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Some zero-balanced terminating hypergeometric series and their applications

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Abstract. Various families of such Special Functions as the hypergeometric functions of one, two and more variables, and their associated summation, transformation and reduction formulas, are potentially useful not only as solutions of ordinary and partial differential equations, but also in the widespread problems in the mathematical, physical, engineering and statistical sciences. The main object of this paper is first to establish four general double-series identities, which involve some suitably-bounded sequences of complex numbers, by using zero-balanced terminating hypergeometric summation theorems for the generalized hypergeometric series ${}_{r+1}F_r(1)\quad(r=1,2,3)$ in conjunction with the series rearrangement technique. The sum (or difference) of two general double hypergeometric functions of the Kampé de Fériet type are then obtained in terms of a generalized hypergeometric function under appropriate convergence conditions. A closed form of the following Clausen hypergeometric function:

$${}_3F_2\left(-\frac{27z}{4(1-z)^3}\right)$$

and a reduction formula for the Srivastava-Daoust double hypergeometric function with the arguments $(z,-\frac{z}{4})$ are also derived. Many of the reduction formulas, which are established in this paper, are verified by using the software program, *Mathematica*. Some potential directions for further researches along the lines of this paper are also indicated.

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1. Introduction and Preliminaries

In our present investigation, we use the following standard notations:

$$\mathbb{N}_0 := \mathbb{N} \cup \{0\} \qquad (\mathbb{N} := \{1, 2, 3, \cdots\})$$

and

$$\mathbb{Z}_0^- := \mathbb{Z}^- \cup \{0\} \qquad (\mathbb{Z}^- := \{-1, -2, -3, \cdots\}).$$

Moreover, as usual, we denote by $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{Z}$ the sets of complex numbers, real numbers and integers, respectively.

The general Pochhammer symbol (or the *shifted* factorial) $(\lambda)_\nu$ $(\lambda, \nu \in \mathbb{C})$ is defined, in terms of the familiar (Euler's) Gamma function, by (see, for example, [15] and [31])

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0;\ \lambda \in \mathbb{C} \setminus \{0\}) \\ \prod\limits_{j=0}^{n-1} (\lambda + j) & (\nu = n \in \mathbb{N};\ \lambda \in \mathbb{C}) \\ \dfrac{(-1)^k\, n!}{(n-k)!} & (\lambda = -n;\ \nu = k;\ n, k \in \mathbb{N}_0;\ 0 \leqq k \leqq n) \\ 0 & (\lambda = -n;\ \nu = k;\ n, k \in \mathbb{N}_0;\ k > n) \\ \dfrac{(-1)^k}{(1-\lambda)_k} & (\nu = -k;\ k \in \mathbb{N};\ \lambda \in \mathbb{C} \setminus \mathbb{Z}), \end{cases} \tag{1}$$

where it is understood conventionally that $(0)_0 := 1$ and assumed tacitly that the Gamma quotient exists.

A natural generalization of the Gauss hypergeometric series ${}_2F_1[\alpha, \beta; \gamma; z]$, that is, the generalized hypergeometric series ${}_pF_q$ with p numerator parameters α_j $(j = 1, \cdots, p) \in \mathbb{C}$ and q denominator parameters β_j $(j = 1, \cdots, q) \in \mathbb{C}$ is defined by

$${}_pF_q\left[\begin{array}{c} (\alpha_j)_{j=1,\cdots,p}; \\ \\ (\beta_j)_{j=1,\cdots,q}; \end{array} z\right] = {}_pF_q\left[\begin{array}{c} \alpha_1, \cdots, \alpha_p; \\ \\ \beta_1, \cdots, \beta_q; \end{array} z\right] = \sum_{n=0}^{\infty} \frac{\prod\limits_{j=1}^{p} (\alpha_j)_n}{\prod\limits_{j=1}^{q} (\beta_j)_n} \frac{z^n}{n!}, \tag{2}$$

provided that $\beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-$ $(j = 1, \cdots, q)$.

Assuming that none of the numerator and denominator parameters is zero or a negative integer, we note that the ${}_pF_q$ series in (2):

(i) converges for $|z| < \infty$, if $p \leqq q$;

(ii) converges for $|z| < 1$, if $p = q + 1$;

(iii) converges absolutely for $|z| = 1$, if $p = q + 1$ and $\Re(\omega) > 0$;

(iv) converges conditionally for $|z| = 1$ $(z \neq 1)$, if $p = q + 1$ and $-1 < \Re(\omega) \leqq 0$;

(v) diverges for all z $(z \neq 0)$, if $p > q + 1$;

(vi) diverges for $|z| = 1$, if $p = q + 1$ and $\Re(\omega) \leqq -1$,

where ω is given by

$$\omega := \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j, \tag{3}$$

$\Re(\omega)$ being the real part of the complex number ω.

In the case when $\omega = 0$, ${}_{q+1}F_q(z)$ is called a *zero-balanced* hypergeometric series.

Just as the Gauss function ${}_2F_1$ was generalized to ${}_pF_q$ by increasing the number of the numerator and denominator parameters, the four Appell functions F_1, F_2, F_3 and F_4 (see [31, p. 53, Eqs. (4), (5), (6) and (7)]) and their seven confluent forms Φ_1, Φ_2, Φ_3, Ψ_1, Ψ_2, Ξ_1 and Ξ_2, which were studied by Humbert (see [10] and [11]), were unified and generalized by Kampé de Fériet (see [12] and [4, p. 150, Eq. (29)]; see also [2] and [3]) who defined a general hypergeometric function of two variables. In the modified notation introduced by Srivastava and Panda [32, p. 423, Eq. (26)] (see also [6, p. 112]), we recall here a more general double hypergeometric function than the one defined by Kampé de Fériet) as follows:

$$F^{p:q;k}_{\ell:m;n}\left[\begin{array}{c}(a_p):(b_q);(c_k);\\ \\ (\alpha_\ell):(\beta_m);(\gamma_n);\end{array} x,y\right] = \sum_{r,s=0}^{\infty} \frac{\prod\limits_{j=1}^{p}(a_j)_{r+s}\prod\limits_{j=1}^{q}(b_j)_r\prod\limits_{j=1}^{k}(c_j)_s}{\prod\limits_{j=1}^{\ell}(\alpha_j)_{r+s}\prod\limits_{j=1}^{m}(\beta_j)_r\prod\limits_{j=1}^{n}(\gamma_j)_s}\,\frac{x^r}{r!}\,\frac{y^s}{s!}, \tag{4}$$

where, for convergence,

(i) $p+q<\ell+m+1$ and $p+k<\ell+n+1$ when $\max\{|x|,|y|\}<\infty$;

(ii) $p+q=\ell+m+1$ and $p+k=\ell+n+1$ when

$$\begin{cases} |x|^{1/(p-\ell)}+|y|^{1/(p-\ell)}<1 & (p>\ell); \\ \\ \max\{|x|,|y|\}<1 & (p\leqq\ell). \end{cases} \tag{5}$$

Throughout this paper, we find it be convenient to abbreviate the parameters $a_1,\cdots,a_p$ by (a_p), , $b_1,\cdots,b_q$ by (b_q), , and so on.

For a detailed and systematic discussion of the absolute and conditional convergence of the Kampé de Fériet double hypergeometric series in (4), the reader can refer to a beautiful research paper by Hài *et al.* [9, pp. 106–107].

