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b_3-subbalancing and b_3-Lucas subbalancing numbers

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Abstract. In this paper, we derive some new algebraic identities on b_3-subbalancing numbers and obtain the functions that generate b_3-subbalancing numbers. We also introduce b_3-Lucas subbalancing numbers and derive some algebraic identities between b_3-Lucas subbalancing numbers and b_3-subbalancing numbers. Further, we give the Binet formulas for b_3-subbalancing and b_3-Lucas subbalancing numbers.

1. Introduction

Behera and Panda [1] introduced balancing numbers n as solutions of the Diophantine equation

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r) \tag{1}$$

for some positive integers r. The positive integer r in (1) is called the balancer of n. For example, $6, 35, 204$ and 1189 are balancing numbers with balancers which are $2, 14, 84$ and 492, respectively. The n^{th} balancing number is denoted by B_n.

If n is a balancing number with balancer r, then from (1)

$$n^2 = \frac{(n+r)(n+r+1)}{2} \quad \text{and} \quad r = \frac{-(2n+1)+\sqrt{8n^2+1}}{2} \tag{2}$$

It follows from (2) that n is a balancing number if and only if $8n^2+1$ is a perfect square. Since $8n^2+1$ is a perfect square for $n = 1$, 1 is accepted as a balancing number.

Balancing numbers satisfy the following recurrence relation

$$B_{n+1} = 6B_n - B_{n-1} \quad , \quad (n \geq 2)$$

with initial terms $B_1 = 1,\ B_2 = 6$.

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By modifying (1) slightly, Panda and Ray [2] introduced cobalancing numbers n as solutions of the Diophantine equation

$$1 + 2 + \cdots + n = (n+1) + (n+2) + \cdots + (n+r) \tag{3}$$

for some positive integers r. The positive integer r in (3) is called the cobalancer of n. For example, $2, 14, 84$ are cobalancing numbers with cobalancers which are $1, 6, 35$, respectively. The n^{th} cobalancing number is denoted by b_n.

Cobalancing numbers satisfy the following recurrence relation

$$b_{n+1} = 6b_n - b_{n-1} + 2 \quad , \quad (n \geq 2)$$

with initial terms $b_1 = 0,\ b_2 = 2$.

If n is a cobalancing number with cobalancer r, then from (3)

$$n(n+1) = \frac{(n+r)(n+r+1)}{2} \quad \text{and} \quad r = \frac{-(2n+1) + \sqrt{8n^2+8n+1}}{2} \tag{4}$$

It follows from (4) that n is a cobalancing number if and only if $8n^2 + 8n + 1$ is a perfect square.

The numbers $C_n = \sqrt{8B_n^2 + 1}$ and $c_n = \sqrt{8b_n^2 + 8b_n + 1}$ are called the n^{th} Lucas-balancing number and the n^{th} Lucas-cobalancing number, respectively ([3, 4]). For example, $3, 17, 99$ are Lucas-balancing numbers and $1, 7, 41$ are Lucas-cobalancing numbers.

Lucas-balancing and Lucas-cobalancing numbers satisfy the following recurrence relations

$$C_{n+1} = 6C_n - C_{n-1} \quad , \quad (n \geq 2)$$
$$c_{n+1} = 6c_n - c_{n-1} \quad , \quad (n \geq 2)$$

with initial terms $C_1 = 3,\ C_2 = 17$ and $c_1 = 1,\ c_2 = 7$, respectively.

Further in [3], the Binet formulas for balancing, cobalancing, Lucas-balancing and Lucas-cobalancing numbers are given as:

$$B_n = \frac{\alpha_1^{2n} - \alpha_2^{2n}}{4\sqrt{2}} \qquad b_n = \frac{\alpha_1^{2n-1} - \alpha_2^{2n-1}}{4\sqrt{2}} - \frac{1}{2}$$

$$C_n = \frac{\alpha_1^{2n} + \alpha_2^{2n}}{2} \qquad c_n = \frac{\alpha_1^{2n-1} + \alpha_2^{2n-1}}{2}$$

where $\alpha_1 = 1 + \sqrt{2}$ and $\alpha_2 = 1 - \sqrt{2}$.

The concept of balancing numbers, cobalancing numbers, Lucas-balancing numbers and Lucas-cobalancing numbers has been extanded and generalized by many authors. In [5], Panda and Ray observed relations between balancing and cobalancing numbers with Pell and Associated Pell numbers. Later in [6], relations between balancing numbers and Pell, Pell-Lucas and oblong numbers were obtained. Further in [7], some identities on the sums, divisibility properties, perfect squares, Pythagorean triples involving Pell, Pell-Lucas and balancing numbers were derived.

There have been studies about investigating balancing numbers in some sequences. Liptai [8] proved that there is no Fibonacci balancing number except 1. Later in [9], he derived that there is no balancing number in the terms of the Lucas sequence. In [10], Szalay also obtained the same result using different method.

Panda and Rout [11], introduced gap balancing numbers and studied specifically on 2-gap balancing numbers. Further in [12], some algebraic identities of k-gap balancing numbers for arbitrary k were given. In [13], Panda and Panda defined almost balancing numbers and obtained some algebraic equations related to these numbers. Later in [14], Tekcan derived algebraic relations between almost balancing numbers and triangular and square triangular numbers. Further in [15], he obtained some algebraic identities on sums of almost balancing numbers.

In [16], Panda generalized the concept of balancing and cobalancing numbers to an arbitrary sequence. Later in [17], (k, l)-balancing numbers were defined and some effective and ineffective finiteness results were proved for (k, l)-balancing numbers. Some properties of generalized balancing numbers were obtained in [18]. Further in [19], a survey about balancing, cobalancing and generalized balancing numbers was given.

