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Refinement on fixed point results in metric and b-metric spaces

Arslan Hojat Ansari Komachali[a], **Yaé Ulrich Gaba**[a,b,c,*], **Maggie Aphane**[a], **Isa Yildirim**[d]

[a] *Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Ga-Rankuwa, Pretoria, Medunsa-0204, South Africa*
[b] *Quantum Leap Africa (QLA), AIMS Rwanda Centre, Remera Sector KN 3, Kigali, Rwanda*
[c] *African Center for Advanced Studies (ACAS), P.O. Box 4477, Yaoundé, Cameroon*
[d] *Department of Mathematics, Faculty of Science, Atatürk University, Erzurum 25240, Turkey*

Abstract. In this manuscript, we introduce some conditions for a sequence to be Cauchy sequence and b-Cauchy sequence in metric spaces and b-metric spaces. After, some fixed point results are obtained regarding them in such spaces. Also, we give two examples to show the usability of our results.

1. Introduction and mathematical preliminaries

Fixed point theory is an interesting field of mathematics that have applications in many sciences like economy, computer sciences, medecine and game theory. Ever since S. Banach proved the Banach fixed point theorem in 1922, many authors have tried to generalize this conclusion. Usually these studies have been obtained by generalizing the concept of metric space or by generalizing the contraction mappings. There are different generalizations of metric space in the literature. One of them is b-metric space was introduced of Bakhtin [19] and Czerwik [20]. Since then, many authors have studied the fixed points of different contraction mappings classes in such spaces (see [2–13, 22]).

The main goal of this manuscript is to give some conditions in order to be Cauchy sequence and b Cauchy sequence of a sequence in the frame of ordinary metric and b-metric spaces. Our results unify and exend some existing fixed point results in the related literature (see [18], [16], [21]).

Now, we will recall some basic definitions and lemma that we will use in our results.

Definition 1.1. *[20] Let U be a (nonempty) set and $b \geq 1$ be a given real number. A function $d : U \times U \to \mathbb{R}^+$ is a b-metric on U if for all $u, v, y \in U$, the following conditions hold:*

(b_1) $d(u, v) = 0$ *if and only if* $u = v$,

(b_2) $d(u, v) = d(v, u)$,

(b_3) $d(u, y) \leq b[d(u, v) + d(v, y)]$.

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* Corresponding author: Yaé Ulrich Gaba
Email addresses: analsisamirmath2@gmail.com, mathanalsisamir4@gmail.com (Arslan Hojat Ansari Komachali), yaeulrich.gaba@gmail.com (Yaé Ulrich Gaba), maggie.aphane@smu.ac.za (Maggie Aphane), isayildirim@atauni.edu.tr (Isa Yildirim)

Then, (U, d) is called a b-metric space.

It is clear that a b-metric space is a metric space for $b = 1$.

Definition 1.2. *[14, 15] Let (U, d) be a b-metric space and $\{u_n\}$ a sequence in U. We say that*
(1) $\{u_n\}$ b-converges to $u \in U$ if $d(u_n, u) \to 0$ as $n \to \infty$;
(2) $\{u_n\}$ is a b-Cauchy sequence if $d(u_m, u_n) \to 0$ as $m, n \to \infty$;
(3) (U, d) is b-complete if every b-Cauchy sequence in U is b-convergent.

Each b-convergent sequence in a b-metric space has a unique limit and it is also a b-Cauchy sequence. In addition, in general, a b-metric is not necessarily continuous.

Lemma 1.3. *[8] Let (U, d) be a b-metric space with coefficient $b \geq 1$ and $f : U \longrightarrow U$ be a mapping. Suppose that $\{u_n\}$ is a sequence in U induced by $u_{n+1} = fu_n$ such that*

$$d(u_n, u_{n+1}) \leq \delta d(u_{n-1}, u_n)$$

for all $n \in \mathbb{N}$ where $\delta \in \left[0, \frac{1}{s}\right)$ is a constant. Then $\{u_n\}$ is a b-Cauchy sequence.

Lemma 1.4. *[17] Let (U, d) be a b-metric space with coefficient $b \geq 1$ and $f : U \longrightarrow U$ be a mapping. Suppose that $\{u_n\}$ is a sequence in U induced by $u_{n+1} = fu_n$ such that*

$$d(u_n, u_{n+1}) \leq \delta d(u_{n-1}, u_n)$$

for all $n \in \mathbb{N}$ where $\delta \in [0, 1)$ is a constant. Then $\{u_n\}$ is a b-Cauchy sequence.