As long ago as 1969, Srivastava and Daoust [24, p. 199] defined a generalization of the Kampé de Fériet function in (4) (see also [4, p. 150]) by means of the following double hypergeometric series (see also [25] and [26]):

$$\begin{aligned} &F^{A:B;B'}_{C:D;D'}\left(\begin{array}{c}[(a_A):\vartheta,\varphi]:[(b_B):\psi];\left[(b'_{B'}):\psi'\right];\\ \\ [(c_C):\delta,\varepsilon]:[(d_D):\eta];\left[(d'_{D'}):\eta'\right];\end{array} x,y\right) \\ &\quad = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\prod\limits_{j=1}^{A}(a_j)_{m\vartheta_j+n\varphi_j}\prod\limits_{j=1}^{B}(b_j)_{m\psi_j}\prod\limits_{j=1}^{B'}(b'_j)_{n\psi'_j}}{\prod\limits_{j=1}^{C}(c_j)_{m\delta_j+n\varepsilon_j}\prod\limits_{j=1}^{D}(d_j)_{m\eta_j}\prod\limits_{j=1}^{D'}(d'_j)_{n\eta'_j}}\,\frac{x^m}{m!}\,\frac{y^n}{n!}, \end{aligned} \tag{6}$$

where the coefficients

$$\begin{cases} \vartheta_1,\cdots,\vartheta_A;\ \ \varphi_1,\cdots,\varphi_A;\ \ \psi_1,\cdots,\psi_B;\ \ \psi'_1,\cdots,\psi'_{B'};\ \ \delta_1,\cdots,\delta_C; \\ \\ \varepsilon_1,\cdots,\varepsilon_C;\ \ \eta_1,\cdots,\eta_D;\ \eta'_1,\cdots,\eta'_{D'} \end{cases} \tag{7}$$

are real and positive. Also, for the sake of brevity, (a_A) is taken to denote the set of A parameters $a_1, a_2, \cdots, a_A$, with similar interpretations for (b_B), $(b'_{B'})$, and so on. Moreover, if we set

$$\Delta_1 := 1 + \sum_{j=1}^{C} \delta_j + \sum_{j=1}^{D} \eta_j - \sum_{j=1}^{A} \vartheta_j - \sum_{j=1}^{B} \psi_j \tag{8}$$

and

$$\Delta_2 := 1 + \sum_{j=1}^{C} \varepsilon_j + \sum_{j=1}^{D'} \eta'_j - \sum_{j=1}^{A} \varphi_j - \sum_{j=1}^{B'} \psi'_j, \tag{9}$$

then we have the following convergence requirements for the double power series in (6):

(i) If $\Delta_1 > 0$ and $\Delta_2 > 0$, then the double power series in (6) converges for all complex values of x and y;

(ii) If $\Delta_1 = 0$ and $\Delta_2 = 0$, then the double power series in (6) is convergent for suitably constrained values of $|x|$ and $|y|$ (see, for details, [26]);

(iii) If $\Delta_1 < 0$ and $\Delta_2 < 0$, then the double power series in (6) would diverge except when $x = y = 0$.

Next, from the following binomial theorem:

$$(1-z)^{-\lambda} = \sum_{n=0}^{\infty} (\lambda)_n \frac{z^n}{n!} = {}_1F_0\left[\begin{matrix} \lambda; \\ \text{———} ; \end{matrix}\ z\right] \qquad \Big(|\arg(1-z)| < \pi;\ \lambda \in \mathbb{C}\Big) \tag{10}$$

it can be deduced that

$${}_1F_0\left[\begin{matrix} \lambda; \\ — ; \end{matrix}\ 1\right] = 0 \qquad \Big(\Re(\lambda) < 0\Big) \tag{11}$$

and

$${}_1F_0\left[\begin{matrix} \lambda; \\ — ; \end{matrix}\ -1\right] = 2^{-\lambda} \qquad (\lambda \in \mathbb{C}). \tag{12}$$

We also have the following result for a terminating Gauss hypergeometric series:

$${}_2F_1\left[\begin{matrix} -1, \frac{3\lambda+2}{2}; \\ \frac{3\lambda}{2}; \end{matrix}\ 1\right] = -\frac{2}{3\lambda} \qquad \left(\frac{3\lambda}{2} \in \mathbb{C} \setminus \mathbb{Z}_0^-\right). \tag{13}$$

The series rearrangement technique, which we shall use in this article, is based upon the following double-series identities [31, p. 100, Lemma 1, Eqs. (1) and (2)]:

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \Xi(m,n) = \sum_{m=0}^{\infty} \sum_{n=0}^{m} \Xi(m-n,n) \tag{14}$$

and

$$\sum_{n=0}^{\infty} \sum_{m=0}^{n} \Xi(n,m) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \Xi(n+m,m), \tag{15}$$

and also on the easily-derivable series identities given by

$$\sum_{m=0}^{n+p} \Xi(m,n) = \sum_{m=0}^{n} \Xi(m,n) + \sum_{m=0}^{p-1} \Xi(m+n+1,n) \tag{16}$$

and

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n+p} \Xi(m,n) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \Xi(m,n+m) + \sum_{n=0}^{\infty}\sum_{m=0}^{(p-1)} \Xi(m+n+1,n) \qquad (p \in \mathbb{N}). \tag{17}$$

The following rather obvious decomposition formula will also be useful in our investigation here:

$$\sum_{n=0}^{\infty} \Xi(n) = \sum_{n=0}^{\infty} \Xi(2n) + \sum_{n=0}^{\infty} \Xi(2n+1). \tag{18}$$

In all of the above series identities, it is each of the series involved is absolutely convergent.

The following zero=balanced hypergeometric summation theorems will also be needed in our investigation.

I. A Zero-Balanced Terminating Hypergeometric ${}_3F_2$ **Series** (see [14, p. 539, Entry 91])

$${}_3F_2\left[\begin{array}{r} -m, \alpha, \beta; \\ \alpha-k, \beta-m+k; \end{array} 1\right] = \frac{m!}{(1-\alpha)_k(1-\beta)_{m-k}} \tag{19}$$

$$\Big(k,m \in \mathbb{N}_0;\ m \geqq k;\ \alpha,\beta,\alpha-k,\beta-m+k \in \mathbb{C} \setminus \mathbb{Z}_0^-\Big).$$

II. A Zero-Balanced Terminating Hypergeometric ${}_4F_3$ **Series** (see [14, p. 555, Entry 19])

$${}_4F_3\left[\begin{array}{r} -m, \alpha, \beta, \gamma; \\ \alpha-k, \beta-s, \gamma-(m-k-s); \end{array} 1\right] = \frac{m!}{(1-\alpha)_k(1-\beta)_s(1-\gamma)_{m-k-s}} \tag{20}$$

$$\Big(k,s,m \in \mathbb{N}_0;\ m \geqq k+s;\ \alpha,\beta,\gamma,\alpha-k,\beta-s,\gamma-m+k+s \in \mathbb{C} \setminus \mathbb{Z}_0^-\Big).$$

III. A Unification and Generalization of the Zero-Balanced Series (19) **and** (20)

$$\begin{aligned} &{}_{r+1}F_r\left[\begin{array}{r} -m, \beta_1, \beta_2, \cdots, \beta_{r-1}, \beta_r; \\ \beta_1-k_1, \beta_2-k_2, \cdots, \beta_{r-1}-k_{r-1}, \beta_r-(m-k_1-k_2-k_3-\cdots-k_{r-1}); \end{array} 1\right] \\ &\quad = \frac{m!}{(1-\beta_1)_{k_1}(1-\beta_2)_{k_2}\cdots(1-\beta_{r-1})_{k_{r-1}}(1-\beta_r)_{m-k_1-k_2-\cdots-k_{r-1}}} \end{aligned} \tag{21}$$

$$\Big(k_1,k_2,\cdots k_{r-1},m \in \mathbb{N}_0;\ m \geqq k_1+k_2+\cdots+k_{r-1}\Big),$$

the remaining numerator and denominator parameters being neither zero nor negative integers.