Davala and Panda [20] defined supercobalancing numbers. Later, in [21] they generalized the concept of almost balancing numbers by defining D-subbalancing numbers n as solutions of the Diophantine equation

$$1 + 2 + \cdots + (n-1) + D = (n+1) + (n+2) + \cdots + (n+r) \tag{5}$$

for some positive integer r, where D is a fixed positive integer. The positive integer r in (5) is called D-subbalancer of n. They observed that the choice of the positive integer D is important for obtaining D-subbalancing numbers and there is no positive integer n that satisfies the equation (5), for every D. They showed that it is possible to find a positive integer n that satisfies the equation (5) when D is chosen as cobalancing numbers. They obtained basic concepts and theorems about b_n-subbalancing numbers. Specifically, they examined b_3-subbalancing numbers and b_5-subbalancing numbers and showed that b_3-subbalancing numbers can be classified in two classes of the form $11B_m + C_m$, $11B_{m+1} - C_{m+1}$, $m \geq 0$.

In the present work, we consider b_3-subbalancing numbers and give some new algebraic identities provided by the b_3-subbalancing numbers and obtain the functions that generate b_3-subbalancing numbers. We also prove that any two consecutive even terms of the sequence of b_3-subbalancing numbers and any two consecutive odd terms of the sequence of b_3-subbalancing numbers are relatively prime. Further, similar to Lucas-balancing numbers, we introduce b_3-Lucas subbalancing numbers. We give the Binet formulas for b_3-subbalancing and b_3-Lucas subbalancing numbers. Later, we obtain some algebraic relations between b_3-Lucas subbalancing numbers and b_3-subbalancing numbers.

2. Preliminaries

Throughout this paper, we denote the m^{th} b_3-subbalancing number and the m^{th} b_3-Lucas subbalancing number as $(Sb_3)_m$ and $(CSb_3)_m$, respectively. We get the following results regarding b_3-subbalancing numbers from [21].

Theorem 2.1. *Let $(Sb_3)_m$ denote the m^{th} b_3-subbalancing number, let B_m denote the m^{th} balancing number and let C_m denote the m^{th} Lucas-balancing number. Then*

$$\begin{aligned}(Sb_3)_{2m} &= 11B_m + C_m\\ (Sb_3)_{2m+1} &= 11B_{m+1} - C_{m+1}\end{aligned}$$

for $m \geq 0$.

Theorem 2.2. *Let $(Sb_3)_m$ denote the m^{th} b_3-subbalancing number. Then, the recurrence relation for b_3-subbalancing numbers is*

$$(Sb_3)_{m+2} = 6(Sb_3)_m - (Sb_3)_{m-2} \qquad (m \geq 2)$$

where $(Sb_3)_0 = 1, (Sb_3)_1 = 8, (Sb_3)_2 = 14, (Sb_3)_3 = 49$.

Corollary 2.1. *$(Sb_3)_m$ is a b_3-subbalancing number if and only if $8(Sb_3)_m^2 + 113$ is a perfect square.*

Theorem 2.3. *Let $(Sb_3)_m$ denote the m^{th} b_3-subbalancing number and B_m denote the m^{th} balancing number. Then*

$$B_m = \frac{11(Sb_3)_{2m} - \sqrt{8(Sb_3)_{2m}^2 + 113}}{113} \qquad (m \geq 0)$$

$$B_m = \frac{11(Sb_3)_{2m-1} + \sqrt{8(Sb_3)_{2m-1}^2 + 113}}{113} \qquad (m \geq 1).$$

3. Main Results

In this section we derive our main results.

3.1. Algebraic Identities for b_3-Subbalancing Numbers

In this subsection, we obtain some new algebraic identities between b_3-subbalancing numbers and balancing and Lucas-balancing numbers.

Theorem 3.1. *Let $(Sb_3)_m$ denote the m^{th} b_3-subbalancing number and B_m denote the m^{th} balancing number . Then*

$$(Sb_3)_{2m+1} = 8(Sb_3)_{2m} - 63B_m \qquad (m \geq 0)$$

and

$$(Sb_3)_{2m} = 22B_m - (Sb_3)_{2m-1} \qquad (m \geq 1).$$

Proof. It follows from Theorem 2.1 that

$$(Sb_3)_{2m+1} = 8B_{m+1} + B_m \tag{6}$$

and

$$(Sb_3)_{2m} = 8B_m + B_{m+1} \tag{7}$$

From (6) and (7), we get

$$(Sb_3)_{2m+1} = 8(Sb_3)_{2m} - 63B_m.$$

Further from (6), we obtain

$$(Sb_3)_{2m-1} = 8B_m + B_{m-1} \tag{8}$$

From (7), we get

$$(Sb_3)_{2m} = 14B_m - B_{m-1} \tag{9}$$

From (8) and (9), we get

$$(Sb_3)_{2m} = 22B_m - (Sb_3)_{2m-1}.$$

Remark 3.1. *We observe that all terms of the sequence of b_3-subbalancing numbers can be written in terms of the sequence of the balancing numbers without involving any other number sequence.*

Corollary 3.1. *The sum of any even term of the sequence of b_3-subbalancing numbers and the preceding one is always even. Equivalently to this, the sum of any odd term of the sequence of b_3-subbalancing numbers and the next one is always even.*

Corollary 3.2. *Let $(Sb_3)_m$ denote the m^{th} b_3-subbalancing number and B_m denote the m^{th} balancing number. Then*

$$4(Sb_3)_{2m} = 7(Sb_3)_{2m-1} - 11B_{m-1}$$

for $m \geq 1$.

Proof. Using (8) and (9), we obtain

$$\begin{aligned} 7(Sb_3)_{2m-1} &= 56B_m + 7B_{m-1} \\ 4(Sb_3)_{2m} &= 56B_m - 4B_{m-1}. \end{aligned}$$

Thus, we get

$$4(Sb_3)_{2m} = 7(Sb_3)_{2m-1} - 11B_{m-1}.$$

Theorem 3.2. *Let $(Sb_3)_m$ denote the m^{th} b_3-subbalancing number. Then*

$$(Sb_3)_{2m} = \frac{113(Sb_3)_{2m-1} - 22(Sb_3)_{2m-2}}{63}$$

and

$$(Sb_3)_{2m+1} = \frac{113(Sb_3)_{2m} - 63(Sb_3)_{2m-1}}{22}$$

for $m \geq 1$.

