



## Pointwise summability of Fourier–Laguerre series of integrable functions

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**Abstract.** We present an approximation version of the results of D. P. Gupta [J. of Approx. Theory, 7 (1973), 226-238] A. N. S. Singroura [Proc. Japan Acad., 39 (4) (1963), 208-210] and G. Szegö [Math. Z., 25 (1926), 87-115]. Some corollaries and examples will also be given.

### 1. Introduction

Let  $L_w$  be the class of all real-valued functions, integrable in the Lebesgue sense over  $\mathbb{R}^+$  with the weight  $w(t) = e^{-t}t^\alpha$  ( $\alpha > -1$ ), i.e.

$$\int_{\mathbb{R}^+} e^{-t}t^\alpha |f(t)| dt < \infty.$$

We will consider the Fourier–Laguerre series

$$S^{(\alpha)}f(x) := \sum_{v=0}^{\infty} a_v^{(\alpha)}(f)L_v^{(\alpha)}(x), \text{ with } \alpha > -1,$$

where

$$L_n^{(\alpha)}(x) = \frac{x^{-\alpha}e^x}{n!} \frac{d^n}{dx^n} (x^{n+\alpha}e^{-x}) = \sum_{v=0}^n \frac{(-1)^v}{v!} \binom{n+\alpha}{n-v} x^v$$

and

$$a_v^{(\alpha)}(f) = \frac{1}{\Gamma(\alpha+1)A_n^{(\alpha)}} \int_0^\infty e^{-y}y^\alpha L_v^{(\alpha)}(y) f(y) dy, \text{ with } A_n^{(\alpha)} = \binom{n+\alpha}{n}.$$

Let define the  $(C, \gamma)$  - means of partial sums

$$S_k^{(\alpha)}f(x) = \sum_{v=0}^n a_v^{(\alpha)}(f)L_v^{(\alpha)}(x)$$

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of  $S^{(\alpha)}f$  as follows

$$S_n^{(\gamma, \alpha)} f(x) = \frac{1}{A_n^{(\gamma)}} \sum_{k=0}^n A_{n-k}^{(\gamma-1)} S_k^{(\alpha)} f(x), \quad (n = 0, 1, 2, \dots)$$

and let

$$\Delta_x f(t) = f(x+t) - f(x).$$

The deviation  $S_n^{(\gamma, \alpha)} f(0) - f(0)$  was examined in the papers [5], [7] and [1] as follows:

**Theorem A.** [1, Theorem 1] *Let  $f \in L_w$ ,  $\alpha > -1$  and  $\gamma > \alpha + \frac{1}{2}$ . If a function  $f$  satisfies the conditions*

$$\int_0^u e^{-t} t^\alpha |\Delta_0 f(t)| dt = o(u^{\alpha+1}) \text{ as } u \rightarrow 0^+ \tag{1}$$

and

$$\int_1^\infty e^{-\frac{t}{2}} t^{\alpha-\gamma-\frac{1}{3}} |\Delta_0 f(t)| dt < \infty, \tag{2}$$

then

$$\left| S_n^{(\gamma, \alpha)} f(0) - f(0) \right| = o(1) \text{ as } n \rightarrow \infty.$$

Similar results in a case of norm approximation due of C. Market and E. L. Poiani in papers [3] and [4] were obtained.

We will say that a nonnegative function  $\omega$  is a function of the modulus of continuity type if it is nondecreasing continuous function on  $[0, \infty)$  having the following conditions:  $\omega(0) = 0$  and  $\omega(\delta_1 + \delta_2) \leq \omega(\delta_1) + \omega(\delta_2)$  for any  $\delta_1, \delta_2 \in [0, \infty)$ .

In this paper, we will study the upper bound of the quantity  $\left| S_n^{(\gamma, \alpha)} f(0) - f(0) \right|$  by some means of a function of the modulus of continuity type  $\omega$ . From our result we will derive some corollaries, remarks and construct some examples.