Remark 1. The unification and generalization (21) follows from the zero-balanced terminating hypergeometric summations (19) and (20), which are recorded by Prudnikov *et al.* [14]. In fact, we have also verified the general result (21) numerically by taking suitable values of the numerator and denominator parameters and using the software, *Mathematica*. The analytic proof of the summation theorem (21) may be left as an exercise for the interested readers.

Our present investigation is motivated essentially by several interesting and widespread developments on various families of zero-balanced hypergeometric functions by (for example) Anderson *et al.* [1], Bühring and Srivastava [5], Evans and Stanton [8], Karp [13], Richards [16], Saigo and Srivastava [17], A. K. Srivastava [18], Srivastava [19], Srivastava and Jain (see [27] and [28]), Wang *et al.* (see [36], [37], [38] and [39]) and Zhao *et al.* [40]. In particular, we express the sum (or difference) of two general double hypergeometric functions of the Kampé de Fériet type in terms of a generalized hypergeometric function under appropriate convergence conditions.

The article is organized as follows. In Section 2, we derive our first general double-series identity (22) by using the zero-balanced terminating hypergeometric summation theorem (19) and the above-mentioned series rearrangement technique. In Section 3, two new results (33) and (35) for the sum (or difference) of two double hypergeometric functions of Kampé de Fériet type are derived by applying the first general double-series identity (22). In Section 4, we establish the second general double-series identity (36) by using the zero-balanced terminating hypergeometric summation theorem (19). In Section 5, we obtain a closed form (40) of the Clausen hypergeometric function:

$$ {}_3F_2\left(-\frac{27z}{4(1-z)^3}\right) $$

and a reduction formula (43) for the Srivastava-Daoust double hypergeometric function with the arguments $(z, -\frac{z}{4})$ in terms of the difference of two generalized hypergeometric functions of one variable with the same argument $4^{D+1-E}\, z^2$ by making use of the second general double-series identity (36). In Section 6, we prove the third general double-series identity (44) by applying the zero-balanced terminating hypergeometric summation theorem (20). In Section 7, the result (45) for the sum (or difference) of two double hypergeometric functions of the Kampé de Fériet type is obtained by using the third general double-series identity (44). In Section 8, a unification and generalization (47) of the first and the third double-series identities (22) and (44) is established by using the above zero-balanced series (21). Finally, in Section 9, we present several concluding remarks and observations.

Remark 2. Throughout this article, we assume that any values of parameters and arguments, which would render the results in Sections 2 to 8 invalid or undefined, are tacitly excluded.

2. First Application of the Zero-Balanced Series (19)

Theorem 1. (First General Double-Series Identity) *Let $\{\Theta(\mu)\}_{\mu=0}^{\infty}$ be a bounded sequence of essentially arbitrary real or complex numbers such that $\Theta(0) \neq 0$. Then the following general double-series identity holds true*:

$$ \begin{aligned} &\frac{p!}{(1-a)_s\,(1-b)_{p-s}}\left(\sum_{n=0}^{\infty}\Theta(n+p)\,\frac{z^n}{(1-b+p-s)_n}\right)\\ &\quad=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\Theta(n+m+p)\,\frac{(1-b+p-s)_n(a)_m(b)_m\,z^n\,z^m}{(1-b+p-s)_{n+m}(1+p)_n(a-s)_m\,m!}-\frac{pab}{(a-s)(b-p+s)}\\ &\qquad\cdot\sum_{n=0}^{\infty}\sum_{m=0}^{p-1}\Theta(n+p)\,\frac{(a+1)_{n+m}(b+1)_{n+m}(1-p)_m\,z^n}{(1+a-s)_{n+m}(2)_{n+m}(1-b+p-s)_n(1+b-p+s)_m} \end{aligned} \tag{22} $$

$$ \Big(p \geqq s;\ p,s\in\mathbb{N}_0;\ 1-a,1-b,a-s,\pm(b-p+s)\in\mathbb{C}\setminus\mathbb{Z}_0^-;\ z\in\mathbb{C}\Big), $$

provided that the single and double series involved in the assertion (22) *are convergent.*

Proof. Let us consider

$$\Omega(z) := \frac{z^p}{(1-a)_s(1-b)_{p-s}} \left(\sum_{n=0}^{\infty} \Theta(n+p) \frac{z^n}{(1-b+p-s)_n} \right)$$
$$= \sum_{n=0}^{\infty} \Theta(n+p) \frac{z^{n+p}}{(1-a)_s(1-b)_{n+p-s}}, \tag{23}$$

which, upon replacing n by $n-p$, yields

$$\Omega(z) = \sum_{n=p}^{\infty} \Theta(n) \frac{z^n}{n!} \left(\frac{n!}{(1-a)_s(1-b)_{n-s}} \right). \tag{24}$$

We now apply the zero-balanced hypergeometric summation theorem (19) in the equation (24). We thus find that

$$\Omega(z) = \sum_{n=p}^{\infty} \Theta(n) \frac{z^n}{n!} \, {}_3F_2 \left[\begin{array}{c} -n, a, b; \\ a-s, b-n+s; \end{array} 1 \right] \tag{25}$$

If we now replace n by $n+p$ in this last equation (25), we find that

$$\Omega(z) = \sum_{n=0}^{\infty} \Theta(n+p) \frac{z^{n+p}}{(n+p)!} \, {}_3F_2 \left[\begin{array}{c} -(n+p), a, b; \\ a-s, b-n-p+s; \end{array} 1 \right]$$
$$= \sum_{n=0}^{\infty} \sum_{m=0}^{n+p} \Theta(n+p) \frac{(-n-p)_m (a)_m (b)_m \, z^{n+p}}{(a-s)_m (b-n-p+s)_m \, (n+p)! \, m!}, \tag{26}$$

which, by virtue of the double-series identity (17), assumes the following form:

$$\Omega(z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \Theta(n+m+p) \frac{(-n-m-p)_m (a)_m (b)_m \, z^{n+m+p}}{(a-s)_m (b-n-m-p+s)_m \, (n+m+p)! \, m!}$$
$$+ \sum_{n=0}^{\infty} \sum_{m=0}^{p-1} \Theta(n+p) \frac{(-n-p)_{m+n+1} (a)_{m+n+1} (b)_{m+n+1} \, z^{n+p}}{(a-s)_{m+n+1} (b-n-p+s)_{m+n+1} (2)_{m+n} \, (n+p)!}. \tag{27}$$

