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Dual Simpson type inequalities for multiplicatively convex functions

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Abstract. In this paper we propose a new identity for multiplicative differentiable functions, based on this identity we establish a dual Simpson type inequality for multiplicatively convex functions. Some applications of the obtained results are also given.

1. Introduction

Let I be an interval of real numbers

Definition 1.1 ([23]). *A function $f : I \to \mathbb{R}$ is said to be convex, if*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and all $t \in [0, 1]$.

The fundamental inequality for convex functions is undoubtedly the Hermite-Hadamard inequality, which can be stated as follows: For every convex function f on the interval $[a, b]$, we have

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

If the function f is concave, then (1) holds in the reverse direction see [23].

The concept of convexity plays an important and very central role in many areas, and has a close relationship in the development of the theory of inequalities, which is an important tool in the study of some properties of solutions of differential equations as well as in the error estimates of quadrature formulas. Concerning some papers dealing with some quadrature see [8, 11, 13–15] and references therein.

In 1967, Grossman and Katz, created the first non-Newtonian calculation system, called geometric calculation. Over the next few years they had created an infinite family of non-Newtonian calculi, thus modifying the classical calculus introduced by Newton and Leibniz in the 17th century each of which differed markedly from the classical calculus of Newton and Leibniz known today as the non-Newtonian

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calculus or the multiplicative calculus, where the ordinary product and ratio are used respectively as sum and exponential difference over the domain of positive real numbers see [10]. This calculation is useful for dealing with exponentially varying functions.

The complete mathematical description of multiplicative calculus was given by Bashirov et al. [5]. Also in the literature, there remains a trace of a similar calculation proposed by the mathematical biologists Volterra and Hostinsky [24] in 1938 called the Volterra calculation which is identified as a particular case of multiplicative calculation.

Recently, Ali et al. [1] gave the analogue of the Hermite-Hadamard inequality for multiplicatively convex functions as follows :

Theorem 1.2. *Let f be a positive and multiplicatively convex function on interval $[a,b]$, then the following inequalities hold*

$$f\left(\frac{a+b}{2}\right) \leq \left(\int_a^b f(x)^{dx}\right)^{\frac{1}{b-a}} \leq \sqrt{f(a)f(b)}. \tag{2}$$

Theorem 1.3. *Let f and g be a positive and multiplicatively convex functions on interval $[a,b]$, then the following inequalities hold*

$$f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \leq \left(\int_a^b f(x)^{dx}\right)^{\frac{1}{b-a}}\left(\int_a^b g(x)^{dx}\right)^{\frac{1}{b-a}} \leq \sqrt{f(a)f(b)}\sqrt{g(a)g(b)} \tag{3}$$

and

$$\frac{f\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)} \leq \left(\frac{\int_a^b f(x)^{dx}}{\int_a^b g(x)^{dx}}\right)^{\frac{1}{b-a}} \leq \frac{\sqrt{f(a)f(b)}}{\sqrt{g(a)g(b)}}. \tag{4}$$

Regarding the generalizations of the results obtained in [1] for different kind of multiplicatively convexity we have: for multiplicatively ϕ-convex and log-ϕ-convex functions [2], for multiplicatively s-convex functions [19], for multiplicatively h-convex functions [21], for multiplicatively preinvex functions [20], for multiplicatively s-preinvex functions [18], for multiplicatively h-preinvex functions [22] and recently the authors in [4] gave the analogue inequality for multiplicatively convex functions on coordinates and for product of two multiplicatively convex functions on coordinates. The fractional analogue of the Hermite-Hadamard inequality via multiplicative convexity was obtained by Budak and Özçelik [7] and for interval-valued multiplicative integrals by Zhang et al. [25]. In the same context, the midpoint and trapezoid-type inequalities were studied in [12], while Ali et al. investigate the Ostrowski and Simpson-type inequalities in [3]. Further literature on multiplicative integral inequalities can be found in [6, 16, 17].

We recall that the dual Simpson quadrature rule, can be stated as follows :

$$\left|\frac{1}{3}\left(2f\left(\frac{3a+b}{4}\right)-f\left(\frac{a+b}{2}\right)+2f\left(\frac{a+3b}{4}\right)\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \leq \frac{7(b-a)^4}{23040}\left\|f^{(4)}\right\|_\infty,$$

where f is four-times continuously differentiable function on (a,b), and $\left\|f^{(4)}\right\|_\infty = \sup_{x\in(a,b)}\left|f^{(4)}(x)\right|$ (see [9]).

Motivated by the aforementioned literature, the present study proposes a novel identity for multiplicative differentiable functions, and leverages this identity to derive a dual Simpson-type inequality for multiplicatively convex functions. To demonstrate the significance of our findings, we present a number of applications of our results.

2. Preliminaries

In this section we begin by recalling some definitions, properties and notions of derivation as well as multiplicative integration

Definition 2.1 ([5]). *Let $f : \mathbb{R} \to \mathbb{R}^+$ be a positive function. The multiplicative derivative of the function f noted by f^* is defined as follows*

$$\frac{d^* f}{dt} = f^*(t) = \lim_{h \to 0} \left(\frac{f(t+h)}{f(t)} \right)^{\frac{1}{h}}.$$

Remark 2.2. *If f has positive values and is differentiable at t, then f^* exists and the relation between f^* and ordinary derivative f' is as follows:*

$$f^*(t) = e^{(\ln f(t))'} = e^{\frac{f'(t)}{f(t)}}.$$

The multiplicative derivative admits the following properties:

Theorem 2.3 ([5]). *Let f and g be multiplicatively differentiable functions, and c is arbitrary constant. Then functions $cf, fg, f+g, f/g$ and f^g are * differentiable and*

- $(cf)^*(t) = f^*(t)$,
- $(fg)^*(t) = f^*(t)\, g^*(t)$,
- $(f+g)^*(t) = f^*(t)^{\frac{f(t)}{f(t)+g(t)}}\, g^*(t)^{\frac{g(t)}{f(t)+g(t)}}$,
- $\left(\frac{f}{g}\right)^*(t) = \frac{f^*(t)}{g^*(t)}$,
- $(f^g)^*(t) = f^*(t)^{g(t)}\, f(t)^{g'(t)}$.