## 2. Statement of the results

First we present the estimate of the quantity  $\left| S_n^{(\gamma, \alpha)} f(0) - f(0) \right|$ .

**Theorem 2.1.** *Let  $f \in L_w$ ,  $\alpha > -1$ ,  $\gamma > \alpha + \frac{1}{2}$  and let a function  $\omega$  of the modulus of continuity type satisfy the conditions:*

$$\frac{u^{-(\alpha+1)}}{\Gamma(\alpha+1)} \int_0^u e^{-\frac{t}{2}} t^\alpha |\Delta_0 f(t)| dt = O(\omega(u)) \quad (u > 0) \tag{3}$$

and

$$\frac{1}{\Gamma(\alpha+1)} \int_u^\infty e^{-\frac{t}{2}} t^{\alpha-\gamma-\frac{1}{3}} |\Delta_0 f(t)| dt = O(\omega(1/u)) \quad (u \geq 1). \tag{4}$$

Then

$$\left| S_n^{(\gamma, \alpha)} f(0) - f(0) \right| = O\left( n^{\eta + \frac{2(\alpha-\gamma)+1}{4}} \right) \sum_{k=1}^n \frac{\omega(1/k)}{k^{\eta + \frac{2(\alpha-\gamma)+1}{4} + 1}} + O(\omega(1/n^\eta))$$

for  $0 < \eta < -\frac{2(\alpha-\gamma)+1}{4}$ .

Now, we formulate some corollaries and remarks.

**Corollary 2.2.** *Under the assumptions of the above theorem*

$$\left| S_n^{(\gamma, \alpha)} f(0) - f(0) \right| = o(1) \text{ as } n \rightarrow \infty.$$

This relation is an immediately consequence of Theorem 2.1, since (3) when  $u \rightarrow 0$  becomes a condition (1) ((1.8) from [1]), (4) when  $u = 1$  is becomes exactly (2) ((1.9) from [1]) and mean

$$n^{\eta + \frac{2(\alpha - \gamma) + 1}{4}} \sum_{k=1}^n \frac{\omega(1/k)}{k^{\eta + \frac{2(\alpha - \gamma) + 1}{4} + 1}}$$

tends to 0 as  $n \rightarrow 0$  for any  $\eta \in \left(0, -\frac{2(\alpha - \gamma) + 1}{4}\right)$  as well as  $\omega(1/n^\eta)$  tends to 0 as  $n \rightarrow 0$  for any  $\eta > 0$  with any function  $\omega$  of modulus of continuity type. Thus we have Theorem A (Theorem 1 from [1]) without any additional conditions and assumptions.

**Remark 2.3.** *Using Theorem 2.1 and Corollary 2.2 we obtain the result of D. P. Gupta from Theorem A.*

**Remark 2.4.** *Similar type of result was obtained by S.P.Yadav [8] for Hölder classes. There was considered Cesàro  $(C, \gamma)$ -means of the order  $\gamma \in \left(\alpha - \frac{1}{2}, \alpha + \frac{1}{2}\right)$  but in our paper  $\gamma > \alpha + \frac{1}{2}$ . D.P.Gupta, S.M.Mazhar [2] also examined similar problems with partial sums ( $\gamma = 0$ ) and  $\alpha \in \left(-\frac{1}{2}, 1\right)$  but in our result we have Cesàro mean of the order  $\gamma > \alpha + \frac{1}{2}$  with an assumption  $\alpha < -\frac{1}{2}$ .*

**Corollary 2.5.** *Analyzing the proof of Theorem 2.1 we can obtain, under the assumptions of this theorem, the following more precise estimate*

$$\left| S_n^{(\gamma, \alpha)} f(0) - f(0) \right| = O\left(n^{\frac{2(\alpha - \gamma) + 1}{4}} \omega(n^\eta)\right) + O(\omega(1/n^\eta)) + O(\omega(1/n)),$$

when  $\frac{2(\alpha - \gamma) + 1}{4} + 1 < 0$ .