Finally, if we appropriately apply the following identities involving the Pochhammer symbols:

$$(-n-p)_{m+n+1} = (-1)^{n+1} p \, (p+1)_n (1-p)_m \qquad (m \leqq p-1), \tag{28}$$

$$(-n-m-p)_m = \frac{(-1)^m (1+p)_{n+m}}{(1+p)_n}, \tag{29}$$

$$(b-n-m-p+s)_m = \frac{(-1)^m (1-b+p-s)_{m+n}}{(1-b+p-s)_n} \tag{30}$$

and

$$(b-n-p+s)_{m+n+1} = (-1)^n (b-p+s)(1-b+p-s)_n (1+b-p+s)_m, \tag{31}$$

in the equation (27), we obtain

$$\Omega(z) = \frac{z^p}{p!}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\Theta(n+m+p)\,\frac{(1-b+p-s)_n(a)_m(b)_m\,z^{n+m}}{(1-b+p-s)_{n+m}(1+p)_n(a-s)_m\,m!}$$
$$-\frac{p\,a\,b\,z^p}{(a-s)(b-p+s)\,p!}\sum_{n=0}^{\infty}\sum_{m=0}^{p-1}\Theta(n+p)$$
$$\cdot\frac{(a+1)_{n+m}(b+1)_{n+m}(1-p)_m\,z^n}{(1+a-s)_{n+m}(2)_{n+m}(1-b+p-s)_n(1+b-p+s)_m}, \tag{32}$$

which leads us readily to the first general double-series identity (22) as asserted by Theorem 1. □

Remark 3. For the precise convergence condition on $z \in \mathbb{C}$ in the assertion (22) of Theorem 1, it is necessary to know the actual form of the sequence $\{\Theta(\mu)\}_{\mu=0}^{\infty}$. In this connection, for the convergence condition on $z \in \mathbb{C}$, see the result (33) in which the value of $\Theta(\mu)$ is known.

3. Application of the First General Double-Series Identity (22)

In this section, we apply Theorem 1 in order to establish the reducibility of the sum (or difference) of two double hypergeometric functions of the Kampé de Fériet type.

Theorem 2. *The following reduction formula holds true:*

$$\frac{p!}{(1-a)_s(1-b)_{p-s}}\;{}_{D+1}F_{E+1}\left[\begin{matrix}(d_D)+p,1;\\ (e_E)+p,1-b+p-s;\end{matrix}\;z\right]$$
$$=F^{D:2;2}_{1+E:1;1}\left[\begin{matrix}(d_D)+p:1,1-b+p-s;a,b;\\ (e_E)+p,1-b+p-s:1+p;a-s;\end{matrix}\;z,z\right]-\frac{pab}{(a-s)(b-p+s)}$$
$$\cdot F^{2:1+D;2}_{2:1+E;1}\left[\begin{matrix}a+1,b+1:1,(d_D)+p;-(p-1),1;\\ 1+a-s,2:(e_E)+p,1-b+p-s;1+b-p+s\,;\end{matrix}\;z,1\right] \tag{33}$$

$$\left(p \geqq s;\; p,s \in \mathbb{N}_0;\; 1-a,1-b,(e_E),a-s,\pm(b-p+s) \in \mathbb{C}\setminus\mathbb{Z}_0^-\right),$$

it being understood that $D \leqq E$ when $|z| < \infty$, and that $D = E+1$ when $|z| < 1$.

Proof. If we set

$$\Theta(\mu) = \frac{(d_1)_\mu(d_2)_\mu\cdots(d_D)_\mu}{(e_1)_\mu(e_2)_\mu\cdots(e_E)_\mu} \qquad (\mu \in \mathbb{N}_0)$$

in both sides of the double-series identity (22), we find after simplification that

$$\frac{p!\,(d_1)_p\cdots(d_D)_p}{(1-a)_s(1-b)_{p-s}(e_1)_p\cdots(e_E)_p}\left(\sum_{n=0}^{\infty}\frac{(d_1+p)_n\cdots(d_D+p)_n\,(1)_n\,z^n}{(e_1+p)_n\cdots(e_E+p)_n(1-b+p-s)_n\,n!}\right)$$
$$=\frac{(d_1)_p\cdots(d_D)_p}{(e_1)_p\cdots(e_E)_p}$$
$$\cdot\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(d_1+p)_{n+m}\cdots(d_D+p)_{n+m}\,(1)_n\,(1-b+p-s)_n(a)_m\,(b)_m\,z^{n+m}}{(e_1+p)_{n+m}\cdots(e_E+p)_{n+m}(1-b+p-s)_{n+m}(1+p)_n(a-s)_m\,n!\,m!}$$
$$-\frac{pab(d_1)_p\cdots(d_D)_p}{(a-s)(b-p+s)\,(e_1)_p\cdots(e_E)_p}\cdot\sum_{n=0}^{\infty}\sum_{m=0}^{p-1}\frac{(a+1)_{n+m}(b+1)_{n+m}}{(1+a-s)_{n+m}\,(2)_{n+m}}$$
$$\cdot\frac{(d_1+p)_n\cdots(d_D+p)_n\,(1)_n\,(1-p)_m\,(1)_m\,z^n\,1^m}{(e_1+p)_n\cdots(e_E+p)_n\,(1-b+p-s)_n(1+b-p+s)_m\,n!\,m!}. \tag{34}$$

By first applying the definition (2) in the left-hand side of the equation (34) and the definition (4) in the right-hand side of the equation (34), and then simplifying the resulting equation, we get the reduction formula (33). □

For $s=0$, the reduction formula (33) yields the following corollary.

Corollary 1. *The following reduction formula holds true*:

$$\frac{p!}{(1-b)_p}\,{}_{D+1}F_{E+1}\left[\begin{array}{c}(d_D)+p,1;\\ \\(e_E)+p,1-b+p;\end{array}\,z\right]$$
$$=F^{D:2;1}_{1+E:1;0}\left[\begin{array}{c}(d_D)+p:1,1-b+p;b;\\ \\(e_E)+p,1-b+p:1+p;-;\end{array}\,z,z\right]$$
$$-\left(\frac{p\,b}{b-p}\right)F^{1:1+D;2}_{1:1+E;1}\left[\begin{array}{c}b+1:1,(d_D)+p;-(p-1),1;\\ \\2:(e_E)+p,1-b+p;b-p+1;\end{array}\,z,1\right] \tag{35}$$
$$\left(p\in\mathbb{N}_0;\ 1-b,(e_E),\pm(b-p)\in\mathbb{C}\setminus\mathbb{Z}_0^-\right),$$

it being understood that $D\leqq E$ *when* $|z|<\infty$, *and that* $D=E+1$ *when* $|z|<1$.