Many of the authors have used Lemma 1.3 to prove some fixed point results in b metric spaces and its generalized versions. Lemma 1.4 expands the range of ([8], Lemma 3.1) from $0 \leq \delta < \frac{1}{s}$ to $0 \leq \delta < 1$. So, we will use the Lemma 1.4 instead of Lemma 3.1 in [8] in order to prove our main results.

2. Results for Picard sequences in b-metric spaces

Proposition 2.1. *Let (U, d) be a b-metric space and $f : U \longrightarrow U$ be a given mapping. Let a Picard sequence $\{u_n\}$ of initial point $u_0 \in X$ be satisfying the following condition:*

$$d(u_n, u_{n+1}) \leq \frac{b[d(u_{n-1}, u_n) + d(u_n, u_{n+1})] + p}{b[d(u_{n-1}, u_n) + d(u_n, u_{n+1})] + m} d(u_{n-1}, u_n) \tag{1}$$

where m, n are fixed positive real numbers such that $p < m$. Then $\{u_n\}$ is a b-Cauchy sequence.

Proof. Let $\{u_n\}$ be Picard sequence which is defined by $u_n = fu_{n-1}$ for all $n \in \mathbb{N}$ and for initial point $u_0 \in U$. Taking $u_{n_0} = u_{n_0 - 1}$ for some $n_0 \in \mathbb{N}$, then u_{n_0} is a fixed point of f and so $\{u_n\}$ is a b-Cauchy sequence.

Suppose that (1) holds for the sequence $\{u_n\}$. If $u_n \neq u_{n-1}$ for all $n \in \mathbb{N}$, from (1), we have that the sequence $\{d(u_{n-1}, u_n)\}$ is decreasing. Thus there exists a non negative real number c such that $d(u_{n-1}, u_n) \to c$ as $n \to \infty$. Now we claim that $c = 0$. If $c > 0$, on taking limit as $n \to +\infty$ on both side of (1), we get that

$$c \leq \frac{b(c + c) + p}{b(c + c) + m} r < r$$

which is a contradiction. It follows that $c = 0$.

Finally, we prove that $\{u_n\}$ is a b-Cauchy sequence. Let $\delta \in [0, 1)$. Since $c = 0$, then there exists $n(\delta) \in \mathbb{N}$ such that

$$\frac{b[d(u_{n-1}, u_n) + d(u_n, u_{n+1})] + p}{b[d(u_{n-1}, u_n) + d(u_n, u_{n+1})] + m} \leq \lambda \quad \text{for all } n \geq n(\delta).$$

This implies that

$$d(u_n, u_{n+1}) \leq \delta d(u_{n-1}, u_n) \quad \text{for all } n \geq n(\delta).$$

By using Lemma 1.4 we obtain that $\{u_n\}$ is a b-Cauchy sequence. □

Considering the previous result, we give the following result.

Theorem 2.2. *Let (U, d) be a complete b-metric space and $f : U \longrightarrow U$ such that*

$$d(fu, fv) \leq \frac{d(u, fv) + d(v, fu) + p}{b[d(u, fu) + d(v, fv)] + m} d(u, v), \text{for all } u, v \in U \tag{2}$$

where $m, n \in \mathbb{R}^+$ are fixed positive real numbers such that

$$p < m\,.$$

Then

(i) f has at least one fixed point $u^* \in U$;

(ii) every Picard sequence of initial point $u_0 \in U$ b-converges to a fixed point of f;

(iii) if $u^*, v^* \in U$ are two distinct fixed points of f, then $d(u^*, v^*) \geq \frac{m-p}{2}$.

