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Guaranteed controls for the control problem of a linear set-valued system

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Abstract. The article considers the control linear differential equation with Hukuhara derivative and the problem of moving a set-valued object to a target set, that is, when at some point in time the cross section of a set-valued solution of the system is contained in the target set. The solvability conditions for this problem are obtained, as well as the time and controls that guarantee the fulfillment of the termination process condition. It is shown that in some cases the given time and controls will be optimal. The results of the article are illustrated by model examples.

1. Introduction

Starting from the 70s of the XXth century, a new approach to the problems of dynamic systems control was formed, based on the analysis of the bunch's trajectory, and not of the individual trajectories - the control problems in the conditions of uncertainty are considered. The cross section of this beam at any moment of time is a certain set and it is necessary to describe the evolution of this set, as well as to determine what moment and why we will consider to be optimal.

The main directions for describing the behavior of such objects are the following:

1) control systems with an inaccurate initial condition:
   a) the behavior of the object is described by a ordinary controlled system (controlled differential equation, controlled integral equation, controlled discrete system, etc.), the initial state belongs to a certain set (see for example [2, 5–7, 21, 22, 29, 30] and the references therein);
   b) the behavior of the system is described by a set-valued controlled system (controlled set-valued differential equations, controlled set-valued integral equations, controlled set-valued discrete systems, etc.) (see for example [1, 11, 14, 19, 33–36, 43, 44] and the references therein);
2) control systems with interference on the right-hand side and with an inaccurate initial condition. In this case, the behavior of the system is described by the ordinary controlled differential (integral, integro-differential, etc.) inclusion, in which the initial state belongs to a certain set (see for example [5, 8, 9, 31, 37, 39, 41, 43] and the references therein);
3) general systems. Such systems are described by controlled quasidifferential equations [38, 43].

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Subsequently, all these approaches began to be applied to fuzzy controlled systems (see for example [13, 17, 26–28, 33, 42, 45, 46] and the references therein).

This article studies one control problem when the behavior of an object is described by a set-valued control system. This system refers to systems of type 1b). When describing the behavior of such systems, differential equations with the Hukuhara derivative are most often used. Such equations were first considered in the work [3]. Further, many authors studied the properties of solutions of set-valued differential, integral and integro-differential equations, set-valued impulse and discrete systems, as well as set-valued differential inclusions (see for example [15, 16, 23, 25, 32, 40, 42, 43] and the references therein). Subsequently, the obtained results and research methods were widely used in the theory of fuzzy systems (see for example [18, 24, 42] and the references therein).

The article considers the problem of moving an object into a target set when the behavior of the object is described by a linear controllable differential equation with the Hukuhara derivative. The conditions for the existence of a solution of such problem, as well as time and control, that guarantee the completion of the process, are obtained. The results are illustrated with model examples.

2. Preliminaries

In this section we recall some results from the literature that are of interest for our work.

Let $\mathbb{R}$ be the set of real numbers and $\mathbb{R}^n$ be the n-dimensional Euclidean space ($n \geq 2$). Denote by $conv(\mathbb{R}^n)$ the set of nonempty compact and convex subsets of $\mathbb{R}^n$ with the Hausdorff metric $h(X, Y) = \min\{r \geq 0 : X \subset Y + B_r(0),\ Y \subset X + B_r(0)\}$, where $X, Y \in conv(\mathbb{R}^n)$, $B_r(c) = \{x \in \mathbb{R}^n : \|x - c\| \leq r\}$ is the closed ball with radius $r > 0$ centered at the point c ($\|\cdot\|$ denotes the Euclidean norm).

In addition to the usual set-theoretic operations, we introduce two operations in the space $conv(\mathbb{R}^n)$: the sum of the sets and the product of the scalar by the set

$$X + Y = \{x + y : x \in X,\ y \in Y\} \qquad \text{and} \qquad \lambda X = \{\lambda x : x \in X,\ \lambda \in \mathbb{R}\}.$$

And also we will add the operation of the product of the matrix on the set:

$$AX = \{Ax : x \in X,\ A \in \mathbb{R}^{n\times n}\}.$$

Further we give the following theorem necessary below.

Theorem 2.1. *[10] For any real $(n \times n)$-matrix A there are two orthogonal $(n \times n)$-matrix B and C such that $B^T AC = M$, where M is the diagonal matrix. We can also choose matrices B and C such that the diagonal elements of the matrix M have the form*

$$\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_r > \sigma_{r+1} = ... = \sigma_n = 0,$$

where r is the rank of the matrix A. That is, if A is a nondegenerate matrix, then

$$\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_n > 0.$$

Columns $b_1, ..., b_n$ of matrix B are called the left singular vectors, columns $c_1, ..., c_n$ of matrix C are called the right singular vectors, and the numbers $\sigma_1, \sigma_2, ..., \sigma_n$ are called the singular numbers of the matrix A. Therefore, this matrix A can be represented as $A = BMC^T$. This decomposition is called singular decomposition.

By [10], the set $Y = \{Ax : x \in B_1(0), A \in \mathbb{R}^{n\times n}\}$ is r-dimensional ellipsoid, in which the semi-axis lengths are equal to the corresponding singular numbers of the matrix A, where $r = rank(A)$.

Hence the following statement is true, if $rank(A) = n$, then

$$B_{\sigma_n}(0) \subset Y \subset B_{\sigma_1}(0), \tag{1}$$

where $B_{\sigma_n}(\mathbf{0})$ is the inscribed ball in set Y, $B_{\sigma_1}(\mathbf{0})$ is the smallest circumscribed ball of the set Y.

The following basic properties are valid [23, 32, 42, 43]:

1. $(conv(\mathbb{R}^n), h)$ is a complete metric space,
2. $h(X+Z, Y+Z) = h(X,Y)$,
3. $h(\lambda X, \lambda Y) = |\lambda| h(X,Y)$ for all $X, Y, Z \in conv(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}$.