In the special case, taking  $\eta = -\frac{2(\alpha - \gamma) + 1}{8}$ , we have

$$\left| S_n^{(\gamma, \alpha)} f(0) - f(0) \right| = O(\omega(1/n^\eta)),$$

when  $\eta \leq 1$ .

### 3. Examples

Let  $f_1(t) = e^{-\frac{t}{2}}$  and  $\omega_1(t) = t$  for  $t \geq 0$ .

It is clear that  $f_1 \in L_w$ . Moreover, applying the Lagrange mean value theorem we get that

$$|\Delta_0 f_1(t)| = |e^{-\frac{t}{2}} - 1| \leq \frac{t}{4}$$

for  $t \geq 0$ . Therefore, by elementary calculations we get

$$\begin{aligned} & \frac{u^{-(\alpha+1)}}{\Gamma(\alpha+1)} \int_0^u e^{-\frac{t}{2}} t^\alpha |\Delta_0 f_1(t)| dt \\ & \leq \frac{u^{-(\alpha+1)}}{2\Gamma(\alpha+1)} \int_0^u t^{\alpha+1} dt = \frac{1}{2\Gamma(\alpha+1)(\alpha+2)} \omega_1(u) \end{aligned}$$

for  $u > 0$  and

$$\begin{aligned} & \frac{1}{\omega_1\left(\frac{1}{u}\right)\Gamma(\alpha+1)} \int_u^\infty e^{-\frac{t}{2}} t^{\alpha-\gamma-\frac{1}{3}} |\Delta_0 f_1(t)| dt \\ & \leq \frac{u}{2\Gamma(\alpha+1)} \int_u^\infty e^{-\frac{t}{2}} t^{\alpha-\gamma+\frac{2}{3}} dt \leq \frac{1}{2\Gamma(\alpha+1)} \int_u^\infty e^{-\frac{t}{2}} t^{\alpha-\gamma+\frac{5}{3}} dt \\ & \leq \frac{1}{2\Gamma(\alpha+1)} \int_0^\infty e^{-\frac{t}{2}} t^{\alpha-\gamma+\frac{5}{3}} dt = \frac{1}{2\Gamma(\alpha+1)} \int_0^\infty e^{-\frac{t}{2}} t^{-1+\alpha-\gamma+\frac{8}{3}} dt \\ & = \frac{1}{\Gamma(\alpha+1)} 2^{\alpha-\gamma+\frac{5}{3}} \Gamma\left(\alpha-\gamma+\frac{8}{3}\right) < \infty \end{aligned}$$

for  $u \geq 1$  and  $\alpha - \gamma + \frac{8}{3} > 0$ .

Hence the function  $f_1$  satisfies the conditions (3) and (4). Using Theorem 2.1 we get the following estimate for  $|S_n^{(\gamma,\alpha)} f_1(0) - f_1(0)|$ :

**Example 3.1.** Let  $\alpha > -1$ ,  $\alpha + \frac{1}{2} < \gamma < \alpha + \frac{8}{3}$  and  $0 < \eta < -\frac{2(\alpha-\gamma)+1}{4}$ . Then

$$|S_n^{(\gamma,\alpha)} f_1(0) - f_1(0)| = O\left(n^{\eta+\frac{2(\alpha-\gamma)+1}{4}}\right) \sum_{k=1}^n \frac{1}{k^{\eta+\frac{2(\alpha-\gamma)+1}{4}+2}} + O\left(\frac{1}{n^\eta}\right).$$