Proof. Suppose that $\{u_n\}$ is a Picard sequence which is defined by $u_n = fu_{n-1}$ where $u_0 \in U$ is an arbitrary point. Taking $u_n = u_{n-1}$ for some $n \in \mathbb{N}$, from definition Picard sequence, we have u_n is a fixed point of f. Assume that $u_n \neq u_{n-1}$ for all $n \in \mathbb{N}$. Taking $u = u_{n-1}$ and $v = v_n$ and using the inequality (2), we obtain

$$\begin{aligned} d(u_n, u_{n+1}) &\leq \frac{d(u_{n-1}, u_{n+1}) + p}{b[d(u_{n-1}, u_n) + d(u_n, u_{n+1})] + m} d(u_{n-1}, u_n), \\ &\leq \frac{b[d(u_{n-1}, u_n) + d(u_n, u_{n+1})] + p}{b[d(u_{n-1}, u_n) + d(u_n, u_{n+1})] + m} d(u_{n-1}, u_n), \end{aligned}$$

that is, condition (1) holds for the sequence $\{u_n\}$. From Proposition 2.1, we say that $\{u_n\}$ is a b-Cauchy sequence. Since U is a b-complete metric space, the sequence $\{u_n\}$ b-converges to some $u^* \in U$. Now, we prove that u^* is a fixed point of f. Taking $u = u_n$ and $v = u^*$ in (2), we get

$$\begin{aligned} d(u_{n+1}, fu^*) = d(fu_n, fu^*) &\leq \frac{d(u_n, fu^*) + d(u^*, fu_n) + p}{b[d(u_n, fu_n) + d(u^*, fu^*)] + m} d(u_n, u^*) \\ &= \frac{d(u_n, fu^*) + d(u^*, u_{n+1}) + p}{b[d(u_n, u_{n+1}) + d(u^*, fu^*)] + m} d(u_n, u^*). \end{aligned} \tag{3}$$

Taking limit on both sides of (3), we obtain

$$d(u^*, fu^*) - bd(u_n, u^*) \leq b\, d(u_n, fu^*) \leq b^2[d(u_n, u^*) + d(u^*, fu^*)]$$

which implies that

$$d(u^*, fu^*) \leq \liminf_{n\to+\infty} b\, d(u_n, fu^*) \leq \limsup_{n\to+\infty}\; b\, d(u_n, fu^*) \leq b^2\, d(u^*, fu^*). \tag{4}$$

Taking lim inf, as $n \to +\infty$, on both sides of (3), by (4) we get

$$\begin{aligned} d(u^*, fu^*) &\leq \liminf_{n\to+\infty} b\, d(u_{n+1}, fu^*) \\ &\leq b \frac{bd(u^*, fu^*) + p}{bd(u^*, fu^*) + m} \limsup_{n\to+\infty} d(u_n, u^*) = 0. \end{aligned}$$

This implies that $d(u^*, fu^*) = 0$, that is, $u^* = fu^*$ and hence u^* is a fixed point of f. Thus (i) and (ii) hold.

Let v^* be another fixed point of f, that is $u^* \neq v^*$. Then using (2) with $u = u^*$ and $v = v^*$, we obtain that

$$\begin{aligned} d(u^*, v^*) = d(fu^*, fv^*) &\leq \frac{d(u^*, fv^*) + d(v^*, fu^*) + p}{m} d(u^*, v^*) \\ &= \frac{2d(u^*, v^*) + p}{m} d(u^*, v^*) \end{aligned}$$

and hence $d(u^*, v^*) \geq \frac{m-p}{2}$, that is, (iii) holds. □

Now we will give the following result for weak contractive condition.

Theorem 2.3. *Let (U, d) be a complete b-metric space and $f : U \longrightarrow U$ be a given mapping. Assume that*

$$d(fu, fv) \leq \frac{d(u, fv) + d(v, fu) + p}{b[d(u, fu) + d(v, fv)] + m} d(u, v) + Ld(v, fu), \text{for all } u, v \in U \tag{5}$$

where m, p, L are fixed positive real numbers such that

$$p < m\,.$$

Then

(i) f has at least one fixed point $u^* \in U$;

(ii) every Picard sequence of initial point $u_0 \in U$ b-converges to a fixed point of f;

(iii) if $u^*, v^* \in U$ are two distinct fixed points of f, then $d(u^*, v^*) \geq \max\{\frac{m(1-L)-p}{2}, 0\}$.