In [5], Bashirov et al. introduced the concept of the * integral called multiplicative integral which is noted $\int_a^b (f(t))^{dt}$, where the ordinary product and ratio are employed as the addition and subtraction in exponential form, correspondingly, across the positive real number range.

The relationship between the Riemann integral and the multiplicative integral is as follows:

Proposition 2.4 ([5]). *If f is Riemann integrable on $[a, b]$, then f is multiplicative integrable on $[a, b]$ and*

$$\int_a^b (f(t))^{dt} = \exp\left(\int_a^b \ln(f(t))\, dt \right).$$

Moreover, Bashirov et al. show that multiplicative integral has the following results and properties:

Theorem 2.5 ([5]). *Let f be a positive and Riemann integrable on $[a, b]$, then f is multiplicative integrable on $[a, b]$ and*

- $\int_a^b \left((f(t))^p\right)^{dt} = \left(\int_a^b (f(t))^{dt} \right)^p$,
- $\int_a^b (f(t)\, g(t))^{dt} = \int_a^b (f(t))^{dt} \int_a^b (g(t))^{dt}$,

- $\int\limits_a^b \left(\frac{f(t)}{g(t)}\right)^{dt} = \frac{\int\limits_a^b (f(t))^{dt}}{\int\limits_a^b (g(t))^{dt}}$,
- $\int\limits_a^b (f(t))^{dt} = \int\limits_a^c (f(t))^{dt} \int\limits_c^b (f(t))^{dt}$, $a < c < b$,
- $\int\limits_a^a (f(t))^{dt} = 1$ and $\int\limits_a^b (f(t))^{dt} = \left(\int\limits_b^a (f(t))^{dt}\right)^{-1}$.

Theorem 2.6 (Multiplicative Integration by Parts [5]). *Let $f : [a,b] \to \mathbb{R}$ be multiplicative differentiable, let $g : [a,b] \to \mathbb{R}$ be differentiable so the function f^g is multiplicative integrable, and*

$$\int\limits_a^b \left(f^*(t)^{g(t)}\right)^{dt} = \frac{f(b)^{g(b)}}{f(a)^{g(a)}} \times \frac{1}{\int\limits_a^b \left(f(t)^{g'(t)}\right)^{dt}}.$$

Lemma 2.7 ([3]). *Let $f : [a,b] \to \mathbb{R}$ be multiplicative differentiable, let $h : [a,b] \to \mathbb{R}$ and let $g : J \subset \mathbb{R} \to \mathbb{R}$ be two differentiable functions. Then we have*

$$\int\limits_a^b \left(f^*(h(t))^{h'(t)g(t)}\right)^{dt} = \frac{f(h(b))^{g(b)}}{f(h(a))^{g(a)}} \times \frac{1}{\int\limits_a^b \left(f(h(t))^{g'(t)}\right)^{dt}}.$$

3. Main results

In order to prove our results, we need the following lemma

Lemma 3.1. *Let $f : [a,b] \to \mathbb{R}^+$ be a multiplicative differentiable mapping on $[a,b]$ with $a < b$. If f^* is multiplicative integrable on $[a,b]$, then we have the following identity for multiplicative integrals*

$$\begin{aligned}
&\left(\left(f\left(\frac{3a+b}{4}\right)\right)^2 \left(f\left(\frac{a+b}{2}\right)\right)^{-1} \left(f\left(\frac{a+3b}{4}\right)\right)^2\right)^{\frac{1}{3}} \left(\int\limits_a^b f(u)^{du}\right)^{\frac{1}{a-b}} \\
&= \left(\int\limits_0^1 \left(f^*\left((1-t)a + t\frac{3a+b}{4}\right)^{\frac{t}{4}}\right)^{dt}\right)^{\frac{b-a}{4}} \left(\int\limits_0^1 \left(f^*\left((1-t)\frac{3a+b}{4} + t\frac{a+b}{2}\right)^{\left(\frac{1}{4}t-\frac{5}{12}\right)}\right)^{dt}\right)^{\frac{b-a}{4}} \\
&\times \left(\int\limits_0^1 \left(f^*\left((1-t)\frac{a+b}{2} + t\frac{a+3b}{4}\right)^{\left(\frac{1}{4}t+\frac{1}{6}\right)}\right)^{dt}\right)^{\frac{b-a}{4}} \left(\int\limits_0^1 \left(f^*\left((1-t)\frac{a+3b}{4} + tb\right)^{\left(\frac{1}{4}t-\frac{1}{4}\right)}\right)^{dt}\right)^{\frac{b-a}{4}}.
\end{aligned}$$