However, $conv(\mathbb{R}^n)$ is not a linear space since it does not contain inverse elements for the addition, and therefore difference is not well defined, i.e. if $X \in conv(\mathbb{R}^n)$ and $X \neq \{x\}$, then $X + (-1)X \neq \{0\}$. As a consequence, alternative formulations for difference have been suggested. One of these alternatives is the Hukuhara difference [12].

Let $X, Y \in conv(\mathbb{R}^n)$. A set $Z \in conv(\mathbb{R}^n)$ such that $X = Y + Z$ is called a Hukuhara difference (H-difference) of the sets X and Y and is denoted by $X \overset{H}{-} Y$ In this case $X \overset{H}{-} X = \{0\}$ and also $(X+Y) \overset{H}{-} Y = X$ for any $X, Y \in conv(\mathbb{R}^n)$.

Simultaneously, M. Hukuhara introduced the concept of Hukuhara differentiability for set-valued mappings by using the Hukuhara difference.

Definition 2.2. *[12] Let $X(\cdot) : [0,T] \to conv(\mathbb{R}^n)$ is set-valued mapping. We say that $X(\cdot)$ has a Hukuhara derivative $D_H X(t) \in conv(\mathbb{R}^n)$ at $t \in (0,T)$, if for all $\Delta > 0$ that are sufficiently closed to 0, the Hukuhara differences and the limits exist*

$$\lim_{\Delta \to 0_+} \Delta^{-1}(X(t+\Delta) \overset{H}{-} X(t)) = \lim_{\Delta \to 0_+} \Delta^{-1}(X(t) \overset{H}{-} X(t-\Delta)) = D_H X(t).$$

Theorem 2.3. *[12] If the mapping $X : [0,T] \to conv(\mathbb{R}^n)$ is Hukuhara differentiable on $[0,T]$, then $X(t) = X(0) + \int_0^t D_H X(s) ds$, where the integral is understood in the sense of M. Hukuhara[12].*

Corollary 2.4. *If the set-valued mapping $X(\cdot)$ is Hukuhara differentiable on $[0,T]$, then $diam(X(\cdot))$ is a non-decreasing function on $[0,T]$.*

Corollary 2.5. *If the function $diam(X(\cdot))$ is a decreasing function on $[0,T]$, then the set-valued mapping $X(\cdot)$ is not Hukuhara differentiable on $[0,T]$.*

The properties of the Hukuhara derivative are discussed in detail in [12, 18, 23, 32, 42, 43].

3. Linear set-valued differential equations

Now, consider the controlled set-valued system

$$\begin{array}{ll} D_H X_1(t) = v(t) A X_1(t), & X_1(0) = B_r(0), \\ \dot{x}_2(t) = v(t)\|A\| x_2(t) + u(t), & x_2(0) = 0, \\ X(t,v,u) = X_1(t) + x_2(t), & \end{array} \tag{2}$$

where $X_1 : \mathbb{R}_+ \to conv(\mathbb{R}^n)$ is the set-valued mapping; $x_2 : \mathbb{R}_+ \to \mathbb{R}^n$ is the vector-valued function; $A \in \mathbb{R}^{n\times n}$ is the constant non-degenerate matrix $(n \times n)$; $\|A\|$ is the spectral norm of matrix A; $v(\cdot)$, $u(\cdot)$ are admissible controls, that is, functions that are measurable by value such that $v(t) \in [0,1]$ and $|u_i(t)| \leq 1$, $i = \overline{1,n}$, for all $t \in \mathbb{R}_+$.

Suppose that the set $X_K = B_R(c)$ (**the target set**).

Consider the following **control problem**: *find time $T^* > 0$ and admissible controls $v^*(\cdot)$, $u^*(\cdot)$, such that set-valued solution of the system (2) satisfies the condition*

$$X(T^*, v^*, u^*) \subset X_K. \tag{3}$$

Remark 3.1. *Note that in article [14] we considered the case when $A = I$ and $X(T^*, v^*, u^*) \equiv X_K$, where I is the identity matrix.*

Obviously, the first equation in system (2) determines the shape and size of the set at each moment of time t, which is a section of the set-valued solution of the system (2), and the second equation in system (2) determines the coordinates of the center of this set at each moment of time t.

As it is known from [19, 32], for any admissible controls $v(\cdot)$ the solution $X_1(\cdot)$ of system (2) exists for all $t \geq 0$ and

$$X_1(t) = \exp\left(\int_0^t v(s)dsA\right)B_1(0).$$

Obviously, the matrix A defines the shape of the ellipse (determines the ratio of the lengths of its main diagonals).

From Theorem 2.1 it follows that the matrix A has singular values $\sigma_1 \geq ... \geq \sigma_n > 0$.

If $\sigma_1 = ... = \sigma_n$, then the cross section of the solution $X_1(t)$ at each moment of time t will be a n-dimensional boll at every moment $t \geq 0$ and $X_1(t) = e^{\sigma_1 \int_0^t v(s)ds} B_1(0)$ [20].

If the matrix A has at least two different singular values, then the cross section of the solution $X_1(t)$ at each moment of time t will be a n-dimensional ellipsoid rotated by some angle. Moreover, the lengths of its main diagonals will be proportional to the singular values of the matrix A.

Also the control $v(\cdot)$ determines the growth rate of this ellipsoid and the function $diam(X_1(t))$ is a non-decreasing function.

Obviously, the solution $x_2(\cdot)$ can be written in the following form:

$$x_2(t) = e^{\int_0^t \|A\| v(s)ds} \int_0^t e^{-\int_0^s \|A\| v(\tau)d\tau} u(s)ds.$$

Hence,

$$X(t, v, u) = e^{\int_0^t v(s)ds\, A} B_1(0) + e^{\int_0^t \|A\| v(s)ds} \int_0^t e^{-\int_0^s \|A\| v(\tau)d\tau} u(s)ds.$$

Obviously, to construct a solution $X(t, v, u)$ to system (2), it is necessary to know all the singular values of the matrix A. However, finding all the singular values causes great difficulties. Therefore, simultaneously with system (2), we consider the following simpler control system:

$$D_H Y(t) = v(t)A_1 Y(t) + u(t), \quad Y(0) = B_r(0), \tag{4}$$

where A_1 is a diagonal matrix, that has the following form

$$A_1 = \begin{pmatrix} \sigma_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \sigma_1 \end{pmatrix};$$

σ_1 is the first singular number of matrix A.