Suppose  $f_2(t) = t^\delta$  and  $\omega_2(t) = t^\delta$  for  $\delta \in (0, 1]$  and  $t \geq 0$ . Obviously  $f_2 \in L_w$ . In addition, it is easy to show that

$$\begin{aligned} & \frac{u^{-(\alpha+1)}}{\Gamma(\alpha+1)} \int_0^u e^{-\frac{t}{2}} t^\alpha |\Delta_0 f_2(t)| dt \\ & \leq \frac{u^{-(\alpha+1)}}{\Gamma(\alpha+1)} \int_0^u t^{\alpha+\delta} dt = \frac{1}{\Gamma(\alpha+1)(\alpha+\delta+1)} \omega_2(u) \end{aligned}$$

for  $u > 0$  and

$$\begin{aligned} & \frac{1}{\omega_2\left(\frac{1}{u}\right)\Gamma(\alpha+1)} \int_u^\infty e^{-\frac{t}{2}} t^{\alpha-\gamma-\frac{1}{3}} |\Delta_0 f_2(t)| dt \\ & \leq \frac{u^\delta}{\Gamma(\alpha+1)} \int_u^\infty e^{-\frac{t}{2}} t^{\alpha-\gamma+\delta-\frac{1}{3}} dt \leq \frac{1}{\Gamma(\alpha+1)} \int_u^\infty e^{-\frac{t}{2}} t^{\alpha-\gamma+2\delta-\frac{1}{3}} dt \\ & \leq \frac{1}{\Gamma(\alpha+1)} \int_0^\infty e^{-\frac{t}{2}} t^{\alpha-\gamma+2\delta-\frac{1}{3}} dt = \frac{1}{\Gamma(\alpha+1)} \int_0^\infty e^{-\frac{t}{2}} t^{-1+\alpha-\gamma+2\delta+\frac{2}{3}} dt \\ & = \frac{1}{\Gamma(\alpha+1)} 2^{\alpha-\gamma+2\delta+\frac{2}{3}} \Gamma\left(\alpha-\gamma+2\delta+\frac{2}{3}\right) < \infty \end{aligned}$$

for  $u \geq 1$  and  $\alpha - \gamma + 2\delta + \frac{2}{3} > 0$ .

Therefore the function  $f_2$  satisfies the conditions (3) and (4). Using Theorem 2.1 we get the following estimate for  $|S_n^{(\gamma,\alpha)} f_2(0) - f_2(0)|$ :

**Example 3.2.** Let  $\alpha > -1$ ,  $\delta \in (0, 1]$ ,  $\alpha + \frac{1}{2} < \gamma < \alpha + 2\delta + \frac{2}{3}$  and  $0 < \eta < -\frac{2(\alpha-\gamma)+1}{4}$ . Then

$$|S_n^{(\gamma,\alpha)} f_2(0) - f_2(0)| = O\left(n^{\eta+\frac{2(\alpha-\gamma)+1}{4}}\right) \sum_{k=1}^n \frac{1}{k^{\eta+\frac{2(\alpha-\gamma)+1}{4}+1+\delta}} + O\left(\frac{1}{n^{\eta\delta}}\right).$$

**4. Auxiliary results**

We begin this section by some notations from [6]. We have

$$L_k^{(\alpha+1)}(y) = \sum_{\nu=0}^k L_\nu^{(\alpha)}(y), \quad L_\nu^{(\alpha)}(0) = \binom{\nu + \alpha}{\nu}$$

and therefore

$$S_k^{(\alpha)} f(0) = \frac{1}{\Gamma(\alpha + 1)} \int_0^\infty e^{-y} y^\alpha L_k^{(\alpha+1)}(y) f(y) dy,$$

$$S_n^{(\gamma, \alpha)} f(0) = \frac{1}{\Gamma(\alpha + 1) A_n^{(\gamma)}} \int_0^\infty e^{-y} y^\alpha L_n^{(\alpha+\gamma+1)}(y) f(y) dy.$$