Proof. Let

$$I_1 = \left(\int\limits_0^1 \left(f^*\left((1-t)a + t\frac{3a+b}{4}\right)^{\frac{1}{4}t}\right)^{dt}\right)^{\frac{b-a}{4}},$$

$$I_2=\left(\int_0^1\left(f^*\left((1-t)\,\frac{3a+b}{4}+t\frac{a+b}{2}\right)^{\left(\frac{1}{4}t-\frac{5}{12}\right)}\right)^{dt}\right)^{\frac{b-a}{4}},$$

$$I_3=\left(\int_0^1\left(f^*\left((1-t)\,\frac{a+b}{2}+t\frac{a+3b}{4}\right)^{\left(\frac{1}{4}t+\frac{1}{6}\right)}\right)^{dt}\right)^{\frac{b-a}{4}}$$

and

$$I_4=\left(\int_0^1\left(f^*\left((1-t)\,\frac{a+3b}{4}+tb\right)^{\left(\frac{1}{4}t-\frac{1}{4}\right)}\right)^{dt}\right)^{\frac{b-a}{4}}.$$

Using the integration by parts for multiplicative integrals, from I_1 we have

$$\begin{aligned}
I_1&=\left(\int_0^1\left(f^*\left((1-t)\,a+t\frac{3a+b}{4}\right)^{\frac{t}{4}}\right)^{dt}\right)^{\frac{b-a}{4}}\\
&=\left(\int_0^1\left(f^*\left((1-t)\,a+t\frac{3a+b}{4}\right)^{\frac{b-a}{4}\frac{t}{4}}\right)^{dt}\right)\\
&=\frac{\left(f\left(\frac{3a+b}{4}\right)\right)^{\frac{1}{4}}}{1}\cdot\frac{1}{\int_0^1\left(f\left((1-t)\,a+t\frac{3a+b}{4}\right)^{\frac{1}{4}}\right)^{dt}}\\
&=\left(f\left(\frac{3a+b}{4}\right)\right)^{\frac{1}{4}}\frac{1}{\left(\int_0^1 f\left((1-t)\,a+t\frac{3a+b}{4}\right)^{dt}\right)^{\frac{1}{4}}}\\
&=\left(f\left(\frac{3a+b}{4}\right)\right)^{\frac{1}{4}}\left(\int_a^{\frac{3a+b}{4}} f\left(u\right)^{du}\right)^{\frac{1}{a-b}}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
I_2&=\left(\int_0^1\left(f^*\left((1-t)\,\frac{3a+b}{4}+t\frac{a+b}{2}\right)^{\left(\frac{1}{4}t-\frac{5}{12}\right)}\right)^{dt}\right)^{\frac{b-a}{4}}\\
&=\int_0^1\left(f^*\left((1-t)\,\frac{3a+b}{4}+t\frac{a+b}{2}\right)^{\frac{b-a}{4}\left(\frac{1}{4}t-\frac{5}{12}\right)}\right)^{dt}\\
&=\frac{\left(f\left(\frac{a+b}{2}\right)\right)^{-\frac{1}{6}}}{\left(f\left(\frac{3a+b}{4}\right)\right)^{-\frac{5}{12}}}\cdot\frac{1}{\int_0^1\left(f\left((1-t)\,\frac{3a+b}{4}+t\frac{a+b}{2}\right)^{\frac{1}{4}}\right)^{dt}}
\end{aligned}$$

$$= \left(f\left(\frac{3a+b}{4}\right)\right)^{\frac{5}{12}} \left(f\left(\frac{a+b}{2}\right)\right)^{-\frac{1}{6}} \cdot \frac{1}{\left(\int\limits_0^1 f\left((1-t)\frac{3a+b}{4}+t\frac{a+b}{2}\right)^{dt}\right)^{\frac{1}{4}}}$$

$$= \left(f\left(\frac{3a+b}{4}\right)\right)^{\frac{5}{12}} \left(f\left(\frac{a+b}{2}\right)\right)^{-\frac{1}{6}} \left(\int\limits_0^1 f\left((1-t)\frac{3a+b}{4}+t\frac{a+b}{2}\right)^{dt}\right)^{-\frac{1}{4}}$$

$$= \left(f\left(\frac{3a+b}{4}\right)\right)^{\frac{5}{12}} \left(f\left(\frac{a+b}{2}\right)\right)^{-\frac{1}{6}} \left(\int\limits_{\frac{3a+b}{4}}^{\frac{a+b}{2}} f(u)^{du}\right)^{\frac{1}{a-b}},$$

$$I_3 = \left(\int\limits_0^1 \left(f^*\left((1-t)\frac{a+b}{2}+t\frac{a+3b}{4}\right)^{\left(\frac{1}{4}t+\frac{1}{6}\right)}\right)^{dt}\right)^{\frac{b-a}{4}}$$

$$= \int\limits_0^1 \left(f^*\left((1-t)\frac{a+b}{2}+t\frac{a+3b}{4}\right)^{\frac{b-a}{4}\left(\frac{1}{4}t+\frac{1}{6}\right)}\right)^{dt}$$

$$= \frac{f\left(\frac{a+3b}{4}\right)^{\frac{5}{12}}}{\left(f\left(\frac{a+b}{2}\right)\right)^{\frac{1}{6}}} \cdot \frac{1}{\int\limits_0^1 \left(f\left((1-t)\frac{a+b}{2}+t\frac{a+3b}{4}\right)^{\frac{1}{4}}\right)^{dt}}$$

$$= \left(f\left(\frac{a+3b}{4}\right)\right)^{\frac{5}{12}} \left(f\left(\frac{a+b}{2}\right)\right)^{-\frac{1}{6}} \cdot \frac{1}{\left(\int\limits_0^1 f\left((1-t)\frac{a+b}{2}+t\frac{a+3b}{4}\right)^{dt}\right)^{\frac{1}{4}}}$$