As it is known from [14, 19, 32], the solution of system (4) can be written in the following form:

$$Y(t, v, u) = e^{\int_0^t \sigma_1 v(s)ds} B_r(0) + e^{\int_0^t \sigma_1 v(s)ds} \int_0^t e^{-\int_0^s \sigma_1 v(\tau)d\tau} u(s)ds. \tag{5}$$

It is clear that formula (5) can be rewritten as the sum of the set-valued mapping $F(t, v)$ and vector-function $g(t, v, u)$, i.e.

$$Y(t, v, u) = F(t, v) + g(t, v, u),$$

where $F(t,v) = e^{\int_0^t \sigma_1 v(s)ds} B_r(0)$, $g(t,v,u) = e^{\int_0^t \sigma_1 v(s)ds} \int_0^t e^{-\int_0^s \sigma_1 v(\tau)d\tau} u(s)ds.$

Therefore, the initial set $B_r(0)$ determines the "shape" of the section of the set-valued solution $Y(t,v,u)$ at time t, and admissible control $v(\cdot)$ defines the change in size. We note that the set-valued mapping $F(t,v)$ has the following properties:

1) $F(0,v) = B_r(0)$;

2) for any $t > 0$ the set $F(t,v)$ will be homothetic to the initial set $B_r(0)$ with constant $k(t) = e^{\int_0^t \sigma_1 v(s)ds} \geq 1$, i.e., will be shaped like a ball.

3) if $v(t) \equiv 0$ for all $t \geq 0$, then $F(t,v) = B_r(0)$.

It is also obvious that the vector function $g(t,v,u)$, that depends on the admissible controls $v(\cdot)$ and $u(\cdot)$, specifying additional shifts of the section of the set-valued mapping $Y(t,v,u)$ at time $t > 0$ relative to the initial set $B_r(0)$.

Remark 3.2. *Because* $\|A\| = \sigma_1$, $e^{\int_0^t v(s)ds\, A} B_1(0) \subseteq e^{\sigma_1 \int_0^t v(s)ds} B_1(0)$ *and*

$$e^{\int_0^t \|A\| v(s)ds} \int_0^t e^{-\int_0^s \|A\| v(\tau)d\tau} u(s)ds = e^{\sigma_1 \int_0^t v(s)ds} \int_0^t e^{-\sigma_1 \int_0^s v(\tau)d\tau} u(s)ds$$

then we have: for all $t \geq 0$ and any admissible controls $v(\cdot)$, $u(\cdot)$ condition $X(t,v,u) \subseteq Y(t,v,u)$ is hold. Also the section of the set $Y(t,v,u)$ will be the sphere described around the ellipsoid $X(t,v,u)$ for all $t > 0$.

Example 3.3. *Let the behavior of the system be described by system (2), where $n = 2$, $r = 1$, $v \equiv 1$, $u = (\frac{1}{3}, 1)^T$, $A = \begin{pmatrix} \frac{6}{5} & \frac{2}{3} \\ \frac{1}{2} & \frac{6}{7} \end{pmatrix}$. The singular numbers of the matrix A are $\sigma_1 = 1.63$, $\sigma_2 = 0.42$ and the solution $X_1(t) =$*

$$\left\{ (x_1, x_2)^T \in \mathbb{R}^2 \,:\, \frac{(x_1 \cos(\psi) + x_2 \sin(\psi))^2}{e^{2\sigma_1 t}} + \frac{(-x_1 \sin(\psi) + x_2 \cos(\psi))^2}{e^{2\sigma_2 t}} \leq 1 \right\},$$

where ψ is angle of rotation of the ellipse (see Figure 1). That is, the matrix A_1 will have the form $A_1 = \begin{pmatrix} 1.63 & 0 \\ 0 & 1.63 \end{pmatrix}$. Then the solutions $X(t,v,u)$ and $Y(t,v,u)$ will have the following form (see Figure 2).

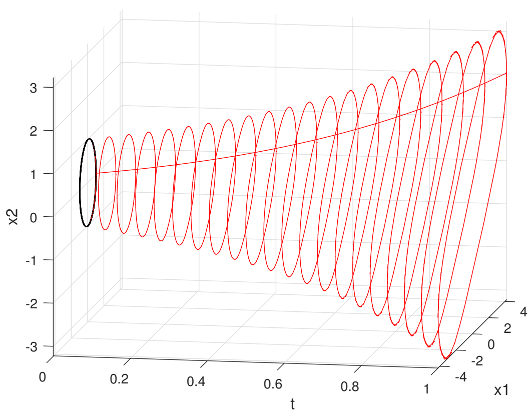

Figure 1. The solution $X_1(t)$, $t \in [0,1]$.

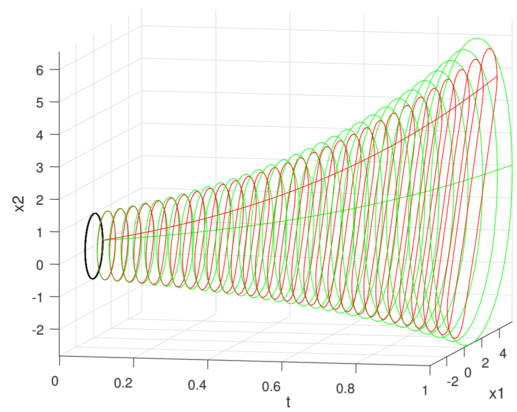

Figure 2. The solution $X(t,v,u)$ - red, the solution $Y(t,v,u)$ - green, $t \in [0,1]$.

It is obvious that the sets $X_0 \equiv B_r(0)$ and $X_K \equiv B_R(c)$ are homothetic, i.e. $X_K = aX_0 + c$, where $a = R/r$.