4. The Second Application of Zero-Balanced Series (19)

Theorem 3. (Second General Double-Series Identity) *Let* $\{\Psi(\mu)\}_{\mu=0}^{\infty}$ *be a bounded sequence of essentially arbitrary real or complex numbers such that* $\Psi(0)\neq 0$. *Then the following general double-series identity holds true*:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^m\,\Psi(n+m)\frac{(3a)_{n+3m}}{(3a)_{2m}}\,\frac{z^{n+m}}{n!\,m!}$$
$$=\Psi(0)-2z\Psi(1)+4z^2\sum_{n=0}^{\infty}\Psi(2n+2)(2z)^{2n}-8z^3\sum_{n=0}^{\infty}\Psi(2n+3)(2z)^{2n} \tag{36}$$
$$\left(3a\in\mathbb{C}\setminus\mathbb{Z}_0^-;\ z\in\mathbb{C}\right),$$

provided that the infinite series occurring on both sides of the assertion (36) *are convergent. The right-hand side of the assertion* (36) *is independent of the parameter a.*

Proof. We denote the left-hand side of the identity (36) by $\Delta(z)$ so that

$$\Delta(z) := \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^m\, \Psi(n+m)\frac{(3a)_{n+3m}}{(3a)_{2m}}\,\frac{z^{n+m}}{n!\,m!} \qquad (|z|<1), \tag{37}$$

which, upon replacing n by $n-m$, followed by some simplications, yields

$$\begin{aligned}\Delta(z) &= \sum_{n=0}^{\infty}(3a)_n\,\Psi(n)\,\frac{z^n}{n!}\,\sum_{m=0}^{n}\frac{(-n)_m(3a+n)_{2m}}{(3a)_{2m}\,m!}\\ &= \sum_{n=0}^{\infty}(3a)_n\,\Psi(n)\,\frac{z^n}{n!}\;{}_3F_2\left[\begin{matrix}-n,\frac{3a+n}{2},\frac{3a+n+1}{2};\\ \frac{3a}{2},\frac{3a+1}{2};\end{matrix}\; 1\right]\\ &= \Psi(0)+3az\Psi(1)\;{}_2F_1\left[\begin{matrix}-1,\frac{3a+2}{2};\\ \frac{3a}{2};\end{matrix}\; 1\right]\\ &\quad+\sum_{n=2}^{\infty}(3a)_n\,\Psi(n)\,\frac{z^n}{n!}\;{}_3F_2\left[\begin{matrix}-n,\frac{3a+n}{2},\frac{3a+n+1}{2};\\ \frac{3a}{2},\frac{3a+1}{2};\end{matrix}\; 1\right].\end{aligned} \tag{38}$$

The rest of our proof would require replacing n by $n+2$ in the equation (38), and then appropriately using the series decomposition identity (18), the terminating Gauss series (13) and the zero-balanced hypergeometric summation theorem (19). This will eventually lead us to the following result:

$$\begin{aligned}\Delta(z) &= \Psi(0)-2z\Psi(1)+\sum_{n=0}^{\infty}\Psi(2n+2)\,\frac{(3a)_{2n+2}\,z^{2n+2}}{\left(1-\frac{3a+2n+2}{2}\right)_{n+1}\left(1-\frac{3a+2n+3}{2}\right)_{n+1}}\\ &\quad+\sum_{n=0}^{\infty}\Psi(2n+3)\,\frac{(3a)_{2n+3}\,z^{2n+3}}{\left(1-\frac{3a+2n+3}{2}\right)_{n+1}\left(1-\frac{3a+2n+4}{2}\right)_{n+2}}.\end{aligned} \tag{39}$$

Finally, in the right-hand side of the equation (39), we apply some easily derivable Pochhammer symbol identities from (1). Then, after some further simplification, the assertion (36) of Theorem 3 follows readily. □

5. Application of the Second General Double-Series Identity (36)

In this section, we establish one more reducible case related with the Clausen hypergeometric function ${}_3F_2$.

Theorem 4. *The following reduction formula holds true for the Clausen hypergeometric function* ${}_3F_2$:

$${}_3F_2\left[\begin{matrix}a,a+\frac{1}{3},a+\frac{2}{3};\\ \frac{3a}{2},\frac{3a+1}{2};\end{matrix}\; -\frac{27z}{4(1-z)^3}\right]=\frac{(1-z)^{3a}}{1+2z} \tag{40}$$

$$\left(27|z|<4|1-z|^3;\ |\arg(1-z)|<\pi;\ |z|<1;\ z\neq-\frac{1}{2};\ 3a\in\mathbb{C}\setminus\mathbb{Z}_0^-\right).$$

Proof. Upon setting $\Psi(\mu)=1$ for all non-negative integer values of μ in both sides of the double-series identity (36), if we make use of the Taylor-Maclaurin expansion, that is, the case $\lambda=1$ of the binomial theorem (10), we get

$$\sum_{m=0}^{\infty}\frac{(3a)_{3m}}{(3a)_{2m}}\,\frac{(-z)^m}{m!}\,\sum_{n=0}^{\infty}(3a+3m)_n\,\frac{z^n}{n!}=1-2z+\frac{4z^2}{1-4z^2}-\frac{8z^3}{1-4z^2}, \tag{41}$$

which can be rewritten as follows:

$$(1-z)^{-3a}\sum_{m=0}^{\infty}\frac{(a)_m\left(a+\frac{1}{3}\right)_m\left(a+\frac{2}{3}\right)_m}{\left(\frac{3a}{2}\right)_m\left(\frac{3a+1}{2}\right)_m}\,\frac{\left(-\frac{27z}{4(1-z)^3}\right)^m}{m!}=\frac{1}{1+2z}. \tag{42}$$

Finally, in view of the definition (2), the last equation (42) yields the required result (40). □

Remark 4. The (presumably new) reduction formula (40) has also been verified numerically by using the software, *Mathematica*, in the interval $-\frac{1}{2}<\Re(z)<1$ and $\Im(z)=0$.

Theorem 5. *The following reduction formula holds true for a general Srivastava-Daoust double hypergeometric function*:

$$\begin{aligned}
&F^{D+1:0;0}_{E:0;2}\left(\begin{array}{c}[(d_D):1,1],[3a:1,3]:\text{—};\text{————————};\\ [(e_E):1,1]:\text{—};\left[\frac{3a}{2}:1\right],\left[\frac{3a+1}{2}:1\right];\end{array}\; z,-\frac{z}{4}\right)\\
&=1-2z\,\frac{\prod_{j=1}^{D}(d_j)}{\prod_{j=1}^{E}(e_j)}+4z^2\,\frac{\prod_{j=1}^{D}(d_j)_2}{\prod_{j=1}^{E}(e_j)_2}\;{}_{2D+1}F_{2E}\left[\begin{array}{c}\frac{(d_D)+2}{2},\frac{(d_D)+3}{2},1;\\ \frac{(e_E)+2}{2},\frac{(e_E)+3}{2};\end{array}\;4^{D+1-E}\,z^2\right]\\
&\quad-8z^3\,\frac{\prod_{j=1}^{D}(d_j)_3}{\prod_{j=1}^{E}(e_j)_3}\;{}_{2D+1}F_{2E}\left[\begin{array}{c}\frac{(d_D)+3}{2},\frac{(d_D)+4}{2},1;\\ \frac{(e_E)+3}{2},\frac{(e_E)+4}{2};\end{array}\;4^{D+1-E}\,z^2\right]
\end{aligned} \tag{43}$$

$$(e_1,e_2,\cdots,e_E,3a\in\mathbb{C}\setminus\mathbb{Z}_0^-).$$

Proof. In order to prove the reduction formula (43), we first put

$$\Psi(\mu)=\frac{(d_1)_\mu(d_2)_\mu\cdots(d_D)_\mu}{(e_1)_\mu(e_2)_\mu\cdots(e_E)_\mu}=\frac{\prod_{j=1}^{D}(d_j)_\mu}{\prod_{j=1}^{E}(e_j)_\mu}\qquad(\mu\in\mathbb{N}_0)$$

Then, by using the definition (6) in the left-hand side and the definition (2) in the right-hand side of the resulting equation, we obtain the required reduction formula (43). □

Remark 5. In the case when $2D+1<2E$, both sides of the equation (43) are convergent for $|z|<\infty$. As a matter of fact, the reduction formula (43) has been numerically verified by using the software, *Mathematica*. It is worth noting that the right-hand side of the equation (43) is independent of the parameter a.

