Filomat 37:22 (2023), 7383–7399
https://doi.org/10.2298/FIL2322383I

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

An improvement of Alzer-Fonseca-Kovačec's type inequalities with applications

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Abstract. In this paper, we start by presenting multiple-term refinement of Young's and Young's type inequalities and its reverse using different weights, which extends and unifies two recent and important results due to M. Khosravi (Math. Slovaca **68** (2018), 803–810) and L. Nasiri et al. (Asian-European Journal Math. **15**, (2022) No. 07). Further, we mainly present some new real power inequalities of Young's inequality, extending a key results of Alzer et al. (Linear Multilinear Algebra **63**(3) (2015), 622–635), D. Q. Huy et al. (Linear Algebra Appl. **656** (2023), 368–384) and J. Zhao (Bull. Malys. Math. Sci. Soc. **46**, No 52 (2023)). As applications of these results, we provide some inequalities for matrices, unitarily invariant norms and traces.

1. Introduction and preliminaries

The classical Young inequality states that

$$a^{\alpha}b^{1-\alpha} \leq \alpha a + (1-\alpha)b, \tag{1}$$

where $a, b > 0$ and $0 \leq \alpha \leq 1$. Equality holds if and only if $a = b$. This inequality can be considered as the weighted arithmetic-geometric mean inequality

$$a\sharp_{\alpha}b \leq a\nabla_{\alpha}b, \tag{2}$$

where $a\sharp_{\alpha}b = a^{\alpha}b^{1-\alpha}$ and $a\nabla_{\alpha}b = \alpha a + (1-\alpha)b$. This inequality, despite its simplicity, has attracted the interest of researchers in operator theory because of its applications in this field. Several studies have focused on refining this inequality and its reverse by finding intermediate terms or adding some positive quantities.

The first refinement of Young's inequality is the squared version, which is formulated and proved in [7] as follows:

$$(a^{\alpha}b^{1-\alpha})^2 + \mathrm{r}_0^2(a-b)^2 \leq (\alpha a + (1-\alpha)b)^2, \tag{3}$$

2020 *Mathematics Subject Classification*. 26D07, 15A45, 47A30.
Keywords. Young's type inequalities; Alzer-Fonseca-Kovačec's inequalities; Matrix inequalities; Weak sub-majorization.
Received: 15 January 2023; Accepted: 30 March 2023
Communicated by Dragan S. Djordjević
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where $r_0 = \min\{\alpha, 1-\alpha\}$.

Later, Kittaneh and Manasrah [15] obtained the following interesting refinement of Young's inequality and its reverse:

$$a^\alpha b^{1-\alpha} + r_0(\sqrt{a} - \sqrt{b})^2 \leq \alpha a + (1-\alpha)b \leq a^\alpha b^{1-\alpha} + R_0(\sqrt{a} - \sqrt{b})^2, \tag{4}$$

where $r_0 = \min\{\alpha, 1-\alpha\}$ and $R_0 = \max\{\alpha, 1-\alpha\}$.

In [14], Kórus gave the following refinement of Young's inequality:

$$\left(1 + L(\alpha)\log^2\left(\frac{a}{b}\right)\right) a^\alpha b^{1-\alpha} \leq \alpha a + (1-\alpha)b. \tag{5}$$

where L is the 1-periodic function defined by

$$L(t) := \begin{cases} \frac{t^2}{2}\left(\frac{1-t}{t}\right)^{2t} & \text{if } t \in (0,1] \\ 0 & \text{if } t = 0. \end{cases} \tag{6}$$

Very recently, Ighachane and Akkouchi established the following multiple-term refinements of Young's inequality.

Theorem 1.1 ([10]). *Let $a, b > 0$, $0 \leq \alpha \leq 1$ and $N \in \mathbb{N}^*$. Then*

$$\begin{aligned} \left(1 + \frac{L(2^N\alpha)}{2^{2N}}\log^2\left(\frac{a}{b}\right)\right) a^\alpha b^{1-\alpha} &+ \sum_{l=0}^{N-1} r_l(\alpha) \sum_{k=1}^{2^l} f_{l,k}(a,b)\chi_{\left(\frac{k-1}{2^l}, \frac{k}{2^l}\right)}(\alpha) \\ &\leq \alpha a + (1-\alpha)b, \end{aligned} \tag{7}$$

where χ_I stands for the characteristic function of an interval I, $r_0 = \min\{\alpha, 1-\alpha\}$, $r_n(\alpha) = \min\{2r_{n-1}(\alpha), 1-2r_{n-1}(\alpha)\}$ and $f_{l,k}(a,b) = \left(\sqrt{a^{\frac{k-1}{2^l}} b^{1-\frac{k-1}{2^l}}} - \sqrt{a^{\frac{k}{2^l}} b^{1-\frac{k}{2^l}}}\right)^2$.

For some related refinements and generalizations of Young's inequality, we refer the reader to recent papers [6, 21] and the references therein.

In [1], Alzer, Fonseca and Kovačec established the following generalization of inequalities (4) for two weights arithmetic and geometric means using two different weights.