Proof. Suppose that $\{u_n\}$ is a Picard sequence which is defined by $u_n = fu_{n-1}$ where $u_0 \in U$ is an arbitrary point. Taking $u_n = u_{n-1}$ for some $n \in \mathbb{N}$, from definition Picard sequence, we have u_n is a fixed point of f. Assume that $u_n \neq u_{n-1}$ for all $n \in \mathbb{N}$. Taking $u = u_{n-1}$ and $v = v_n$ and using the inequality (5), we obtain

$$\begin{aligned} d(u_n, u_{n+1}) &\leq \frac{d(u_{n-1}, u_{n+1}) + p}{b[d(u_{n-1}, u_n) + d(u_n, u_{n+1})] + m} d(u_{n-1}, u_n) + Ld(u_n, u_n), \\ &\leq \frac{b[d(u_{n-1}, u_n) + d(u_n, u_{n+1})] + p}{b[d(u_{n-1}, u_n) + d(u_n, u_{n+1})] + m} d(u_{n-1}, u_n), \end{aligned}$$

that is, condition (1) holds for the sequence $\{u_n\}$. From Proposition 2.1, we have that $\{u_n\}$ is a b-Cauchy sequence. Since U is a b-complete metric space, the sequence $\{u_n\}$ b-converges to some $u^* \in U$. Now, we prove that u^* is a fixed point of f. Taking $u = u_n$ and $v = u^*$ in (5), we obtain

$$\begin{aligned} d(u_{n+1}, fu^*) = d(fu_n, fu^*) &\leq \frac{d(u_n, fu^*) + d(u^*, fu_n) + p}{b[d(u_n, fu_n) + d(u^*, fu^*)] + m} d(u_n, u^*) + Ld(u^*, fu_n) \\ &= \frac{d(u_n, fu^*) + d(u^*, u_{n+1}) + p}{b[d(u_n, u_{n+1}) + d(u^*, fu^*)] + m} d(u_n, u^*) + Ld(u^*, u_{n+1}). \end{aligned} \tag{6}$$

Taking lim inf as $n \to +\infty$ on both sides of (6), by (4), we get that

$$\begin{aligned} d(u^*, fu^*) &\leq \liminf_{n\to+\infty} b\, d(u_{n+1}, fu^*) \\ &\leq b \frac{b\, d(u^*, fu^*) + p}{b\, d(u^*, fu^*) + m} \limsup_{n\to+\infty} d(u_n, u^*) + \limsup_{n\to+\infty} Ld(u^*, u_{n+1}) = 0. \end{aligned}$$

This implies that $d(u^*, fu^*) = 0$, that is, $u^* = fu^*$ and hence u^* is a fixed point of f. Thus (i) and (ii) hold.

Let v^* be another fixed point of f, that is $u^* \neq v^*$. Then using (5) with $u = u^*$ and $v = v^*$, we obtain that

$$\begin{aligned} d(u^*, v^*) = d(fu^*, fv^*) &\leq \frac{d(u^*, fv^*) + d(v^*, fu^*) + p}{m} d(u^*, v^*) + Ld(v^*, fu^*) \\ &= \frac{2d(u^*, v^*) + p}{m} d(u^*, v^*) + Ld(u^*, v^*) \end{aligned}$$

and hence $1 \leq \frac{2d(u^*,v^*)+p}{m} + L$, that is, (iii) holds. □

If we take $p = 0$ and $m = 1$ in Theorems 2.2-2.3, we have the following corollaries.

Corollary 2.4. *([16]) Let (U, d) be a b-metric space and $f : U \longrightarrow U$ be a given mapping. Assume that*

$$d(fu, fv) \leq \frac{d(u, fv) + d(v, fu)}{b[d(u, fu) + d(v, fv)] + 1} d(u, v), \quad \text{for all } u, v \in U.$$

Then

(i) f has at least one fixed point $u^* \in U$;
(ii) every Picard sequence of initial point $u_0 \in U$ $b-$converges to a fixed point of f;
(iii) if $u^*, v^* \in U$ are two distinct fixed points of f, then $d(u^*, v^*) \geq \frac{1}{2}$.

Corollary 2.5. *([16]) Let (U, d) be a b-metric space and $f : U \longrightarrow U$ be a given mapping. Assume that*

$$d(fu, fv) \leq \frac{d(u, fv) + d(v, fu)}{b[d(u, fu) + d(v, fv)] + 1} d(u, v) + Ld(v, fu), \quad \text{for all } u, v \in U.$$

where L is a positive real numbers. Then

(i) f has at least one fixed point $u^* \in U$;
(ii) every Picard sequence of initial point $u_0 \in U$ b-converges to a fixed point of f;
(iii) if $u^*, v^* \in U$ are two distinct fixed points of f, then $d(u^*, v^*) \geq \max\{\frac{1-L}{2}, 0\}$.

