \int_\Omega |u(x)|^{p(x)}dx < +\infty\Big\},$$

the norm for $L^{p(x)}(Q_T)$ is defined below:

$$\|u\|_{p(\cdot)} = \inf\Big\{\tau > 0; \int_\Omega |\frac{u(x)}{\tau}|^{p(x)}dx \leq 1\Big\}.$$

Note that the inequality below will be used later

$$\min\Big\{\|u\|^{p^-}_{p(\cdot)} \ ; \ \|u\|^{p^+}_{p(\cdot)}\Big\} \leq \int_\Omega |u(x)|^{p(x)}dx \leq \max\Big\{\|u\|^{p^-}_{p(\cdot)} \ ; \ \|u\|^{p^+}_{p(\cdot)}\Big\}.$$

It should be noted that if $1 < p^- < \infty$, then $L^{p(\cdot)}(\Omega)$ is reflexive and its dual is $L^{p'(\cdot)}(\Omega)$, where $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$, and then for any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$ the inequality of type Hölder is given by

$$\int_\Omega |uv|\mathrm{d}x \leq \Big(\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)}\Big)\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)}.$$

Then, if $p(\cdot), p'(\cdot) \in C_+(\overline{\Omega})$, the Young's inequality is established by the following formula:

$$ab \leq \frac{a^{p(x)}}{p(x)} + \frac{b^{p'(x)}}{p'(x)},$$

such that $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$ and for each $a,\ b > 0$. By extending a variable exponent p: $\overline{\Omega} \to [1, +\infty)$ to $\overline{Q}_T = \overline{\Omega} \times [0, T]$ by defining $p(x) := p(t, x)$ for each $(x, t) \in \overline{Q}_T$, we can also consider the generalized Lebesgue space

$$L^{p(\cdot)}(Q_T) = \Big\{u : Q_T \to \mathbb{R};\ \text{measurable such that } \int_{Q_T} \Big|u(x, t)\Big|^{p(x)} dxdt < \infty\Big\},$$

under the norm

$$\|u\|_{L^{p(\cdot)}(Q_T)} = \inf\Big\{\tau > 0; \int_{Q_T} \Big|\frac{u(x, t)}{\tau}\Big|^{p(x)} dxdt < 1\Big\},$$

retains the same properties as $L^{p(\cdot)}(\Omega)$. Furthermore, the variable exponent Sobolev space given by

$$W^{1,p(\cdot)}(\Omega) = \Big\{u \in L^{p(\cdot)}(\Omega) \ ; \ |\nabla u| \in L^{p(\cdot)}(\Omega)\Big\},$$

is a Banach space with the following norm

$$\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)},$$

such that

$$\|u\|_{1,p(\cdot)} = \inf\Big\{\tau > 0; \int_\Omega \Big(\Big|\frac{\nabla u(x)}{\tau}\Big|^{p(x)} + \Big|\frac{u(x)}{\tau}\Big|^{p(x)}\Big)dx \leq 1\Big\}. \tag{2}$$

We define the functional space $W^{1,p(\cdot)}_0(\Omega)$ as the closure of $C^\infty_c(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$ with respect to the norm (2). Note that $W^{1,p(\cdot)}_0(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$ are separable and reflexive Banach spaces if $1 \leq p^- < \infty$ and $1 < p^- < \infty$ respectively. At last, we shall employ the standard notation for Bochner spaces, i.e., $L^q(0, T; X)$ is the space of strongly measurable function $u : (0, T) \to X$ for which $t \mapsto \|u(t)\|_X \in L^q(0, T)$. In addition, $C([0, T]; X)$ represents the space of continuous function u: $[0, T] \to X$ according to the norm $\|u\|_{C([0,T];X)} = \max_{t\in[0,T]} \|u(t)\|_X$, where X is a Banach space and $q \geq 1$.

$$L^{p^-}(0,\ T; W^{1,p(x)}_0(\Omega)) = \Big\{u : (0, T) \to W^{1,p(x)}_0(\Omega) \text{ measurable with } \Big(\int_0^T \|u(t)\|^{p^-}_{W^{1,p(x)}_0(\Omega)}\Big)^{\frac{1}{p^-}} dt < +\infty\Big\}.$$

2.2. Measures and Parabolic capacity

Let $Q_T = \Omega\times(0,T)$ for each fixed $T > 0$, and recall that $V = W_0^{1,p(\cdot)}(\Omega)\cap L^2(\Omega)$ has the norm $\|.\|_{W_0^{1,p(\cdot)}} + \|.\|_{L^2(\Omega)}$. The space $W_{p(\cdot)}(0,T)$ is defined as

$$W_{p(\cdot)}(0,T) = \Big\{u \in L^{p^-}(0,T,V);\ \nabla u \in (L^{p(\cdot)}(Q_T))^N \text{ and } u_t \in L^{(p^-)'}(0,T,V')\Big\}$$

with the following standard

$$\|u\|_{W_{p(\cdot)}(0,T)} = \|u\|_{L(0,T,V)} + \|\nabla u\| + \|u_t\|_{L(0,T,V')}.$$

Note that $W_{p(\cdot)}(0,T) \hookrightarrow C([0,T],L^2(\Omega))$ continuously. Let $O \subseteq Q_T$ be an open set, we define the (generalized) parabolic capacity of O as

$$cap_{p(\cdot)}(O) = \inf\Big\{\|u\|_{W_{p(\cdot)}(0,T)} : O \in W_{p(\cdot)}(0,T), s \geq \chi_O \text{a.e. in } Q_T\Big\},$$

where as usual we set $\inf\{\emptyset\} = +\infty$, then for any Borel set $B \subseteq Q_T$, the definition of (generalized) parabolic capacity can be extended by setting

$$cap_{p(\cdot)}(B) = \inf\Big\{cap_{p(\cdot)}(O) : O \text{ open subset of } Q_T, B \subseteq O\Big\}.$$

Since we are interested by using some regular properties, we need to define the following space

$$\mathcal{V} = \Big\{u \in L^{p^-}(0,T,W_0^{1,p(\cdot)}(\Omega))\ :\ \nabla u \in (L^{p(\cdot)}(Q_T))^N \textit{and } u_t \in L^{(p^-)'}(0,T,W^{1,p'(\cdot)}(\Omega)) + L^1(Q_T)\Big\},$$

endowed with its natural norm

$$\|u\|_{\mathcal{V}} = \|u\|_{L^{p^-}(0,T,W_0^{1,p(\cdot)}(\Omega))} + \|\nabla u\|_{(L^{p(\cdot)}(Q_T))^N} + \|u_t\|_{L(0,T,W^{1,p'(\cdot)}(\Omega))+L^1(Q_T)}.$$

In the following, $\mathcal{M}_b(Q_T)$ denotes the set of all Radon measures with bounded variation on Q_T, and $\mathcal{M}_0(Q_T)$ designates

$$\mathcal{M}_0(Q_T) = \Big\{\mu \in \mathcal{M}_b(Q_T) : \mu(E) = 0 \text{ for every } E \subset Q_T \text{ such that } cap_{p(\cdot)}(E) = 0\Big\}.$$

To better specify the nature of a measure in $\mathcal{M}_0(Q_T)$, we need then to detail the structure of the dual space $(W_{p(\cdot)}(0,T))'$

Lemma 2.1. *[37, lemma 4.2] Let $g \in (W_{p(\cdot)}(0,T))'$ then there exists $g_1 \in L^{(p^-)'}(0,T;W^{-1,p'(\cdot)}(\Omega))$, $g_2 \in L^{p^-}(0,T;V)$, $H \in (L^{p'(\cdot)}(Q_T))^N$ and $g_3 \in L^{(p^-)'}(0,T;L^2(\Omega))$ such that*

$$\ll g,u \gg = \int_0^T \langle g_1,u\rangle dt + \int_0^T \langle u_t,g_2\rangle dt + \int_{Q_T} H.\nabla u\, dxdt + \int_{Q_T} g_3 u\, \mathrm{d}x\mathrm{d}t,$$

for every $u \in W_{p(\cdot)}(0,T)$. Moreover, we can choose (g_1,g_2,H,g_3) such that

$$\|g_1\|_{L^{(p^-)'}(0,T;W^{-1,p'(\cdot)}(\Omega))} + \|g_2\|_{L^{p^-}(0,T;V)} + \||H|\|_{(L^{p'(\cdot)}(Q_T))^N}$$

$$+\|g_3\|_{L(0,T;L^2(\Omega))} \leq C\|g\|_{(W_{p(\cdot)}(0,T))'},$$

with C not depending on g.

One of the decomposition results of elements of $\mathcal{M}_0(Q_T)$ is the following

Theorem 2.2. *[37, Theorem 4.4] Let $\mu \in \mathcal{M}_0(Q_T)$, then there exists $h \in L^1(Q_T)$ and $g \in (W_{p(\cdot)}(0,T))'$ such that $h + g = \mu$ in the sense that*

$$\int_{Q_T} h\varphi \, dxdt + \ll g, \varphi \gg = \int_{Q_T} \varphi \, d\mu, \ \forall \varphi \in C_c^\infty([0,T]\times\Omega).$$

We obtain the following decomposition theorem as a result of Lemma 2.1 and Theorem 2.2.

Theorem 2.3. *[37, Theorem 4.5] Let $\mu \in \mathcal{M}_0(Q_T)$, then there exists (f, H, g_1, g_2) such that , $g_1 \in L^{(p^-)'}(0,T; W^{-1,p'(\cdot)}(\Omega))$, $g_2 \in L^{p^-}(0,T;V)$, $H \in (L^{p'(\cdot)}(Q_T))^N$; $f \in L^1(Q_T)$, in the sense that*

$$\int_{Q_T} f\varphi dxdt + \int_{Q_T} H.\nabla u dxdt + \int_0^T \langle g_1, \varphi\rangle dt - \int_0^T \langle \varphi_t, g_2\rangle dt = \int_{Q_T} \varphi d\mu,$$

for any $\varphi \in C_c^\infty([0,T]\times\Omega)$.