$$= \left(f\left(\frac{a+3b}{4}\right)\right)^{\frac{5}{12}} \left(f\left(\frac{a+b}{2}\right)\right)^{-\frac{1}{6}} \left(\int\limits_0^1 f\left((1-t)\frac{a+b}{2}+t\frac{a+3b}{4}\right)^{dt}\right)^{-\frac{1}{4}}$$

$$= \left(f\left(\frac{a+3b}{4}\right)\right)^{\frac{5}{12}} \left(f\left(\frac{a+b}{2}\right)\right)^{-\frac{1}{6}} \left(\int\limits_{\frac{a+b}{2}}^{\frac{a+3b}{4}} f(u)^{du}\right)^{\frac{1}{a-b}},$$

and

$$I_4 = \left(\int\limits_0^1 \left(f^*\left((1-t)\frac{a+3b}{4}+tb\right)^{\left(\frac{1}{4}t-\frac{1}{4}\right)}\right)^{dt}\right)^{\frac{b-a}{4}}$$

$$= \int\limits_0^1 \left(f^*\left((1-t)\frac{a+3b}{4}+tb\right)^{\frac{b-a}{4}\left(\frac{1}{4}t-\frac{1}{4}\right)}\right)^{dt}$$

$$= \frac{1}{\left(f\left(\frac{a+3b}{4}\right)\right)^{-\frac{1}{4}}} \cdot \frac{1}{\int\limits_0^1 \left(f\left((1-t)\frac{a+3b}{4}+tb\right)^{\frac{1}{4}}\right)^{dt}}$$

$$=\left(f\left(\frac{a+3b}{4}\right)\right)^{\frac{1}{4}}.\frac{1}{\left(\int_0^1 f\left((1-t)\frac{a+3b}{4}+tb\right)^{dt}\right)^{\frac{1}{4}}}$$

$$=\left(f\left(\frac{a+3b}{4}\right)\right)^{\frac{1}{4}}\left(\int_0^1 f\left((1-t)\frac{a+3b}{4}+tb\right)^{dt}\right)^{-\frac{1}{4}}$$

$$=\left(f\left(\frac{a+3b}{4}\right)\right)^{\frac{1}{4}}\left(\int_{\frac{a+3b}{4}}^{b} f(u)^{du}\right)^{\frac{1}{a-b}}.$$

Multiplying above equalities we get

$$I_1\times I_2\times I_3\times I_4$$

$$=\left(f\left(\frac{3a+b}{4}\right)\right)^{\frac{1}{4}}\left(\int_a^{\frac{3a+b}{4}} f(u)^{du}\right)^{\frac{1}{a-b}}\left(f\left(\frac{3a+b}{4}\right)\right)^{\frac{5}{12}}\left(f\left(\frac{a+b}{2}\right)\right)^{-\frac{1}{6}}\left(\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} f(u)^{du}\right)^{\frac{1}{a-b}}$$

$$\times\left(f\left(\frac{a+3b}{4}\right)\right)^{\frac{5}{12}}\left(f\left(\frac{a+b}{2}\right)\right)^{-\frac{1}{6}}\left(\int_{\frac{a+b}{2}}^{\frac{a+3b}{4}} f(u)^{du}\right)^{\frac{1}{a-b}}\left(f\left(\frac{a+3b}{4}\right)\right)^{\frac{1}{4}}\left(\int_{\frac{a+3b}{4}}^{b} f(u)^{du}\right)^{\frac{1}{a-b}}$$

$$=\left(f\left(\frac{3a+b}{4}\right)\right)^{\frac{1}{4}}\left(f\left(\frac{3a+b}{4}\right)\right)^{\frac{5}{12}}\left(f\left(\frac{a+b}{2}\right)\right)^{-\frac{1}{6}}\left(f\left(\frac{a+3b}{4}\right)\right)^{\frac{5}{12}}\left(f\left(\frac{a+b}{2}\right)\right)^{-\frac{1}{6}}\left(f\left(\frac{a+3b}{4}\right)\right)^{\frac{1}{4}}$$

$$\times\left(\int_a^{\frac{3a+b}{4}} f(u)^{du}\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} f(u)^{du}\int_{\frac{a+b}{2}}^{\frac{a+3b}{4}} f(u)^{du}\int_{\frac{a+3b}{4}}^{b} f(u)^{du}\right)^{\frac{1}{a-b}}$$

$$=\left(f\left(\frac{3a+b}{4}\right)\right)^{\frac{8}{12}}\left(f\left(\frac{a+b}{2}\right)\right)^{-\frac{2}{6}}\left(f\left(\frac{a+3b}{4}\right)\right)^{\frac{8}{12}}\left(\int_a^{\frac{a+b}{2}} f(u)^{du}\int_{\frac{a+b}{2}}^{\frac{a+3b}{4}} f(u)^{du}\int_{\frac{a+3b}{4}}^{b} f(u)^{du}\right)^{\frac{1}{a-b}}$$

$$=\left(f\left(\frac{3a+b}{4}\right)\right)^{\frac{2}{3}}\left(f\left(\frac{a+b}{2}\right)\right)^{-\frac{1}{3}}\left(f\left(\frac{a+3b}{4}\right)\right)^{\frac{2}{3}}\left(\int_a^{\frac{a+3b}{4}} f(u)^{du}\int_{\frac{a+3b}{4}}^{b} f(u)^{du}\right)^{\frac{1}{a-b}}$$