Theorem 1.2 (Alzer-Fonseca-Kovačec). *Let $a, b > 0$, $0 \leq \alpha < \beta \leq 1$ and $\lambda \geq 1$. Then*

$$\left(\frac{\alpha}{\beta}\right)^\lambda \left((a\nabla_\beta b)^\lambda - (a\sharp_\beta b)^\lambda\right) \leq (a\nabla_\alpha b)^\lambda - (a\sharp_\alpha b)^\lambda \leq \left(\frac{1-\alpha}{1-\beta}\right)^\lambda \left((a\nabla_\beta b)^\lambda - (a\sharp_\beta b)^\lambda\right). \tag{8}$$

The significance of Theorem 1.2 is as follows: when $\beta = \frac{1}{2}$ or $\alpha = \frac{1}{2}$, then Theorem 1.2 retrieves inequalities (4). In fact, Theorem 1.2 implies better estimates than the main result of [2]. The Alzer-Fonseca-Kovačec inequalities can be regarded as a major development concerning the Young's inequality.

The Kantorovich constant is defined by $K(t,2) = \frac{(t+1)^2}{4t}$ for $t > 0$. Recently, Khosravi has obtained in [16] the following refinement of Alzer-Fonseca-Kovačec inequalities (8) for $\lambda = 1$ using the Kantorovich constant as follows

Theorem 1.3. *Let $a, b > 0$ and $0 \leq \alpha < \beta \leq 1$. Then*

$$K(h^\beta, 2)^{r'} a\sharp_\alpha b \leq a\nabla_\alpha b - \left(\frac{\alpha}{\beta}\right)(a\nabla_\beta b - a\sharp_\beta b) \leq K(h^\beta, 2)^{R'} a\sharp_\alpha b, \tag{9}$$

where $r' = \min\{\frac{\alpha}{\beta}, 1-\frac{\alpha}{\beta}\}$, $R' = \max\{\frac{\alpha}{\beta}, 1-\frac{\alpha}{\beta}\}$ *and* $h = \frac{b}{a}$. *In addition,*

$$K(h^{1-\alpha},2)^{r''} a\sharp_\beta b \le a\nabla_\beta b - \left(\frac{1-\beta}{1-\alpha}\right)(a\nabla_\alpha b - a\sharp_\alpha b) \le K(h^{1-\alpha},2)^{R''} a\sharp_\beta b, \tag{10}$$

where $r'' = \min\{\frac{1-\beta}{1-\alpha}, 1-\frac{1-\beta}{1-\alpha}\}$, $R'' = \max\{\frac{1-\beta}{1-\alpha}, 1-\frac{1-\beta}{1-\alpha}\}$ *and* $h = \frac{b}{a}$.

Very recently, Nasiri et al. obtained in [18] the following refinement of Theorem 1.3 using the Kantorovich constant.

Theorem 1.4. *Let* $a, b \ge 0$ *and* $0 \le \alpha < \beta \le 1$. *Then*

$$\begin{aligned} r\left(\sqrt{b} - \sqrt{a\sharp_\beta b}\right)^2 + K(\sqrt{h^\beta},2)^{r'} a\sharp_\alpha b &\le a\nabla_\alpha b - \left(\frac{\alpha}{\beta}\right)(a\nabla_\beta b - a\sharp_\beta b) \\ &\le R(\sqrt{b} - \sqrt{a\sharp_\beta b})^2 + K(\sqrt{h^\beta},2)^{R'} a\sharp_\alpha b, \end{aligned}$$

where $r = \min\{\frac{\alpha}{\beta}, 1-\frac{\alpha}{\beta}\}$, $R = \max\{\frac{\alpha}{\beta}, 1-\frac{\alpha}{\beta}\}$, $r' = \min\{2r, 1-2r\}$, $R' = \max\{2r, 1-2r\}$ *and* $h = \frac{b}{a}$. *Moreover,*

$$\begin{aligned} r\left(\sqrt{a} - \sqrt{a\sharp_\alpha b}\right)^2 + K(\sqrt{h^{1-\alpha}},2)^{r'} a\sharp_\beta b &\le a\nabla_\beta b - \left(\frac{1-\beta}{1-\alpha}\right)(a\nabla_\alpha b - a\sharp_\alpha b) \\ &\le R(\sqrt{a} - \sqrt{a\sharp_\alpha b})^2 + K(\sqrt{h^{1-\alpha}},2)^{R'} a\sharp_\beta b, \end{aligned}$$

where $r = \min\{\frac{1-\beta}{1-\alpha}, 1-\frac{1-\beta}{1-\alpha}\}$, $R = \max\{\frac{1-\beta}{1-\alpha}, 1-\frac{1-\beta}{1-\alpha}\}$, $r' = \min\{2r, 1-2r\}$ *and* $R' = \max\{2r, 1-2r\}$ *and* $h = \frac{b}{a}$.