Remark 2.4. *Note that, according to Theorem 2.3, for any $\mu \in \mathcal{M}_b(Q_T)$. Then, there is (f, h) such that $f \in L^1(Q_T)$, $H \in (L^{p'(\cdot)}(Q_T))^N$, in the sense that*

$$\int_{Q_T} \varphi d\mu = \int_{Q_T} f\varphi dxdt + \int_{Q_T} H.\nabla u dxdt + \int_{Q_T} \varphi d\mu_c.$$

for each $\varphi \in C_c^\infty([0,T]\times\Omega)$.

Note that the decomposition of $\mu \in \mathcal{M}_0(Q_T)$ in the previous theorem is not unique. A well-known decomposition result can be found in [37, Lemma 4.6] and [29, Lemma 2.1]. Every μ in $\mathcal{M}_b(Q_T)$ can be expressed as a unique sum of its absolutely continuous part μ_0 with respect to $p(\cdot)$-capacity and its singular part μ_c focused on a set E with zero p-capacity. Therefore, if $\mu \in \mathcal{M}_b(Q_T)$, thanks to theorem 2.3, we have

$$\mu = f - div(H) + g_t + \mu_c^+ - \mu_c^-,$$

In the distributional sense, where $H \in (L^{p'(\cdot)}(Q_T))^N$, $f \in L^1(Q_T)$, $g \in L^{p^-}(0,T;V)$ and (μ_c^-, μ_c^+) are the positive and negative parts of μ_c. To investigate the existence of a solution and to verify the density results, we need to consider the following preliminary result, which involves some relevant data approximation.

Proposition 2.5. *[24, Proposition 2.31] Let $\mu \in \mathcal{M}_0(Q_T)$, then there exists a decomposition $(f, div(H), g)$ of μ in the sense of Theorem 2.3 and an approximation μ_m of μ satisfying*

$$||\mu_m||_{L^1(Q_T)} \leq C, \ \mu_m \in C_c^\infty(Q_T)$$

and

$$\int_{Q_T} \mu_m\varphi \, d\mu = \int_{Q_T} f_m\varphi \, dx \, dt + \int_0^T \langle div(H_m), \varphi\rangle dxdt - \int_0^T \langle \varphi_t, g_m\rangle \, dt,$$

with

$$\begin{cases} f_m \in C_c^\infty(Q_T) : f_m \to f \text{ in } L^1(Q_T) \, , \\ H_m \in C_c^\infty(Q_T) : H_m \to H \text{ in } L^{p'(\cdot)}(Q_T)^N \\ g_m \in C_c^\infty(Q_T) : g_m \to g \text{ in } L^{p^-}(0,T,V) \, . \end{cases}$$

as m tends to 0, for very $\varphi \in C_c^\infty([0,T]\times\Omega)$.

Remark 2.6. *Let us recall the following function of $\omega_n(r) = re^{\Lambda r^2}$ which had this useful property:*

$$a\omega_n'(r) - b|\omega_n(r)| \geq 1, \ \forall r \in \mathbb{R}, \ \forall a, \ b > 0, \ \forall \Lambda > \frac{b^2}{8a^2}. \tag{3}$$

The truncation function and the following functions will be used in the following:

$$T_k(r) = \max\{-k, min(k, r)\}, \qquad \Theta_k(r) = T_1(r - T_k(r)).$$

We will be interested in a specific type of positive bump functions C_c^∞ known as "cut-off" functions during the proof of our principal result $\omega_n : \mathbb{R}^{N+1} \to \mathbb{R}$ satisfy

$$\begin{cases} \varphi_\gamma(r) \equiv 1 \quad if \quad r \in K_\gamma, \\ \varphi_\gamma(r) = 0 \quad if \quad r \in Q_T \backslash K_\gamma, \\ 0 \leq \varphi_\gamma \leq 1, \quad \forall r \in Q_T. \end{cases}$$

let us define, for every $0 < q(x) < \infty$, the Marcinkiewicz space $\mathcal{M}^{q(x)}(Q_T)$ as the space of every measurable function g where

$$\exists C > 0 \text{ with } meas\Big\{(t, x) \in Q_T \quad |g(t, x)| \geq h\Big\} \leq \frac{C}{h^{q^-}}.$$

for every positive k, endowed with the semi-norm

$$||g||_{\mathcal{M}^{q(x)}(Q)} = \inf\Big\{C > 0 : meas\Big\{(t, x) : |g(t, x)| \geq h\Big\} \leq \Big(\frac{C}{k}\Big)^{q(x)}\Big\}.$$

Note that, if $q(x) \geq q^- > 1$, then we obtain the following continuous embedding

$$L^{q(x)}(Q_T) \hookrightarrow \mathcal{M}^{q(x)}(Q_T) \hookrightarrow L^{q(x)-\varepsilon}(Q_T), \ \forall \epsilon \in (0, q(x) - 1].$$

3. Assumptions and Technical Lemmas

The following assumptions are assumed throughout the work. We take a look at a Leray-Lions operator defined by the formula:

$$Au = -div[\phi(x, t, u, \nabla u)],$$

where $\phi : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function, satisfying the following condition, there exist $k \in L^{p(.)}(Q_T)$ and $\alpha > 0, \beta > 0$ such that, for each $(t, x) \in Q_T$ all $(u, \xi) \in \mathbb{R} \times \mathbb{R}^N$.

$$\phi(t, x, u, \xi) \cdot \zeta \geq L(|u|)|\xi|^{p(x)}, \tag{4}$$

$$|\phi(t, x, u, \xi)| \leq \beta[k(t, x) + L(|u|)|\xi|^{p(x)-1}], \tag{5}$$

$$[\phi(x, t, u, \xi) - \phi(x, t, u, \eta)](\xi - \eta) > 0 \quad \forall \ \xi \neq \eta. \tag{6}$$

Moreover, the function L satisfies

$$L(|u|) \geq \alpha, \ \forall u \in \mathbb{R}. \tag{7}$$

where α, λ, Λ are fxed real numbers. Here

$$b : \mathbb{R} \to \mathbb{R} \text{ is a strictly increasing } C^1\text{-function with } b(0) = 0, \tag{8}$$

and there exist $b_0 > 0$ and $b_1 > 0$ such that

$$b_0 \le b'(s) \le b_1, \; for\ every\ s \in \mathbb{R}. \tag{9}$$

and

$$\mu \in \mathcal{M}_b(Q_T). \tag{10}$$

This part introduces several fundamental technical concepts and results that will be used throughout this article. For some details concerning their related contents, the reader can consult (see[37]).

Lemma 3.1. *Let $0 \le \Lambda \in \mathcal{M}_b(Q_T)$ be concentrated on a set E such that $cap_{p(x)}(E) = 0$. Then, for each 0, there exist $\varphi_\gamma \in C_c^\infty(Q_T)$ and $K_\gamma \subset E$ a compact subset such that*

$$\begin{cases} 0 \le \varphi_\gamma \le 1, \quad \varphi \equiv 0 \; in \; K_\gamma, \; \Lambda(E \backslash K_\gamma) < \gamma, \\ \lim_{\gamma \to 0} \|\varphi_\gamma\|_{\mathcal{V}} = 0, \; \int_{Q_T} (1 - \varphi_\gamma) d\Lambda = \varpi(\gamma), \end{cases} \tag{11}$$

and, in particular, a decomposition $[(\varphi_\gamma)_t^1, \; (\varphi_\gamma)_t^2]$ such that

$$\begin{cases} \varphi_\gamma \to 0 \; \text{*-weakly in} \; L^\infty(Q_T), \; a.e. in \; Q_T \; and \; in \; L^1(Q_T) \\ \| (\varphi_\gamma)_t^1 \|_{L^{p'^-}(0,T,W^{-1,p'(x)}(\Omega))} \le \dfrac{\gamma}{3}, \qquad \| (\varphi_\gamma)_t^2 \|_{L^1(Q_t))} \le \dfrac{\gamma}{3} \end{cases} \tag{12}$$

Remark 3.2. *Let $\mu = f - div(H) + g_t + \mu_c^+ - \mu_c^-$ concentrated on two disjoint sets $E^\pm$ by applying two compact sets $K_\gamma^\pm \subseteq E^\pm$ such that$\mu_c^-(E^- \backslash K_\gamma^-) \le \gamma$, $\mu_c^+(E^+ \backslash K_\gamma^+) \le \gamma$ and four cut-off functions where $\varphi_\eta^\pm$ and $\varphi_\gamma^\pm$ are in $C_c^1(Q_T)$ such that*

$$\begin{cases} \varphi_\gamma^\pm \equiv 1 \; on \; K_\gamma^\pm, 0 \le \varphi_\gamma^\pm \le 1, Supp(\varphi_\gamma^+) \cap Supp(\varphi_\gamma^- \equiv \emptyset) \\ \|\varphi_\gamma^\pm\|_{\mathcal{V}} \le \gamma, \end{cases} \tag{13}$$

and,

$$\begin{cases} (\varphi_\gamma^\pm)_t \; such\ that \; \|(\varphi_\gamma^\pm)_r^1\|_{L^{p'^-}(0,T;W^{-1,p(x)}(\Omega))} \le \dfrac{\gamma}{3} \\ \|(\varphi_\gamma^\pm)_t^2\|_{L^1(Q_T)} \le \dfrac{\gamma}{3}, \end{cases} \tag{14}$$

additionally , if $\mu_{c,m}^{\oplus\ominus}$ are as in (21) we obtain

$$\begin{cases} \int_{Q_T} \varphi_\gamma^\pm d\mu_{c,m}^{\oplus\ominus} = \varpi(m,\gamma), \quad \int_{Q_T} \varphi_\gamma^\pm d\mu_c^\pm \le \gamma, \\ \int_{Q_T} (1 - \varphi_\gamma^\pm \varphi_\eta^\pm) d\mu_{c,m}^{\oplus\ominus} = \varpi(m,\gamma,\eta) \quad and \quad \int_{Q_T} (1 - \varphi_\gamma^\pm \varphi_\eta^\pm) d\mu_c^\pm \le \gamma + \eta. \end{cases} \tag{15}$$

Moreover, if $\varphi_\gamma^\pm, \varphi_\eta^\pm$ in $W^{2,\infty}(Q_T)$ we have

$$\begin{cases} 0 \le \int_{Q_T} \varphi_\eta^+ d\mu_c^- \le \eta \; and \; 0 \le \int_{Q_T} \varphi_\eta^- d\mu_c^+ \le \eta, \\ 0 \le \varphi_\gamma^+ \le 1, 0 \le \varphi_\eta^- \le 1. \end{cases} \tag{16}$$

Lemma 3.3. *[13] Suppose that(4) - (10) are satisfied and let (u_m) be a sequence in $L^{p^-}(0,T;L^{p(x)}(\Omega))$ such that $u_m \to u$ weakly in $L^{p^-}(0,T,L^{p(x)}(\Omega))$ and*

$$\int_{Q_T} (\phi(x,t,u_m),\nabla u_m) - \phi(x,t,u_m,\nabla u))\nabla(u_m - u)dx \to 0.$$

Then, $u_m \to u$ strongly in $L^{p^-}(0,T;L^{p(.)}(\Omega))$.