6. Application of the Zero-Balanced Series (20)

Theorem 6. **(Third General Double-Series Identity)** *Let* $\{\Phi(\mu)\}_{\mu=0}^{\infty}$ *be a bounded sequence of essentially arbitrary* *real or complex numbers such that* $\Phi(0) \neq 0$. *Then the following general double-series identity holds true*:

$$\begin{aligned}
&\frac{p!}{(1-a)_k(1-b)_s(1-c)_{p-k-s}} \sum_{n=0}^{\infty} \Phi(n+p) \, \frac{z^n}{(1-c+p-k-s)_n} \\
&\quad = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \Phi(n+m+p) \, \frac{(1-c+p-k-s)_n (a)_m (b)_m (c)_m}{(1-c+p-k-s)_{n+m}(1+p)_n(a-k)_m(b-s)_m} \, \frac{z^{n+m}}{m!} \\
&\qquad - \frac{p\,a\,b\,c}{(c-p+k+s)(a-k)(b-s)} \sum_{n=0}^{\infty}\sum_{m=0}^{p-1} \Phi(n+p) \\
&\qquad \cdot \frac{(a+1)_{n+m}(b+1)_{n+m}(c+1)_{n+m}(1-p)_m z^n}{(1+a-k)_{n+m}(1+b-s)_{n+m}(2)_{n+m}(1-c+p-k-s)_n(1+c-p+k+s)_m}
\end{aligned} \tag{44}$$

$$\Big(p \geqq k+s;\ k,s,p \in \mathbb{N}_0;\ 1-a, 1-b, 1-c, a-k, b-s,\ \pm(c-p+k+s) \in \mathbb{C}\setminus\mathbb{Z}_0^-;\ z \in \mathbb{C}\Big),$$

provided that the infinite series occurring on both sides of the assertion (44) *are convergent.*

Proof. The demonstration of Theorem 6 follows the same lines as those of our derivation of Theorem 1.(22). Use is made here of the zero-balanced series (20). □

7. Application of the Third General Double-Series Identity (44)

Theorem 7. *For* $p \geqq k+s, k,s,p \in \mathbb{N}_0$ *and* $1-a, 1-b, 1-c, (e_E), a-k, b-s, \pm(c-p+k+s) \in \mathbb{C}\setminus\mathbb{Z}_0^-$, *the following sum* (*or difference*) *of two double hypergeometric functions of the Kampé de Fériet type holds true*:

$$\begin{aligned}
&\frac{p!}{(1-a)_k(1-b)_s(1-c)_{p-k-s}} \; {}_{D+1}F_{E+1}\left[\begin{array}{c} (d_D)+p, 1; \\ (e_E)+p, 1-c+p-k-s; \end{array} z\right] \\
&\quad = F^{D:2;3}_{E+1:1;2}\left[\begin{array}{c} (d_D)+p : 1-c+p-k-s, 1; a, b, c; \\ (e_E)+p, 1-c+p-k-s : 1+p; a-k, b-s; \end{array} z, z\right] \\
&\qquad - \frac{p\,a\,b\,c}{(a-k)(b-s)(c-p+k+s)} \\
&\qquad \cdot F^{3:D+1;2}_{3:E+1;1}\left[\begin{array}{c} a+1, b+1, c+1 : (d_D)+p, 1; -(p-1), 1; \\ a-k+1, b-s+1, 2 : (e_E)+p, 1-c+p-k-s; c-p+k+s+1; \end{array} z, 1\right],
\end{aligned} \tag{45}$$

provided that $D \leqq E$ *when* $|z| < \infty$ *and that* $D = E+1$ *when* $|z| < 1$.

Proof. Just as in our demonstration of Theorem 5, we first set

$$\Phi(\mu) = \frac{(d_1)_\mu (d_2)_\mu \cdots (d_D)_\mu}{(e_1)_\mu (e_2)_\mu \cdots (e_E)_\mu} = \frac{\prod_{j=1}^{D} (d_j)_\mu}{\prod_{j=1}^{E} (e_j)_\mu} \qquad (\mu \in \mathbb{N}_0) \tag{46}$$

in both sides of the double-series identity (44) After some simplification, we obtain the hypergeometric representation given on the right-hand side of the reduction formula (45). □

Corollary 2. *The cases of the reduction formula* (45) *when* $k = 0$ *and* $k = s = 0$ *yield the corresponding results in the forms of* (33) *and* (35), *respectively.*

8. Unification and Generalization of the Double-Series Identities (22) and (44)

Theorem 8. (Fourth General Double-Series Identity) *Let* $\{\nabla(\mu)\}_{\mu=0}^{\infty}$ *be a bounded sequence of essentially arbitrary real or complex numbers such that* $\nabla(0) \neq 0$. *Then the following general double-series identity holds true*:

$$\frac{p!}{(1-\beta_r)_{p-k_1-k_2-\cdots-k_{r-1}}\prod_{j=1}^{r-1}(1-\beta_j)_{k_j}}\sum_{n=0}^{\infty}\nabla(n+p)\,\frac{z^n}{(1-\beta_r+p-k_1-k_2-\cdots-k_{r-1})_n}$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\nabla(n+m+p)\,\frac{(1)_n(1+p-\beta_r-k_1-k_2-\cdots-k_{r-1})_n\prod_{j=1}^{r}(\beta_j)_m}{(1+p)_n(1+p-\beta_r-k_1-k_2-\cdots-k_{r-1})_{n+m}\prod_{j=1}^{r-1}(\beta_j-k_j)_m}$$
$$\cdot\frac{z^{n+m}}{n!\;m!}-\frac{p\prod_{j=1}^{r}(\beta_j)}{(\beta_r-p+k_1+k_2+\cdots+k_{r-1})\prod_{j=1}^{r-1}(\beta_j-k_j)}$$
$$\cdot\sum_{n=0}^{\infty}\sum_{m=0}^{p-1}\nabla(n+p)\frac{\prod_{j=1}^{r}(1+\beta_j)_{n+m}\Big(-(p-1)\Big)_m}{(2)_{n+m}\prod_{j=1}^{r-1}(1+\beta_j-k_j)_{n+m}}$$
$$\cdot\frac{z^n}{(1+p-\beta_r-k_1-k_2-\cdots-k_{r-1})_n(1-p+\beta_r+k_1+k_2+\cdots+k_{r-1})_m},\tag{47}$$

provided that $z\in\mathbb{C}$,

$$p \geqq k_1+k_2+\cdots+k_{r-1}\quad(p,k_1,k_2,\cdots,k_{r-1}\in\mathbb{N}_0),$$

$$1-\beta_1,1-\beta_2,\cdots,1-\beta_{r-1},1-\beta_r\in\mathbb{C}\setminus\mathbb{Z}_0^-,$$

$$\beta_1-k_1,\beta_2-k_2,\cdots,\beta_{r-1}-k_{r-1}\in\mathbb{C}\setminus\mathbb{Z}_0^-$$

and

$$\pm(\beta_r-p+k_1+k_2+\cdots+k_{r-1})\in\mathbb{C}\setminus\mathbb{Z}_0^-.$$

The infinite series, which occur on both sides of the assertion (47), *are also assumed to be convergent.*