Hence, by evidence equality

$$\frac{1}{\Gamma(\alpha + 1)} \int_0^\infty e^{-y} y^\alpha L_\nu^{(\alpha+1)}(y) dy = \begin{cases} 1 & \text{if } \nu = 0, \\ 0 & \text{if } \nu \neq 0, \end{cases}$$

we have

$$S_k^{(\alpha)} f(0) - f(0) = \frac{1}{\Gamma(\alpha + 1)} \int_0^\infty e^{-y} y^\alpha L_k^{(\alpha+1)}(y) \Delta_0 f(y) dy,$$

$$S_n^{(\gamma, \alpha)} f(0) - f(0) = \frac{1}{\Gamma(\alpha + 1) A_n^{(\gamma)}} \int_0^\infty e^{-y} y^\alpha L_n^{(\alpha+\gamma+1)}(y) \Delta_0 f(y) dy.$$

Next, we present the useful estimates:

**Lemma 4.1.** [6, p. 172] Let  $\beta$  be an arbitrary real number,  $c$  and  $\delta$  be fixed positive constants. Then

$$\left| L_n^{(\beta)}(x) \right| = \begin{cases} O(n^\beta) & \text{if } 0 \leq x \leq \frac{c}{n}, \\ O(x^{-(2\beta+1)/4} n^{(2\beta-1)/4}) & \text{if } \frac{c}{n} \leq x \leq \delta. \end{cases}$$

**Lemma 4.2.** [6, p. 235] Let  $\beta$  and  $\lambda$  be arbitrary real numbers,  $\delta > 0$  and  $0 < \theta < 4$ . Then

$$\max_x e^{-x/2} x^\lambda \left| L_n^{(\beta)}(x) \right| = \begin{cases} O\left(n^{\max(\lambda - \frac{1}{2}, \frac{\beta}{2} - \frac{1}{4})}\right) & \text{if } \delta \leq x \leq (4 - \theta)n, \\ O\left(n^{\max(\lambda - \frac{1}{3}, \frac{\beta}{2} - \frac{1}{4})}\right) & \text{if } x \geq \delta. \end{cases}$$

**Lemma 4.3.** [9, Vol. I, (1.15) and Theorem 1.17] If  $\gamma > -1$ , then

$$A_n^{(\gamma)} = \binom{n + \gamma}{n} \approx O((n + 1)^\gamma)$$

and  $A_n^{(\gamma)}$  is positive for  $\gamma > -1$  increasing (as a function of  $n$ ) for  $\gamma > 0$  and decreasing for  $-1 < \gamma < 0$ .

**5. Proof of Theorem 2.1**

It is clear that if

$$S_n^{(\gamma, \alpha)} f(0) - f(0) = \frac{1}{\Gamma(\alpha + 1) A_n^{(\gamma)}} \int_0^\infty e^{-y} y^\alpha L_n^{(\alpha+\gamma+1)}(y) \Delta_0 f(y) dy$$

$$= \left( \int_0^{1/n} + \int_{1/n}^1 + \int_1^{n^\eta} + \int_{n^\eta}^\infty \right) = J_1 + J_2 + J_3 + J_4,$$

then

$$\left| S_n^{(\gamma, \alpha)} f(0) - f(0) \right| \leq |J_1| + |J_2| + |J_3| + |J_4|.$$

By Lemma 4.1, Lemma 4.33 and (3)

$$\begin{aligned} |J_1| &\leq \frac{1}{\Gamma(\alpha + 1) A_n^{(\gamma)}} \int_0^{1/n} e^{-\frac{y}{2}} y^\alpha \left| L_n^{(\alpha+\gamma+1)}(y) \right| |\Delta_0 f(y)| dy \\ &= \frac{O(n^{\alpha+\gamma+1})}{\Gamma(\alpha + 1) A_n^{(\gamma)}} \int_0^{1/n} e^{-\frac{y}{2}} y^\alpha |\Delta_0 f(y)| dy \\ &= \frac{O(n^{\alpha+1})}{\Gamma(\alpha + 1)} \int_0^{1/n} e^{-\frac{y}{2}} y^\alpha |\Delta_0 f(y)| dy = O(\omega(1/n)) \\ &\leq O(n^{(2\alpha-2\gamma+1)/4}) \sum_{k=1}^n \frac{\omega(1/k)}{k^{(2\alpha-2\gamma+1)/4+1}} \\ &\leq O\left(n^{\eta + \frac{2(\alpha-\gamma)+1}{4}}\right) \sum_{k=1}^n \frac{\omega(1/k)}{k^{\eta + \frac{2(\alpha-\gamma)+1}{4} + 1}}, \end{aligned}$$

with  $0 < \eta < -\frac{2(\alpha-\gamma)+1}{4}$ .