In a recent work, Kai [13] gave the following Young type inequality:

$$\left[\alpha^{2\alpha}a^\alpha b^{1-\alpha}\chi_{(0,\frac{1}{2}]}(\alpha) + (1-\alpha)^{2-2\alpha}a^\alpha b^{1-\alpha}\chi_{(\frac{1}{2},1)}(\alpha)\right] + r_0^2(\sqrt{a}-\sqrt{b})^2 \le \alpha^2 a + (1-\alpha)^2 b, \tag{11}$$

where $r_0 = \min\{\alpha, 1-\alpha\}$ and χ_I the charcteristic function.

Very recently, Nasiri et al. obtained in [19] the following refinement of Young's type inequality (11) using the Kantorovich constant.

Theorem 1.5. *Let* $a, b \ge 0$ *and* $0 \le \alpha < \beta \le 1$.

1. *If* $0 \le \alpha < \beta \le \frac{1}{2}$ *then*

$$\begin{aligned} \alpha^2 a + (1-\alpha)^2 b &\ge \alpha^{2\alpha}a^\alpha b^{1-\alpha} + \alpha^2\left(\sqrt{a}-\sqrt{b}\right)^2 + r_0\sqrt{b}\left(\sqrt{(\alpha\sqrt{a})\sharp_{2\beta}\sqrt{b}} - \sqrt[4]{b}\right)^2 \\ &+ \left(\frac{\alpha\sqrt{b}}{\beta}\right)\left(\left(\alpha\sqrt{a}\right)\nabla_{2\beta}\sqrt{b} - \left(\alpha\sqrt{a}\right)\sharp_{2\beta}\sqrt{b}\right), \end{aligned}$$

   where $r_0 = \min\{2\alpha, 1-2\alpha\}$.

2. *If* $\frac{1}{2} \le \alpha < \beta \le 1$ *then*

$$\begin{aligned} \alpha^2 a + (1-\alpha)^2 b &\ge (1-\alpha)^{2-2\alpha}a^\alpha b^{1-\alpha} + (1-\alpha)^2\left(\sqrt{a}-\sqrt{b}\right)^2 \\ &+ r_0\sqrt{a}\left(\sqrt{\sqrt{a}\sharp_{2\beta-1}(1-\alpha)\sqrt{b}} - \sqrt{(1-\alpha)\sqrt{b}}\right)^2 \\ &+ \left(\frac{2\alpha-1}{2\beta-1}\right)\sqrt{a}\left(\sqrt{a}\nabla_{2\beta-1}(1-\alpha)\sqrt{b} - \sqrt{a}\sharp_{2\beta-1}((1-\alpha)\sqrt{b})\right), \end{aligned}$$

   where $r_0 = \min\{2\alpha-1, 2-2\alpha\}$.

The main aim of this work is to establish refinements and generalisations of Alzer-Fonseca-Kovačec type inequalities. In Section 2, we obtain the improvement and reverse improvement of Alzer-Fonseca-Kovačec inequalities (8) and Alzer-Fonseca-Kovačec type inequalities (11) for $\lambda = 1$ by adding as many refining terms as we wish. In Section 3, via the weak sub-majorization theory we present a new refinement of Alzer-Fonseca-Kovačec inequalities mentioned in Theorem 1.2. Further, we establish some Alzer-Fonseca-Kovačec type inequalities for inequality (11). In Section 4 and 5, we apply the obtained results of the previous sections to matrix inequalities.

2. Multiple-term refinement of Young's and Young's type inequalities and its reverse using different weights

We start this section by the following refinement of the first inequality in Theorem 1.2 for $\lambda = 1$ by adding as much refining terms as desired.

Theorem 2.1. *Let $a, b > 0$, $0 < \alpha \leq \beta < 1$ and $N \in \mathbb{N}^*$. Then*

$$\begin{aligned}\left(1+\frac{L\left(2^N\frac{\alpha}{\beta}\right)}{2^{2N}}\log^2\left(\frac{a\sharp_\beta b}{b}\right)\right)a\sharp_\alpha b &+ \frac{\alpha}{\beta}\left(a\nabla_\beta b - a\sharp_\beta b\right)\\ &+ \sum_{l=0}^{N-1}\mathrm{r}_l\left(\frac{\alpha}{\beta}\right)\sum_{k=1}^{2^l} f_{l,k}(a\sharp_\beta b, b)\chi_{\left(\frac{k-1}{2^l},\frac{k}{2^l}\right)}\left(\frac{\alpha}{\beta}\right)\\ &\leq a\nabla_\alpha b.\end{aligned}$$