$$=\left(\left(f\left(\frac{3a+b}{4}\right)\right)^{2}\left(f\left(\frac{a+b}{2}\right)\right)^{-1}\left(f\left(\frac{a+3b}{4}\right)\right)^{2}\right)^{\frac{1}{3}}\left(\int_a^{b} f(u)^{du}\right)^{\frac{1}{a-b}},$$

which is the required result. The proof is completed. □

Theorem 3.2. *Let $f:[a,b]\to\mathbb{R}^+$ be a multiplicative differentiable mapping on $[a,b]$ with $a<b$. If f^* is multiplicative convex on $[a,b]$, then we have*

$$\left|\left(\left(f\left(\frac{3a+b}{4}\right)\right)^{2}\left(f\left(\frac{a+b}{2}\right)\right)^{-1}\left(f\left(\frac{a+3b}{4}\right)\right)^{2}\right)^{\frac{1}{3}}\left(\int_a^{b} f(u)^{du}\right)^{\frac{1}{a-b}}\right|$$

$$\leq \left((f^*(a)) \left(f^*\left(\frac{3a+b}{4}\right)\right)^6 \left(f^*\left(\frac{a+b}{2}\right)\right)^6 \left(f^*\left(\frac{a+3b}{4}\right)\right)^6 (f^*(b)) \right)^{\frac{b-a}{96}}.$$

Proof. From Lemma 3.1, properties of multiplicative integral, we have

$$\left| \left(\left(f\left(\frac{3a+b}{4}\right)\right)^2 \left(f\left(\tfrac{a+b}{2}\right)\right)^{-1} \left(f\left(\tfrac{a+3b}{4}\right)\right)^2 \right)^{\frac{1}{3}} \left(\int_a^b f(u)^{du} \right)^{\frac{1}{a-b}} \right|$$
$$\leq \left(\exp \frac{b-a}{4} \int_0^1 \left| \ln\left(f^*\left((1-t)a + t\tfrac{3a+b}{4}\right)^{\frac{1}{4}t}\right) \right| dt \right) \left(\exp \frac{b-a}{4} \int_0^1 \left| \ln\left(f^*\left((1-t)\tfrac{3a+b}{4} + t\tfrac{a+b}{2}\right)^{\left(\frac{1}{4}t-\frac{5}{12}\right)}\right) \right| dt \right)$$
$$\times \left(\exp \frac{b-a}{4} \int_0^1 \left| \ln\left(f^*\left((1-t)\tfrac{a+b}{2} + t\tfrac{a+3b}{4}\right)^{\left(\frac{1}{4}t+\frac{1}{6}\right)}\right) \right| dt \right) \left(\exp \frac{b-a}{4} \int_0^1 \left| \ln\left(f^*\left((1-t)\tfrac{a+3b}{4} + tb\right)^{\left(\frac{1}{4}t-\frac{1}{4}\right)}\right) \right| dt \right)$$
$$= \left(\exp \frac{b-a}{4} \int_0^1 \tfrac{1}{4}t \left| \ln\left(f^*\left((1-t)a + t\tfrac{3a+b}{4}\right)\right) \right| dt \right) \left(\exp \frac{b-a}{4} \int_0^1 \left|\tfrac{1}{4}t - \tfrac{5}{12}\right| \left| \ln\left(f^*\left((1-t)\tfrac{3a+b}{4} + t\tfrac{a+b}{2}\right)\right) \right| dt \right)$$
$$\times \left(\exp \frac{b-a}{4} \int_0^1 \left|\tfrac{1}{4}t + \tfrac{1}{6}\right| \left| \ln\left(f^*\left((1-t)\tfrac{a+b}{2} + t\tfrac{a+3b}{4}\right)^{\left(\frac{1}{4}t+\frac{1}{6}\right)}\right) \right| dt \right)$$
$$\times \left(\exp \frac{b-a}{4} \int_0^1 \left|\tfrac{1}{4}t - \tfrac{1}{4}\right| \left| \ln\left(f^*\left((1-t)\tfrac{a+3b}{4} + tb\right)^{\left(\frac{1}{4}t-\frac{1}{4}\right)}\right) \right| dt \right)$$
$$= \left(\exp \frac{b-a}{4} \int_0^1 \tfrac{1}{4}t \left| \ln\left(f^*\left((1-t)a + t\tfrac{3a+b}{4}\right)\right) \right| dt \right) \left(\exp \frac{b-a}{4} \int_0^1 \left(\tfrac{5}{12} - \tfrac{1}{4}t\right) \left| \ln\left(f^*\left((1-t)\tfrac{3a+b}{4} + t\tfrac{a+b}{2}\right)\right) \right| dt \right)$$
$$\times \left(\exp \frac{b-a}{4} \int_0^1 \left(\tfrac{1}{4}t + \tfrac{1}{6}\right) \left| \ln\left(f^*\left((1-t)\tfrac{a+b}{2} + t\tfrac{a+3b}{4}\right)\right) \right| dt \right)$$
$$\times \left(\exp \frac{b-a}{4} \int_0^1 \left(\tfrac{1}{4} - \tfrac{1}{4}t\right) \left| \ln\left(f^*\left((1-t)\tfrac{a+3b}{4} + tb\right)\right) \right| dt \right).$$