Proof. From Theorem 3.1, we deduce that

$$8(Sb_3)_{2m-2} = (Sb_3)_{2m-1} + 63B_{m-1} \tag{10}$$

Using (10) and Corollary 3.2, we obtain

$$252(Sb_3)_{2m} + 88(Sb_3)_{2m-2} = 452(Sb_3)_{2m-1} \tag{11}$$

From (11), we get

$$63(Sb_3)_{2m} = 113(Sb_3)_{2m-1} - 22(Sb_3)_{2m-2}.$$

Thus, we get

$$(Sb_3)_{2m} = \frac{113(Sb_3)_{2m-1} - 22(Sb_3)_{2m-2}}{63}.$$

The other case can be proved similarly.

In the following two theorems the relations between b_3-subbalancing numbers and Lucas-balancing numbers are given.

Theorem 3.3. *Let $(Sb_3)_m$ denote the m^{th} b_3-subbalancing number and C_m denote the m^{th} Lucas-balancing number. Then*

$$11(Sb_3)_{2m+1} = 25(Sb_3)_{2m} + 63C_m$$

for $m \geq 0$.

Proof. Using Theorem 2.1 and relations between balancing numbers and Lucas-balancing numbers, we obtain

$$(Sb_3)_{2m+1} = 25B_m + 8C_m.$$

From Theorem 2.1, we get

$$11(Sb_3)_{2m+1} = 25(Sb_3)_{2m} + 63C_m.$$

Theorem 3.4. *Let $(Sb_3)_m$ denote the m^{th} b_3-subbalancing number and C_m denote the m^{th} Lucas-balancing number. Then*

$$(Sb_3)_{2m} = (Sb_3)_{2m-1} + 2C_m$$

for $m \geq 1$.

Proof. It follows from Theorem 2.1 that

$$(Sb_3)_{2m-1} = 11B_m - C_m.$$

Thus, we get

$$(Sb_3)_{2m} = (Sb_3)_{2m-1} + 2C_m.$$

Theorem 3.5. *Let $(Sb_3)_m$ denote the m^{th} b_3-subbalancing number. Then*

$$(Sb_3)_m^2 = (Sb_3)_{m-2}(Sb_3)_{m+2} + 113$$

for $m \geq 2$.

Proof. This theorem is proved by induction. It is easily seen that the assertion is true for $m = 2$.
Assuming the assertion is true for $m \leq k$, we have

$$\begin{aligned}
(Sb_3)_{k+1}^2 &= (Sb_3)_{k+1}(Sb_3)_{k+1} \\
&= (6(Sb_3)_{k-1} - (Sb_3)_{k-3})(Sb_3)_{k+1} \\
&= 6(Sb_3)_{k-1}(Sb_3)_{k+1} - (Sb_3)_{k-1}^2 + 113 \\
&= (Sb_3)_{k-1}(6(Sb_3)_{k+1} - (Sb_3)_{k-1}) + 113 \\
&= (Sb_3)_{k-1}(Sb_3)_{k+3} + 113.
\end{aligned}$$

Thus it is shown that the assertion is true for $m = k + 1$.

Theorem 3.6. *Let $(Sb_3)_m$ denote the m^{th} b_3-subbalancing number and B_m denote the m^{th} balancing number. Then*

$$(Sb_3)_{m+1}(Sb_3)_m - (Sb_3)_{m-1}(Sb_3)_{m-2} = 113B_m$$

for $m \geq 2$.

Proof. This theorem is proved by induction. It is easily seen that the assertion is true for $m = 2$.
Assuming the assertion is true for $m \leq k$, we have

$$\begin{aligned}
(Sb_3)_{k+2}(Sb_3)_{k+1} - (Sb_3)_k(Sb_3)_{k-1} &= (6(Sb_3)_k - (Sb_3)_{k-2})(Sb_3)_{k+1} - (Sb_3)_k(Sb_3)_{k-1} \\
&= 6(Sb_3)_{k+1}(Sb_3)_k - (Sb_3)_{k+1}(Sb_3)_{k-2} - (Sb_3)_k(Sb_3)_{k-1} \\
&= 6(Sb_3)_{k+1}(Sb_3)_k - 6(Sb_3)_{k-1}(Sb_3)_{k-2} \\
&\quad + (Sb_3)_{k-2}(Sb_3)_{k-3} - (Sb_3)_k(Sb_3)_{k-1} \\
&= 6((Sb_3)_{k+1}(Sb_3)_k - (Sb_3)_{k-1}(Sb_3)_{k-2}) \\
&\quad - ((Sb_3)_k(Sb_3)_{k-1} - (Sb_3)_{k-2}(Sb_3)_{k-3}) \\
&= 6(113B_k) - 113B_{k-1} \\
&= 113B_{k+1}.
\end{aligned}$$

Thus it is shown that the assertion is true for $m = k + 1$.

Theorem 3.7. *Let $(Sb_3)_m$ denote the m^{th} b_3-subbalancing number and B_n denote the n^{th} balancing number. Then*

$$(Sb_3)_{2n+m-1} = (Sb_3)_{m-1}B_{n+1} - (Sb_3)_{m-3}B_n$$

for any two positive integers m and n.

Proof. This theorem is proved by induction. It is easily seen that the assertion is true for $n = 1$. Assuming the assertion is true for $n \leq k$, we have

$$\begin{aligned} (Sb_3)_{2k+m+1} &= 6(Sb_3)_{2k+m-1} - (Sb_3)_{2k+m-3} \\ &= 6(B_{k+1}(Sb_3)_{m-1} - B_k(Sb_3)_{m-3}) - (B_k(Sb_3)_{m-1} - B_{k-1}(Sb_3)_{m-3}) \\ &= (Sb_3)_{m-1}(6B_{k+1} - B_k) - (Sb_3)_{m-3}(6B_k - B_{k-1}) \\ &= B_{k+2}(Sb_3)_{m-1} - B_{k+1}(Sb_3)_{m-3}. \end{aligned}$$

Thus it is shown that the assertion is true for $n = k + 1$.