Remark 3.4. *If $0 < a < 1$, then $diam(X_0) > diam(X_K)$ and this control problem does not make sense.*

Remark 3.5. *If $a \geq 1$, $||c|| \leq R - r$, then $X_0 \subseteq X_K$. Hence, the initial set X_0 a subset of the target set X_K, i.e. this control problem does not make sense.*

Next, we consider two possible cases.:
1. $a = 1$, $c \neq 0$;
2. $a > 1$, $||c|| > R - r$.

Case 1. $a = 1$, $c \neq 0$, i.e. $X_K = B_r(c)$.

Hence, $X_K = X_0 + c = B_r(0) + c$. Taking into account property 3) of the solution of the system (4), we get $v^*(t) \equiv 0$. Then systems (2) and (4) have the same form:

$$\begin{array}{ll} D_H X_1(t) = \{0\}, & X_1(0) = B_r(0), \\ \dot{x}_2(t) = u(t), & x_2(0) = 0, \\ X(t,0,u) = X_1(t) + x_2(t), & \end{array}$$

and the solution can be written in the following form: $X(t,0,u) = B_r(0) + \int\limits_0^t u(s)ds$.

Therefore, it is easy to find some minimum T^* and admissible control $u^*(\cdot)$ such that

$$X(T^*,0,u^*) = B_r(0) + \int\limits_0^{T^*} u^*(s)ds = B_r(0) + c,$$

i.e.

$$\int\limits_0^{T^*} u_i^*(s)ds = c_i,\ i = \overline{1,n}. \tag{6}$$

Hence $T^* = \max\limits_{i=\overline{1,n}} |c_i|$. Also optimal control $u^*(\cdot) = (u_1^*(\cdot), ..., u_n^*(\cdot))^T$ such that $|u_i^*(t)| \leq 1$, $i = \overline{1,n}$, and there exists at least one $j \in \{1, ..., n\}$, that $|c_j| = \max\limits_{i=\overline{1,n}} |c_i|$ and $u_j^*(t) \equiv \begin{cases} 1, & c_j > 0 \\ -1, & c_j < 0 \end{cases}$ for all $t \in [0, T^*]$ and the condition (6) is hold.

Obviously, in the class of constant functions, such an optimal control is $u^*(\cdot) = (u_1^*, ..., u_n^*)^T$, such that $u_i^* \equiv \frac{c_i}{c_{max}}$, $i = \overline{1,n}$, where $c_{max} = \max\limits_{i=\overline{1,n}} |c_i|$.

Example 3.6. *Consider the system from Example 3.3 and $X_K = B_1(c)$, $c = (-1.2, 0.9)^T$.*

It is obvious that the set X_K is homothetic to the initial set X_0, i.e. $X_K = aX_0 + c$ and $a = 1$, $c = (-1.2, 0.9)^T$. Then $T^* = \max\{1.2, 0.9\} = 1.2$, $v^* \equiv 0$, $u^* = (-1, 0.75)^T$ and the solution has the form (see Figure 3).

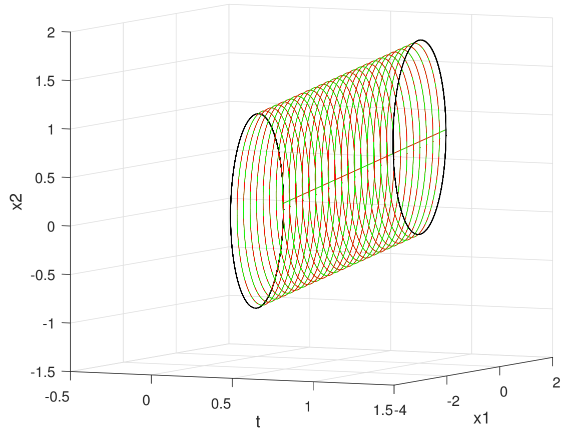

Figure 3. The solution $X(t, v^*, u^*)$, $t \in [0, 1.2]$.

Case 2. $a > 1$, $\|c\| > R - r$. Then $X_K = B_{ar}(c)$, i.e. $X_K = aX_0 + c$.

We take $v(t) \equiv 1$, since in this case this control will increase the diameter of the solution section of the system (2) as quickly as possible and shift the center of this section as quickly as possible in the space $\mathbb{R}^n$.

Then systems (2) and (4) have the following form

$$\begin{array}{ll} D_H X_1(t) = AX_1(t), & X_1(0) = B_r(0), \\ \dot{x}_2(t) = \|A\| x_2(t) + u(t), & x_2(t) = 0, \\ X(t, 1, u) = X_1(t) + x_2(t), & \end{array} \tag{7}$$

$$D_H Y(t) = A_1 Y(t) + u(t), \quad Y(0) = B_r(0), \tag{8}$$

and the solution of system (8) can be written in the following form

$$Y(t, 1, u) = e^{\sigma_1 t} B_r(0) + e^{\sigma_1 t} \int_0^t e^{-\sigma_1 s} u(s) ds \tag{9}$$

or

$$Y(t, 1, u) = B_{re^{\sigma_1 t}}(0) + e^{\sigma_1 t} \int_0^t e^{-\sigma_1 s} u(s) ds. \tag{10}$$

Obviously, at time $T_1 = \frac{\ln(a)}{\sigma_1}$ the diameter of the section of the solution $Y(T_1, 1, u)$ will be equal to the length of the diameter of the target set X_K.

Also, by (10) we have, that the solution center of system (8) will move in the space $\mathbb{R}^n$ by trajectory

$$y(t, 1, u) = e^{\sigma_1 t} \int_0^t e^{-\sigma_1 s} u(s) ds. \tag{11}$$

Then when choosing the optimal control $u^*(\cdot)$ at some point in time T_2 the condition

$$e^{\sigma_1 T_2} \int_0^{T_2} e^{-\sigma_1 s} u^*(s) ds = c \tag{12}$$

will be satisfied.

Therefore, there are three possible cases:

a) $T_1 = T_2$; b) $T_1 > T_2$; c) $T_1 < T_2$.