Proof. By using Theorem 1.1, we have

$$\begin{aligned}a\nabla_\alpha b - \frac{\alpha}{\beta}\left(a\nabla_\beta b - a\sharp_\beta b\right) &= \left(1-\frac{\alpha}{\beta}\right)b + \frac{\alpha}{\beta}a\sharp_\beta b\\ &\geq \left(1+\frac{L\left(2^N\frac{\alpha}{\beta}\right)}{2^{2N}}\log^2\left(\frac{a\sharp_\beta b}{b}\right)\right)b^{1-\frac{\alpha}{\beta}}(a\sharp_\beta b)^{\frac{\alpha}{\beta}}\\ &+ \sum_{l=0}^{N-1}\mathrm{r}_l\left(\frac{\alpha}{\beta}\right)\sum_{k=1}^{2^l} f_{l,k}(a\sharp_\beta b, b)\chi_{\left(\frac{k-1}{2^l},\frac{k}{2^l}\right)}\left(\frac{\alpha}{\beta}\right)\\ &= \left(1+\frac{L\left(2^N\frac{\alpha}{\beta}\right)}{2^{2N}}\log^2\left(\frac{a\sharp_\beta b}{b}\right)\right)a\sharp_\alpha b\\ &+ \sum_{l=0}^{N-1}\mathrm{r}_l\left(\frac{\alpha}{\beta}\right)\sum_{k=1}^{2^l} f_{l,k}(a\sharp_\beta b, b)\chi_{\left(\frac{k-1}{2^l},\frac{k}{2^l}\right)}\left(\frac{\alpha}{\beta}\right).\end{aligned}$$

□

As a consequence of Theorem 2.1 is the following refinement of the second inequality in Theorem 1.2 for $\lambda = 1$, by adding as many refining terms as we wish.

Theorem 2.2. *Let $a, b > 0$, $0 < \alpha \leq \beta < 1$ and $N \in \mathbb{N}^*$. Then*

$$\begin{aligned}\left(1+\frac{L\left(2^N\frac{1-\beta}{1-\alpha}\right)}{2^{2N}}\log^2\left(\frac{a\sharp_\alpha b}{a}\right)\right)a\sharp_\beta b &+ \frac{1-\beta}{1-\alpha}\left(a\nabla_\alpha b - a\sharp_\alpha b\right)\\ &+ \sum_{l=0}^{N-1}\mathrm{r}_l\left(\frac{1-\beta}{1-\alpha}\right)\sum_{k=1}^{2^l} f_{l,k}(a\sharp_\alpha b, a)\chi_{\left(\frac{k-1}{2^l},\frac{k}{2^l}\right)}\left(\frac{1-\beta}{1-\alpha}\right)\\ &\leq a\nabla_\beta b.\end{aligned}$$

Proof. Notice that, if $0 \leq \alpha \leq \beta \leq 1$ then $0 \leq 1-\beta \leq 1-\alpha \leq 1$. Hence, by changing respectively a, b, α and β by $b, a, 1-\beta$ and $1-\alpha$ in Theorem 2.1 we obtain the desired result. □

Remark 2.3. *Observe that, for $\lambda = 1$ the second inequality in Theorem 1.2 can be written as follows:*

$$\left(\frac{1-\beta}{1-\alpha}\right)[(a\nabla_\alpha b) - (a\sharp_\alpha b)] \leq \left(a\nabla_\beta b\right) - (a\sharp_\beta b), \tag{12}$$

where $0 \leq \alpha < \beta \leq 1$. Consequently, Theorem 2.2 present multiple refining terms of inequality (12).

Using Theorem 1.1, we obtain the following version of Alzer-Fonseca-Kovačec's type inequalities for inequality (11) for $\lambda = 1$ and its refinement.

Theorem 2.4. *Let $a, b > 0$ and α, β be two real numbers.*

1. *If $0 < \alpha < \beta \leq \frac{1}{2}$. Then*

$$\begin{aligned}\alpha^2 a + (1-\alpha)^2 b &\geq \alpha^{\frac{\alpha}{\beta}} a\sharp_\alpha b + \frac{\alpha^2}{\beta}(a\nabla_\beta b - a\sharp_\beta b)\\ &+ \sum_{l=0}^{N-1} \mathrm{r}_l\Big(\frac{\alpha}{\beta}\Big)\sum_{k=1}^{2^l} f_{l,k}(\alpha(a\sharp_\beta b), b)\chi_{\left(\frac{k-1}{2^l}, \frac{k}{2^l}\right)}\Big(\frac{\alpha}{\beta}\Big).\end{aligned}$$

2. *If $\frac{1}{2} \leq \alpha < \beta < 1$. Then*

$$\begin{aligned}\beta^2 a + (1-\beta)^2 b &\geq (1-\beta)^{\frac{1-\beta}{1-\alpha}} a\sharp_\beta b + \frac{(1-\beta)^2}{1-\alpha}(a\nabla_\alpha b - a\sharp_\alpha b)\\ &+ \sum_{l=0}^{N-1} \mathrm{r}_l\left(\frac{1-\beta}{1-\alpha}\right)\sum_{k=1}^{2^l} f_{l,k}((1-\beta)(a\sharp_\alpha b), b)\chi_{\left(\frac{k-1}{2^l}, \frac{k}{2^l}\right)}\left(\frac{1-\beta}{1-\alpha}\right).\end{aligned}$$

Proof. We first claim that

$$(1-\alpha)^2 - \alpha^2\left(\frac{1-\beta}{\beta}\right) \geq (1-\alpha) - \alpha\left(\frac{1-\beta}{\beta}\right) = \left(1 - \frac{\alpha}{\beta}\right).$$