Theorem 3.8. *Any two consecutive even terms of the sequence of b_3-subbalancing numbers and any two consecutive odd terms of the sequence of b_3-subbalancing numbers are relatively prime, that is*

$$((Sb_3)_{2m}, (Sb_3)_{2m-2}) = 1 \qquad \text{and} \qquad ((Sb_3)_{2m+1}, (Sb_3)_{2m-1}) = 1.$$

Proof. Using the Euclidean Algorithm and Theorem 2.1, we deduce that

$$\begin{aligned} (Sb_3)_{2m} &= 6(Sb_3)_{2m-2} + [-(Sb_3)_{2m-2} + (Sb_3)_{2m-2}] - (Sb_3)_{2m-4} \\ (Sb_3)_{2m-2} &= [(Sb_3)_{2m-2} - (Sb_3)_{2m-4}] \times 1 + (Sb_3)_{2m-4} \\ (Sb_3)_{2m-2} - (Sb_3)_{2m-4} &= (6-2)(Sb_3)_{2m-4} + [(Sb_3)_{2m-4} - (Sb_3)_{2m-6}] \\ (Sb_3)_{2m-4} &= [(Sb_3)_{2m-4} - (Sb_3)_{2m-6}] \times 1 + (Sb_3)_{2m-6} \\ (Sb_3)_{2m-4} - (Sb_3)_{2m-6} &= (6-2)(Sb_3)_{2m-6} + [(Sb_3)_{2m-6} - (Sb_3)_{2m-8}] \\ &\vdots \\ (Sb_3)_4 - (Sb_3)_2 &= (6-2)(Sb_3)_2 + [(Sb_3)_2 - (Sb_3)_0] \\ (Sb_3)_2 &= [(Sb_3)_2 - (Sb_3)_0] \times 1 + (Sb_3)_0 \\ (Sb_3)_2 - (Sb_3)_0 &= [(Sb_3)_2 - (Sb_3)_0](Sb_3)_0 + 0. \end{aligned}$$

Since $((Sb_3)_2 - (Sb_3)_0, (Sb_3)_0) = 1$, we get $((Sb_3)_{2m}, (Sb_3)_{2m-2}) = 1$.

The other case can be proved similarly.

In the following theorem the Binet formula for b_3-subbalancing numbers is given.

Theorem 3.9. *Let $(Sb_3)_m$ denote the m^{th} b_3-subbalancing number. Then*

$$(Sb_3)_{2m} = \frac{\left(11\sqrt{2}+4\right)\alpha_1^m - (11\sqrt{2}-4)\alpha_2^m}{8}$$

and

$$(Sb_3)_{2m+1} = \frac{\left(11\sqrt{2}-4\right)\alpha_1^{m+1} - (11\sqrt{2}+4)\alpha_2^{m+1}}{8}$$

where $\alpha_1 = 3 + 2\sqrt{2}$, $\alpha_2 = 3 - 2\sqrt{2}$.

Proof. From the Binet formulas for balancing and Lucas-balancing numbers, we get

$$
\begin{aligned}
(Sb_3)_{2m} &= 11\left(\frac{\alpha_1^m - \alpha_2^m}{4\sqrt{2}}\right) + \left(\frac{\alpha_1^m + \alpha_2^m}{2}\right) \\
&= \frac{(4 + 11\sqrt{2})\alpha_1^m + (4 - 11\sqrt{2})\alpha_2^m}{8} \\
&= \frac{\left(11\sqrt{2} + 4\right)\alpha_1^m - (11\sqrt{2} - 4)\alpha_2^m}{8}.
\end{aligned}
$$

The other case can be proved similarly.

3.2. Functions Genarating b_3-Subbalancing Numbers

In this subsection, we present some functions that generate b_3-subbalancing numbers.

Theorem 3.10. *For any even term of the sequence of b_3-subbalancing numbers x, $g(x) = \frac{129x-22\sqrt{8x^2+113}}{113}$ is the b_3-subbalancing number just prior to it and $\tilde{g}(x) = \frac{211x+63\sqrt{8x^2+113}}{113}$ is the b_3-subbalancing number next to it.*

Proof. Since x even term of the sequence of b_3-subbalancing numbers, $x = (Sb_3)_{2m}$

for some positive integer m.

From Theorem 2.3 and Theorem 3.1, we get

$$
\begin{aligned}
(Sb_3)_{2m-1} &= 22\left(\frac{11(Sb_3)_{2m} - \sqrt{8(Sb_3)_{2m}^2 + 113}}{113}\right) - (Sb_3)_{2m} \\
&= \frac{129(Sb_3)_{2m} - 22\sqrt{8(Sb_3)_{2m}^2 + 113}}{113}
\end{aligned}
$$

which is equivalent to

$$(Sb_3)_{2m-1} = g((Sb_3)_{2m})$$

proving that the b_3-subbalancing number just prior to x.
Similarly, we get

$$
\begin{aligned}
(Sb_3)_{2m+1} &= 8(Sb_3)_{2m} - 63\left(\frac{11(Sb_3)_{2m} - \sqrt{8(Sb_3)_{2m}^2 + 113}}{113}\right) \\
&= \frac{211(Sb_3)_{2m} + 63\sqrt{8(Sb_3)_{2m}^2 + 113}}{113}.
\end{aligned}
$$

Thus the b_3-subbalancing number next to $(Sb_3)_{2m}$ is

$$\tilde{g}((Sb_3)_{2m}) = (Sb_3)_{2m+1}.$$

Theorem 3.11. *For any odd term of the sequence of b_3-subbalancing numbers x, $g(x) = \frac{211x-63\sqrt{8x^2+113}}{113}$ is the b_3-subbalancing number just prior to it and $\tilde{g}(x) = \frac{129x+22\sqrt{8x^2+113}}{113}$ is the b_3-subbalancing number next to it.*

Proof. Since x any odd term of the sequence of b_3-subbalancing numbers, $x = (Sb_3)_{2m+1}$ for some positive integer m.