Next, we look at all of these cases sequentially.

a) $T_1 = T_2 = T^* = \frac{\ln(a)}{\sigma_1}$. By (9) and (12) we have

$$re^{\sigma_1 T_2} = re^{\sigma_1 T_1} = re^{\sigma_1 T^*} = ar, \qquad a\int_0^{T^*} e^{-\sigma_1 s}u^*(s)ds = c,$$

i.e. there is control $u^*(\cdot) = (u_1^*(\cdot), ..., u_n^*(\cdot))^T$ such that $\int_0^{T^*} e^{-\sigma_1 s}u_i^*(s)ds = \frac{c_i}{a}$, $|u_i^*(t)| \leq 1$, for all $t \in [0, T^*]$ and $i = \overline{1, n}$.

Since $u^*(\cdot)$ is optimal control, then there is at least one $j \in \{1, ..., n\}$ such that $|u_j^*(t)| \equiv 1$ and $|c_j| = \max_{i=\overline{1,n}} |c_i|$.

Therefore, $a\int_0^{T^*} e^{-\sigma_1 s}ds = |c_j|$, i.e. $\frac{a-1}{\sigma_1} = |c_j| = \max_{i=\overline{1,n}} |c_i|$.

Also note that the optimal control $u^*(\cdot) = (u_1^*(\cdot), ..., u_n^*(\cdot))^T$ from the class of constant functions will be $u_i^*(t) = \frac{c_i}{c_{max}}$ for all $t \in [0, T^*]$ and $i = \overline{1, n}$.

Example 3.7. *Consider the system of Example 3.3 and* $X_K = B_R(c)$, $R = 2.96$, $c = (-1.2, 0.9)^T$.

It is obvious that the set X_K *homothetic to the initial set* X_0, *i.e.* $X_K = aX_0 + c$ *and* $a = 2.96$, $c = (-1.2, 0.9)^T$. *Then* $\frac{a-1}{\sigma_1} = (2.96 - 1)/1.63 = 1.2$ *and* $\max_{i=\overline{1,2}} |c_i| = 1.2$, *i.e.* $\frac{a-1}{\sigma_1} = \max_{i=\overline{1,2}} |c_i|$. *Here,* $T^* = 0.665$, $v^* \equiv 1$, $u^* = (-1, 0.75)^T$. *Then the solution will be as follows (see Figure 4).*

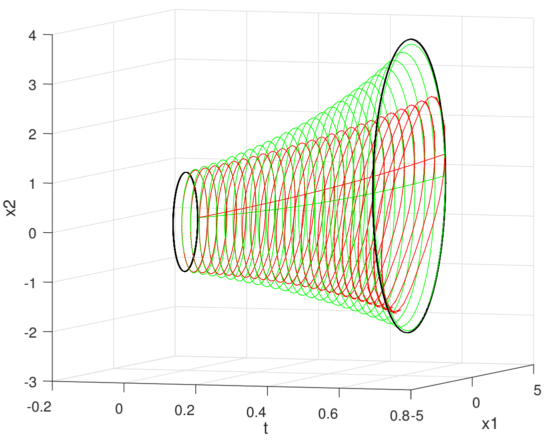

Figure 4. The solution $X(t, v^*, u^*)$ - red, the solution $Y(t, v^*, u^*)$ - green, $t \in [0, 0.665]$.

b) $T_1 > T_2$**.** By (9) and (12) we have $ce^{\sigma_1 T_2} < ce^{\sigma_1 T_1} = ar = R$, $e^{\sigma_1 T_2}\int_0^{T_2} e^{-\sigma_1 s}u^*(s)ds = c$, i.e. $\frac{a-1}{\sigma_1} > \max_{i=\overline{1,n}} |c_i|$.

Therefore, when we choose the optimal control $u^*(\cdot)$, then the center of the initial set X_0 is transferred, according to the equation (11), to the center of the target set X_K in the time $T_2 < T_1$. Thus, the section of the solution $Y(t, v, u*)$ of the system (8) at time T_2 will have a radius less than the radius of the target set X_K, i.e. $Y(T_2, v, u^*) \subset X_K$.

So in this case we take the time $T^* = T_2 = \frac{ln(\sigma_1 \max\limits_{i=\overline{1,n}} |c_i|+1)}{\sigma_1}$ and controls $v^*(t) \equiv 1$ and $u^*(\cdot) = (u_1^*(\cdot), ..., u_n^*(\cdot))^T$ such that $|u_i^*(t)| \leq 1$ and $e^{\sigma_1 T^*} \int\limits_0^{T^*} e^{-\sigma_1 s} u_i^*(s)ds = c_i$ for all $t \in [0, T^*]$ and $i = \overline{1, n}$.

For example, such control in the class of constant functions will be $u^*(\cdot) = (u_1^*, ..., u_n^*)^T$ such that $u_i^* = \frac{c_i}{\max\limits_{i=\overline{1,n}} |c_i|}$, $i = \overline{1, n}$.

Example 3.8. *Consider the system of Example 3.3 and* $X_0 = B_r(0)$, $X_K = B_R(c)$, $r = \frac{1}{3}$, $R = \frac{2}{3}$, $c = (0.5, -0.2)^T$.

It is obvious that the set X_K *is homothetic to the initial set* X_0, *i.e.* $X_K = aX_0 + c$ *and* $a = 2$, $c = (0.5, -0.2)^T$. *Then* $\frac{a-1}{\sigma_1} = (2-1)/1.63 = 0.61$ *and* $\max\limits_{i=\overline{1,2}} |c_i| = 0.5$, *i.e.* $\frac{a-1}{\sigma_1} > \max\limits_{i=\overline{1,2}} |c_i|$. *So,* $T^* = 0.365$, $v^* \equiv 1$, $u^* = (1, -0.4)^T$. *Then the solutions will be as follows (see Figure 5).*