Using Lemma 4.1, we get

$$\begin{aligned} |J_2| &\leq \frac{1}{\Gamma(\alpha + 1) A_n^{(\gamma)}} \int_{1/n}^1 e^{-y} y^\alpha \left| L_n^{(\alpha+\gamma+1)}(y) \right| |\Delta_0 f(y)| dy \\ &= \frac{O\left(n^{\frac{2(\alpha+\gamma)+1}{4}}\right)}{\Gamma(\alpha + 1) A_n^{(\gamma)}} \int_{1/n}^1 e^{-\frac{y}{2}} y^\alpha |\Delta_0 f(y)| y^{-\frac{2(\alpha+\gamma)+3}{4}} dy. \end{aligned}$$

Let  $F_\alpha(y) = \frac{y^{-(\alpha+1)}}{\Gamma(\alpha+1)} \int_0^y e^{-\frac{u}{2}} u^\alpha |\Delta_0 f(u)| du$ . Applying Lemma 4.3 and integrating by parts with  $\gamma > \alpha + \frac{1}{2}$  and  $\alpha > -1$  we have

$$\begin{aligned} |J_2| &= \frac{O(n^{(2\alpha+2\gamma+1)/4})}{\Gamma(\alpha + 1) A_n^{(\gamma)}} \int_{1/n}^1 e^{-y/2} y^{(2\alpha-2\gamma-3)/4} |\Delta_0 f(y)| dy \\ &= O(n^{(2\alpha-2\gamma+1)/4}) \left\{ \left[ F_\alpha(y) y^{\frac{2(\alpha-\gamma)+1}{4}} \right]_{y=1/n}^1 \right. \\ &\quad \left. + \frac{2(\alpha + \gamma) + 3}{4} \int_{1/n}^1 F_\alpha(y) y^{\frac{2(\alpha-\gamma)+1}{4}-1} dy \right\} \\ &\leq O(n^{(2\alpha-2\gamma+1)/4}) \left\{ F_\alpha(1) + \int_{1/n}^1 F_\alpha(y) y^{\frac{2(\alpha-\gamma)+1}{4}-1} dy \right\} \\ &= O(n^{(2\alpha-2\gamma+1)/4}) \left\{ F_\alpha(1) + \int_1^n F_\alpha(1/y) y^{-\frac{2(\alpha-\gamma)+1}{4}-1} dy \right\} \\ &= O(n^{(2\alpha-2\gamma+1)/4}) \left\{ F_\alpha(1) + \sum_{k=1}^{n-1} \int_k^{k+1} F_\alpha(1/y) y^{-\frac{2(\alpha-\gamma)+1}{4}-1} dy \right\} \end{aligned}$$

$$\begin{aligned} &\leq O\left(n^{(2\alpha-2\gamma+1)/4}\right)\left\{F_\alpha(1) + \sum_{k=1}^{n-1} F_\alpha(1/k) k^{-\frac{2(\alpha-\gamma)+1}{4}-1}\right\} \\ &\leq O\left(n^{(2\alpha-2\gamma+1)/4}\right)\sum_{k=1}^{n-1} 2F_\alpha(1/k) k^{-\frac{2(\alpha-\gamma)+1}{4}-1} \\ &\leq O\left(n^{(2\alpha-2\gamma+1)/4}\right)\sum_{k=1}^n \frac{F_\alpha(1/k)}{k^{(2\alpha-2\gamma+1)/4+1}}. \end{aligned}$$