For $0 < \alpha < \beta \leq \frac{1}{2}$, we have

$$\begin{aligned}
& (1-\alpha)^2 - \alpha^2\left(\frac{1-\beta}{\beta}\right) \geq (1-\alpha) - \alpha\left(\frac{1-\beta}{\beta}\right)\\
\iff\ & (1-\alpha)^2 - (1-\alpha) \geq \alpha^2\left(\frac{1-\beta}{\beta}\right) - \alpha\left(\frac{1-\beta}{\beta}\right)\\
\iff\ & \frac{(1-\alpha)^2 - (1-\alpha)}{1-\beta} \geq \frac{\alpha^2 - \alpha}{\beta}\\
\iff\ & \frac{(\alpha-1)\alpha}{1-\beta} \geq \frac{\alpha(\alpha-1)}{\beta}\\
\iff\ & \frac{1}{1-\beta} \leq \frac{1}{\beta}\\
\iff\ & 0 < \beta \leq \frac{1}{2}.
\end{aligned}$$

Now, by using Theorem 1.1, we have

$$\begin{aligned}\alpha^2 a+(1-\alpha)^2 b &- \frac{\alpha^2}{\beta}(a\nabla_\beta b-a\sharp_\beta b)=\left((1-\alpha)^2-\alpha^2\left(\frac{1-\beta}{\beta}\right)\right)b+\frac{\alpha}{\beta}\Big(\alpha(a\sharp_\beta b)\Big)\\
&\geq \left((1-\alpha)-\alpha\left(\frac{1-\beta}{\beta}\right)\right)b+\frac{\alpha}{\beta}\left(\alpha(a\sharp_\beta b)\right)\\
&= \left(1-\frac{\alpha}{\beta}\right)b+\frac{\alpha}{\beta}\left(\alpha(a\sharp_\beta b)\right)\\
&\geq \alpha^{\frac{\alpha}{\beta}}b^{1-\frac{\alpha}{\beta}}(a\sharp_\beta b)^{\frac{\alpha}{\beta}}+\sum_{l=0}^{N-1}\mathrm{r}_l\Big(\frac{\alpha}{\beta}\Big)\sum_{k=1}^{2^l}f_{l,k}(\alpha(a\sharp_\beta b),b)\chi_{\left(\frac{k-1}{2^l},\frac{k}{2^l}\right)}\left(\frac{\alpha}{\beta}\right)\\
&= \alpha^{\frac{\alpha}{\beta}}a\sharp_\alpha b+\sum_{l=0}^{N-1}\mathrm{r}_l\left(\frac{\alpha}{\beta}\right)\sum_{k=1}^{2^l}f_{l,k}(\alpha(a\sharp_\beta b),b)\chi_{\left(\frac{k-1}{2^l},\frac{k}{2^l}\right)}\left(\frac{\alpha}{\beta}\right).\end{aligned}$$

This prove the first inequality.
Notice that, if $\frac{1}{2}\leq\alpha<\beta<1$ then $0<\alpha<\beta\leq\frac{1}{2}$. Hence, by changing respectively a,b,α and β by $b,a,1-\beta$ and $1-\alpha$ in the first inequality in Theorem 2.4, we obtain the desired results. □

3. An improvement of Alzer-Fonseca-Kovačec's inequalities via weak sub-majorization

In this section, we give an improved version of Theorem 1.2, due to Alzer-Fonseca-Kovačec's inequalities and Alzer-Fonseca-Kovačec's type inequalities for inequality (11).

The method used in [11, 12] to prove a refinement of Alzer-Fonseca-Kovačec's type inequalities has the AM-GM inequality approach, that we cannot use her to prove the general case for a real power λ. To prove the general case we begin with recalling the theory of weak sub-majorization. Throughout this section, we denote by $u^*=\left(u_1^*,\dots,u_n^*\right)$ the vector obtained from the vector $u=(u_1,\dots,u_n)\in\mathbb{R}^n$ by rearranging the components of it in decreasing order. Then, for two vectors $u=(u_1,\dots,u_n)$ and $v=(v_1,\dots,v_n)$ in $\mathbb{R}^n, u$ is said to be weakly sub-majorized by v, written $u\prec_w v$, if

$$\sum_{i=1}^k u_i^*\leq\sum_{i=1}^k v_i^*$$

for all $k=1,\dots,n$. An important feature of the theory of weak sub-majorization which will be used in proofs of our results is given by the following lemma.

Lemma 3.1. *[17, pp. 13] Let $u=(u_i)_{i=1}^n, v=(v_i)_{i=1}^n\in\mathbb{R}^n$ and $J\subset\mathbb{R}$ be an interval containing the components of u and v. If $u\prec_w v$ and $\phi:J\to\mathbb{R}$ is a continuous increasing convex function, then*

$$\sum_{i=1}^n\phi\left(u_i\right)\leq\sum_{i=1}^n\phi\left(v_i\right).$$

3.1. An improvement of Alzer-Fonseca-Kovačec's inequalities

In order to accomplish our results, we need the following Lemmas.
The following Lemma is a direct consequence of Theorem 2.1 for $N=1$.