Lemma 3.4. *Let h' is zero away from a compact set of $\mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ be a continuous piecewise C^1-function where $h(0) = 0$, It should be noted that $H(r) = \int_0^r h(\sigma)d\sigma$. If $u \in L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$, $u_m \in L^{p'^-}(0,T;W^{-1,p(x)}(\Omega)) + L^1(Q_T)$ and $\varphi \in C^\infty(\overline{Q_T})$, we have then*

$$\int_0^T \langle u_m, h(u)\varphi\rangle dt = \int_\Omega H(u(T))\varphi(T)dx - \int_\Omega H(u(0))\varphi(0)dx - \int_{Q_T} \varphi_t H(u)dxdt. \tag{17}$$

In general, we will work with measurable functions and truncations in the energy space $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$. For this, we consider the notion of "generalized gradient", whose fundamental result is contained in the following lemma.

Lemma 3.5. *[13] For every $u \in \mathcal{T}_0^{1,p(x)}(Q_T)$, there exists a unique measurable function $v : Q_T \mapsto \mathbb{R}^N$ such that, $\nabla T_k(u) = v\chi_{\{|u|\leq k\}}$, a.e. in Q_T for each $k > 0$, where E is the characteristic function of the measurable set E.*
Moreover, if

$$\int_{Q_T} |\nabla T_k(u)|^{p(x)} dxdt \leq C(k+1), \tag{18}$$

then, v coincides with the classical gradient of u and is denoted by $\nabla u = v$. with u is $cap_{p(x)}$- a.e. finite, i.e. $cap_{p(x)}\{(t,x) \in Q_T \ : \ |u(t,x)| = +\infty\} = 0$, and there exists a $cap_{p(x)} - q.c.r.$ of u, namely a function $\tilde{u}$ such that $\tilde{u} = u$ a.e. in Q_T and $\tilde{u}$ is $cap_{p(x)}$-quasi continuous.

4. Existence results

In this section we shall present the notion of a weak solution to problem $(\mathcal{P})$ and we shall give the existence result for such solution.

Definition 4.1. *Let Ω be a bounded open subset of $\mathbb{R}^N, N \geq 2$. For each $\mu \in \mathcal{M}_b(Q_T)$, we define a "weak" solution to the problem $(\mathcal{P})$ as a measurable function $s \in C([0,T];L^1(\Omega))$ such that $\phi(t,x,u,\nabla u) \in L^1(Q_T)^N$, $T_k(u) \in L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$, and it verifies*

$$\begin{aligned}
&\int_0^T \left\langle \left(b(u_m)\right)_t, \varphi \right\rangle dt \\
&\quad + \int_{Q_T} \phi(t,x,u_m)(1+|u_m|)^{s(x)}|\nabla u_m|^{p(x)-2}\nabla u_m \nabla\varphi dxdt + \int_{Q_T} \zeta(x,t)(1+|u_m|)^{s(x)}|\nabla u_m dxdt \\
&\quad = \int_{Q_T} f_m \varphi dxdt + \int_{Q_T} H_m \nabla\varphi dxdt + \int_{Q_T} \varphi d\mu_{m,c}, \ \forall \varphi \in C_c^\infty(Q_T).
\end{aligned}$$

Theorem 4.2. *Let $q(x) < s(x) - 1$, $s(x) \geq 0$, and $\mu \in \mathcal{M}_b(Q_T)$ suppose that $\phi(t,x,u)$ is a Carathéodory function verifying the following hypothesis*

$$0 < \alpha \leq \phi(t,x,\xi) \leq \beta \text{ and } \quad 0 < \lambda \leq \zeta(t,x) \leq \Lambda \qquad a.e.\ (t,x) \in Q_T, \ for\ all\ \xi \in \mathbb{R}. \tag{19}$$

where α, β, λ, Λ are fixed real numbers.
Then, the problem $(\mathcal{P})$ has a positive weak solution u such that.

- *if* $u(x) > 1$, *then* $u \in W_0 \cap L^{\eta(x)}(Q_T)$ *for every* $\eta(x) < \frac{(p(x)(N+1)-N)(s(x)+1)}{N+1}$,
- *if* $0 \le u(x) \le 1$, *then* $s \in L^{(r^-)}(0,T;W_0^{1,r(x)}(\Omega))$ *for every* $r(x) < \frac{N(p(x)-1+s(x))}{N-(1-s(x))}$.

Proof. The proof of Theorem 4.2 will be completed in 5 steps.
Step1: Approximate problem. We begin by proving the existence of a weak solution in the presence of regular data, i.e., assuming that μ is a limit of bounded sequences μ_m in $L^\infty(Q_T)$. We present the following approximate problem

$$(\mathcal{P}_m)\begin{cases} \Big(b(u_m)\Big)_t - div[\phi(t,x,u_m)(1+|u_m|)^{s(x)}|\nabla u_m|^{p(x)-2}\nabla u_m] + \zeta(x,t)(1+|u_m|)^{q(x)-1}u_m|\nabla u_m|^{p(x)} = \mu_m \\ in \quad Q_T = (0,T)\times\Omega, \\ b(u_m)(0,x) = b(u_0^m)(x) \ in\ \Omega, \quad u_m(t,x) = 0 \quad on\ (0,T)\times\Omega, \end{cases} \tag{20}$$

where $\mu_m = \mu_{m,d} + \mu_{m,c} = f_m - div(H_m) + g_{m,t} + \mu_{m,c}$. Accoring to [37], we suppose that

$$\begin{cases} H_m \in C_c^\infty(Q_T) \quad : \quad H_m \to H \ in\ (L^{p'(x)}(Q_T))^N, \\ 0 \le \mu_{m,c} \in C_c^\infty(Q_T) \quad : \quad \mu_{m,c} \to \mu_c \text{ in } \mathcal{M}_b(Q_T), \\ f_m \in C_c^\infty(Q_T) \quad : \ f_m \to f \quad \text{weakly} \quad in \quad L^l(Q_T) \end{cases} \tag{21}$$

Furthermore, it follows that $\|\mu_m\|_{L^1(Q_T)} \le C$. On the other hand, as ϕ verifying the conditions (19)with $1 < p(x) < N$, then $(\mathcal{P}_m)$ admits a weak solution $b(u_m) \in L^{p^-}(0,T;W_0^{1,p(x)}(\Omega)) \cap L^\infty(Q)$ with $(b(u_m))_t \in L^{p^-}(0,T;W^{-1,p(x)}(\Omega))$ by using Schauder fix point.

Step.2: This step is dedicated to check the a priori estimates.

Considering $\varphi_{1,k}(u_m) = T_1(u_m - T_k(u))$ as test function in the weak formulation of $(\mathcal{P}_m)$, we get by the integration by parts formula and a virtue of Young's inequality that

$$\begin{aligned} &\int_\Omega \Theta_{1,k}(u_m)(T)dx + \alpha\int_{\{k\le u_m<k+1\}}(1+u_m)^{s(x)}|\nabla u_m|^{p(x)}dxdt \\ &\le \|f_m\|_{L^1(Q_T)} + C\int_{Q_T}|H_m|^{p'(x)}dxdt + \frac{1}{2}\int_{\{k\le u_m<k+1\}}\zeta(x,t)|\nabla u_m|^{p(x)}dxdt + \|\mu_{c,m}\|_{L^1(Q_T)} + \int_\Omega \Theta_{1,k}(u_m)(0,x)dx, \end{aligned}$$

where $\Theta_{1,k}(s) = \displaystyle\int_0^s \varphi(\sigma)b'(\sigma)d\sigma$. Remarking that $\Theta_{1,k}(u)$ is nonnegative and that $\Theta_{1,k}(u_m)(0,x) \le |b(u_0^m)(x)|$, as H_m is bounded in $L^{p'(x)}(Q_T)$, f_m, $\mu_{c,m}$ and $b(u_0^m)$ are, respectively, bounded in $L^1(Q_T)$ and in $L^1(\Omega)$, we get

$$\int_\Omega \Theta_{1,k}(u_m)(T)\,dx \le \ C \quad for\ each\ t\ \in [0,T], \tag{22}$$

and

$$\int_{\{k\le u_m<k+1\}}\zeta(x,t)(|1+u_m|^{q(x)-1})u_m|\nabla u_m|^{p(x)}dxdt \le C \quad for\ each\ k > 0, \tag{23}$$

which gives the estimate of u_m in $L^\infty(0,T;L^1(\Omega))$, and the estimate if $(s(x) > 1)$.