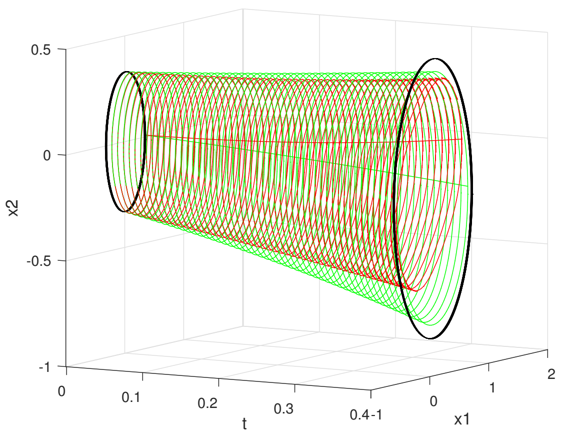

Figure 5. The solution $X(t, v^*, u^*)$ - red, the solution $Y(t, v^*, u^*)$ - green, $t \in [0, 0.365]$.

c) $T_1 < T_2$. By (9) and (12) we have $re^{\sigma_1 T_2} > re^{\sigma_1 T_1} = ar = R$, and $e^{\sigma_1 T_2} \int\limits_0^{T_2} e^{-\sigma_1 s} u^*(s)ds = c$, i.e. $\frac{a-1}{\sigma_1} < \max\limits_{i=\overline{1,n}} |c_i|$.

Therefore, when we choose the optimal control $u^*(\cdot)$, then the center of the initial set X_0 is transferred, according to the equation (11), to the center of the target set X_K in the time $T_2 > T_1$. Thus, the section of the solution $Y(t, v^*, u^*)$ of system (8) at time T_2 will have a radius greater than the radius of the target set, i.e. $X_K \subset Y(T_2, v^*, u^*)$.

Therefore, in this case we cannot select $v^*(t) \equiv 1$ for all $t \in [0, T_2]$. Therefore, we must choose $v^*(\cdot)$ such that $0 \leq v^*(t) \leq 1$ for all $t \geq 0$ and $v^*(t) \not\equiv 1$. We also note that in this case the time T of transfer of the center of the section of the solution of system (10) to the center of the target set X_K will be more than T_2.

We will write the following system

$$\begin{cases} e^{\int\limits_0^T \sigma_1 v^*(s)ds} = a, \\ re^{\int\limits_0^T \sigma_1 v^*(s)ds} + e^{\int\limits_0^T \sigma_1 v^*(s)ds} \int\limits_0^T e^{-\int\limits_0^t \sigma_1 v^*(s)ds} u^*(t)dt = c. \end{cases}$$

Hence we have

$$e^{\int\limits_0^T \sigma_1 v^*(s)ds} = a, \qquad \int\limits_0^T e^{-\int\limits_0^t \sigma_1 v^*(s)ds} u^*(t)dt = \frac{c}{a}.$$

Considering that for some $j \in \{1, ..., n\}$ we have $|u_j^*(t)| \equiv 1$ and $|c_j| = \max\limits_{i=\overline{1,n}} |c_i|$, then

$$e^{\int_0^{T^*} \sigma_1 v^*(s)ds} = a, \qquad \int_0^{T^*} e^{-\int_0^t \sigma_1 v^*(s)ds} dt = \frac{\max\limits_{i=\overline{1,n}} |c_i|}{a}.$$

If $v^*(t) = v^* = const$, then $\quad e^{T^*\sigma_1 v^*} = a, \qquad \int_0^{T^*} e^{-t\sigma_1 v^*} dt = \frac{\max_{i=\overline{1,n}} |c_i|}{a}.$

As a result

$$T^* = \frac{\ln(a) \max\limits_{i=\overline{1,n}} |c_i|}{a-1} \quad \text{and} \quad v^* = \frac{a-1}{\sigma_1 \max\limits_{i=\overline{1,n}} |c_i|}. \tag{13}$$

Then the optimal control $u^*(\cdot) = (u_1^*(\cdot), ..., u_n^*(\cdot))^T$ must satisfy the condition

$$|u_i^*(t)| \le 1, \qquad \int_0^{T^*} e^{-t\sigma_1 v^*} u_i^*(t)dt = \frac{c_i}{a}, \quad i = \overline{1, n}. \tag{14}$$

For example, such control in the class of constant functions will be $u^*(\cdot) = (u_1^*, ..., u_n^*)^T$, such that $u_i^* = \frac{c_i}{c_{max}}, \quad i = \overline{1, n}$.

Example 3.9. *Consider the system of Example 3.3 and* $X_K = B_R(c)$, $R = 2$, $c = (-1.2, 0.9)^T$.

It is obvious that the set X_K *is homothetic to the initial set* X_0, *i.e.* $X_K = aX_0 + c$ *and* $a = 2$, $c = (-1.2, 0.9)^T$. *Then* $\frac{a-1}{\sigma_1} = (2-1)/1.63 = 0.61$ *and* $\max\limits_{i=\overline{1,2}} |c_i| = 1.2$, *i.e.* $\frac{a-1}{\sigma_1} < \max\limits_{i=\overline{1,2}} |c_i|$. *So,* $T^* = \frac{\ln(a) c_{max}}{a-1} = \frac{1.2\ln(2)}{2-1} = 0.83$, $v^* = \frac{a-1}{\sigma_1 c_{max}} = \frac{2-1}{1.63 \cdot 1.2} = 0.51$, $u^* = (-1, 0.75)^T$ *(see Figure 8). Its also easy to get that* $T_1 = \frac{\ln(a)}{\sigma_1} = 0.43$ *and* $T_2 = \frac{\ln(\sigma_1 c_{max}+1)}{\sigma_1} = 0.665$, *i.e.* $T_1 < T_2 < T^*$. *We also note that the section of the solution* $Y(t, 1, u^*)$ *of system (8) at time* T_1 *is equal in size to the target set* S_K, *but their centers do not coincide. The center of the initial set* X_0 *did not manage to move to the center of the target set* X_K *(see Figure 6). We also note that at time* T_2, *the center of the initial set* X_0 *has moved to the center of the target set* X_K, *but the section of the solution* $Y(t, 1, u^*)$ *to system (8) at moment* T_2 *is larger than the target set* X_K, *i.e.* $X_K \subset Y(T_2, 1, u^*)$ *(see Figure 7).*

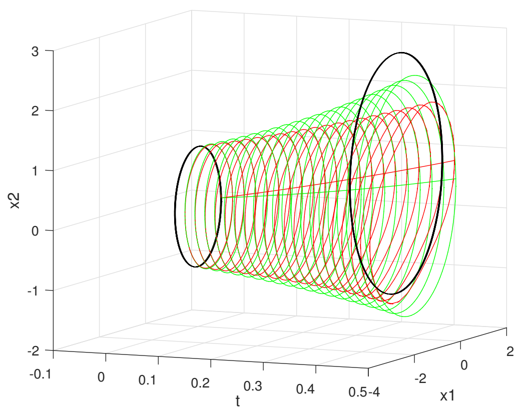

Figure 6. $X(t, 1, u^*)$ - red, $Y(t, 1, u^*)$ - green, $t \in [0, 0.43]$.

