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Approximation of functions by wavelet expansions with dilation matrix

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Abstract. In this paper, we obtain the degree of approximation of a function f in $L^p(1 \leq p \leq \infty)$ norm under general conditions of the pointwise and uniform convergence of wavelet expansions associated with the multiresolution analysis with dilation matrix. Our results show that the degree has the exponential decay (faster than any polynomial) for the function f in $L^p(R)$ on a finite interval (a, b).

1. Introduction

Wavelets with local support in the time and frequency domains were defined by Grossman and Morlet [1]. Mallat [8] and Meyer [10] evolved the framework of multiresolution analysis in order to recognize the underlying structure and to generate examples of orthogonal bases for $L^2(R)$.

Meyer [10] was among the first to study convergence results for wavelet expansions. Mayer [10] was followed by Walter [3, 4] who obtained results on pointwise and uniform convergence of wavelet expansions in the $L^1 \cap L^2$ norm. Kostadinova and Vindas [7] extend and improve the result of Walter [4] and study the pointwise behaviour of Schwartz distributions in several variables via multiresolution expansions. Zhao et al. [6] studied convergence of wavelet expansions of the function in $L^2(R)$ to the mean value of its both sides limits at a generalized continuous point. Junjian [11] studied the convergence of wavelet expansion with divergent free properties in vector-valued Besov spaces function using biorthogonal B-spline wavelets. Xiehua [9] obtained results on pointwise and uniform convergence of wavelet expansions in L^2 norm. Mallat [8] and Meyer [10] have also shown that the Sobolev class of a function is determined by the L^2 norm of its wavelet expansion. One can also see [5] for more details in the direction of present work.

Since above studies clearly suggest that nothing seems to have been done so far to obtain the degree of approximation of the function f in $L^p(R)$ spaces by wavelet expansions associated with multiresolution analysis with dilation matrix; therefore, in this article, we obtain quite new results on the degree of approximation of functions by wavelet and multiresolution-type expansions with dilation matrix. In fact, we will obtain the degree of approximation of the function f in $L^p(1 \leq p \leq \infty)$ norm under general conditions of pointwise and uniform convergence of wavelet expansions associated with the multiresolution analysis with dilation matrix. Our estimates show that for the function $f \in L^p(R)$ on a finite interval (a, b), the degree has the exponential decay.

Remaining part of this paper is organized as follows: In section 2, we give definitions and some examples related to the presented work. In section 3, we state and prove our main results.

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2. Definitions and Preliminaries

In this section, we give important definitions related to the present work.

2.1. Multiresolution Analysis (MRA)

Let A be any real $n \times n$ matrix. A wavelet set associated with a dilation matrix A, called dilation matrix, is a finite set of functions $\psi^r(x) \in L^2(R),\ r = 1, 2, 3, ...s$ such that the system

$$|\mathrm{det}A|^{j/2}\psi^r(A^j x - \gamma);\, j \in \mathbb{Z},\ \gamma \in \mathbb{Z} \tag{1}$$

forms an orthogonal basis in $L^2(R)$.
We define a function $F(\phi, \psi, etc..)$ on R denoted by $F_{j\gamma}$ as

$$F_{j\gamma}(x) = |\mathrm{det}A|^{j/2}F(A^j x - \gamma);\ j \in \mathbb{Z},\ \gamma \in \mathbb{Z}. \tag{2}$$

and omit r.
A multiresolution is a sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(R)$ associated with dilation matrix A if the followings are satisfied:
(i) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$;
(ii) $\bigcup\limits_{j\in\mathbb{Z}} V_j$ is dense in $L^2(R)$;
(iii) $\bigcap\limits_{j\in\mathbb{Z}} V_j = \{0\}$;
(iv) $f \in V_j$ if and only if $f(Ax) \in V_{j+1}$ for all $j \in \mathbb{Z}$;
(v) $f \in V_0$ if and only if $f(x - \gamma) \in V_0$ for all $\gamma \in \mathbb{Z}$;
(vi) there exists a scaling function $\phi \in V_0$ such that $\{\phi(t - \gamma)\}_{\gamma\in\mathbb{Z}}$ is an orthogonal basis in V_0.