Using the multiplicative convexity of f^*, we get

$$\left| \left(\left(f\left(\frac{3a+b}{4}\right)\right)^2 \left(f\left(\frac{a+b}{2}\right)\right)^{-1} \left(f\left(\frac{a+3b}{4}\right)\right)^2 \right)^{\frac{1}{3}} \left(\int_a^b f(u)^{du} \right)^{\frac{1}{a-b}} \right|$$
$$\leq \left(\exp \frac{b-a}{4} \int_0^1 \frac{1}{4}t \ln\left(f^*(a)^{1-t} f^*\left(\frac{3a+b}{4}\right)^t \right) dt \right) \left(\exp \frac{b-a}{4} \int_0^1 \left(\frac{5}{12} - \frac{1}{4}t\right) \ln\left(f^*\left(\frac{3a+b}{4}\right)^{1-t} f^*\left(\frac{a+b}{2}\right)^t \right) dt \right)$$
$$\times \left(\exp \frac{b-a}{4} \int_0^1 \left(\tfrac{1}{4}t + \tfrac{1}{6}\right) \ln\left(f^*\left(\tfrac{a+b}{2}\right)^{1-t} f^*\left(\tfrac{a+3b}{4}\right)^t \right) dt \right) \left(\exp \frac{b-a}{4} \int_0^1 \left(\tfrac{1}{4} - \tfrac{1}{4}t\right) \ln\left(f^*\left(\tfrac{a+3b}{4}\right)^{1-t} f^*(b)^t \right) dt \right)$$

$$
\begin{aligned}
&= \left(\exp \frac{b-a}{4} \int_0^1 \frac{1}{4} t \left((1-t) \ln f^*(a) + t \ln f^* \left(\frac{3a+b}{4} \right) \right) dt \right) \\
&\times \left(\exp \frac{b-a}{4} \int_0^1 \left(\frac{5}{12} - \frac{1}{4} t \right) \left((1-t) \ln f^* \left(\frac{3a+b}{4} \right) + t \ln f^* \left(\frac{a+b}{2} \right) \right) dt \right) \\
&\times \left(\exp \frac{b-a}{4} \int_0^1 \left(\frac{1}{4} t + \frac{1}{6} \right) \left((1-t) \ln f^* \left(\frac{a+b}{2} \right) + t \ln f^* \left(\frac{a+3b}{4} \right) \right) dt \right) \\
&\times \left(\exp \frac{b-a}{4} \int_0^1 \left(\frac{1}{4} - \frac{1}{4} t \right) \left((1-t) \ln f^* \left(\frac{a+3b}{4} \right) + t \ln f^*(b) \right) dt \right) \\
&= \left(\exp \frac{b-a}{4} \left(\ln (f^*(a)) \int_0^1 \frac{1}{4} t (1-t) dt + \ln \left(f^* \left(\frac{3a+b}{4} \right) \right) \int_0^1 \frac{1}{4} t^2 dt \right) \right) \\
&\times \left(\exp \frac{b-a}{4} \left(\ln \left(f^* \left(\frac{3a+b}{4} \right) \right) \int_0^1 \left(\frac{5}{12} - \frac{1}{4} t \right) (1-t) dt \ + \ \ln \left(f^* \left(\frac{a+b}{2} \right) \right) \int_0^1 \left(\frac{5}{12} - \frac{1}{4} t \right) t dt \right) \right) \\
&\times \left(\exp \frac{b-a}{4} \left(\ln \left(f^* \left(\frac{a+b}{2} \right) \right) \int_0^1 \left(\frac{1}{4} t + \frac{1}{6} \right) (1-t) dt \ + \ \ln \left(f^* \left(\frac{a+3b}{4} \right) \right) \int_0^1 \left(\frac{1}{4} t + \frac{1}{6} \right) t dt \right) \right) \\
&\times \left(\exp \frac{b-a}{4} \left(\ln \left(f^* \left(\frac{a+3b}{4} \right) \right) \int_0^1 \frac{1}{4} (1-t)^2 dt + \ln (f^*(b)) \int_0^1 \frac{1}{4} (1-t) t dt \right) \right) \\
&= \left(\exp \frac{b-a}{4} \left(\frac{1}{24} \ln (f^*(a)) + \frac{1}{24} \ln \left(f^* \left(\frac{3a+b}{4} \right) \right) \right) \right) \left(\exp \frac{b-a}{4} \left(\frac{1}{6} \ln \left(f^* \left(\frac{3a+b}{4} \right) \right) + \frac{1}{8} \ln \left(f^* \left(\frac{a+b}{2} \right) \right) \right) \right) \\
&\times \left(\exp \frac{b-a}{4} \left(\frac{1}{8} \ln \left(f^* \left(\frac{a+b}{2} \right) \right) + \frac{1}{6} \ln \left(f^* \left(\frac{a+3b}{4} \right) \right) \right) \right) \left(\exp \frac{b-a}{4} \left(\frac{1}{24} \ln \left(f^* \left(\frac{a+3b}{4} \right) \right) + \frac{1}{24} \ln (f^*(b)) \right) \right) \\
&= \left(\exp \left(\ln (f^*(a))^{\frac{b-a}{96}} \ln \left(f^* \left(\frac{3a+b}{4} \right) \right)^{\frac{b-a}{48}} \right) \right) \left(\exp \left(\ln \left(f^* \left(\frac{3a+b}{4} \right) \right)^{\frac{b-a}{24}} \ln \left(f^* \left(\frac{a+b}{2} \right) \right)^{\frac{b-a}{32}} \right) \right) \\
&\times \left(\exp \left(\ln \left(f^* \left(\frac{a+b}{2} \right) \right)^{\frac{b-a}{32}} \ln \left(f^* \left(\frac{a+3b}{4} \right) \right)^{\frac{b-a}{24}} \right) \right) \left(\exp \left(\ln \left(f^* \left(\frac{a+3b}{4} \right) \right)^{\frac{b-a}{4}} \ln (f^*(b))^{\frac{b-a}{96}} \right) \right) \\
&= (f^*(a))^{\frac{b-a}{96}} \left(f^* \left(\frac{3a+b}{4} \right) \right)^{\frac{b-a}{16}} \left(f^* \left(\frac{a+b}{2} \right) \right)^{\frac{b-a}{16}} \left(f^* \left(\frac{a+3b}{4} \right) \right)^{\frac{b-a}{16}} (f^*(b))^{\frac{b-a}{96}} \\
&= \left((f^*(a)) \left(f^* \left(\frac{3a+b}{4} \right) \right)^6 \left(f^* \left(\frac{a+b}{2} \right) \right)^6 \left(f^* \left(\frac{a+3b}{4} \right) \right)^6 (f^*(b)) \right)^{\frac{b-a}{96}} .
\end{aligned}
$$