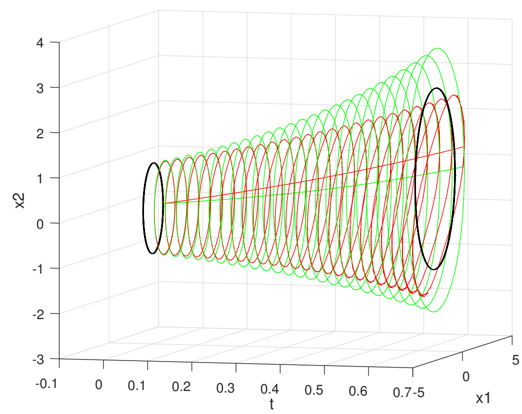

Figure 7. $X(t, 1, u^*)$ - red, $Y(t, 1, u^*)$ - green, $t \in [0, 0.665]$.

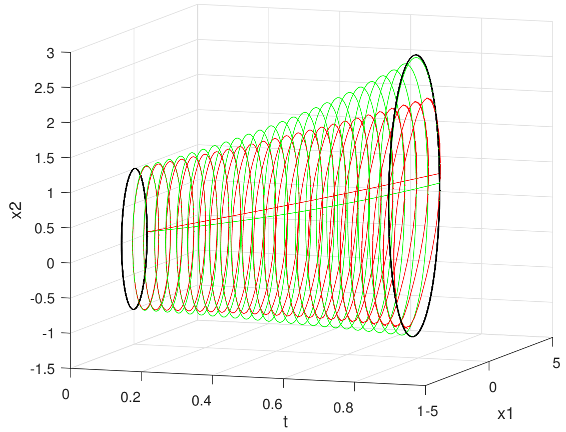

Figure 8. The solution $X(t, v^*, u^*)$ - red, the solution $Y(t, v^*, u^*)$ - green, $t \in [0, 0.83]$

Taking into account all the previous arguments, we state the following theorem.

Theorem 3.10. *If the sets $X_0 = B_r(0)$ and $X_K = B_R(c)$ satisfy the condition $\|c\| \geq R - r$, then the control problem (2),(3) has a solution, and time T^* and controls v^*, $u^* = (u_1^*, ..., u_n^*)^T$ from the class of constant functions will be*

$$T^* = \begin{cases} c_{max}, & a = 1, \\ \frac{ln(a)}{\sigma_1}, & a > 1 \text{ and } \frac{a-1}{\sigma_1} = c_{max}, \\ \frac{ln(\sigma_1 c_{max}+1)}{\sigma_1}, & a > 1 \text{ and } \frac{a-1}{\sigma_1} > c_{max}, \\ \frac{\ln(a) c_{max}}{a-1}, & a > 1 \text{ and } \frac{a-1}{\sigma_1} < c_{max}, \end{cases} \quad v^* = \begin{cases} 0, & a = 1, \\ 1, & a > 1 \text{ and } \frac{a-1}{\sigma_1} \geq c_{max}, \\ \frac{a-1}{\sigma_1 c_{max}}, & a > 1 \text{ and } \frac{a-1}{\sigma_1} < c_{max}, \end{cases}$$

$u_i^* = \frac{c_i}{c_{max}}$, $i = \overline{1, n}$, *where* $a = \frac{R}{r}$, $c_{max} = \max\limits_{i=\overline{1,n}} |c_i|$.

Remark 3.11. *Since in cases 1), 2a) and 2c) the intersection of the solution of system (4) coincides with the target set X_K, and the problem is solved as a time-optimal problem for system (4), then for the control problem for system (2) in cases 1), 2a) and 2c) the time will be minimal, and controls will be optimal. In case 2b), the time may not be optimal for system (2).*

Example 3.12. *Let the behavior of the system be described by equation (2), where $X_0 = B_{\frac{1}{3}}(0)$, $X_K = B_{\frac{2}{3}}(c)$, $c = (-0.5, 0.2)^T$, $A = \begin{pmatrix} 1 & \frac{1}{6} \\ \frac{1}{2} & \frac{8}{7} \end{pmatrix}$.*

It is easy to verify that this is case 2b), i.e. $\frac{a-1}{\sigma_1} = \frac{2-1}{1.43} = 0.699 > c_{max} = 0.5$. Then $v^* \equiv 1$, $u^* = (-1, 0.4)^T$. But at the moment $T^* = \frac{\ln(\sigma_1 c_{max}+1)}{\sigma_1} = \frac{\ln(1.43 \times 0.5+1)}{1.43} = 0.275$ there will be the touch of the boundary of the solution section of system (4) with the boundary of the target set X_K, but not of the ellipsoid, that is the section of the solution of system (2) (see Figure 9). The touch of the ellipsoid, that is the section of the solution of system (2) with the boundary of the target set X_K, will occur earlier at time $T = 0.248$ (see Figure 10).