From (6) and Theorem 2.3, we get

$$\begin{aligned} B_m &= (Sb_3)_{2m+1} - 8\left(\frac{11(Sb_3)_{2m+1} + \sqrt{8(Sb_3)^2_{2m+1} + 113}}{113}\right) \\ &= \frac{25(Sb_3)_{2m+1} - 8\sqrt{8(Sb_3)^2_{2m+1} + 113}}{113} \end{aligned} \tag{12}$$

It follows from (7) and (12) that

$$\begin{aligned} (Sb_3)_{2m} &= 8\left(\frac{25(Sb_3)_{2m+1} - 8\sqrt{8(Sb_3)^2_{2m+1} + 113}}{113}\right) + \left(\frac{11(Sb_3)_{2m+1} + \sqrt{8(Sb_3)^2_{2m+1} + 113}}{113}\right) \\ &= \frac{211(Sb_3)_{2m+1} - 63\sqrt{8(Sb_3)^2_{2m+1} + 113}}{113}. \end{aligned}$$

Thus the b_3-subbalancing number just prior to $(Sb_3)_{2m+1}$ is

$$g((Sb_3)_{2m+1}) = (Sb_3)_{2m}.$$

Using Theorem 2.3 and Theorem 3.1, we get

$$\begin{aligned} (Sb_3)_{2m+2} &= 22\left(\frac{11(Sb_3)_{2m+1} + \sqrt{8(Sb_3)^2_{2m+1} + 113}}{113}\right) - (Sb_3)_{2m+1} \\ &= \frac{129(Sb_3)_{2m+1} + 22\sqrt{8(Sb_3)^2_{2m+1} + 113}}{113}. \end{aligned}$$

Thus the b_3-subbalancing number next to $(Sb_3)_{2m+1}$ is

$$\tilde{g}((Sb_3)_{2m+1}) = (Sb_3)_{2m+2}.$$

Theorem 3.12. *If x is a b_3-subbalancing number, then the functions $f(x) = 3x + \sqrt{8x^2 + 113}$ and $g(x) = 17x + 6\sqrt{8x^2 + 113}$ are also b_3-subbalancing numbers.*

Proof.

$$\begin{aligned} 8(f(x))^2 + 113 &= 8(3x + \sqrt{8x^2 + 113})^2 + 113 \\ &= 136x^2 + 48x\sqrt{8x^2 + 113} + 1017 \\ &= (8x + 3\sqrt{8x^2 + 113})^2 \end{aligned} \tag{13}$$

Since $8(f(x))^2 + 113$ is a perfect square, $f(x)$ is a b_3-subbalancing number.

$$\begin{aligned} g(x) &= 17x + 6\sqrt{8x^2 + 113} \\ &= 3(3x + \sqrt{8x^2 + 113}) + 8x + 3\sqrt{8x^2 + 113}. \end{aligned}$$

From (13), we get

$$\begin{aligned} g(x) &= 3(3x + \sqrt{8x^2 + 113}) + \sqrt{8(f(x))^2 + 113} \\ &= 3f(x) + \sqrt{8(f(x))^2 + 113} \\ &= f(f(x)). \end{aligned}$$

Since $g(x) = f(f(x))$, $g(x)$ is a b_3-subbalancing number.

Remark 3.2. *We observe that when x is even $f(x)$ generates odd b_3-subbalancing numbers while $g(x)$ generates even b_3-subbalancing numbers. Similarly, when x is odd $f(x)$ generates even b_3-subbalancing numbers while $g(x)$ generates odd b_3-subbalancing numbers.*

The following theorem shows that the function $f(x)$ in Theorem 3.12 is equal to the $(m+2)^{th}$ b_3-subbalancing number when x is equal to the m^{th} b_3-subbalancing number.

Theorem 3.13. *Let $(Sb_3)_m$ denote the m^{th} b_3-subbalancing number and $f(x) = 3x + \sqrt{8x^2 + 113}$. Then*

$$f((Sb_3)_m) = (Sb_3)_{m+2}$$

for $m \geq 0$.

Proof. Using the recurrence relation for b_3-subbalancing numbers, we get

$$\begin{aligned} (Sb_3)_{m+2} &= 6(Sb_3)_m - (Sb_3)_{m-2} \\ &= 3(Sb_3)_m + (3(Sb_3)_m - (Sb_3)_{m-2}) \end{aligned} \tag{14}$$

Then

$$\begin{aligned} (3(Sb_3)_m - (Sb_3)_{m-2})^2 &= 9(Sb_3)_m^2 + (Sb_3)_{m-2}^2 - ((Sb_3)_{m+2} + (Sb_3)_{m-2})(Sb_3)_{m-2} \\ &= 9(Sb_3)_{m-2} - (Sb_3)_{m+2}(Sb_3)_{m-2} \\ &= 8(Sb_3)_m^2 + 113. \end{aligned}$$

Hence

$$3(Sb_3)_m - (Sb_3)_{m-2} = \sqrt{8(Sb_3)_m^2 + 113} \tag{15}$$

From (14) and (15), we obtain

$$(Sb_3)_{m+2} = 3(Sb_3)_m + \sqrt{8(Sb_3)_m^2 + 113} \tag{16}$$

Thus, we get

$$f((Sb_3)_m) = (Sb_3)_{m+2}.$$

Theorem 3.14. *Let $(Sb_3)_m$ denote the m^{th} b_3-subbalancing number and $\tilde{f}(x) = 3x - \sqrt{8x^2 + 113}$. Then*

$$\tilde{f}((Sb_3)_m) = (Sb_3)_{m-2}$$

for $m \geq 2$.

Proof. Since x any b_3-subbalancing number, then $x = (Sb_3)_m$ for $m \geq 2$.

From (15), we get

$$(Sb_3)_{m-2} = 3(Sb_3)_m - \sqrt{8(Sb_3)_m^2 + 113} \tag{17}$$

Thus, we get

$$\tilde{f}((Sb_3)_m) = (Sb_3)_{m-2}.$$

3.3. b_3-Lucas Subbalancing Numbers

In this subsection, first we introduce the concept of b_3-Lucas subbalancing numbers. Then, we deduce some algebraic identities between b_3-Lucas subbalancing numbers and balancing, Lucas-balancing numbers.