2.1.1. Examples

(1) Wavelets arise from MRA generated by the scaling functions (see [2]):

(i) The Haar wavelet is constructed from MRA generated by the scaling function

$$\phi(x) = \chi_{[-1,0]}(x)$$

associated with MRA $\left\{V_j : j \in \mathbb{Z}\right\}$, where V_j be the space of all functions in $L^2(R)$ which are constant on intervals of the form $[2^{-j}\gamma, 2^{-j}(\gamma + 1)]$, $\gamma \in \mathbb{Z}$. Since

$$\frac{1}{2}\phi\left(\frac{1}{2}x\right) = \frac{1}{2}\chi_{[-2,0]}(x) = \frac{1}{2}\phi(x) + \frac{1}{2}\phi(x + 1),$$

then we can have

$$\psi(x) = \phi(2x + 1) - \phi(2x) = \chi_{[-1,-\frac{1}{2})} - \chi_{[-\frac{1}{2},0)} \tag{3}$$

and the low-pass filter for the Haar wavelet is

$$k_0(\xi) = \frac{1}{2}(1 + e^{i\xi}).$$

Since,

$$\hat{\phi}(\xi) = e^{i\frac{\xi}{2}}\frac{\sin(\frac{\xi}{2})}{\frac{\xi}{2}},$$

then we can have

$$\hat{\psi}(\xi) = ie^{i\frac{\xi}{2}}\frac{\sin^2(\frac{\xi}{4})}{\frac{\xi}{4}}.$$

(ii) The Shannon wavelet is constructed from MRA generated by the scaling function

$$\phi(x) = \frac{\sin \pi x}{\pi x}$$

associated with MRA $\left\{V_j : j \in \mathbb{Z}\right\}$, where V_j be the closed span of $\left\{\phi_{j,\gamma} = 2^{j/2}\phi(2^j \cdot -\gamma), \gamma \in \mathbb{Z}\right\}$, for all $j \in \mathbb{Z}$. The Shannon wavelet is

$$\psi(x) = -2\frac{\sin(2\pi x) + \cos(\pi x)}{\pi(2x+1)}, \tag{4}$$

and the low-pass filter for the Shannon wavelet is

$$k_0(\xi) = \begin{cases} 1 & \text{if} - \frac{\pi}{2} \leq \xi < \frac{\pi}{2}, \\ 0 & \text{if} - \pi \leq \xi < -\frac{\pi}{2} \text{ or } \frac{\pi}{2} \leq \xi < \pi. \end{cases}$$

Since,

$$\hat{\phi}(\xi) = \chi_{[-\pi,\pi]}(\xi),$$

then we can have

$$\hat{\psi}(\xi) = e^{i\frac{\xi}{2}}\chi_I(\xi), \text{where } I = [-2\pi, \pi) \cup (\pi, 2\pi].$$

(2) Wavelet which does not arise from MRA (see [8]):
The wavelet consisting of a function ψ which satisfies

$$\hat{\psi}(\xi) = \chi_D(\xi),$$

where

$$D = [-(32/7)\pi, -4\pi) \cup [-\pi, -(4/7)\pi) \cup ((4/7)\pi, \pi] \cup (4\pi, (32/7)\pi].$$

2.2. Wavelet Expansion Associated with Multiresolution Analysis with Dilation Matrix

Associated with the V_j spaces, we additionally define W_j to be the orthogonal complement of V_j in V_{j+1}, so that $V_{j+1} = V_j \oplus W_j$. Thus, $L^2(R) = \overline{\sum \oplus W_j}$. We define P_j and $Q_j = P_{j+1} - P_j$ respectively, to be the orthogonal projections onto the spaces V_j and W_j, with kernels $P_j(x, y)$ and $Q_j(x, y)$.