$$\int_{\{k\le u_m<k+1\}}|\nabla u_m|^{p(x)}dxdt \le \frac{C(k+1)}{(1+k)^{s^-}} \quad for\ each\ k > 0, \tag{24}$$

which involves that u_m is bounded in $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$.
Let us note that, according to Theorem 4.2 and if $0 \leq s^- \leq s(x) \leq s^+ \leq 1$, that u_m is bounded in $L^{q^-}(0,T;W_0^{1,q(x)}(\Omega))$ for each $q(x) < \frac{Ns(x)+N}{N-1+s(x)}$. Furthermore, we infer that $(1+u_m)^{s(x)}|\nabla u_m|$ is bounded in $L^{r(x)}(Q_T)$ for every $r(x) < p(x) - \dfrac{N}{N+1}$.
As a result, in the corresponding space, there exists a function u_m converges to u and a.e. in Q_T and weakly in the related spaces. Additionally, we may derive from (22)-(24) that $T_k(u)$ is a Cauchy sequence in $L^{p(x)}(Q_T)$ for all $k > 0$, from the fact that $T_k(u_n)$ is a Cauchy sequence in $L^{p(x)}(Q_T)$ for all $k > 0$, we can deduce that it is a Cauchy sequence in measure for each $k > 0$. This means that for any $k > 0$ and for any $\varepsilon > 0$, there exists an N such that for all $m, n \geq N$, the measure of the set $x \in Q_T : |T_k(u_m)(x) - T_k(u_n)(x)|_{L^{p(x)}(Q_T)} > \varepsilon$ is smaller than ε. Hence, by means of the related Marcinkiewicz estimates on u_m, we get that u_m is a Cauchy sequence in measure. Indeed, we first notice that for any $k,\ \sigma > 0$ and for each $s,\ t \in \mathbb{N}$, for each $s,\ t \in \mathbb{N}$,

$$\Big\{|u_s - u_t| > \sigma\Big\} \subseteq \{|u_s| \geq k\} \cup \{|u_t| \geq k\} \cup \Big\{|T_k(u_s) - T_k(u_t)| > \sigma\Big\}. \tag{25}$$

At present, if $\varepsilon > 0$ is fixed, Marcinkiewicz's estimates lead to the existence of k such that

$$meas(\{|u_s| > k\}) < \frac{\varepsilon}{3},\ meas(\{|u_t| > k\}) < \frac{\varepsilon}{3},\ for\ each\ s,t \in \mathbb{N},\ \ for\ each\ k > \mathrm{k}',$$

since, $T_k(u)$ for every fixed $s > 0$ is a Cauchy sequence in measure, We establish that there exists a value of $\eta_\varepsilon > 0$ such that

$$maes(\{|T_k(u_s) - T_k(u_t) > \sigma\}| < \frac{\varepsilon}{3} \quad for\ each\ s,t > \eta_\varepsilon,\ \ for\ each\ \sigma > 0.$$

Also, if $k > \mathrm{k}'$, from (25) we conclude that

$$\{|u_s - u_t| > \sigma\} < \varepsilon,\ for\ each\ s,t \geq \eta_\varepsilon,\ \ for\ each\ \sigma > 0,$$

As a result, u_s is a Cauchy sequence in measure. In this situation, there is a measurable function $u : Q_T \to \mathbb{R}$, such that u_s converges a.e. in Q_T, resulting in a finite limit function u. As a consequence, for all $k > 0$, we obtain

$$T_k(u_m) \to T_k(u) \text{ weakly in } L^{p^-}(0,T;W_0^{1,p(x)}(\Omega)) \text{ and a.e. in } Q_T. \tag{26}$$

At last, by the weak lower semi continuity and from (24) and (22), we find

$$\int_\Omega \Theta_{1,k}(u)(t)dx \leq C \quad \text{and} \quad \int_{\{k \leq u_m < k\}} |\nabla u|^{p(x)} dxdt \leq C(k+1),\ for\ each\ k > 0. \tag{27}$$

and

$$\int_{\{k \leq u_m < k+1\}} (|1+u_m|^{q(x)-1})u_m |\nabla u_m|^{p(x)} dxdt \leq \frac{C}{\mathcal{A}_n} \quad for\ each\ k > 0, \tag{28}$$

We may deduce that the function u is $cap_{p(x)}$-a. e. finite and $cap_{p(x)}$-quasi-continuous based on what has been mentioned and the lemma 3.5. The above results ensure only weak convergence of $T_k(u_m)$ to $T_k(u)$ in $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$.

Step.3: We will also show the strong convergence of the truncation in $L^{p^-}(0,T;W_0^{1,p}(\Omega))$.
In this part, which will ensure the convergence of ∇u_m to ∇u in Q_T. Using the same procedure of [41] to prove that

$$\lim_{n\to\infty} \int_{Q_T} |\nabla T_k(u_m) - \nabla T_k(u)|^{p(x)} dxdt = 0, \tag{29}$$

and thus use [21] to complete the result.
(i) : Near E. If $\mu_m = f_m - div(H) + \mu_{c,m}$ then, in the weak formulation of u_m by choosing $\omega_n((k-u_m)^+)\varphi_\gamma$, with ω_n defined in (3) and $k > 0$, as the test function, we obtain

$$\begin{aligned}&\int_0^T \langle (b(u_m))_t, \omega_n((k-u_m)^+)\varphi_\gamma\rangle dt \\ &\qquad + \int_{Q_T} \phi(t,x,u_m)(1+|u_m|)^{s(x)}|\nabla u_m|^{p(x)-2}\nabla u_m \cdot \nabla(\omega_n((k-u_m)^+)\varphi_\gamma)dxdt \\ &\qquad + \int_{Q_T} \zeta(x,t)(1+u_m)^{q(x)-1}u_m|\nabla u_m|^{p(x)}\omega_n((k-u_m)^+)\varphi_\gamma dxdt = \int_{Q_T} f_m\omega_n((k-u_m)^+)\varphi_\gamma dxdt \\ &\qquad + \int_{Q_T} H_m \cdot \nabla(\omega_n((k-u_m)^+)\varphi_\gamma)dxdt + \int_{Q_T} \omega_n((k-u_m)^+)\varphi_\gamma d\mu_{c,m}.\end{aligned}$$

Thus, by means (19) and the fact that $\omega_n((k-u_m)^+)\,) = 0$ if $u_m > k$, we obtain

$$\begin{aligned}&\int_0^T \Big\langle b\Big((u_m)\Big)_t, \omega_n((k-u_m)^+))\varphi_\gamma\Big\rangle dt + \alpha\int_{Q_T} \omega_n'((k-u_m)^+)|\nabla T_k(u_m)|^{p(x)}\varphi_\gamma dxdt \\ &\qquad - \max\{1,(1+k)^{q(x)}\}\int_{Q_T}\omega_n((k-u_m)^+)dxdt + \int_{Q_T}\omega_n((k-u_m)^+)\varphi_\gamma d\mu_{c,m}dxdt \\ &\qquad \le -\int_{Q_T} f_m\omega_n((k-u_m)^+)\varphi_\gamma dxdt - \int_{Q_T} H_m\cdot\nabla(\omega_n((k-u_m)^+)\varphi_\gamma)dxdt \\ &\qquad + \int_{Q_T}\phi(t,x,u_m)(1+T_k(u_m))^{s(x)}|\nabla T_k(u_m)|^{p(x)-2}\nabla T_k(u_m)\cdot\nabla\varphi_\gamma\omega_n((k-u_m)^+)dxdt,\end{aligned}$$

Now, since n, which depends on k, verifies (3), we have that

$$\begin{aligned}&\int_\Omega \Phi_{k,n}(u_m(0,x))\varphi_\gamma(0,x)dx + \frac{\alpha}{2}\int_{Q_T}|\nabla T_k(u_m)|^{p(x)}\varphi_\gamma dxdt + \int_{Q_T}\omega_n((k-u_m)^+)\varphi_\gamma d\mu_{c,m} \\ &\le \int_\Omega \Phi_{k,n}(u_m(T,x))\varphi_\gamma(T,x)dx - \int_{Q_T}\Phi_{k,n}(u_m(t,x))(\varphi_\gamma)_t dxdt \\ &- \int_{Q_T} f_m\omega_n((k-u_m)^+)\varphi_\gamma dxdt - \int_{Q_T} H_m\cdot\nabla(\omega_n((k-u_m)^+)\varphi_\gamma)dxdt \\ &+ \int_{Q_T}\omega_n((k-u_m)^+)\phi(t,x,u_m)(1+T_k(u_m))^{s(x)}|\nabla T_k(u_m)|^{p(x)-2}\nabla T_k(u_m)\nabla\varphi_\gamma dxdt,\end{aligned} \tag{30}$$

where $\Phi_{k,n}(\ell) = \int_0^\ell \omega_n((k-y)^+)b'(y)dy$ is a primitive of $\omega_n((k-l)^+)b'(l)$.