By (3) we obtain

$$\begin{aligned} |J_2| &= O\left(n^{(2\alpha-2\gamma+1)/4}\right)\sum_{k=1}^n \frac{\omega(1/k)}{k^{(2\alpha-2\gamma+1)/4+1}} \\ &\leq O\left(n^{\eta+\frac{2(\alpha-\gamma)+1}{4}}\right)\sum_{k=1}^n \frac{\omega(1/k)}{k^{\eta+\frac{2(\alpha-\gamma)+1}{4}+1}}, \end{aligned}$$

with  $0 < \eta < -\frac{2(\alpha-\gamma)+1}{4}$ .

Applying Lemma 4.2 with  $\alpha + \gamma + 1$  instead of  $\beta$  and  $\lambda = \frac{2\alpha+2\gamma+3}{4}$  (since  $\max\left(\lambda - \frac{1}{2}, \frac{\alpha+\gamma+1}{2} - \frac{1}{4}\right) = \frac{2\alpha+2\gamma+1}{4}$ ) we have

$$\begin{aligned} &|J_3| \\ &\leq \frac{1}{\Gamma(\alpha+1)A_n^{(\gamma)}} \int_1^{n^\eta} e^{-y/2} y^{(2\alpha-2\gamma-3)/4} |\Delta_0 f(y)| e^{-y/2} y^{(2\alpha+2\gamma+3)/4} \left|L_n^{(\alpha+\gamma+1)}(y)\right| dy \\ &= \frac{O\left(n^{(2\alpha+2\gamma+1)/4}\right)}{\Gamma(\alpha+1)A_n^{(\gamma)}} \int_1^{n^\eta} e^{-y/2} y^{(2\alpha-2\gamma-3)/4} |\Delta_0 f(y)| dy. \end{aligned}$$

Using Lemma 4.3 and integrating by parts with  $\gamma > \alpha + \frac{1}{2}$  and  $\alpha > -1$ , we get

$$\begin{aligned} |J_3| &= \frac{O\left(n^{(2\alpha-2\gamma+1)/4}\right)}{\Gamma(\alpha+1)} \int_1^{n^\eta} \left[e^{-y/2} y^\alpha |\Delta_0 f(y)|\right] y^{(-2\alpha-2\gamma-3)/4} dy \\ &= O\left(n^{(2\alpha-2\gamma+1)/4}\right) \int_1^{n^\eta} \frac{d}{dy} \left[ \int_0^y \frac{e^{-u/2} u^\alpha |\Delta_0 f(u)|}{\Gamma(\alpha+1)} du \right] y^{(-2\alpha-2\gamma-3)/4} dy \\ &= O\left(n^{(2\alpha-2\gamma+1)/4}\right) \left\{ \left[ y^{(-2\alpha-2\gamma-3)/4} \int_0^y \frac{e^{-u/2} u^\alpha |\Delta_0 f(u)|}{\Gamma(\alpha+1)} du \right]_1^{n^\eta} \right. \\ &\quad \left. + \frac{2\alpha+2\gamma+3}{4} \int_1^{n^\eta} \frac{d}{dy} \left[ \int_0^y \frac{e^{-u/2} u^\alpha |\Delta_0 f(u)|}{\Gamma(\alpha+1)} du \right] y^{\frac{-2\alpha-2\gamma-3}{4}-1} dy \right\} \\ &\leq O\left(n^{(2\alpha-2\gamma+1)/4}\right) \left\{ F_\alpha(n^\eta) n^{\eta \frac{2(\alpha-\gamma)+1}{4}} + \frac{2\alpha+2\gamma+3}{4} \int_1^{n^\eta} F_\alpha(y) y^{\frac{2(\alpha-\gamma)+1}{4}-1} dy \right\}. \end{aligned}$$