For $f \in L^p(R)(1 \leq p \leq \infty)$, we define the following related expansions of f:
(i) a sequence of projections $\{P_j f(x)\}_j$ is called the multiresolution expansions of f;
(ii) the scaling expansion of f is defined as

$$f \sim \sum_{\gamma}^{\infty} b_{j,\gamma}|detA|^{j/2}\phi(A^j x - \gamma) + \sum_{k=j,\gamma}^{\infty} a_{k,\gamma}|detA|^k\psi(A^k x - \gamma), \tag{5}$$

where the coefficients $a_{j,\gamma}$ and $b_{j,\gamma}$ are $L^2(R)$ expansion coefficients of f.
(iii) the wavelet expansion of f associated with dilation matrix A is given by

$$f \sim \sum_{j,\gamma} a_{j,\gamma}F_{j,\gamma}(x)dx, \tag{6}$$

where the coefficients $a_{j,\gamma}$ are the $L^2(R)$ expansion coefficients of f.
Remark 1.

(i) In the above definition 2.2 (i), it can be shown that the projection $\{P_j f(x)\}$ extend to bounded operators on $L_p, 1 \leq p \leq \infty$.
(ii) In the above definition 2.2 (ii) and 2.2 (iii), the L_2 expansion coefficients are defined and uniformly bounded for any $f \in L^p (1 \leq p \leq \infty)$.

Considering the convergence in the sense of $L^2(R)$, one can write

$$f(t) = \sum_j \sum_\gamma a_{j,\gamma} F_{j,\gamma}(t) \tag{7}$$

and

$$f(t) = \sum_\gamma^\infty b_{j,\gamma} |detA|^{j/2} \phi(A^j t - \gamma) + \sum_{k=j}^\infty \sum_\gamma a_{k,\gamma} F_{j,\gamma}(t) \tag{8}$$

$$= f_j(t) + f_n(t), \tag{9}$$

The function f_n is the projection f onto V_n, can be defined as

$$f_n(x) = \int_R q_n(x,t) f(t) dt, \tag{10}$$

where $q_n(x,t)$ is called the reproducing kernel of V_n, given by

$$q_n(x,t) = |detA|^{n/2} q(A^n x, A^n t) \tag{11}$$

and

$$q(x,t) = \sum_\gamma \phi(x-\gamma)\phi(t-\gamma), \gamma \in Z. \tag{12}$$

The scaling function $\phi(x)$ is r-regular ([10]) i. e. $\phi \in C^r(R)$ and

$$\left|\phi^\gamma(t)\right| \leq \frac{C_{\gamma p}}{(1+|t|)^p}; \gamma = 0,1....r;\ p = 0,1,2.... \tag{13}$$

2.3. Pointwise Modulus of Continuity

The pointwise modulus of continuity of the function $f(x)$ at the point x is given by

$$w_x(f,t) = \sup_{|u-x|\leq t} |f(u) - f(x)|. \tag{14}$$

3. Main Results

Theorem 3.1. *If $\phi(x)$ satisfies*

$$\phi(x) = O\left\{\frac{1}{(1+|x|)^N}\right\},\ N > 1. \tag{15}$$

and function f is continuous at x, then

$$|f_n(x) - f(x)| = O(1)\left\{\left(\|f\|_{L^p(R)} + \frac{|f(x)|}{(N-1)}\right)\frac{1}{A^{n(N-1)}} + \sum_{\gamma=0}^{A^n} \frac{\omega_x(f,\frac{\gamma}{A^n})}{\gamma^N}\right\}, \tag{16}$$

where A is a dilation matrix.