Remark 3.13. *If the initial set X_0 is an arbitrary nonempty convex compact set, then we can describe a ball around it $B_r(0)$ and for system (4) obtain a solution (T^*, v^*, u^*), if the conditions of Theorem 2 are satisfied. However, the obtained value T^* will only guarantee the fulfillment of the condition (3), since it may not be minimal.*

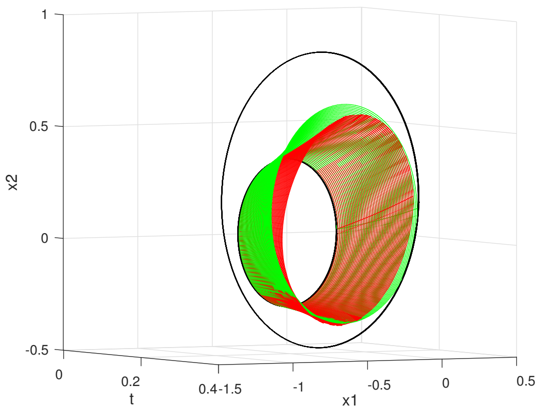

Figure 9. $X(t, v^*, u^*)$ - red, $Y(t, v^*, u^*)$ - green, $t \in [0, 0.275]$

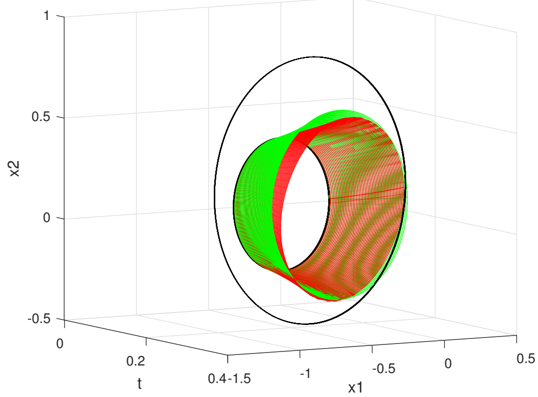

Figure 10. $X(t, v^*, u^*)$ - red, $Y(t, v^*, u^*)$ - green, $t \in [0, 0.248]$

Example 3.14. *Let the object's behavior be described by the system*

$$\begin{array}{lll} D_H X_1(t) = v(t) A X_1(t), & X_1(0) = X_0, & \\ \dot{x}_2(t) = v(t)\|A\| x_2(t) + u(t), & x_2(0) = 0, & X(T) \subseteq X_K, \\ X(t, v, u) = X_1(t) + x_2(t), & & \end{array} \tag{15}$$

where $A = \begin{pmatrix} \frac{6}{5} & \frac{2}{3} \\ \frac{1}{2} & \frac{4}{7} \end{pmatrix}$, $X_0 = \left\{ x \in \mathbb{R}^2 \, : \, \frac{16x_1^2}{9} + x_2^2 \leq 1 \right\}$, $X_K = B_R(c)$, $R = \frac{5}{4}$, $c = (\frac{4}{3}, \frac{2}{3})^T$.

Consider the auxiliary system

$$\begin{array}{lll} D_H Y_1(t) = v(t) A Y_1(t), & Y_1(0) = Y_0, & \\ \dot{x}_2(t) = v(t)\|A\| x_2(t) + u(t), & x_2(0) = 0, & Y(T) \subseteq X_K, \\ Y(t, v, u) = Y_1(t) + x_2(t), & & \end{array} \tag{16}$$

where $Y_0 = \left\{ x \in \mathbb{R}^2 \, : \, x_1^2 + x_2^2 \leq 1 \right\}$ *is a ball described around the set* X_0.

Since X_K *is homothetically* Y_0 *with parameters* $a = \frac{5}{4}$, $c = (\frac{4}{3}, \frac{2}{3})^T$ *and* $\sigma_1 = 1.55$ *is the maximum singular number of the matrix* A, *then* $\frac{a-1}{\sigma_1} = 0.215 < \frac{4}{3} = \max\{|c_1|, |c_2|\}$. *Then* $T^* = 1.19$, $v^* = 0.12$, $u^* = (1, 0.5)^T$ *(see Figure 11).*

We also consider system (15) with the initial set $X_0 = \left\{ x \in \mathbb{R}^2 \, : \, x_1^2 + \frac{16x_2^2}{9} \leq 1 \right\}$. *Obviously, the initial system in this case will be system (16), so too* $T^* = 1.19$, $v^* = 0.12$, $u^* = (1, 0.5)^T$ *(see Figure 12).*

Obviously, in the first case (Figure 10) the time $T^* = 1.19$ *is minimal (because the set* $X(T^*, v^*, u^*)$ *touch the set* X_K*). In the second case (Figure 11) the time* $T^* = 1.19$ *is not minimal (the set* $X(T^*, v^*, u^*)$ *is strictly inside the set* X_K*).*

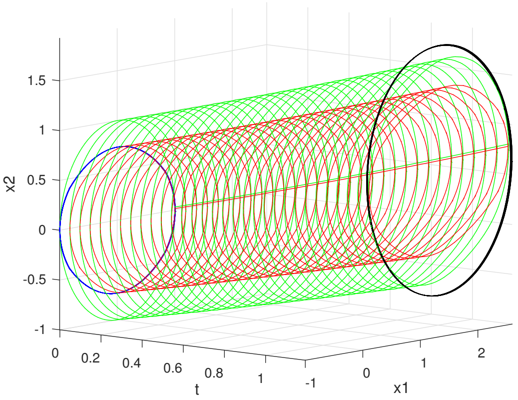

Figure 11. X_0 - blue, X_K - black, $X(t, v^*, u^*)$ - red, $Y(t, v^*, u^*)$ - green, $t \in [0, 1.19]$

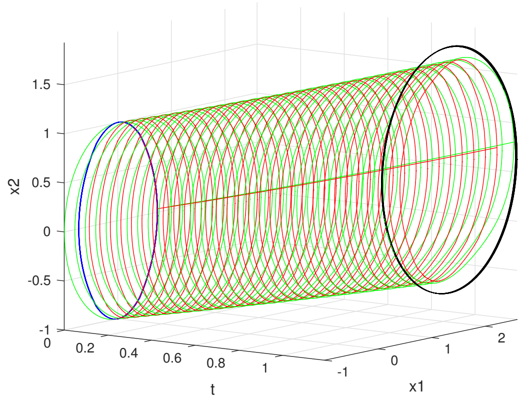

Figure 12. X_0 - blue, X_K - black, $X(t, v^*, u^*)$ - red, $Y(t, v^*, u^*)$ - green, $t \in [0, 1.19]$

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