By (3) we obtain

$$|J_3| \leq O\left(n^{(2\alpha-2\gamma+1)/4}\right) \left\{ \omega(n^\eta) n^{\eta \frac{2(\alpha-\gamma)+1}{4}} + \frac{2\alpha+2\gamma+3}{4} \int_1^{n^\eta} \omega(y) y^{\frac{2(\alpha-\gamma)+1}{4}-1} dy \right\}$$

$$\begin{aligned} &\leq O(1) \left\{ n^{(2\alpha-2\gamma+1)/4} n^\eta \omega(1) n^{\eta \frac{2(\alpha-\gamma)+1}{4}} + n^{(2\alpha-2\gamma+1)/4} \omega(n^\eta) \int_1^{n^\eta} y^{\frac{2(\alpha-\gamma)+1}{4}-1} dy \right\} \\ &\leq O\left( n^{(2\alpha-2\gamma+1)/4} n^\eta n^{\eta \frac{2(\alpha-\gamma)+1}{4}} + n^{\frac{2(\alpha-\gamma)+1}{4}} n^\eta \right) \omega(1) \leq O\left( n^{\frac{2(\alpha-\gamma)+1}{4}} n^\eta \right) \omega(1) \\ &\leq O\left( n^{\eta + \frac{2(\alpha-\gamma)+1}{4}} \right) \sum_{k=1}^n \frac{\omega(1/k)}{k^{\eta + \frac{2(\alpha-\gamma)+1}{4} + 1}}, \end{aligned}$$

with  $0 < \eta < -\frac{2(\alpha-\gamma)+1}{4}$ .

If  $\lambda = \gamma + \frac{1}{3}$  then  $\lambda - \frac{1}{3} = \gamma > \frac{\alpha+\gamma+1}{2} - \frac{1}{4}$  since  $\gamma > \alpha + \frac{1}{2}$ . So, applying Lemma 4.2 with  $\alpha + \gamma + 1$  instead of  $\beta$  and  $\lambda = \gamma + \frac{1}{3}$  we obtain

$$\begin{aligned} &|J_4| \\ &\leq \frac{1}{\Gamma(\alpha + 1) A_n^{(\gamma)}} \int_{n^\eta}^\infty e^{-y/2} y^{(3\alpha-3\gamma-1)/3} |\Delta_0 f(y)| e^{-y/2} y^{(3\gamma+1)/3} \left| L_n^{(\alpha+\gamma+1)}(y) \right| dy \\ &= \frac{O(n^\gamma)}{\Gamma(\alpha + 1) A_n^{(\gamma)}} \int_{n^\eta}^\infty e^{-y/2} y^{(3\alpha-3\gamma-1)/3} |\Delta_0 f(y)| dy. \end{aligned}$$

Next, using Lemma 4.3 and (4) we get

$$|J_4| = \frac{O(1)}{\Gamma(\alpha + 1)} \int_{n^\eta}^\infty e^{-y/2} y^{(3\alpha-3\gamma-1)/3} |\Delta_0 f(y)| dy = O(\omega(1/n^\eta)).$$

Finally, collecting the above estimates we have

$$\left| S_n^{(\gamma,\alpha)} f(0) - f(0) \right| = O\left( n^{\eta + \frac{2(\alpha-\gamma)+1}{4}} \right) \sum_{k=1}^n \frac{\omega(1/k)}{k^{\eta + \frac{2(\alpha-\gamma)+1}{4} + 1}} + O(\omega(1/n^\eta)).$$

and our proof is completed. ■

### 6. Conclusion

We investigated pointwise approximation of real-valued functions, integrable in the Lebesgue sense over  $\mathbb{R}^+$  with the weight  $w(t) = e^{-t} t^\alpha$  ( $\alpha > -1$ ) by the  $(C, \gamma)$ -means of partial sums of their Fourier-Laguerre series. In particular, we estimated the deviation  $\left| S_n^{(\gamma,\alpha)} f(0) - f(0) \right|$  by means of a function of the modulus of continuity type  $\omega$ . From our result some corollaries were derived and some examples were constructed.

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