Proof. From (15), we have

$$|q(x,t)| \leq \frac{C}{(1+|x-t|)^N}, \tag{17}$$

where C is a positive constant. Form [4, 10], we have

$$\int\limits_{-\infty}^{\infty} |q_n(x,t)|dt = 1. \tag{18}$$

Thus,

$$\begin{aligned}
f_n(x) - f(x) &= \int_{-\infty}^{\infty} q_n(x,t)\{f(t) - f(x)\}dt \\
&= \int_{x-1}^{x+1} q_n(x,t)\{f(t) - f(x)\}dt + \int_{x+1}^{\infty} q_n(x,t)\{f(t) - f(x)\}dt \\
&\quad + \int_{-\infty}^{x-1} q_n(x,t)\{f(t) - f(x)\}dt \\
&= J_1 + J_2 + J_3.
\end{aligned} \tag{19}$$

First, we consider J_1,

$$|J_1| \leq \int_{x-1}^{x+1} |q_n(x,t)||f(t) - f(x)|dt \tag{20}$$

$$\begin{aligned}
&\leq C \int_{x-1}^{x+1} \frac{A^n}{(1+A^n|x-t|)^N} \omega_x(f, |t-x|)dt \\
&= C \int_{-A^n}^{A^n} \frac{\omega_x\left(f, \frac{|u|}{A^n}\right)}{(1+|u|)^N} du \\
&= 2C \int_{0}^{A^n} \frac{\omega_x\left(f, \frac{u}{A^n}\right)}{(1+u)^N} du \\
&= 2C \sum_{\gamma=0}^{A^n-1} \int_{\gamma}^{\gamma+1} \frac{\omega_x\left(f, \frac{u}{A^n}\right)}{(1+u)^N} du \\
&\leq 2C \sum_{\gamma=0}^{A^n} \frac{\omega_x\left(f, \frac{\gamma}{A^n}\right)}{\gamma^N}.
\end{aligned} \tag{21}$$

Now, we consider J_2,

$$\begin{aligned}
|J_2| &\leq \int\limits_{x+1}^{\infty} |q_n(x,t)||f(t) - f(x)|dt \\
&\leq \int\limits_{x+1}^{\infty} |f(t)||q_n(x,t)|dt + |f(x)| \int\limits_{x+1}^{\infty} |q_n(x,t)|dt \\
&\leq C \int\limits_{x+1}^{\infty} |f(t)| \frac{A^n}{(1+A^n|x-t|)^N} dt + C|f(x)| \int\limits_{x+1}^{\infty} \frac{A^n}{(1+A^n|x-t|)^N} dt.
\end{aligned} \tag{22}$$

Using Hölder's inequality in first term of (22), we get

$$
\begin{aligned}
|J_2| &\leq C\left(\int_{x+1}^{\infty}|f(t)|^p dt\right)^{\frac{1}{p}}\left(\int_{x+1}^{\infty}\frac{A^{nq}}{(1+A^n|x-t|)^{Nq}}dt\right)^{\frac{1}{q}}+C|f(x)|\int_{x+1}^{\infty}\frac{A^n}{(1+A^n|x-t|)^N}dt\\
&\leq C\|f\|_{L^p(R)}\left(\int_{1}^{\infty}\frac{A^{nq}}{(1+A^n u)^{Nq}}du\right)^{\frac{1}{q}}+C|f(x)|\int_{1}^{\infty}\frac{A^n}{(1+A^n u)^N}du\\
&\leq C\|f\|_{L^p(R)}\left(\int_{1}^{\infty}\frac{A^{nq}}{A^{nNq}u^{Nq}}du\right)^{\frac{1}{q}}+C|f(x)|\int_{1}^{\infty}\frac{A^n}{A^{nN}u^N}du\\
&\leq C\left(\frac{\|f\|_{L^p(R)}}{A^{n(N-1)}}+\frac{|f(x)|}{(N-1)A^{n(N-1)}}\right)\\
&= C\left(\|f\|_{L^p(R)}+\frac{|f(x)|}{(N-1)}\right)\frac{1}{A^{n(N-1)}}. \qquad (23)
\end{aligned}
$$