Remark that, as $T_k(u)$ converges weakly in $L^{p^-}(0,T;W_0^{\ell,p(x)}(\Omega)$ and $\Phi'_{k,\Lambda}(\ell) = \omega_n((k-\ell)^+)b'(\ell)$, which means that $\Phi_{k,n}(\ell)$ is a bounded function with compact support and $\Phi_{k,\Lambda}(\ell) \ge 0$ if $\ell \ge 0$, we get that

$$\Phi'_{k,n}(u_m) \to \Phi'_{k,n}(u) \text{ weakly in } L^{p^-}(0,T;W_0^{1,p(x)}(\Omega)), \text{weakly}^* \textit{ in } L^\infty(Q_T) \textit{ and a.e.in } Q_T.$$

So, let's look at each term individually, according to Lemma 3.1 and Steps.1-2 we treat the second integral on the right side of (30) as well as the fourth by applying (26), and as $\nabla(\omega_n((k-u_m)^+)\varphi_\gamma$ converges to

$\nabla(\omega_n((k-u)^+b'(u_m))$ weakly in $L^{p(x)}(Q_T)^N$ and $\omega_n((k-u_m)^+)b'(u_m) \le b_1\omega_n(k)$ we then have

$$\begin{aligned}
\lim_{n\to}\sup_{\infty} \frac{\alpha}{2}\int_{Q_T} |\nabla T_k(u_m)|^{p(x)}\varphi_\gamma dxdt + b_1 \int_{Q_T} \omega_n(k-u_m)^+)\varphi_\gamma d\mu_{c,m}^{\oplus} \\
\le \int_\Omega \Phi_{k,\rho}(u(T,x))\varphi_\gamma(T,x)dx - b_0\int_{Q_T}\Phi_{k,\rho}(u(t,x))(\varphi_\gamma)_t dxdt \\
- \int_{Q_T} f\omega_n((k-u)^+)\varphi_\gamma dxdt - \int_{Q_T} H\cdot\nabla(\omega_n((k-u)^+)\varphi_\gamma)dxdt \\
+ \int_{Q_T}\phi(t,x,u)(1+T_k(u))^{s(x)}|\nabla T_k(u_m)|^{p(x)-2}\nabla T_k(u)\cdot\nabla\varphi_\gamma\omega_n((k-u)^+)dxdt.
\end{aligned} \tag{31}$$

However, as $\Phi_{k,\rho}(u) \in L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$, and taking into account the convergence properties of φ_γ in the lemma 3.1, we get

$$\begin{aligned}
&- \int_{Q_T} f\omega_n((k-u)^+)\varphi_\gamma dxdt - \int_{Q_T} H\cdot\nabla(\omega_n((k-u)^+)\varphi_\gamma)dxdt \\
&+ \int_{Q_T}\phi(t,x,u)(1+|T_k(u)|)^{s(x)}|\nabla T_k(u_m)|^{p(x)-2}\nabla T_k(u)\nabla\varphi_\gamma\omega_n((k-u)^+)dxdt \\
&\qquad \le C(k)[\int_{Q_T}(|f|+|\nabla T_k(u)||H|)\varphi_\gamma dxdt + \int_{Q_T}(|H|+|\nabla T_k(u)|)|\nabla\varphi_\gamma|dxdt],
\end{aligned}$$

Hence, thanks to Lebesgue's theorem and the Lemma 3.1 and (29), we easily obtain

$$\begin{cases} \int_{Q_T} |\nabla T_k(u_m)|^{p(x)}\varphi_\gamma dxdt = \varpi(m,\gamma), \\ \int_{Q_T} |\omega_n((k-u_m)^+|\varphi_\gamma d\mu_{c,m} = \varpi(m,\gamma). \end{cases} \tag{32}$$

(ii) : Far from E(1). We note the Landes time regularization of the truncation function $T_k(u)$ by the symbol $T_k(u)_\theta$. Let x_θ be a sequence of functions such that

$$\begin{cases} x_\theta \in W_0^{1,p(x)}(\Omega)\cap L^\infty(\Omega), \|x_\theta\|_{L^\infty(\Omega)} \le k, \\ x_\theta \to T_k(u_0) \quad \text{a.e.in } \Omega \text{ as } \theta \text{ tends to infinity }, \\ \frac{1}{\theta}\|x_\theta\|^{p(x)}_{W_0^{1,p(x)}(\Omega)} \to 0 \text{ as } \theta \text{ tends to infinity.} \end{cases}$$

Next, for $\theta > 0$ and $k > 0$ fixed, we designate by $T_k(u)_\theta$ the unique solution of the problem

$$\begin{cases} \dfrac{\partial T_k(u)_\theta}{\partial t} = v(T_k(u) - \partial T_k(u)_\theta) \quad \textit{in the sense of distributions}, \\ T_k(u)_\theta(0) = x_\theta \text{ in } \Omega. \end{cases}$$

So, $T_k(u)_\theta$ *belongs to* $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))\cap L^\infty(Q_T)$ and $\dfrac{\partial T_k(u)}{\partial t}$ belongs to $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$. As a result, we can demonstrate that when θ diverges then, there existe a subsequences (as in [34]).

$$\begin{cases} \|T_k(u)_\theta\|_{L^\infty(Q_T)} \le k, \quad \textit{for each } k > 0, \\ T_k(u)_\theta \to T_k(u) \text{ strongly in } L^{p^-}(0,T;W_0^{1,p(x)}(\Omega)) \text{ and a.e. in } Q_T. \end{cases}$$

Let us start by proving a result that is crucial for dealing with the second term of the right-hand side (iv).

Lemma 4.3. *Let $k, h > 0$, φ_γ and u_m are defined as previously, hence*

$$\int_{\{h\le|u_m|<k+h\}} |\nabla u_m|^{p(x)}(1-\varphi_\gamma)dxdt = \varpi(m,h,\gamma). \tag{33}$$

Proof. Let us choose $\varphi(u_m)(\ell-\varphi_\gamma)$ as test function in weak formulation of u_m, where $\varphi(\ell) = T_{2k}(\ell - T_h(\ell))$. Integrating, if $\Theta_{k,h}(\ell) = \int_0^\ell \varphi(\xi)b'(\xi)d\xi$, we obtain

$$\begin{aligned}
&\int_{Q_T} \Theta_{k,h}(u_m)_t(1-\varphi_\gamma)dxdt \\
&\qquad + \int_{Q_T} \phi(t,x,u_m)(1+u_m)^{s(x)}|\nabla u_m|^{p(x)-2}\nabla u_m \nabla T_{2k}(u_m - T_h(u_m))(1-\varphi_\gamma)dxdt \\
&\qquad - \int_{Q_T} \phi(t,x,u_m)(1+u_m)^{s(x)}|\nabla u_m|^{p(x)-2}\nabla u_m \cdot \nabla\varphi_\gamma T_{2k}(u_m - T_h(u_m))dxdt \\
&\qquad = \int_{Q_T} f_m T_{2k}(u_m - T_h(u_m))(1-\varphi_\gamma)dxdt + \int_{Q_T} H_m \cdot \nabla(T_{2k}(u_m - T_h(u_m))(1-\varphi_\gamma)dxdt \\
&\qquad + \int_{Q_T} T_{2k}(u_m - T_h(u_m))(1-\varphi_\gamma)d\mu_c^m.
\end{aligned} \tag{34}$$

To arrive at the result, we use the property of equi-integrability and Young's inequality.

$$\begin{aligned}
\Big|\int_{Q_T} H_m \cdot \nabla T_{2k}(u_m - T_h(u_m))(1-\varphi_\gamma)dxdt\Big| &\le C_1 \int_{\{h\le|u_m|<k+h\}} |\nabla u_m|^{p(x)}(1-\varphi_\gamma)dxdt \\
+ C_2 \int_{\{h\le| \ u_m|<k+h\}} |\nabla u_m|^{p(x)}(1-\varphi_\gamma)dxdt &\le \varpi(m,h) + C_2 \int_{\{h\le u_m<2k+h\}} |\nabla u_m|^{p(x)}dxdt
\end{aligned}$$

and

$$\int_{Q_T} f_m(1-\varphi_\gamma)T_{2k}(u_m - T_h(u_m))dxdt = \varpi(m,h)\ ,$$

where, apply Young's inequality, one can take C_2 as small as one chooses (e.g. $C_2 < \frac{\alpha}{2}$); thus, by the hypothesis (19) on "ϕ" in the second term of (32), we get

$$\begin{aligned}
&\int_{Q_T} \phi(t,x,u_m)(1+u_m)^{s(x)}|\nabla u_m|^{p(x)-2}\nabla u_m \cdot \nabla T_{2k}(u_m - T_h(u_m))(1-\varphi_\gamma)dxdt \\
&= \int_{\{h\le|u_m|<h+2k\}} \phi(t,x,u_m)(1+u_m)^{s(x)}|\nabla u_m|^{p(x)-2}\nabla u_m \cdot \nabla u_m(1-\varphi_\gamma)dxdt \\
&\qquad\qquad\qquad \ge \alpha \int_{\{h\le u_m<h+2k\}} |\nabla u_m|^{p(x)}(1-\varphi_\gamma)dxdt,
\end{aligned}$$

Remark that $\Theta_{k,h}(u)$ is non-negative for all $s \in \mathbb{R}$ thus, integration by parts, we have

$$\int_{Q_T} \Theta_{k,h}(u_m)_t(1-\varphi_\gamma)dxdt = \int_{Q_T} \Theta_{k,h}(u_m)\frac{\partial\varphi_\gamma}{\partial t}dxdt - \int_\Omega \Theta_{k,h}(u_0^m)dx,$$

which gives, by Vitali's theorem and the definition of $\Theta_{k,h}(\ell)$ and the strong compactness in $L^1(Q_T)$ of $b(u_m)$ and $b(u_0^m)$, that

$$\int_{Q_T} \Theta_{k,h}(u_m)_t(\ell-\varphi_\gamma)dxdt = \varpi(m,h),$$

Finally, from the lemma (3.1) and the tight convergence of $\mu_{c,m}$, we have

$$\Big|\int_{Q_T}(1-\varphi_\gamma)T_{2k}(u_m-T_h(u_m))dxdt\Big|\leq 2k\Big|\int_{Q_T}(1-\varphi_\gamma)d\mu_{c,m}\Big|=\varpi(m,\gamma),$$

and the third term of (32) can be computed, for any $r(x)<p(x)-\frac{N}{N+1}$, as the following

$$\int_{Q_T}\phi(t,x,u_m)(1+u_m)^{s(x)}|\nabla u_m|^{p(x)-2}\nabla u_m\cdot\nabla\varphi_\gamma T_{2k}(u_m-T_h(u))dxdt$$

$$\leq 2k\beta C(\gamma)\Big(\int_{Q_T}(1+|u_m|)^{s(x)}\nabla u_m|^{r(x)}dxdt\Big)^{(\frac{q}{r})^-}\Big(meas\{(t,x):u_m(t,x)\geq h\}\Big)^{1-\frac{1}{(r^-)}}+\varpi(m,h,\gamma).$$

$$\leq\varpi(m,h,\gamma)+\frac{C(k,\gamma)}{h^{1-\frac{1}{s^-}}}.$$

Putting all these points to gather, we get (33). □

In the following, we apply a method presented in the parabolic case in [42], we can chosen $2k<h$

$$z_m=T_{2k}(u_m-T_h(u_m)+T_k(u_m)-T_k(u)_\theta);$$

We remark that $\nabla z_m=0$ if $|u_m|>h+4k$, so the estimate on $T_k(u)$ of step.2 means that z_m is bounded in $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$; thus, it is obvious to get

$$z_m\to T_{2k}(u-T_h(u)+T_k(u)-T_k(u)_\theta)\text{ weakly in }L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))\text{ and a.e. in }Q_T.$$