Proof. Consider the function $\mho(z)$ given by

$$\mho(z):=\frac{1}{(1-\beta_r)_{p-k_1-k_2-\cdots-k_{r-1}}\prod_{j=1}^{r-1}(1-\beta_j)_{k_j}}$$
$$\cdot\sum_{n=0}^{\infty}\nabla(n+p)\,\frac{z^{n+p}}{(1+p-\beta_r-k_1-k_2-\cdots-k_{r-1})_n}.\tag{48}$$

Upon replacing n by $n-p$ in the equation (48), if we make use of an elementary Pochhammer-symbol identity derivable from (1), we obtain

$$\mho(z) = \frac{1}{\prod_{j=1}^{r-1}(1-\beta_j)_{k_j}} \sum_{n=p}^{\infty} \nabla(n) \frac{z^n}{(1-\beta_r)_{n-k_1-k_2-\cdots-k_{r-1}}}$$

$$= \sum_{n=p}^{\infty} \nabla(n) \left(\frac{n!}{(1-\beta_r)_{n-k_1-k_2-\cdots-k_{r-1}} \prod_{j=1}^{r-1}(1-\beta_j)_{k_j}} \right) \frac{z^n}{n!} \tag{49}$$

We now apply the generalized zero-balanced hypergeometric summation theorem (21) for ${}_{r+1}F_r(1)$ in the equation (49), and replace n by $n+p$. We thus find that

$$\mho(z) = \sum_{n=0}^{\infty} \nabla(n+p) \frac{z^{n+p}}{(n+p)!}$$

$$\cdot {}_{r+1}F_r \left[\begin{array}{r} -(n+p), \beta_1, \beta_2, \cdots, \beta_{r-1}, \beta_r; \\ \beta_1 - k_1, \beta_2 - k_2, \cdots, \beta_{r-1} - k_{r-1}, \beta_r - (n+p-k_1-k_2-\cdots-k_{r-1}); \end{array} 1 \right]$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{n+p} \nabla(n+p) \frac{z^{n+p}}{(n+p)!}$$

$$\cdot \frac{(-n-p)_m (\beta_1)_m (\beta_2)_m \cdots (\beta_{r-1})_m (\beta_r)_m}{(\beta_1-k_1)_m(\beta_2-k_2)_m \cdots (\beta_{r-1}-k_{r-1})_m (\beta_r - n - p + k_1 + k_2 + \cdots + k_{r-1})_m \, m!}. \tag{50}$$

Finally, we use the double-series identity (17) in the equation (50), and apply several Pochhammer-symbol identities resulting easily from the definition (1), we are led eventually to the following result for the function $\mho(z)$ defined by (48):

$$\mho(z) = \frac{z^p}{p!} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \nabla(n+m+p) \frac{(1)_n (1+p-\beta_r-k_1-k_2-\cdots-k_{r-1})_n}{(1+p-\beta_r-k_1-k_2-\cdots-k_{r-1})_{n+m}(1+p)_n}$$

$$\cdot \frac{\prod_{j=1}^{r}(\beta_j)_m \, z^{n+m}}{\prod_{j=1}^{r-1}(\beta_j-k_j)_m \, n! \, m!} - \frac{p z^p \prod_{j=1}^{r}(\beta_j)}{p! \, (\beta_r - p + k_1 + k_2 + \cdots + k_{r-1}) \prod_{j=1}^{r-1}(\beta_j - k_j)}$$

$$\cdot \sum_{n=0}^{\infty} \sum_{m=0}^{(p-1)} \nabla(n+p) \frac{\prod_{j=1}^{r}(1+\beta_j)_{n+m} \Big(-(p-1) \Big)_m (1)_n (1)_m}{(2)_{n+m} \prod_{j=1}^{r-1}(1+\beta_j-k_j)_{n+m}(1+p-\beta_r-k_1-k_2-\cdots-k_{r-1})_n}$$

$$\cdot \frac{z^n \, (1)^m}{(1+\beta_r-p+k_1+k_2+\cdots+k_{r-1})_m \, n! \, m!}. \tag{51}$$

The required identity (47), which is asserted by Theorem 8, would now follow upon multiplying the right-hand sides of the equations (48) and (51) by $\dfrac{p!}{z^p}$. □

Remark 6. By applying the fourth general double-series identity asserted by Theorem 8, one can derive several further results, which are analogous to those in Section 3 and Section 7, involving the Kampé de Fériet type double hypergeometric functions and the generalized hypergeometric function of one variable.

9. Concluding Remarks and Observations

In our present investigation is motivated essentially by the widely- and extensively-demonstrated usefulness of various families of such Special Functions as the hypergeometric functions of one, two and more variables, and their associated summation, transformation and reduction formulas, are potentially useful not only as solutions of ordinary and partial differential equations, but also in the widespread problems in the mathematical, physical, engineering and statistical sciences (see, for example, [29], [30]; see also [20]). Here, in this paper, we have established four general double-series identities by using some zero-balanced terminating summation theorems for ${}_3F_2(1)$, ${}_4F_3(1)$ and ${}_{r+1}F_r(1)$. by using these general double-series identities, we have deduced the sum (or difference) of two double hypergeometric functions of the Kampé de Fériet type. We have also derived closed-form reduction formulas for the following Clausen hypergeometric function:

$$ {}_3F_2\left(-\frac{27z}{4(1-z)^3}\right) $$

and the Srivastava-Daoust double hypergeometric function with the arguments $(z, -\frac{z}{4})$. The various results, which we have presented in this article, are potentially useful in mathematical analysis and applied mathematics.

With a view to encouraging and motivating further researches emerging from the present investigation, we have chosen to draw the attention of the interested readers toward some related recent developments (see, for example, [7], [20], [21], [22], [23], [33], [34] and [35]) on hypergeometric functions, hypergeometric polynomials and other families of higher transcendental functions, and also various basic (or quantum or q-) series and basic (or quantum or q-) polynomials.

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References

[1] G. D. Anderson, R. W. Barnard, K. C. Richards, M. K. Vamanamurthy and M. Vuorinen, Inequalities for zero-balanced hypergeometric functions, *Trans. Amer. Math. Soc.* **347** (1995), 1713–1723.

[2] P. Appell, Sur les séries hypergéoméetriques de deux variables, et sur des équations différentialles linéaires aux dérivées partielles, *C. R. Acad. Sci. Paris* **90** (1880), 296–298.

[3] P. Appell, *Sur les Fonctions Hypergéométriques de Plusieurs Variables*, Mémor. Sci. Math. Fasc. **3**, Gauthier-Villars, Paris, 1925.

[4] P. Appell and J. Kampé de Fériet, *Fonctions Hypergéométriques et Hypersphériques*; *Polynômes d'Hermite*, Gauthier-Villars, Paris, 1926.