Now, we consider J_3,

$$
\begin{aligned}
|J_3| &\leq \int_{-\infty}^{x-1}|q_n(x,t)||f(t)-f(x)|dt\\
&\leq \int_{-\infty}^{x-1}|q_n(x,t)||f(t)|dt+|f(x)|\int_{-\infty}^{x-1}|q_n(x,t)|dt\\
&\leq C\int_{-\infty}^{x-1}|f(t)|\frac{A^n}{(1+A^n|x-t|)^N}dt+C|f(x)|\int_{-\infty}^{x-1}\frac{A^n}{(1+A^n|x-t|)^N}dt. \qquad (24)
\end{aligned}
$$

Using Hölder's inequality in first term of (24), we have

$$
\begin{aligned}
|J_3| &\leq C\left(\int_{-\infty}^{x-1}|f(t)|^p dt\right)^{\frac{1}{p}}\left(\int_{-\infty}^{x-1}\frac{A^{nq}}{(1+A^n|x-t|)^{Nq}}dt\right)^{\frac{1}{q}}+C|f(x)|\int_{-\infty}^{x-1}\frac{A^n}{(1+A^n|x-t|)^N}dt\\
&\leq C\|f\|_{L^p(R)}\left(\int_{1}^{\infty}\frac{A^{nq}}{(1+A^n u)^{Nq}}du\right)^{\frac{1}{q}}+C|f(x)|\int_{1}^{\infty}\frac{A^n}{(1+A^n u)^N}du\\
&= C\left(\|f\|_{L^p(R)}+\frac{|f(x)|}{(N-1)}\right)\frac{1}{A^{n(N-1)}}. \qquad (25)
\end{aligned}
$$

Combining (19) to (25), we get

$$
|f_n(x)-f(x)| = O(1)\left\{\left(\|f\|_{L^p(R)}+\frac{|f(x)|}{(N-1)}\right)\frac{1}{A^{n(N-1)}}+\sum_{\gamma=0}^{A^n}\frac{\omega_x(f,\frac{\gamma}{A^n})}{\gamma^N}\right\}.
$$

□

Theorem 3.2. *If $\phi(x)$ satisfies (15) and $f \in L^p(R)$ is continuous on (a,b), then*

$$
|f_n(x)-f(x)| \leq B\left\{\omega(f,(b-a)A^{-\frac{n}{2}})+A^{-\frac{n}{2}(N-1)}\right\} \qquad (26)
$$

for $x \in [a+2\delta, b-2\delta]$, $\delta < \frac{1}{8}(b-a)$, where $w(f,.)$ is the modulus of continuity of $f(x)$ on $[a+\delta, b-\delta]$, A is a dilation matrix; and B depends on $C, N, \delta, (b-a), \|f\|$ and M, where

$$
M = \sup\{|f(x)| : x \in [a+2\delta, b-2\delta]\}. \qquad (27)
$$

Proof. We have

$$
\begin{aligned}
f_n(x) - f(x) &= \int_{-\infty}^{\infty} q_n(x,t)\{f(t) - f(x)\}dt \\
&= \int_{a+\delta}^{b-\delta} |q_n(x,t)|\{f(t) - f(x)\}dt + \int_{b-\delta}^{\infty} |q_n(x,t)|\{f(t) - f(x)\}dt \\
&\quad + \int_{-\infty}^{a+\delta} |q_n(x,t)|\{f(t) - f(x)\}dt \\
&= K_1 + K_2 + K_3.
\end{aligned} \tag{28}
$$