Lemma 3.2. *Let $a,b>0$ and $0<\alpha\leq\beta<1$. Then*

$$a\nabla_\alpha b\geq\left(1+\frac{L\left(2\frac{\alpha}{\beta}\right)}{4}\beta^2\log^2\left(\frac{a}{b}\right)\right)(a\sharp_\alpha b)+\frac{\alpha}{\beta}\left(a\nabla_\beta b-(a\sharp_\beta b)\right)+2\mathrm{r}_0\left(b\nabla(a\sharp_\beta b)-a\sharp_{\frac{\beta}{2}}b\right),$$

where $\mathrm{r}_0=\min\{\frac{\alpha}{\beta},1-\frac{\alpha}{\beta}\}$.

Lemma 3.3. *Let $a, b > 0$, $0 < \alpha \le \beta < 1$ and let $u = (u_1, u_2, u_3), v = (v_1, v_2, v_3) \in \mathbb{R}^3$ be two vectors with components*

$$u_1 = \left(1 + \lambda_1 \log^2\left(\frac{a}{b}\right)\right) a\sharp_\alpha b,\ u_2 = \frac{\alpha}{\beta}\left(a\nabla_\beta b\right), u_3 = 2\mathrm{r}_0\left(b\nabla(a\sharp_\beta b)\right),$$

and

$$v_1 = a\nabla_\alpha b,\ v_2 = \frac{\alpha}{\beta}(a\sharp_\beta b),\ v_3 = 2\mathrm{r}_0 a\sharp_{\frac{\beta}{2}} b,$$

where $\mathrm{r}_0 = \min\{\frac{\alpha}{\beta}, 1 - \frac{\alpha}{\beta}\}$ *and* $\lambda_1 = \min\left\{\frac{L\left(2\frac{\alpha}{\beta}\right)}{4}\beta^2, L(\frac{\alpha}{\beta}), L(\alpha)\right\}$. *Then, we have $u \prec_w v$, namely, the vectors u^* and v^* have components satisfying that*

$$u_1^* \le v_1^*, \tag{13}$$

$$u_1^* + u_2^* \le v_1^* + v_2^*, \tag{14}$$

$$u_1^* + u_2^* + u_3^* \le v_1^* + v_2^* + v_3^*. \tag{15}$$

Proof. In order to prove (13) remark that v_1^* is exactly v_1. Indeed, on one hand we have $v_1 \ge v_2$, since $v_1 \ge u_2$ and $u_2 \ge v_2$. On the other hand, we have

$$v_1 - u_3 = (1 - \alpha - \mathrm{r}_0)b + \alpha a - \mathrm{r}_0 a\sharp_\beta b \ge 0. \tag{16}$$

In fact, first of all remark that

$$\{(\alpha,\beta) \in [0,1]^2 \ :\ 0 < \alpha \le \beta < 1\} = \left\{(\alpha,\beta) \in [0,1]^2 \ :\ 0 < \alpha \le \frac{\beta}{2}\right\} \bigcup \left\{(\alpha,\beta) \in [0,1]^2 \ :\ \frac{\beta}{2} \le \alpha < 1\right\}.$$

At this point, we distinguish two situations.
If $\alpha \in \left(0, \frac{\beta}{2}\right]$, then $\mathrm{r}_0 = \frac{\alpha}{\beta}$, and so (16) becomes

$$\begin{aligned} v_1 - u_3 &= \left(1 - \alpha - \frac{\alpha}{\beta}\right)b + \alpha a - \frac{\alpha}{\beta} a\sharp_\beta b \\ &= \left(1 - \alpha - \frac{\alpha}{\beta}\right)b + \alpha a - \frac{\alpha}{\beta} a\sharp_\beta b + \frac{\alpha}{\beta}\left(a\nabla_\beta b\right) - \frac{\alpha}{\beta}\left(a\nabla_\beta b\right) \\ &= \left(1 - \alpha - \frac{\alpha}{\beta} - \frac{\alpha}{\beta}(1-\beta)\right)b + \frac{\alpha}{\beta}\left(a\nabla_\beta b - (a\sharp_\beta b)\right) \\ &= \left(\frac{\beta - 2\alpha}{\beta}\right)b + \frac{\alpha}{\beta}\left(a\nabla_\beta b - (a\sharp_\beta b)\right) \\ &\ge 0. \end{aligned}$$