Definition 3.1. *Let $(Sb_3)_m$ denote the m^{th} b_3-subbalancing number. The square root of the number $8(Sb_3)_m^2+113$ is called the m^{th} b_3-Lucas subbalancing number and denoted by $(CSb_3)_m$. That is,*

$$(CSb_3)_m = \sqrt{8(Sb_3)_m^2+113}.$$

For example, $(CSb_3)_0 = 11$ and $(CSb_3)_1 = 25$, since $(Sb_3)_0 = 1$ and $(Sb_3)_1 = 8$.

Corollary 3.3. *Using definition of b_3-Lucas subbalancing numbers, (16) and (17), we get*

$$\begin{aligned}(CSb_3)_m &= (Sb_3)_{m+2} - 3(Sb_3)_m \quad (m \geq 0)\\ (CSb_3)_m &= 3(Sb_3)_m - (Sb_3)_{m-2} \quad (m \geq 2).\end{aligned}$$

It is obvious that these identities shows the relations between b_3-Lucas subbalancing number and b_3-subbalancing number.

Theorem 3.15. *Let $(CSb_3)_m$ denote the m^{th} b_3-Lucas subbalancing number and $(CSb_3)_0 = 11, (CSb_3)_2 = 41$*

$$(CSb_3)_{2m} = 6(CSb_3)_{2m-2} - (CSb_3)_{2m-4}$$

for $m \geq 2$.

Proof. Using the equation (16), we get

$$\begin{aligned}(CSb_3)_{2m}^2 &= 8(Sb_3)_{2m}^2 + 113\\ &= 8\left(3(Sb_3)_{2m-2} + (CSb_3)_{2m-2}\right)^2 + 113\\ &= (3(CSb_3)_{2m-2} + 8(Sb_3)_{2m-2})^2.\end{aligned}$$

Thus, we get

$$(CSb_3)_{2m} = 3(CSb_3)_{2m-2} + 8(Sb_3)_{2m-2} \tag{18}$$

Similarly, from the equation (17), we get

$$(CSb_3)_{2m-4} = 3(CSb_3)_{2m-2} - 8(Sb_3)_{2m-2} \tag{19}$$

From (18) and (19), we get

$$(CSb_3)_{2m} = 6(CSb_3)_{2m-2} - (CSb_3)_{2m-4}.$$

Theorem 3.16. *Let $(CSb_3)_m$ denote the m^{th} b_3-Lucas subbalancing number and $(CSb_3)_1 = 25, (CSb_3)_3 = 139$*

$$(CSb_3)_{2m+1} = 6(CSb_3)_{2m-1} - (CSb_3)_{2m-3}$$

for $m \geq 2$.

Proof. Using the equation (16), we get

$$\begin{aligned}(CSb_3)_{2m+1}^2 &= 8(Sb_3)_{2m+1}^2 + 113\\ &= 8\left(3(Sb_3)_{2m-1} + \sqrt{8(Sb_3)_{2m-1}^2 + 113}\right)^2 + 113\\ &= (3(CSb_3)_{2m-1} + 8(Sb_3)_{2m-1})^2.\end{aligned}$$

Thus, we get

$$(CSb_3)_{2m+1} = 3(CSb_3)_{2m-1} + 8(Sb_3)_{2m-1} \tag{20}$$

Similarly, from the equation (17), we get

$$(CSb_3)_{2m-3} = 3(CSb_3)_{2m-1} - 8(Sb_3)_{2m-1} \tag{21}$$

From (20) and (21), we get

$$(CSb_3)_{2m+1} = 6(CSb_3)_{2m-1} - (CSb_3)_{2m-3}.$$

The following theorem shows that the odd and even terms of the sequence of b_3-Lucas subbalancing numbers can be written depending on the balancing and Lucas-balancing numbers.

Theorem 3.17. *Let $(CSb_3)_m$ denote the m^{th} b_3-Lucas subbalancing number, B_m denote the m^{th} balancing number and C_m denote the m^{th} Lucas-balancing number . Then*

$$\begin{aligned} (CSb_3)_{2m} &= 11C_m + 8B_m \\ (CSb_3)_{2m+1} &= 11C_{m+1} - 8B_{m+1} \end{aligned}$$

for $m \geq 0$.

Proof. This theorem is proved by induction. It is easily seen that the assertion is true for $m = 0$. Assuming the assertion is true for $m \leq k$, we have

$$\begin{aligned} (CSb_3)_{2k+2} &= 6(CSb_3)_{2k} - (CSb_3)_{2k-2} \\ &= 6(11C_k + 8B_k) - (11C_{k-1} + 8B_{k-1}) \\ &= 11(6C_k - C_{k-1}) + 8(6B_k - B_{k-1}) \\ &= 11C_{k+1} + 8B_{k+1}. \end{aligned}$$

Thus it is shown that the assertion is true for $m = k + 1$. The other case can be proved similarly.

From Theorem 3.17, we can give the following two results that show the connection between b_3-Lucas subbalancing numbers and Lucas-balancing and balancing numbers.

Theorem 3.18. *Let $(CSb_3)_m$ denote the m^{th} b_3-Lucas subbalancing number and C_m denote the m^{th} Lucas-balancing number. Then*

$$\begin{aligned} (CSb_3)_{2m} &= 8C_m + C_{m+1} \\ (CSb_3)_{2m+1} &= C_m + 8C_{m+1} \end{aligned}$$

for $m \geq 0$.

Theorem 3.19. *Let $(CSb_3)_m$ denote the m^{th} b_3-Lucas subbalancing number and B_m denote the m^{th} balancing number. Then*

$$\begin{aligned} (CSb_3)_{2m} &= 11B_{m+1} - 25B_m \\ (CSb_3)_{2m+1} &= 25B_{m+1} - 11B_m \end{aligned}$$

for $m \geq 0$.

Theorem 3.20. *Let $(CSb_3)_m$ denote the m^{th} b_3-Lucas subbalancing number and C_m denote the m^{th} Lucas-balancing number. Then*

$$(CSb_3)_{2m+1} = 8(CSb_3)_{2m} - 63C_m$$

for $m \geq 0$.