First, we consider K_1,

$$
\begin{aligned}
|K_1| &\le \int_{a+\delta}^{b-\delta} |q_n(x,t)||f(t) - f(x)|dt \\
&\le C \int_{a+\delta}^{b-\delta} \frac{A^n}{(1 + A^n|x-t|)^N} \omega(f, |x-t|)dx \ \text{ by (14)} \\
&\le 2C \int_0^{A^{n(b-a-3\delta)}} \frac{\omega(f, uA^{-n})}{(1+u)^N} du \\
&\le 2C \left(\int_0^{A^{\frac{n}{2}(b-a-3\delta)}} \frac{\omega(f, uA^{-n})}{(1+u)^N} du + \int_0^{A^{\frac{n}{2}(b-a-3\delta)}} \frac{\omega(f, uA^{-n})}{(1+u)^N} du \right) \\
&\le 2C \left(\omega(f, (b-a-3\delta)A^{-\frac{n}{2}} \int_{\zeta_1}^{A^{\frac{n}{2}(b-a-3\delta)}} \frac{du}{(1+u)^N} + \omega(f, b-a-3\delta) \int_{\zeta_2}^{A^{n(b-a-3\delta)}} \frac{du}{(1+u)^N}) \right),
\end{aligned}
$$

where $0 < \zeta_1 < A^{\frac{n}{2}}(b-a-3\delta)$ and $A^{\frac{n}{2}}(b-a-3\delta) < \zeta_2 < A^n(b-a-3\delta)$.

$$
\begin{aligned}
|K_1| &\le 2C \left(\omega(f, (b-a-3\delta)A^{-\frac{n}{2}} \int_{\zeta_1}^{A^{\frac{n}{2}}(b-a-3\delta)} \frac{du}{u^N} + \omega(f, b-a-3\delta) \int_{\zeta_2}^{A^n(b-a-3\delta)} \frac{du}{u^N} \right) \\
&\le 2C \left(\frac{\omega(f, (b-a)A^{-\frac{n}{2}})}{N-1} + \frac{2M\, A^{-\frac{n}{2}(N-1)}}{(N-1)(b-a)^{N-1}} \right).
\end{aligned} \tag{29}
$$

Now, we consider K_2,

$$
\begin{aligned}
|K_2| &\le \int_{b-\delta}^{\infty} |f(t)||q_n(x,t)|dt + |f(x)| \int_{b-\delta}^{\infty} |q_n(x,t)|dt \\
&\le C \int_{b-\delta}^{\infty} |f(t)| \frac{A^n}{(1 + A^n|x-t|)^N} dt + C|f(x)| \int_{b-\delta}^{\infty} \frac{A^n}{(1 + A^n|x-t|)^N} dt.
\end{aligned} \tag{30}
$$

Using Hölder's inequality in first term of (30), we have

$$
\begin{aligned}
|K_2| &\leq C\left(\int\limits_{b-\delta}^{\infty}|f(x)|^p dt\right)^{\frac{1}{p}}\left(\int\limits_{b-\delta}^{\infty}\frac{A^{nq}}{(1+A^n|x-t|)^{Nq}}dt\right)^{\frac{1}{q}}+C|f(x)|\int\limits_{b-\delta}^{\infty}\frac{A^n}{(1+A^n|x-t|)^N}dt \\
&\leq C\|f\|_{L^p(R)}\left(\int_{b-\delta-x}^{\infty}\frac{A^{nq}}{(1+A^nu)^{Nq}}du\right)^{\frac{1}{q}}+CM\int_{b-\delta-x}^{\infty}\frac{A^n}{(1+A^nu)^N}du \\
&\leq C\|f\|_{L^p(R)}\left(\int_{b-\delta-x}^{\infty}\frac{A^{nq}}{(A^nu)^{Nq}}du\right)^{\frac{1}{q}}+CM\int_{b-\delta-x}^{\infty}\frac{A^n du}{(A^nu)^N} \\
&\leq C\left(\frac{\|f\|_{L^p(R)}}{A^{n(N-1)}(b-\delta-x)^{N-\frac{1}{q}}}+\frac{M}{(N-1)(b-\delta-x)^{N-1}A^{n(N-1)}}\right) \\
&\leq C\left(\frac{\|f\|_{L^p(R)}}{\delta^{(N-\frac{1}{q})}}+\frac{M}{(N-1)\delta^{(N-1)}}\right)\frac{1}{A^{n(N-1)}}. \qquad (31)
\end{aligned}
$$