If $\alpha \in \left[\frac{\beta}{2}, 1\right)$, then $\mathrm{r}_0 = 1 - \frac{\alpha}{\beta}$. Therefore

$$\begin{aligned} v_1 - u_3 &= \left(\frac{\alpha}{\beta} - \alpha\right)b + \alpha a - \left(1 - \frac{\alpha}{\beta}\right)(a\sharp_\beta b) \\ &= \left(\frac{\alpha}{\beta} - \alpha\right)b + \alpha a - \left(1 - \frac{\alpha}{\beta}\right)(a\sharp_\beta b) + \left(1 - \frac{\alpha}{\beta}\right)\left(a\nabla_\beta b\right) \\ &- \left(1 - \frac{\alpha}{\beta}\right)\left(a\nabla_\beta b\right) \\ &= \left(\frac{(1-\beta)(2\alpha - \beta)}{\beta}\right)b + (2\alpha - \beta)a + \left(1 - \frac{\alpha}{\beta}\right)\left((a\nabla_\beta b) - (a\sharp_\beta b)\right) \\ &\ge 0. \end{aligned}$$

This implies that $v_1 \geq v_3$, because $v_1 \geq u_3$ and $u_3 \geq v_3$. Hence, $v_1^* = v_1$. Moreover, from the previous note we have $u_i \leq v_1$ for every $i = 1, 2, 3$. In particular, $u_1^* \leq v_1^*$.

The third inequality comes from Lemma 3.2 and we have

$$u_1 + u_2 + u_3 \leq v_1 + v_2 + v_3. \tag{17}$$

To prove the second inequality (14), the following inequalities should be shown.

$$\begin{aligned} u_1 + u_2 &\leq v_1 + v_2, &(18)\\ u_1 + u_3 &\leq v_1 + v_3, &(19)\\ u_2 + u_3 &\leq v_1 + v_2. &(20)\end{aligned}$$

The first inequality (18) comes easily from Theorem 2.1 for $N = 0$. For the second inequality (19), since $v_2 \leq u_2$ together with (17), we get that

$$u_1 + u_3 \leq v_1 + v_3 - (u_2 - v_2) \leq v_1 + v_3.$$

Now let us treat our last inequality (20). We discuss the following two cases.
If $\alpha \in \left(0, \frac{\beta}{2}\right]$, then $\mathrm{r}_0 = \frac{\alpha}{\beta}$. We have

$$\begin{aligned} v_1 + v_2 - (u_2 + u_3) &= a\nabla_\alpha b + \frac{\alpha}{\beta}(a\sharp_\beta b) - \left(\frac{\alpha}{\beta}(a\nabla_\beta b) + \frac{2\alpha}{\beta}(b\nabla(a\sharp_\beta b))\right)\\ &= \left(\frac{\beta - 2\alpha}{\beta}\right) b\\ &\geq 0.\end{aligned}$$

If $\alpha \in \left[\frac{\beta}{2}, 1\right)$, then $\mathrm{r}_0 = 1 - \frac{\alpha}{\beta}$. Hence

$$\begin{aligned} v_1 + v_2 - (u_2 + u_3) &= a\nabla_\alpha b + \frac{\alpha}{\beta}(a\sharp_\beta b) - \left(\frac{\alpha}{\beta}(b\nabla_\beta a) + 2\left(1 - \frac{\alpha}{\beta}\right)(b\nabla(a\sharp_\beta b))\right)\\ &= \left(\frac{2\alpha - \beta}{\beta}\right)(a\sharp_\beta b)\\ &\geq 0.\end{aligned}$$

This completes the proof. □

We are now able to present our first main result of this section. Our arguments are inspired by some ideas from [8]. As mentioned before, the following results generalize those of Alzer et al. [1], Duong et al. [8] and J. Zhao [23].

Theorem 3.4. *Let $a, b > 0$, $0 < \alpha \leq \beta < 1$ and ϕ be a strictly increasing convex function defined on $(0, +\infty)$. Then*

$$\begin{aligned} \phi\left(a\nabla_\alpha b\right) &\geq \phi\left(\left(1 + \lambda_1 \log^2\left(\frac{a}{b}\right)\right) a\sharp_\alpha b\right) + \phi\left(\frac{\alpha}{\beta}(a\nabla_\beta b)\right) - \phi\left(\frac{\alpha}{\beta}(a\sharp_\beta b)\right)\\ &+ \phi\Big(2\mathrm{r}_0(b\nabla(a\sharp_\beta b))\Big) - \phi\left(2\mathrm{r}_0\left(a\sharp_{\frac{\beta}{2}} b\right)\right). \end{aligned} \tag{21}$$

where $\mathrm{r}_0 = \min\left\{\frac{\alpha}{\beta}, 1 - \frac{\alpha}{\beta}\right\}$ *and* $\lambda_1 = \min\left\{\frac{L\left(2\frac{\alpha}{\beta}\right)}{4}\beta^2, L\left(\frac{\alpha}{\beta}\right), L(\alpha)\right\}$.

Proof. Let $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ two vectors in $\mathbb{R}^3$ where the component are the same as Lemma 3.3. Since $u \prec_w v$, by applying Lemma 3.1, we get that

$$\phi(u_1) + \phi(u_2) + \phi(u_3) \leq \phi(v_1) + \phi(v_2) + \phi(v_3),$$

or equivalently,

$$\phi(v_1) \geq \phi(u_1) + \Big(\phi(u_2) - \phi(v_2)\Big) + \Big(\phi(u_3) - \phi(v_3)\Big).$$

This completes the proof. □

The reverse version of Theorem 3.4 is presented in the following theorem. To achieve this result, we use the previous theorem and some special variable changes. We also mention that the following theorem can be proved using the same arguments of Theorem 3.4.