Therefore, before integrating by parts we multiply by $z_m(1-\varphi_\gamma)$ the equation solved by u_m to obtain

$$\mathcal{A}+\mathcal{B}\leq C+\mathcal{D}+\mathcal{E}+\mathcal{F},\tag{35}$$

where

$$\mathcal{A}=\int_0^T\Big\langle\Big(b(u_m)\Big)_t,z_m(1-\varphi_\gamma)\Big\rangle dt$$

$$\mathcal{B}=\int_{Q_T}\phi(t,x,u_m)(1+T_M(u_m))^{s(x)}|\nabla T_M(u_m)|^{p(x)-2}\nabla T_M(u_m)\cdot\nabla z_m(1-\varphi_\gamma)dxdt$$

$$C=\int_{Q_T}f_m z_m(1-\varphi_\gamma)dxdt$$

$$\mathcal{D}=\int_{Q_T}H\cdot\nabla(z_m(1-\varphi_\gamma))+2k\,\mathcal{E}\,d\mu_{c,m}$$

$$\mathcal{E}=\int_{Q_T}(1-\varphi_\gamma)$$

$$\mathcal{F}=\int_{Q_T}\phi(t,x,u_m)(1+u_m)^{s(x)}|\nabla u_m|^{p(x)-2}\nabla u_m\cdot\nabla\varphi_\gamma z_m dxdt$$

Now consider the member $\mathcal{B}$, if we choose $M:=h+4k$, we get

$$\int_{Q_T}\phi(t,x,u_m)(1+u_m)^{s(x)}|\nabla u_m|^{p(x)-2}|\nabla u_m|^{p(x)-2}\nabla u_m\cdot\nabla z_m(1-\varphi_\gamma)dxdt$$

$$=\int_{Q_T}\phi(t,x,u_m)(1+u_m\chi_{\{|u_m|\leq M\}})^{s(x)}\,|\nabla u_m\chi_{\{|u_m|\leq M\}}|^{p(x)-2}\nabla u_m\chi_{\{|u_m|\leq M\}}\cdot\nabla z_m(1-\varphi_\gamma)dxdt.$$

Next, if $E_m = \{|u_m - T_h(u_m) + T_k(u_m) - T_k(u)_\theta| \leq 2k\}$ and $h \geq 2k$ it can be divided as follows

$$\begin{aligned}
&\int_{Q_T} \phi(t,x,u_m)(1+u_m\chi_{\{|u_m|\leq M\}})^{s(x)}|\nabla u_m\chi_{\{|u_m|\leq M\}}|^{p(x)-2}\nabla u_m\chi_{\{|u_m|\leq M\}}\cdot\nabla z_m(1-\varphi_\gamma)dxdt\\
&\quad= \int_{Q_T} \phi(t,x,u_m)(1+u_m\chi_{\{|u_m|\leq M\}})^{s(x)}|\nabla u_m\chi_{\{|u_m|\leq M\}}|^{p(x)-2}\nabla u_m\chi_{\{|u_m|\leq M\}}\nabla(u_m - T_h(u)_\theta)(1-\varphi_\gamma)dxdt\\
&\quad+ \int_{\{|u_m|>k\}} \phi(t,x,u_m)(1+u_m\chi_{\{|u_m|\leq M\}})^{s(x)}|\nabla u_m\chi_{\{|u_m|\leq M\}}|^{p(x)-2}\nabla u_m\nabla(u_m - T_h(u_m))(1-\varphi_\gamma)\chi_E dxdt \qquad (36)\\
&\quad- \int_{\{|u_m|>k\}} \phi(t,x,u_m)(1+u_m\chi_{\{|u_m|\leq M\}})^{s(x)}|\nabla u_m\chi_{\{|u_m|\leq M\}}|^{p(x)-2}\nabla u_m\chi_{\{|u_m|\leq M\}}\nabla T_k(u)_\theta(1-\varphi_\gamma)\chi_{E_m}dxdt.
\end{aligned}$$

Consider the second member of (36), as $u_m - T_h(u_m) = 0$ if $|u_m| \leq h$, we find

$$\begin{aligned}
&\Big|\int_{\{|u_m|>k\}} \phi(t,x,u_m)(1+u_m\chi_{\{|u_m|\leq M\}})^{s(x)}|\nabla u_m\chi_{\{|u_m|\leq M\}}|^{p(x)-2}\nabla u_m\chi_{\{|u_m|\leq M\}}\nabla(u_m - T_h(u_m))(1-\varphi_\gamma)\chi_E dxdt\Big|\\
&\qquad\leq \int_{\{h\leq|u_m|\leq h+4k\}} |\phi(t,x,u_m)(1+u_m\chi_{\{|u_m|\leq M\}})^{s(x)}|\nabla u_m\chi_{\{|u_m|\leq M\}}|^{p(x)-2}||\nabla u_m|dxdt,
\end{aligned}$$

while, applying Lemma (3.1) and the hypotheses (19), we have immediately

$$\begin{aligned}
&\int_{\{h\leq|u_m|<h+4k\}} |\phi(t,x,u_m)(1+u_m)^{s(x)}|\nabla u_m|^{p(x)-2}|\nabla u_m(1-\varphi_\gamma)dxdt|\\
&\qquad\leq \beta\int_{\{h\leq|u_m|<h+4k\}} (1+u_m)^{s(x)}|\nabla u_m|^{p(x)}(1-\varphi_\gamma)dxdt\\
&\qquad\leq C(h,k)\beta\int_{\{h\leq|u_m|<h+4k\}} |\nabla u_m|^{p(x)}(1-\varphi_\gamma)dxdt\\
&\qquad\leq \varpi(m,h,\gamma).
\end{aligned}$$

Thus, by applying Lemma (3.1)and the iqui-integrity, we get

$$\int_{\{|u_m|>k\}} \phi(t,x,u_m)(1+u_m\chi_{\{|u_m|\leq M\}})^{s(x)}|\nabla u_m|^{p(x)-2}\nabla u_m\nabla(u_m - T_h(u_m))(1-\varphi_\gamma)\chi_{E_m}dxdt = \varpi(m,h,\gamma). \qquad (37)$$

Now consider the third member of the right-hand side of (3.39); so, thanks to **Step.1**, we get

$$\int_{\{|u_m|>k\}} \phi(t,x,u_m)(1+u_m\chi_{\{|u_m|\leq M\}})^{s(x)}|\nabla u_m\chi_{\{|u_m|\leq M\}}|^{p(x)-2}\nabla u_m\chi_{\{|u_m|\leq M\}}\nabla T_k(u)(1-\varphi_\gamma)\chi_{E_m}dxdt = \varpi(m),$$

hence

$$\begin{aligned}
&\int_{\{|u_m|>k\}} \phi(t,x,u_m)(1+(u_m\chi_{\{|u_m|\leq M\}})^{s(x)}|\nabla u_m\chi_{\{|u_m|\leq M\}}|^{p(x)-2}\nabla u_m\chi_{\{|u_m|\leq M\}}\nabla T_k(u)_\theta(1-\varphi_\gamma)\chi_{E_m}dxdt \qquad (38)\\
&= \int_{\{|u_m|>k\}} \phi(t,x,u_m)(1+u_m\chi_{\{|u_m|\leq M\}})^{s(x)}|\nabla u_m\chi_{\{|u_m|\leq M\}}|^{p(x)-2}\nabla u_m\chi_{\{|u_m|\leq M\}}\\
&\qquad\qquad\times\nabla(T_k(u)_\theta - T_k(u))(1-\varphi_\gamma)\chi_{E_m}dxdt + \varpi(m)\ ,
\end{aligned}$$

Thus, since $T_k(u)_\theta$ converges strongly to $T_k(u)$ in $L^{p^-}(0,T,W_0^{1,p(x)}(\Omega))$ and using again **Step.1**, we can easily obtain

$$\begin{aligned}
&\int_{\{|u_m|>k\}} \phi(t,x,u_m)(1+u_m\chi_{\{|u_m|\leq M\}})^{s(x)}|\nabla u_m\chi_{\{|u_m|\leq M\}}|^{p(x)-2}\nabla u_m\chi_{\{|u_m|\leq M\}}\\
&\qquad\qquad\times\nabla(T_k(u)_\theta - T_k(u))(1-\varphi_\gamma)\chi_{E_m}dxdt = \varpi(m,v),
\end{aligned}$$

from, and(38), we have

$$\int_{\{|u_m|>k\}} \phi(t,x,u_m)(1+|u_m|\chi_{\{|u_m|\le M\}})^{s(x)}|\nabla u_m \chi_{\{|u_m|\le M\}}|^{p(x)-2}\nabla u_m \chi\{|u_m|\le M\}\nabla T_k(u)_\theta(1-\varphi_\gamma)\chi_{E_m} dxdt = \varpi(m,\theta).$$

According to the last result, (38) and (37), we get

$$\mathcal{B} = \int_{Q_T} \phi(t,x,u_m)(1+T_k(u_m)^{s(x)})|\nabla T_k(u_m)|^{p(x)-2}\nabla T_k(u_m)\nabla(u_m - T_k(u)_\theta)(1-\varphi_\gamma)dxdt = \varpi(m,\theta,h,\gamma).$$