Similarly,

$$
|K_3| \leq C\left(\frac{\|f\|_{L^p(R)}}{\delta^{(N-\frac{1}{q})}}+\frac{M}{(N-1)\delta^{(N-1)}}\right)\frac{1}{A^{n(N-1)}}. \qquad (32)
$$

Combining (28) to (32), we get

$$
|f_n(x)-f(x)| \leq B\left\{\omega(f,(b-a))A^{-\frac{n}{2}}+A^{-\frac{n}{2}(N-1)}\right\}.
$$

□

Theorem 3.3. *If $\phi(x)$ satisfies*

$$
\phi(x) = O\left\{\frac{1}{1+\eta(|x|)^N}\right\},\ N>1, \qquad (33)
$$

where η is a positive monotonic increasing function of x and if the function f is continuous at x, then

$$
|f_n(x)-f(x)| = O(1)\left\{\left(\|f\|_{L^p(R)}+|f(x)|\right)+\int\limits_{A^n}^{\infty}\frac{du}{1+(\eta(u))^N}+\sum_{\gamma=0}^{A^n}\frac{\omega_x(f,\frac{\gamma}{A^n})}{1+(\eta(\gamma))^N}\right\}, \qquad (34)
$$

where A is a dilation matrix.

Proof. From (33), we have

$$
|q(x,t) \leq \frac{C}{1+(\eta(|x-t|))^N},\ N>1, \qquad (35)
$$

where C is a positive constant. From Walter [4] and Meyer[10], we have

$$
\int\limits_{-\infty}^{\infty}|q_n(x,t)|dt = 1. \qquad (36)
$$

Thus,

$$
\begin{aligned}
f_n(x) - f(x) &= \int_{-\infty}^{\infty} q_n(x,t)\{f(t) - f(x)\}dt \\
&= \int_{x-1}^{x+1} q_n(x,t)\{f(t) - f(x)\}dt + \int_{x+1}^{\infty} |q_n(x,t)\{f(t) - f(x)\}dt \\
&\quad + \int_{-\infty}^{x-1} |q_n(x,t)\{f(t) - f(x)\}dt \\
&= L_1 + L_2 + L_3.
\end{aligned} \tag{37}
$$

First, we consider L_1,

$$
\begin{aligned}
|L_1| &\leq \int_{x-1}^{x+1} |q_n(x,t)||f(t) - f(x)|dt \\
&\leq C \int_{x-1}^{x+1} \frac{A^n}{1 + (\eta(A^n|x-t|))^N} \omega_x(f, |t-x|)dt \\
&\leq C \int_{-A^n}^{A^n} \frac{\omega_x(f, \frac{|u|}{A^n})}{1 + (\eta(|u|))^N} du \\
&= 2C \int_0^{A^n} \frac{\omega_x(f, \frac{u}{A^n})}{1 + (\eta(u))^N} du \\
&= 2C \sum_{\gamma=0}^{A^n-1} \int_{\gamma}^{\gamma+1} \frac{\omega_x(f, \frac{u}{A^n})}{1 + (\eta(u))^N} du \\
&\leq 2C \sum_{\gamma=0}^{A^n} \frac{\omega_x(f, \frac{\gamma}{A^n})}{1 + (\eta(\gamma))^N}.
\end{aligned} \tag{38}
$$