[5] W. Bühring and H. M. Srivastava, Analytic continuation of the generalized hypergeometric series near unit argument with emphasis on the zero-balanced series, in *Approximation Theory and Applications* (Th. M. Rassias, Editor), Hadronic Press, Palm Harbor, Florida, 1998, pp. 17–35.

[6] J. L. Burchnall and T. W. Chaundy, Expansions of Appell's double hypergeometric functions (II), *Quart. J. Math. Oxford Ser.* **12** (1941), 112–128.

[7] J. Cao, H. M. Srivastava, H.-L. Zhou and S. Arjika, Generalized q-difference equations for q-hypergeometric polynomials with double q-binomial coefficients, *Mathematics* **10** (2022), Article ID 556, 1–17.

[8] R. J. Evans and D. Stanton, Asymptotic formulas for zero-balanced hypergeometric series, *SIAM J. Math. Anal.* **15** (1984), 1010–1020.

[9] N. T. Hài, O. I. Marichev and H. M. Srivastava, A note on the convergence of certain families of multiple hypergeometric series, *J. Math. Anal. Appl.* **164** (1992), 104–115.

[10] P. Humbert, La fonction $W_k, \mu_1, \mu_2, \cdots, \mu_n(x_1, x_2, \cdots, xn)$, *C. R. Acad. Sci. Paris* **171** (1920), 428–430.

[11] P. Humbert, The confluent hypergeometric functions of two variables, *Proc. Royal Soc. Edinburgh Sect. A* **41** (1922), 73–96.

[12] J. Kampé de Fériet, Les Fonctions hypergéométriques d'ordre supérieur à deux variables, *C. R. Acad. Sci. Paris* **173** (1921), 401–404.

[13] D. Karp, An approximation for zero-balanced Appell function F_1 near $(1, 1)$, *J. Math. Anal. Appl.* **339** (2008), 1332–1341.

[14] A. P. Prudnikov, Yu. A. Brychkov and O. I. Marichev, *Integrals and Series*, Vol. **III**, *More Special Functions*, Nauka, Moscow, 1986 (in Russian); (Translated from the Russian by G. G. Gould), Gordon and Breach Science Publishers, New York, Philadelphia, London, Paris, Montreux, Tokyo and Melbourne, 1990.

[15] E. D. Rainville, *Special Functions*, The Macmillan Company, New York, 1960; Reprinted by Chelsea Publishing Company, Bronx, New York, 1971.

[16] K. C. Richards, A note on inequalities for the ratio of zero-balanced hypergeometric functions, *Proc. Amer. Math. Soc. Ser. B* **6** (2019), 15–20.

[17] M. Saigo and H. M. Srivastava, The behavior of the zero-balanced hypergeometric series ${}_pF_{p-1}$ near the boundary of its convergence region, *Proc. Amer. Math. Soc.* **110** (1990), 71–76.
[18] A. K. Srivastava, Asymptotic behaviour of certain zero-balanced hypergeometric series, *Proc. Indian Acad. Sci. Math. Sci.* **106** (1996), 39–51.
[19] H. M. Srivastava, A transformation for an n-balanced ${}_3\Phi_2$, *Proc. Amer. Math. Soc.* **101** (1987), 108–112.
[20] H. M. Srivastava, A survey of some recent developments on higher transcendental functions of analytic number theory and applied mathematics, *Symmetry* **13** (2021), Article ID 2294, 1–22.
[21] H. M. Srivastava, An introductory overview of fractional-calculus operators based upon the Fox-Wright and related higher transcendental functions, *J. Adv. Engrg. Comput.* **5** (2021), 135–166.
[22] H. M. Srivastava, An introductory overview of the Bessel polynomials, the generalized Bessel polynomials and the q-Bessel polynomials. *Symmetry* **15** (2023), Article ID 822, 1–28.
[23] H. M. Srivastava, R. C. S. Chandel and H. Kumar, Some general Hurwitz-Lerch type zeta functions associated with the Srivastava-Daoust multiple hypergeometric functions, *J. Nonlinear Var. Anal.* **6** (2022), 299–315.
[24] H. M. Srivastava and M. C. Daoust, On Eulerian integrals associated with Kampé de Fériet's function, *Publ. Inst. Math.* (*Beograd*) (*Nouvelle Sér.*) **9** (**23**) (1969), 199–202.
[25] H. M. Srivastava and M. C. Daoust, Certain generalized Neumann expansions associated with the Kampé de Fériet function, *Nederl. Akad. Wetensch. Proc. Ser. A* **72** = *Indag. Math.* **31** (1969), 449–457.
[26] H. M. Srivastava and M. C. Daoust, A note on the convergence of Kampé de Fériet's double hypergeometric series, *Math. Nachr.* **53** (1972), 151–159.
[27] H. M. Srivastava and V. K. Jain, q-Series identities and reducibility of basic double hypergeometric functions, *Canad. J. Math.* **38** (1986), 215–231.
[28] H. M. Srivastava and V. K. Jain, An elementary proof of a certain transformation for an n-balanced hypergeometric ${}_3\Phi_2$ series, *Proc. Japan Acad. Ser. A Math. Sci.* **64** (1988), 198–200.
[29] H. M. Srivastava and P. W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1985.
[30] H. M. Srivastava and B. R. K. Kashyap, *Special Functions in Queuing Theory and Related Stochastic Processes*, Academic Press, New York and London, 1982.
[31] H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester Brisbane and Toronto, 1984.
[32] H. M. Srivastava and R. Panda, An integral representation for the product of two Jacobi polynomials, *J. London Math. Soc.* (*Ser.* 2) **12** (1976), 419–425.
[33] H. M. Srivastava, M. I. Qureshi and S. Jabee, Some general series identities and summation theorems for Clausen's hypergeometric function with negative integer numerator and denominator parameters, *J. Nonlinear Convex Anal.* **21** (2020), 805–819.
[34] H. M. Srivastava, M. I. Qureshi and S. H. Malik, Some hypergeometric transformations and reduction formulas for the Gauss function and their applications involving the Clausen function, *J. Nonlinear Var. Anal.* **5** (2021), 981–987.
[35] H. M. Srivastava, M. I. Qureshi and S. H. Malik, Some modified reduction formulas for the Gauss and Clausen hypergeometric functions, *Proc. Jangjeon Math. Soc.* **24** (2021), 275–283.
[36] L.-M. Wang, Mapping properties of the zero-balanced hypergeometric functions, *J. Math. Anal. Appl.* **505** (2022), Article ID 125448, 1–13.
[37] M.-K. Wang and Y.-M. Chu, Refinements of transformation inequalities for zero-balanced hypergeometric functions, *Acta Math. Sci.* **37** (2017), 607–622.
[38] M.-K. Wang, Y.-M. Chu and Y.-P. Jiang, Ramanujan's cubic transformation inequalities for zero-balanced hypergeometric functions, *Rocky Mountain J. Math.* **46** (2016), 679–691.
[39] M.-K. Wang, Y.-M. Chu and W. Zhang, Monotonicity and inequalities involving zero-balanced hypergeometric function, *Math. Inequal. Appl.* **22** (2019), 601–617.
[40] T.-H. Zhao, Z.-Y. He and Y.-M. Chu, On some refinements for inequalities involving zero-balanced hypergeometric function, *AIMS Math.* **5** (2020), 6479–6495.