Here, where we have used

$$
\int_0^1 \frac{1}{4} t (1-t) dt = \frac{1}{24},
$$

$$\int_0^1 \frac{1}{4}t^2 dt = \int_0^1 \frac{1}{4}(1-t)^2 dt = \frac{1}{12},$$

$$\int_0^1 \left(\frac{5}{12} - \frac{1}{4}t\right)(1-t)\, dt = \int_0^1 \left(\frac{1}{6} + \frac{1}{4}t\right) t dt = \frac{1}{6}$$

and

$$\int_0^1 \left(\frac{5}{12} - \frac{1}{4}t\right) t dt = \int_0^1 \left(\frac{1}{6} + \frac{1}{4}t\right)(1-t)\, dt = \frac{1}{8}.$$

□

Corollary 3.3. *In Theorem 3.2, using the multiplicative convexity of f^*, we obtain*

$$\left|\left(\left(f\left(\frac{3a+b}{4}\right)\right)^2 \left(f\left(\frac{a+b}{2}\right)\right)^{-1} \left(f\left(\frac{a+3b}{4}\right)\right)^2\right)^{\frac{1}{3}} \left(\int_a^b f(u)^{du}\right)^{\frac{1}{a-b}}\right| \leq \left(f^*(a)\left(f^*\left(\frac{a+b}{2}\right)\right)^3 f^*(b)\right)^{\frac{b-a}{24}}$$
$$\leq \left(f^*(a)\, f^*(b)\right)^{\frac{5(b-a)}{48}}.$$

Corollary 3.4. *In Theorem 3.2, If we assume that $f^* \leq M$, we get*

$$\left|\left(\left(f\left(\frac{3a+b}{4}\right)\right)^2 \left(f\left(\frac{a+b}{2}\right)\right)^{-1} \left(f\left(\frac{a+3b}{4}\right)\right)^2\right)^{\frac{1}{3}} \left(\int_a^b f(u)^{du}\right)^{\frac{1}{a-b}}\right| \leq M^{\frac{5(b-a)}{24}}.$$

4. Applications to special means

We shall consider the means for arbitrary real numbers a, b
The Arithmetic mean: $A(a_1, a_2, ..., a_n) = \frac{a_1+a_2+...+a_n}{n}$.
The logarithmic means: $L(a,b) = \frac{b-a}{\ln b - \ln a}$, $a, b > 0$ and $a \neq b$.
The p-Logarithmic mean: $L_p(a,b) = \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}$, $a, b > 0, a \neq b$ and $p \in \mathbb{R} \setminus \{-1, 0\}$.

Proposition 4.1. *Let $a, b \in \mathbb{R}$ with $0 < a < b$, then we have*

$$e^{\frac{2}{3}A^p(a,a,a,b) - \frac{1}{3}A^p(a,b) + \frac{2}{3}A^p(a,b,b,b) - L_p^p(a,b)} \leq \left(e^{A\left(a^{p-1}, b^{p-1}\right) + 3A^{p-1}(a,a,a,b) + 3A^{p-1}(a,b) + 3A^{p-1}(a,b,b,b)}\right)^{p\frac{b-a}{48}}.$$

Proof. The assertion follows from Theorem 3.2 applied to the function $f(t) = e^{t^p}$ with $p \geq 2$ whose $f^*(x) = e^{pt^{p-1}}$ and $\left(\int_a^b f(u)^{du}\right)^{\frac{1}{a-b}} = \exp\left(-L_p^p(a,b)\right)$. □

Proposition 4.2. *Let $a, b \in \mathbb{R}$ with $0 < a < b$, then we have*

$$e^{\frac{2}{3}A^{-1}(a,a,a,b) - \frac{1}{3}A^{-1}(a,b) + \frac{2}{3}A^{-1}(a,b,b,b) - L^{-1}(a,b)} \leq e^{-\frac{5(b-a)}{24b^2}}.$$

Proof. The assertion follows from Corollary 3.4 applied to the function $f(t) = e^{\frac{1}{t}}$ whose $f^*(x) = e^{-\frac{1}{t^2}}$, $M = e^{-\frac{1}{b^2}}$ and $\left(\int_a^b f(u)^{du}\right)^{\frac{1}{a-b}} = \exp\left(-L^{-1}(a,b)\right)$. □

5. Conclusion

In this work, we focused error bounds of the dual Simpson rule in the framework of multiplicative calculus, which represents a different type of calculation from that of Newton's and has shown significance and effectiveness in several areas of applied sciences. After introducing a new integral identity, we established some dual Simpson-type inequalities for functions with multiplicatively convex derivatives. The study concludes with several applications.