If we choose $p < 1$ and $m = 1$ in Theorems 2.2-2.3, we have the following results.

Corollary 2.6. *Let (U, d) be a b-metric space and $f : U \longrightarrow U$ be a given mapping. Assume that*

$$d(fu, fv) \leq \frac{d(u, fv) + d(v, fu) + p}{b[d(u, fu) + d(v, fv)] + 1} d(u, v), \quad \text{for all } u, v \in U.$$

Then

(i) f has at least one fixed point $u^* \in U$;
(ii) every Picard sequence of initial point $u_0 \in U$ b-converges to a fixed point of f;
(iii) if $u^*, v^* \in U$ are two distinct fixed points of f, then $d(u^*, v^*) \geq \frac{1-p}{2}$.

Corollary 2.7. *Let (U, d) be a b-metric space and $f : U \longrightarrow U$ be a given mapping. Assume that*

$$d(fu, fv) \leq \frac{d(u, fv) + d(v, fu) + p}{b[d(u, fu) + d(v, fv)] + 1} d(u, v) + Ld(v, fu), \quad \text{for all } u, v \in U.$$

Then

(i) f has at least one fixed point $u^* \in U$;
(ii) every Picard sequence of initial point $u_0 \in U$ b-converges to a fixed point of f;
(iii) if $u^*, v^* \in U$ are two distinct fixed points of f, then $d(u^*, v^*) \geq \max\left\{\frac{1-p-L}{2}, 0\right\}$.

Example 2.8. *Let $U = [1, 3]$, $\frac{1}{4} \leq \frac{p}{m} < 1$ and $d : U \times U \to \mathbb{R}^+$ defined by*

$$d(u, v) = |u - v|^2, \quad \text{for all } u, v \in U.$$

Then (U, d) is a b-complete metric space with $b = 2$. Let $f : U \to U$ be defined by

$$f(u) = \frac{3u + 1}{u + 3} \quad \text{for all } u \in U.$$

Note that

$$1 \leq \frac{3u + 1}{u + 3} \leq 3.$$

For all $u, v \in U$, we have

$$d(fu, fv) = d(\frac{3u+1}{u+3}, \frac{3v+1}{v+3}) = \frac{64\,|u-v|^2}{(u+3)^2(v+3)^2} \leq \frac{|u-v|^2}{4}$$

Now, if $u, v \in U$ and $v > u$, we have

$$\begin{aligned}\frac{d(u,fv)+d(v,fu)}{b\,[d(u,fu)+d(v,fv)]} &= \frac{d(u,fv)+d(v,fu)}{2\,[d(u,fu)+d(v,fv)]}\\ &\geq \frac{(uv+3v-3u-1)^2(v+3)^2}{4(v^2-1)^2(v+3)^2}\\ &\geq \frac{(uv+3v-3u-1)^2}{4(v^2-1)^2} \geq \frac{1}{4}.\end{aligned}$$

Then

$$\begin{aligned}d(fu,fv) &\leq \frac{|u-v|^2}{4}\\ &\leq \frac{d(u,fv)+d(v,fu)+p}{b\,[d(u,fu)+d(v,fv)]+m}\,|u-v|^2\end{aligned}$$

for all $u, v \in U$. Thus we have that the condition (2) *of Theorem 2.2 is fulfilled. That is f has a fixed point $u = 1$.*

3. Results for Picard sequences in metric space

Since a metric space is a particular type of b-metric space.Thus the results of this Section are as particular cases of the results of Section 2,so we tell them here without giving the detailed proof.

Proposition 3.1. *Let (U, d) be a metric space and $f : U \longrightarrow U$ be a given mapping. Suppose that a Picard sequence $\{u_n\}$ of initial point $u_0 \in U$ satisfies the following condition:*

$$d(u_n, u_{n+1}) \leq \frac{d(u_{n-1}, u_n) + d(u_n, u_{n+1}) + p}{d(u_{n-1}, u_n) + d(u_n, u_{n+1}) + m} d(u_{n-1}, u_n) \tag{7}$$

where m, p are fixed positive real numbers such that $p < m$ .Then $\{u_n\}$ is a Cauchy sequence.