Let us first examine the term $\mathcal{F}$ when m tends to infinity: we obtain for $1 < r(x) < p(x) - \frac{N}{N+1}$ and according the convergence results from **step.1** that

$$\begin{aligned}
\mathcal{F} &\le \beta \int_{Q_T} (1+T_h(u_m))^{s(x)}|\nabla T_h(u_m)||\nabla\varphi_\gamma|(T_k(u_m)-T_k(u))^+ dxdt \\
&+ 2k\beta \int_{\{lt_m>h\}} (1+u_m)^{s(x)}|\nabla T_h(u_m)|^{p(x)-2}|\nabla u_m||\nabla\varphi_\gamma| dxdt \\
&\le \beta \int_{Q_T} (1+T_h(u_m))^{s(x)}|\nabla T_h(u_m)||\nabla\varphi_\gamma|(T_k(u_m)-T_k(u))^+ dxdt \\
&+ 2C(\gamma)k\beta ||(1+u_m)^{s(x)}|| ||\nabla u_m||_{L^1(Q_T)} meas\{(t,x) : u_m(t,x) > h\}^{1-\frac{1}{r^-}} \\
&\le \varpi(m,h,\gamma) + \frac{C(k,\gamma)}{h^{1-\frac{1}{r^-}}}.
\end{aligned}$$

According to the properties of z_m and Lebesgue's theorem, we get that $\mathcal{F} = \varpi(m,\theta,h)$; on the other hand, we get

$$\mathcal{D} = \int_{\{h\le u_m<h+2k\}} H\cdot\nabla u(1-\varphi_\gamma)dxdt + \varpi(m,\theta,h)\,,$$

then, by applying Lemma 3.1 and Young's inequality, we have

$$\Big|\int_{Q_T} H\cdot\nabla u(1-\varphi_\gamma)\Big| dxdt = \varpi(h,\gamma).$$

Here, we proceed in the same way as in the proof of Lemma 4.3, from Lemma 3.1 and applying the fact that $|z_m| \le 2k$ we can easily see that $\mathcal{E} = \varpi(m,\gamma)$; then from step.1 and the definition of z_m we get that $\mathcal{F} = \varpi(m,\theta,h)$, and recalling that, by a similar reasoning of the proof of [39, Inequality (7.35)] we have $\mathcal{A} \ge \varpi(m,\theta,h)$, then combining all these facts, we arrive at the conclusion that

$$\limsup_{n,\theta,\gamma} \int_{Q_T} |\nabla(T_k(u_m) - T_k(u)_\theta)^+|^{p(x)}(1-\varphi_\gamma)dxdt \le 0.$$

(iii): Far from E(2) Next, consider the time regularization $T_k(u)_\theta$ chosen in (ii), which converges strongly to $T_k(u)$ in $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$, let us take in the weak formulation of the problem 20 the test function $\omega_n((u_m - T_k(u))^-)_\theta(1-\varphi_\gamma)$ (noting that $\varphi_\Lambda \ge 0$, since $\varphi_\Lambda(0) = 0$, $T_k(u)_\theta \le k$, and $\varphi_\Lambda(l)\chi_{\{l>k\}} = 0$), so that $\omega_n((u_m - T_k(u)_\theta)^-) = \omega_n((T_k(u_m) - T_k(u)_\theta)^-)$, and all integrals intervening in the weak formulation are taken only on the subset $\{(t,x) : u_m \le k\}$, we obtain

$$\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4 = \mathcal{I}_5 + \mathcal{I}_6 + \mathcal{I}_7, \tag{39}$$

where

$$\mathcal{I}_1 = \int_0^T \Big\langle \big(b(u_m)\big)_t,\ \omega_n((u_m - T_k(u)_\theta)^-)(1-\varphi_\gamma)\Big\rangle dt$$

$$\mathcal{I}_2 = \int_{Q_T} \phi(t,x,u_m)(1+u_m)^{s(x)}|\nabla u_m|^{p(x)-2}\nabla u_m\cdot\nabla(\omega_n((u_m - T_k(u)_\theta)^-)(1-\varphi_\gamma)dxdt$$

$$\mathcal{I}_3 = -\int_{Q_T} \phi(t,x,u_m)(1+T_k(u_m))^{s(x)}|\nabla T_k(u_m)|^{p(x)-2}\nabla T_k(u_m)\cdot\nabla\varphi_\gamma\omega_n((u_m - T_k(u)_\theta)^-)dxdt$$

$$\mathcal{I}_4 = \int_{Q_T} \zeta(x,t)(1+T_k(u_m))^{q(x)-1}T_k(u_m)|\nabla u_m|^{p(x)}\omega_n((u_m - T_k(u)_\theta)^-)(1-\varphi_\gamma)dxdt$$

$$\mathcal{I}_5 = \int_{Q_T} f_m\omega_n((u_m - T_k(u)_\theta)^-)(1-\varphi_\gamma)dxdt$$

$$\mathcal{I}_6 = \int_{Q_T} H\cdot\nabla(\omega_n((u_m - T_k(u)_\theta)^-)(1-\varphi_\gamma))dxdt$$

$$\mathcal{I}_7 = \int_{Q_T} \omega_n((u_m - T_k(u)_\theta)^-)(1-\varphi_\gamma)d\mu_{c,m}.$$

First, we analyze the behavior of the derived term in time. By the definition of $T_k(u)_\theta$, we obtain

$$\begin{aligned}
&\int_0^T \Big\langle \frac{\partial b(u_m)}{\partial t},\ \omega_n((u_m - T_k(u)_\theta)^-)(1-\varphi_\gamma)\Big\rangle dt \\
&\quad = \int_0^T \Big\langle \frac{\partial (u_m - T_k(u)_\theta)}{\partial t},\ \omega_n((u_m - T_k(u)_\theta)^-)(1-\varphi_\gamma)\Big\rangle dt \\
&\quad + \theta\int_{Q_T} (T_k(u) - T_k(u)_\theta)\omega_n((u_m - T_k(u)_\theta)^-)(1-\varphi_\gamma)dxdt.
\end{aligned}$$

We put $\omega_n^-(u) = \displaystyle\int_0^u \omega_n((\xi - T_k(\xi)^-)b'(\xi)d\xi$; so, since $\omega_n^-(u)\le 0$ and $0\le\varphi_\gamma\le 1$, using the integration by part, we get

$$\begin{aligned}
&\int_0^T \Big\langle \frac{\partial b(u_m)}{\partial t},\ \omega_n((u_m - T_k(u)_\theta)^-)(1-\varphi_\gamma)\Big\rangle dt \le -\int_\Omega \omega_n^-(u_0^m - x_\theta)(1-\varphi_\gamma)dx \\
&\quad + \theta\int_{Q_T} (T_k(u) - T_k(u)_\theta)\omega_n((u_m - T_k(u)_\theta)^-)(1-\varphi_\gamma)dxdt \\
&\quad + \int_{Q_T} \frac{\partial\varphi_\gamma}{\partial t}\omega_n^-(u_m - T_k(u)_\theta)dxdt.
\end{aligned}$$

Next, passing to the limit when m tends to zero by the Lebesgue's theorem, by the fact that u_0^m converges to u_0 in $L^1(\Omega)$ and that $\omega_n^-(u_0^m - x_\theta)$ is uniformly bounded in m. Then, as $\omega_n(s^-)u\le 0$, we obtain

$$\begin{aligned}
&\limsup_{n\to\infty}\int_0^T \Big\langle \frac{\partial b(u_m)}{\partial t},\ \omega_n((u_m - T_k(u)_\theta)^-)(1-\varphi_\gamma)\Big\rangle dt \\
&\quad \le -\int_\Omega \omega_n^-(u_0 - x_\theta)(1-\varphi_\gamma)dx + \int_{Q_T}\frac{\partial\varphi_\gamma}{\partial t}\omega_n^-(u - T_k(u)_\theta)
\end{aligned}$$

which implies, by tending θ towards infinity and d by to the definition of x_θ, we have

$$\begin{aligned}
&\limsup_{\theta\to\infty}\limsup_{n\to\infty}\int_0^T \Big\langle \frac{\partial b(u_m)}{\partial t},\ \omega_n((u_m - T_k(u)_\theta)^-)(1-\varphi_\gamma)\Big\rangle dt \\
&\qquad \le -\int_\Omega \omega_n^-(u_0 - T_k(u_0))(1-\varphi_\gamma)dx + \int_{Q_T}\frac{\partial\varphi_\gamma}{\partial t}\omega_n^-(u - T_k(u))dxdt.
\end{aligned}$$

As $\omega_n^-(u - T_k(u)) = 0$ for any l, we have

$$\limsup_{\theta u\to\infty}\limsup_{n\to\infty}(\mathcal{I}_1)\le 0.$$

Dealing with $\mathcal{I}_2$, we have

$$\mathcal{I}_2 = -\int_{Q_T} \phi(t,x,u_m)(1+u_m)^{s(x)}|\nabla(u_m - T_k(u))^-|^{p(x)}\omega_n'((u_m - T_k(u)_\theta)^-)(1-\varphi_\gamma)dxdt$$

$$+\int_{Q_T} \phi(t,x,u_m)(1+T_k(u_m))^{s(x)}\nabla T_k(u)\nabla(\omega_n((u_m - T_k(u)_\theta)^-)(1-\varphi_\gamma)dxdt,$$

since $(u_m - T_k(u)_\theta)^-$ converges weakly to $(u - T_k(u)_\theta)^-$ in $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$ and u_m converges to u a.e. in Q_T, which is equal to zero, so

$$\mathcal{I}_2 = \varpi(m) - \int_{Q_T} \phi(t,x,u_m)(1+u_m)^{s(x)}|\nabla(u_m - T_k(u))^-|^{p(x)}\omega_n'((u_m - T_k(u)_\theta)^-)(1-\varphi_\gamma)dxdt$$

$$\leq \varpi(m) - \alpha\int_{Q_T} |\nabla(u_m - T_k(u))^-|^{p(x)}\omega_n'((u_m - T_k(u)_\theta)^-)(1-\varphi_\gamma)dxdt.$$