The findings of this study could prompt further investigation into this fascinating subject, as well as extensions to other forms of generalized convexity, weighted formulas, and higher dimensions.

References

[1] M. A. Ali, M. Abbas, Z. Zhang, I. B. Sial and R. Arif, On integral inequalities for product and quotient of two multiplicatively convex functions. Asian research journal of mathematics, 12 (2019), no. 3, 1-11.

[2] M. A. Ali, M. Abbas and A. A. Zafer, On some Hermite-Hadamard integral inequalities in multiplicative calculus. J. Ineq. Special Func. 10 (2019), no.1, 111-122.

[3] M. A. Ali, H. Budak, M. Z. Sarikaya and Z. Zhang, Ostrowski and Simpson type inequalities for multiplicative integrals. Proyecciones 40 (2021), no. 3, 743-763.

[4] M. A. Ali, M. Abbas, H. Budak, A. Kashuri, Some new Hermite-Hadamard integral inequalities in multiplicative calculus, TWMS Journal of Applied and Engineering Mathematics, 11(4), (2021), pp. 1183-1193.

[5] A.E. Bashirov, E.M. Kurpinar and A. Özyapici, Multiplicative calculus and its applications. J. Math. Anal. Appl. 337 (2008), no. 1, 36–48.

[6] H. Boulares, B. Meftah, A. Moumen, R. Shafqat, H. Saber, T. Alraqad and E. Ahmad, Fractional multiplicative Bullen type inequalities for multiplicative differentiable functions. Symmetry, 15(2), (2023), 451.

[7] H. Budak and K. Özçelik, On Hermite-Hadamard type inequalities for multiplicative fractional integrals. Miskolc Math. Notes 21 (2020), no. 1, 91–99.

[8] T. Chiheb, N. Boumaza and B. Meftah, Some new Simpson-like type inequalities via preqausiinvexity. Transylv. J. Math. Mech.12 (2020), no.1, 1-10.

[9] Lj. Dedić, M. Matić and J. Pečarić, On dual Euler-Simpson formulae. Bull. Belg. Math. Soc. Simon Stevin 8 (2001), no. 3, 479–504.

[10] M. Grossman and R. Katz, Non-Newtonian calculus. Lee Press, Pigeon Cove, Mass., 1972.

[11] A. Kashuri, B. Meftah and P.O. Mohammed, Some weighted Simpson type inequalities for differentiable s-convex functions and their applications. J. Frac. Calc. & Nonlinear Sys.1 (2021) no. 1, 75-94.

[12] S. Khan and H. Budak, On midpoint and trapezoid type inequalities for multiplicative integrals. Mathematica, 64 (87), (2022), pp. 95–108

[13] B. Meftah and A. Azaizia, Ostrowski type inequalities for functions whose derivatives are strongly *beta*-convex. Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. 39 (2019), no. 4, Mathematics, 126-147.

[14] B. Meftah, M. Merad, N. Ouanas and A. Souahi, Some new Hermite-Hadamard type inequalities for functions whose n^{th} derivatives are convex. Acta Comment. Univ. Tartu. Math. 23 (2019), no. 2, 163-178.

[15] B. Meftah and K. Mekalfa, Some weighted trapezoidal type inequalities via h-preinvexity. Rad Hrvat. Akad. Znan. Umjet. Mat. Znan. 24 (2020), 81-97.

[16] B. Meftah, Maclaurin type inequalities for multiplicatively convex functions. Proc. Amer. Math. Soc. 151 (2023), 2115-2125.

[17] A. Moumen, H. Boulares, B. Meftah, R. Shafqat, T. Alraqad, E. Ahmad and A. Abouelregal, Multiplicatively Simpson Type Inequalities via Fractional Integral. Symmetry, 15(2), (2023), 460.

[18] S. Özcan, Some integral inequalities of Hermite-Hadamard type for multiplicatively s-preinvex functions. International Journal of Mathematical Modelling & Computations, 9 (2019), no. 4, 253-266.

[19] S. Özcan, Hermite-hadamard type inequalities for multiplicatively s-convex functions. Cumhuriyet Science Journal, 41 (2020), no. 1, 245-259.

[20] S. Özcan, Some integral inequalities of Hermite-Hadamard type for multiplicatively preinvex functions. AIMS Math. 5 (2020), no. 2, 1505–1518.

[21] S. Özcan, Hermite-Hadamard type inequalities for multiplicatively h-convex functions. Konuralp J. Math. 8 (2020), no. 1, 158–164.

[22] S. Özcan, Hermite-Hadamard type inequalities for multiplicatively h-preinvex functions. Turk. J. Anal. Number Theory. 9 (2021), no.3, 65-70.

[23] J. E. Pečarić, F. Proschan and Y. L. Tong, Convex functions, partial orderings, and statistical applications. Mathematics in Science and Engineering, 187. Academic Press, Inc., Boston, MA, 1992.

[24] V. Volterra and B. Hostinsky, Operations Infinitesimales Lineaires, Gauthier-Villars, Paris, 1938.

[25] Z. Zhang, M.A. Ali, H. Budak, M.Z. Sarikaya, On Hermite-Hadamard type inequalities for interval-valued multiplicative integrals, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. 69 (2020), 1428–1448.