Considering the result on the above, we can give the following theorem.

Theorem 3.2. *Let (U, d) be a complete metric space and $f : U \longrightarrow U$ such that*

$$d(fu, fv) \leq \frac{d(u, fv) + d(v, fu) + p}{d(u, fu) + d(v, fv) + m} d(u, v), \text{for all } u, v \in U \tag{8}$$

where m, p are fixed positive real numbers such that

$$p < m.$$

Then

(i) f has at least one fixed point $u^* \in U$;

(ii) every Picard sequence of initial point $u_0 \in U$ converges to a fixed point of f;

(iii) if $u^*, v^* \in U$ are two distinct fixed points of f, then $d(u^*, v^*) \geq \frac{m-p}{2}$.

Remark 3.3. *Note that, for all $p \in \mathbb{R}^+$, we have*

$$\frac{d(u,fv)+d(v,fu)+p}{d(u,fu)+d(v,fv)+1} \geq \frac{d(u,fv)+d(v,fu)}{d(u,fu)+d(v,fv)+1},$$

From the result on the above, we have the following result in [18].

Theorem 3.4. *[18] Let (U,d) be a complete metric space and $f : U \longrightarrow U$ such that*

$$d(fu,fv) \leq \frac{d(u,fv)+d(v,fu)}{d(u,fu)+d(v,fv)+1} d(u,v), \text{for all } u,v \in U \tag{9}$$

Then

(i) f has at least one fixed point $u^* \in U$;

(ii) every Picard sequence of initial point $u_0 \in U$ converges to a fixed point of f;

(iii) if $u^*, v^* \in U$ are two distinct fixed points of f, then $d(u^*,v^*) \geq \frac{1}{2}$.

Proof. By Remark 3.3 we see that

$$\begin{aligned} d(fu,fv) &\leq \frac{d(u,fv)+d(v,fu)}{d(u,fu)+d(v,fv)+1} d(u,v) \\ &\leq \frac{d(u,fv)+d(v,fu)+p}{d(u,fu)+d(v,fv)+1} d(u,v) \end{aligned}$$

and this satisfy in Theorem 3.2 with $m = 1$ for all $n \in \mathbb{R}^+$ and $n = 0$, □

Example 3.5. *Let $U = [1,2]$, $\frac{1}{2} \leq \frac{p}{m} < 1$ and $d : U \times U \to \mathbb{R}^+$ defined by*

$$d(u,v) = |u-v|, \quad \text{for all } u,v \in U.$$

We know that (U,d) is a complete metric space . Assume that $f : U \to U$ is defined by

$$f(u) = \frac{2u+1}{u+2} \quad \text{for all } u \in U.$$

Since

$$1 \leq \frac{2u+1}{u+2} \leq 2,$$

we have

$$d(fu,fu) = d(\frac{2u+1}{u+2}, \frac{2v+1}{v+2}) = \frac{3\,|u-v|}{(u+2)(v+2)} \leq \frac{|u-v|}{3}$$

for all $u,v \in U$.

If $v > u$, (so $\frac{v^2-1}{v+2} = d(v,fv) \geq d(u,fu) = \frac{u^2-1}{u+2}$), we obtain that

$$\begin{aligned} \frac{d(u,fv)+d(v,fu)}{d(u,fu)+d(v,fv)} &\geq \frac{d(u,fv)}{d(u,fu)+d(v,fv)} \\ &\geq \frac{\frac{(uv+2v-2u-1)}{(v+2)}}{2\frac{(v^2-1)}{v+2}} \\ &= \frac{(uv+2v-2u-1)}{2(v^2-1)} \geq \frac{1}{2}. \end{aligned}$$

Therefore

$$\begin{aligned} d(fu, fv) &\leq \frac{|u-v|}{3} \leq \frac{|u-v|}{2} \\ &\leq \frac{d(u, fv) + d(v, fu) + p}{d(u, fu) + d(v, fv) + m} |u - v| \\ &= \frac{d(u, fv) + d(v, fu) + p}{d(u, fu) + d(v, fv) + m} d(u, v) \end{aligned}$$

for all $u, v \in U$. That is the condition (2) *of Theorem 2.2 is fulfilled. In this case f has a fixed point $u = 1$.*

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