Now, we consider L_2,

$$
\begin{aligned}
|L_2| &\leq \int_{x+1}^{\infty} |q_n(x,t)||f(t) - f(x)|dt \\
&\leq \int_{x+1}^{\infty} |f(t)||q_n(x,t)|dt + |f(x)| \int_{x+1}^{\infty} |q_n(x,t)|dt \\
&\leq C \int_{x+1}^{\infty} |f(t)| \frac{A^n}{1 + (\eta(A^n|x-t|))^N} dt + C|f(x)| \int_{x+1}^{\infty} \frac{A^n}{1 + (\eta(A^n|x-t|))^N} dt.
\end{aligned} \tag{39}
$$

Using Hölder's inequality in the first term of (41), we get

$$
\begin{aligned}
|L_2| &\leq C \left(\int_{x+1}^{\infty} |f(x)|^p dt \right)^{\frac{1}{p}} \left(\int_{x+1}^{\infty} \frac{A^{nq}}{1 + (\eta(A^n|x-t|))^{Nq}} dt \right)^{\frac{1}{q}} + C|f(x)| \int_{x+1}^{\infty} \frac{A^n}{1 + (\eta(A^n|x-t|))^N} dt \\
&\leq C\|f\|_{L^p(R)} \left(\int_1^{\infty} \frac{A^{nq}}{1 + (\eta(A^n u))^{Nq}} du \right)^{\frac{1}{q}} + C|f(x)| \int_1^{\infty} \frac{A^n}{1 + (\eta(A^n u))^N} du \\
&\leq C \left(\|f\|_{L^p(R)} + |f(x)| \right) \int_1^{\infty} \frac{A^n}{1 + (\eta(A^n u))^N} du \\
&\leq C \left(\|f\|_{L^p(R)} + |f(x)| \right) \int_{A^n}^{\infty} \frac{1}{1 + (\eta(u))^N} du.
\end{aligned} \tag{40}
$$

Now, we consider L_3,

$$|L_3| \leq \int_{-\infty}^{x-1} |q_n(x,t)||f(t)-f(x)|dt$$
$$\leq \int_{-\infty}^{x-1} |f(t)||q_n(x,t)|dt + |f(x)|\int_{-\infty}^{x-1} |q_n(x,t)|dt$$
$$\leq C\int_{-\infty}^{x-1} |f(t)|\frac{A^n}{1+(\eta(A^n|x-t|))^N}dt + C|f(x)|\int_{-\infty}^{x-1}\frac{A^n}{1+(\eta(A^n|x-t|))^N}dt. \tag{41}$$

Using Hölder's inequality in the first term of (41), we get

$$|L_3| \leq C\left(\int_{-\infty}^{x-1}|f(x)|^p dt\right)^{\frac{1}{p}}\left(\int_{-\infty}^{x-1}\frac{A^{nq}}{1+(\eta(A^n|x-t|))^{Nq}}dt\right)^{\frac{1}{q}} + C|f(x)|\int_{-\infty}^{x-1}\frac{A^n}{1+(\eta(A^n|x-t|))^N}dt$$
$$\leq C\|f\|_{L^p(R)}\left(\int_1^\infty \frac{A^{nq}}{1+(\eta(A^n u))^{Nq}}du\right)^{\frac{1}{q}} + C|f(x)|\int_1^\infty\frac{A^n}{1+(\eta(A^n u))^N}du$$
$$\leq C\left(\|f\|_{L^p(R)} + |f(x)|\right)\int_1^\infty \frac{A^n}{1+(\eta(A^n u))^N}du$$
$$\leq C\left(\|f\|_{L^p(R)} + |f(x)|\right)\int_{A^n}^\infty \frac{1}{1+(\eta(u))^N}du. \tag{42}$$

Combining (37) to (42), we get

$$|f_n(x)-f(x)| = O(1)\left\{\left(\|f\|_{L^p(R)} + |f(x)|\right) + \int_{A^n}^{\infty}\frac{du}{1+(\eta(u))^N} + \sum_{\gamma=0}^{A^n}\frac{\omega_x(f,\frac{\gamma}{A^n})}{1+(\eta(\gamma))^N}\right\}.$$

□

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