Proof. From Theorem 3.18, we get

$$\begin{aligned}(CSb_3)_{2m+1} &= C_m + 8C_{m+1}\\ &= C_m + 8C_{m+1} + 8(CSb_3)_{2m} - 64C_m - 8C_{m+1}\\ &= 8(CSb_3)_{2m} - 63C_m.\end{aligned}$$

Theorem 3.21. *Let $(CSb_3)_m$ denote the m^{th} b_3-Lucas subbalancing number and C_m denote the m^{th} Lucas-balancing number. Then*

$$(CSb_3)_{2m} = 22C_m - (CSb_3)_{2m-1}$$

for $m \geq 1$.

Proof. From Theorem 3.18, we obtain

$$(CSb_3)_{2m} = 14C_m - C_{m-1} \tag{22}$$

and

$$(CSb_3)_{2m-1} = 8C_m + C_{m-1} \tag{23}$$

From (22) and (23), we get

$$(CSb_3)_{2m} = 22C_m - (CSb_3)_{2m-1}.$$

Corollary 3.4. *The sum of any even term of the sequence of b_3-Lucas subbalancing numbers and the preceding term of the sequence of b_3-Lucas subbalancing number is always even. Similarly, the sum of any odd term of the sequence of b_3-Lucas subbalancing numbers and the next term of the sequence of b_3-Lucas subbalancing number is always even.*

Theorem 3.22. *Let $(CSb_3)_m$ denote the m^{th} b_3-Lucas subbalancing number. Then*

$$22(CSb_3)_{2m+1} = 113(CSb_3)_{2m} - 63(CSb_3)_{2m-1}$$

for $m \geq 1$.

Proof. Using Theorem 3.20 and Theorem 3.21, we obtain

$$\begin{aligned}22(CSb_3)_{2m+1} &= 176(CSb_3)_{2m} - 1386C_m\\ 63(CSb_3)_{2m} &= 1386C_m - 63(CSb_3)_{2m-1}.\end{aligned}$$

Thus

$$22(CSb_3)_{2m+1} = 113(CSb_3)_{2m} - 63(CSb_3)_{2m-1}.$$

Theorem 3.23. *Let $(CSb_3)_m$ denote the m^{th} b_3-Lucas subbalancing number and C_m denote the m^{th} Lucas-balancing number.Then*

$$4(CSb_3)_{2m} = 7(CSb_3)_{2m-1} - 11C_{m-1}$$

for $m \geq 1$.

Proof. Using (22) and (23), we get

$$7(CSb_3)_{2m-1} = 56C_m + 7C_{m-1}$$

and

$$4(CSb_3)_{2m} = 56C_m - 4C_{m-1}.$$

Then

$$4(CSb_3)_{2m} = 7(CSb_3)_{2m-1} - 11C_{m-1}.$$

Theorem 3.24. *Let $(CSb_3)_m$ denote the m^{th} b_3-Lucas subbalancing number. Then*

$$63(CSb_3)_{2m} = 113(CSb_3)_{2m-1} - 22(CSb_3)_{2m-2}$$

for $m \geq 1$.

Proof. From Theorem 3.20 and Theorem 3.23, we obtain

$$8(CSb_3)_{2m-2} = (CSb_3)_{2m-1} + 63C_{m-1}$$

and

$$252(CSb_3)_{2m} + 88(CSb_3)_{2m-2} = 452(CSb_3)_{2m-1}.$$

Then

$$63(CSb_3)_{2m} = 113(CSb_3)_{2m-1} - 22(CSb_3)_{2m-2}.$$

In the following theorem the Binet formula for b_3-Lucas subbalancing numbers is given.

Theorem 3.25. *Let $(CSb_3)_m$ denote the m^{th} b_3-Lucas subbalancing number. Then*

$$(CSb_3)_{2m} = \frac{\left(8\sqrt{2}+44\right)\alpha_1^m - \left(8\sqrt{2}-44\right)\alpha_2^m}{8}$$

and

$$(CSb_3)_{2m+1} = \frac{\left(-8\sqrt{2}+44\right)\alpha_1^{m+1} + \left(8\sqrt{2}+44\right)\alpha_2^{m+1}}{8}$$

where $\alpha_1 = 3 + 2\sqrt{2}$, $\alpha_2 = 3 - 2\sqrt{2}$.

Proof. Using Theorem 3.17 and the Binet formulas for balancing and Lucas-balancing numbers, we obtain

$$\begin{aligned}
(CSb_3)_{2m} &= 11\left(\frac{\alpha_1^m + \alpha_2^m}{2}\right) + 8\left(\frac{\alpha_1^m - \alpha_2^m}{4\sqrt{2}}\right) \\
&= \frac{(44+8\sqrt{2})\alpha_1^m + (44-8\sqrt{2})\alpha_2^m}{8} \\
&= \frac{(8\sqrt{2}+44)\alpha_1^m + (44-8\sqrt{2})\alpha_2^m}{8}.
\end{aligned}$$

The other case can be proved similarly.

Theorem 3.26. *Let $(CSb_3)_m$ denote the m^{th} b_3-Lucas subbalancing number. Then*

$$(CSb_3)_m^2 = (CSb_3)_{m-2}(CSb_3)_{m+2} - 904$$

for $m \geq 2$.

Proof. This theorem is proved by induction. It is easily seen that the assertion is true for $m = 2$.
Assuming the assertion is true for $m \leq k$, we have

$$\begin{aligned}
(CSb_3)_{k+1}^2 &= (CSb_3)_{k+1}(CSb_3)_{k+1} \\
&= (6(CSb_3)_{k-1} - (CSb_3)_{k-3})(CSb_3)_{k+1} \\
&= 6(CSb_3)_{k-1}(CSb_3)_{k+1} - (CSb_3)_{k-1}^2 - 904 \\
&= (CSb_3)_{k-1}(6(CSb_3)_{k+1} - (CSb_3)_{k-1}) - 904 \\
&= (CSb_3)_{k-1}(CSb_3)_{k+3} - 904.
\end{aligned}$$

Thus it is shown that the assertion is true for m = k + 1.