Furthermore, according to step.1 and as $\omega_n((u - T_k(u))^-) = \omega_n(0) = 0$, we obtain

$$\mathcal{I}_3 = \varpi(m,\theta),$$

when θ tends to infinity then, $\omega_n((u - T_k(u)_\theta)^-)$ converges a.e. (while weakly-* *in* $L^\infty(Q_T)$) to $\omega_n((u - T_k(u))^-) \equiv 0$ and $\omega_n((u - T_k(u))^-) = \omega_n(0) = 0$, next, by reminding us that $\omega_n((u_m - T_k(u))^-)$ is bounded by $\omega_n(k)$, we have

$$\begin{aligned}\mathcal{I}_4 &\leq \Lambda C(k)\int_{Q_T} |\nabla u_m|^{p(x)}\omega_n((u_m - T_k(u)_\theta)^-)(1-\varphi_\gamma)dxdt\\
&\leq 2\Lambda C(k)\int_{Q_T} |\nabla(u - T_k(u))^-|^{p(x)}\omega_n((u_m - T_k(u)_\theta)^-)(1-\varphi_\gamma)dxdt\\
&+ 2\Lambda C(k)\int_{Q_T} |\nabla T_k(u)|^{p(x)}\omega_n((u_m - T_k(u)_\theta)^-)(1-\varphi_\gamma)dxdt\\
&\leq \varpi(m,\gamma) + 2\Lambda C(k)\omega_n(k)\int_{Q_T} |\nabla(u_m - T_k(u)_\theta)^-|^{p(x)}(1-\varphi_\gamma)dxdt.\end{aligned}$$

Thus to finish, as $\mu_{n,c}$ is positive and $\nabla(\omega_n((u_m - T_k(u)_\theta)^-)(1-\varphi_\gamma)) \to 0$ in $L^{p'(\cdot)}(Q_T)$, from Properties of f_m, H_m and according to Lebesgue's theorem, we obtain

$$\mathcal{I}_5 = \varpi(m,\theta,\gamma), \qquad \mathcal{I}_6 = \varpi(m,\theta,\gamma) \qquad \textit{and} \qquad \mathcal{I}_7 = \varpi(m,\theta,\gamma).$$

Hence, we can readily infer, by first tending m to infinity, θ to infinity and then γ to zero in (39), by means the fact that $\omega_n'((u_m - T_k(u)_\theta)^-)$ is bounded by $\omega_n'(k)$ and by taking an appropriate choice of Λ satisfying (3), we get

$$\int_{Q_T} |\nabla(u_m - T_k(u)_\theta)^-|^{p(x)}(1-\varphi_\gamma)dxdt = \varpi(m,\theta,\gamma).$$

(iv): Near and far-from E

Here, we serve to demonstrate the strong convergence of truncations in $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$; to do this, we can describe

$$\begin{aligned}\limsup_{n\to\infty}\int_{Q_T} |\nabla T_k(u_m) - \nabla T_k(u)|^{p(x)}dxdt &\leq \limsup_{n\to\infty}\int_{Q_T} |\nabla(T_k(u_m) - T_k(u))^+|^{p(x)}(1-\varphi_\gamma)dxdt\\
&\leq \limsup_{n\to\infty}\int_{Q_T} |\nabla(T_k(u_m) - T_k(u)_\theta)^+|^{p(x)}(1-\varphi_\gamma)dxdt + \int_{Q_T} |\nabla(T_k(u)_\theta - T_k(u))^+|^{p(x)}dxdt\\
&\leq \varpi(\gamma,\theta) + \limsup_{n\to\infty} 2\int_{Q_T} |\nabla T_k(u_m)|^{p(x)}\varphi_\gamma dxdt + 2\int_{Q_T} |\nabla T_k(u)|^{p(x)}\varphi_\gamma dxdt.\end{aligned}$$

Thus, using the properties of φ_γ, and the lemma (3.1), we arrive at the following

$$\int_{Q_T} |\nabla T_k(u_m) - \nabla T_k(u)|^{p(x)} dxdt = \varpi(m)\,, \tag{40}$$

so, we deduce that

$$T_k(u_m) \to T_k(u) \text{ strongly in } L^{p^-}(0,T;W_0^{1,p(x)}(\Omega)), \tag{41}$$

which also implies, by similar steps of [4, 18], that

$$\nabla u_m \to \nabla u \text{ a.e. in } Q_T. \tag{42}$$

Step.4: Equi-integrability of the lower order term in $L^1(Q_T)$.
To show that

$$\zeta(x,t)(1+u_m)^{q(x)-1}u_m|\nabla u_m|^{p(x)} \longrightarrow \zeta(x,t)(1+u)^{q(x)-1}u|\nabla u|^{p(x)} \text{ strongly in } L^1(Q_T),$$

we must ensure that the sequence $\{(1+u_m)^{q(x)-1}u_m|\nabla u_m|^{p(x)}\ in\ L^1(Q_T)\}$ is equi-integrable(as we easily know, from (42), the convergence a.e.the lower order term). In this case, let us put $n(x) > 0$ with $n(x) < \frac{s(x)-q(x)-1}{2}$ and let be a measurable subset of Q_T, we get

$$\begin{aligned}
&\int_B \zeta(x,t)(1+u_m)^{q(x)-1}u_m|\nabla u_m|^{p(x)} dxdt\\
&\le \max\{\Lambda,(1+k)^{q^-}\}\int_B |\nabla u_m \chi_{\{u_m\le k\}}|^{p(x)} dxdt + \int_{\{u_m>k\}} (1+u_m)^{q^- -1}u_m|\nabla u_m|^{p(x)} dxdt\\
&\le \max\{\Lambda,(1+k)^{q^-}\}\int_B |\nabla T_k(u_m)|^{p(x)} dxdt + \frac{\Lambda}{k^{n^-}}\int_{Q_T} (1+u_m)^{q^- -1}u_m^{1+n(x)}|\nabla u_m|^{p(x)} dxdt\\
&\le \max\{\Lambda,(1+k)^{q^-}\}\int_B |\nabla T_k(u_m)|^{p(x)} dxdt + \frac{\Lambda}{k^{n^-}}\sum_{k=0}^{+\infty}\int_{\{k\le u_m<k+1\}} (1+u_m)^{q^- +n^-}|\nabla u_m|^{p(x)} dxdt\\
&\le \max\{\Lambda,(1+k)^{q^-)}\}\int_B |\nabla T_k(u_m)|^{p(x)} dxdt + \frac{C}{k^{n^-}}\sum_{k=0}^{+\infty}\frac{1}{(1+k)^{1+n^-}}\\
&\le C(k)\int_B |\nabla T_k(u_m)|^{p(x)} dxdt + \frac{C}{k^{n^-}}.
\end{aligned} \tag{43}$$

Next, taking k_0 such that $\dfrac{c}{k_0^{n^-}} \le \varepsilon$ (where $\varepsilon > 0$ is yielded), hence, according to (41), there is $\eta > 0$ such that for any measurable subset $B \subset Q_T < \eta$ we obtain

$$\int_{Q_T} |\nabla T_{l_0}(u_m)|^{p(x)} dxdt \le \frac{\varepsilon}{C(k_0)},\ \forall\ m \in \mathbb{N}.$$

Therefore, from (43), it follows that $\zeta(x,t)(1+u_m)^{q(x)-1}|u_m|^{p(x)}$ is equi-integrable in Q_T, which yields under the Vitali theorem that

$$\zeta(x,t)(1+u_m)^{q(x)-1}u_m|\nabla u_m|^{p(x)} \text{ strongly converges to } \zeta(x,t)(1+u)^{q(x)-1}u|\nabla u|^{p(x)} \text{ in } L^1(Q_T).$$

Step.5: Passage to the limit
Let us now take the weak formulation of the approximate problem (20) and consider the limit, when m tends to ∞, as $(1+u_m)^{s(x)}\nabla u_m$ is bounded in $L^{q(x)}(Q_T)^N$ for all $q(x) < p(x) - \frac{N}{N+1}$, converges strongly to $(1+u)^{s(x)}\nabla u$ in $L^1(Q_T)^N$, ∇u_m converges to ∇u a. e. in Q_T.

On the other hand, since for $s(x) > 1$, the sequence $\{u_m\}$ converges to u in $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$; thus, choosing $\Gamma_k(u_m) = u_m - T_k(u)$ for each $i > 0$ and $k > 0$, we obtain

$$\int_{Q_T} |\nabla \Gamma_i(u_m)|^{p(x)} dxdt = \sum_{k=i}^{+\infty} \int_{\{k \le u_m < k+1\}} |\nabla u_m|^{p(x)} dxdt \le \sum_{k=i}^{+\infty} \frac{C}{(1+k)^{s^-}}.$$

Therefore, we can take i, ε strictly positive such that $\Big(\int_{Q_T} |\nabla \Gamma_i(u_m)|^{p(x)} dxdt \Big)^{\frac{1}{p^-}} \le \frac{\varepsilon}{3}$ for each $m \in \mathbb{N}$, which gives, from the strong convergences of the truncations and the weak lower semi-continuity (41) that there exists $\theta_\varepsilon > 0$ verifying, for each $m \ge \theta_\varepsilon$, such that

$$\begin{aligned} \|u_m - u\|_{L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))} &\le \|T_i(u_m) - T_i(u)\|_{L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))} \\ &+ \|\Gamma_i(u_m)\|_{L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))} + \|\Gamma_i(u)\|_{L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))} \\ &\le \varepsilon, \end{aligned}$$

as a result,

$$u_m \to u \qquad strongly \quad in \qquad L^{p^-}(0,T;W_0^{1,p(x)}(\Omega)).$$

Therefore, the problem $(\mathcal{P})$ admits u as a weak solution (see the definition 4.1). □

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