Theorem 3.27. *Let $(CSb_3)_m$ denote the m^{th} b_3-Lucas subbalancing number and B_m denote the m^{th} balancing number. Then*

$$(CSb_3)_{m+2}(CSb_3)_{m+1} - (CSb_3)_m(CSb_3)_{m-1} = 904B_{m+1}$$

for $m \geq 1$.

Proof. This theorem is proved by induction. It is easily seen that the assertion is true for $m = 1$. Assuming the assertion is true for $m \leq k$ and using Theorem 3.6 and Corollary 3.3, we have

$$\begin{aligned}(CSb_3)_{k+3}(CSb_3)_{k+2} - (CSb_3)_{k+1}(CSb_3)_k &= (3(Sb_3)_{k+3} - (Sb_3)_{k+1})(3(Sb_3)_{k+2} - (Sb_3)_k)\\ &\quad - ((Sb_3)_{k+3} - 3(Sb_3)_{k+1})((Sb_3)_{k+2} - 3(Sb_3)_k)\\ &= 8((Sb_3)_{k+3}(Sb_3)_{k+2} - (Sb_3)_{k+1}(Sb_3)_k)\\ &= 904B_{m+1}.\end{aligned}$$

Thus it is shown that the assertion is true for m = k + 1.

3.4. Relationship Between b_3-Lucas Subbalancing Numbers and b_3-Subbalancing Numbers

In this subsection, some relations between b_3-Lucas subbalancing numbers and b_3-subbalancing numbers are given.

Theorem 3.28. *Let $(CSb_3)_m$ denote the m^{th} b_3-Lucas subbalancing number, $(Sb_3)_m$ denote the m^{th} b_3-subbalancing number and B_m denote the m^{th} balancing number. Then*

$$(CSb_3)_m(Sb_3)_{m-1} + (CSb_3)_{m-1}(Sb_3)_m = 113B_m$$

for $m \geq 1$.

Proof. From Theorem 3.6 and Corollary 3.3, we get

$$\begin{aligned}(CSb_3)_m(Sb_3)_{m-1} + (CSb_3)_{m-1}(Sb_3)_m &= (3(Sb_3)_m - (Sb_3)_{m-2})(Sb_3)_{m-1}\\ &\quad + ((Sb_3)_{m+1} - 3(Sb_3)_{m-1})(Sb_3)_m\\ &= 3(Sb_3)_m(Sb_3)_{m-1} - (Sb_3)_{m-1}(Sb_3)_{m-2}\\ &\quad + (Sb_3)_{m+1}(Sb_3)_m - 3(Sb_3)_m(Sb_3)_{m-1}\\ &= (Sb_3)_{m+1}(Sb_3)_m - (Sb_3)_{m-1}(Sb_3)_{m-2}\\ &= 113B_m.\end{aligned}$$

Theorem 3.29. *Let $(CSb_3)_m$ denote the m^{th} b_3-Lucas subbalancing number and $(Sb_3)_m$ denote the m^{th} b_3-subbalancing number. Then*

$$(CSb_3)_m(Sb_3)_{m+2} - (CSb_3)_{m+2}(Sb_3)_m = 113$$

for $m \geq 0$.

Proof. If $m = 0$, we have

$$(CSb_3)_0(Sb_3)_2 - (CSb_3)_2(Sb_3)_0 = 113.$$

Thus the assertion is true for $m = 0$. If $m = 1$, we have

$$(CSb_3)_1(Sb_3)_3 - (CSb_3)_3(Sb_3)_1 = 113.$$

Thus the assertion is true for $m = 1$.
Assuming the assertion is true for $m \leq k$, we have

$$\begin{aligned}(CSb_3)_{k+1}(Sb_3)_{k+3} - (CSb_3)_{k+3}(Sb_3)_{k+1} &= (CSb_3)_{k+1}(6(Sb_3)_{k+1} - (Sb_3)_{k-1}) \\ &\quad - (Sb_3)_{k+1}(6(CSb_3)_{k+1} - (CSb_3)_{k-1}) \\ &= (Sb_3)_{k+1}(CSb_3)_{k-1} - (CSb_3)_{k+1}(Sb_3)_{k-1} \\ &= 113.\end{aligned}$$

Thus it is shown that the assertion is true for m = k + 1.

Theorem 3.30. *Let $(CSb_3)_m$ denote the m^{th} b_3-Lucas subbalancing number and $(Sb_3)_m$ denote the m^{th} b_3-subbalancing number. Then*

$$(CSb_3)_{2m}(Sb_3)_{2m+1} - (CSb_3)_{2m+1}(Sb_3)_{2m} = 63$$

and

$$(CSb_3)_{2m+1}(Sb_3)_{2m+2} - (CSb_3)_{2m+2}(Sb_3)_{2m+1} = 22$$

for $m \geq 0$.

Proof. If $m = 0$, we have

$$(CSb_3)_0(Sb_3)_1 - (CSb_3)_1(Sb_3)_0 = 63.$$

Thus the assertion is true for $m = 0$. If $m = 1$, we have

$$(CSb_3)_2(Sb_3)_3 - (CSb_3)_3(Sb_3)_2 = 63.$$

Thus the assertion is true for $m = 1$.
Assuming the assertion is true for $m \leq k$ and using Theorem 3.18 and equations (6), (7), we get

$$\begin{aligned}(CSb_3)_{2k+2}(Sb_3)_{2k+3} - (CSb_3)_{2k+3}(Sb_3)_{2k+2} &= (8C_{k+1} + C_{k+2})(8B_{k+2} + B_{k+1}) \\ &\quad - (8C_{k+2} + C_{k+1})(8B_{k+1} + B_{k+2}) \\ &= 63B_{k+2}C_{k+1} - 63C_{k+2}B_{k+1} \\ &= 63(B_{k+2}C_{k+1} - C_{k+2}B_{k+1}) \\ &= 63.\end{aligned}$$

Thus it is shown that the assertion is true for m = k + 1. The other case can be proved similarly.

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