Theorem 3.5. *Let $a, b > 0$, $0 < \alpha \leq \beta < 1$ and ϕ be a strictly increasing convex function defined on $(0, +\infty)$. Then*

$$\begin{aligned} \phi\left(a\nabla_\beta b\right) &\geq \phi\left(\left(1 + \lambda_2 \log^2\left(\frac{a}{b}\right)\right) a\sharp_\beta b\right) + \phi\left(\frac{1-\beta}{1-\alpha}(a\nabla_\alpha b)\right) - \phi\left(\frac{1-\beta}{1-\alpha}(a\sharp_\alpha b)\right) \\ &+ \phi\Big(2\mathrm{R}_0(a\nabla(a\sharp_\alpha b))\Big) - \phi\left(2\mathrm{R}_0\left(a\sharp_{\frac{1+\alpha}{2}} b\right)\right), \end{aligned} \tag{22}$$

where $\mathrm{R}_0 = \min\left\{\frac{1-\beta}{1-\alpha}, 1 - \frac{1-\beta}{1-\alpha}\right\}$ *and* $\lambda_2 = \min\left\{\frac{L\left(2\frac{1-\beta}{1-\alpha}\right)}{4}(1-\alpha)^2, L\left(\frac{1-\beta}{1-\alpha}\right), L(1-\beta)\right\}$.

Proof. It is easy to see that $0 \leq 1 - \beta \leq 1 - \alpha \leq 1$ whenever $0 \leq \alpha \leq \beta \leq 1$. Hence, by changing respectively a, b, α and β by $b, a, 1-\beta$ and $1-\alpha$ in Theorem 3.4, we get the desired result. □

Now, by selecting $\phi(x) = x^\lambda$ for $\lambda \geq 1$, in Theorems 3.4 and 3.5, we get the following corollary.

Corollary 3.6. *Let $a, b > 0$, $0 < \alpha \leq \beta < 1$ and $\lambda > 1$. Then, we have*

$$\begin{aligned} (a\nabla_\alpha b)^\lambda &\geq \left(1 + \lambda_1 \log^2\left(\frac{a}{b}\right)\right)^\lambda (a\sharp_\alpha b)^\lambda + \left(\frac{\alpha}{\beta}\right)^\lambda \left(\left(a\nabla_\beta b\right)^\lambda - (a\sharp_\beta b)^\lambda\right) \\ &+ (2\mathrm{r}_0)^\lambda \left(\left(b\nabla(a\sharp_\beta b)\right)^\lambda - \left(a\sharp_{\frac{\beta}{2}} b\right)^\lambda\right), \end{aligned} \tag{23}$$

where $\mathrm{r}_0 = \min\left\{\frac{\alpha}{\beta}, 1 - \frac{\alpha}{\beta}\right\}$, *and* $\lambda_1 = \min\left\{\frac{L\left(2\frac{\alpha}{\beta}\right)}{4}\beta^2, L(\frac{\alpha}{\beta}), L(\alpha)\right\}$. *And*

$$\begin{aligned} \left(a\nabla_\beta b\right)^\lambda &\geq \left(1 + \lambda_2 \log^2\left(\frac{a}{b}\right)\right)^\lambda (a\sharp_\beta b)^\lambda + \left(\frac{1-\beta}{1-\alpha}\right)^\lambda \left((a\nabla_\alpha b)^\lambda - (a\sharp_\alpha b)^\lambda\right) \\ &+ (2\mathrm{R}_0)^\lambda \left((a\nabla(a\sharp_\alpha b))^\lambda - \left(a\sharp_{\frac{1+\alpha}{2}} b\right)^\lambda\right), \end{aligned} \tag{24}$$

where $\mathrm{R}_0 = \min\left\{\frac{1-\beta}{1-\alpha}, 1 - \frac{1-\beta}{1-\alpha}\right\}$ *and* $\lambda_2 = \min\left\{\frac{L\left(2\frac{1-\beta}{1-\alpha}\right)}{4}(1-\alpha)^2, L\left(\frac{1-\beta}{1-\alpha}\right), L(1-\beta)\right\}$.

Remark 3.7. *Before proceeding to further results, we explain a little about the relation among the Corollary 3.6 and Theorem 1.2. Notice that the first inequality in Theorem 1.2 can be written as follows:*

$$\left(\frac{\alpha}{\beta}\right)^\lambda \left[\left(a\nabla_\beta b\right)^\lambda - (a\sharp_\beta b)^\lambda\right] \leq (a\nabla_\alpha b)^\lambda - (a\sharp_\alpha b)^\lambda, \tag{25}$$

with $0 \leq \alpha < \beta \leq 1$ *and* $\lambda \geq 1$. *While the second inequality in the same theorem can be stated in the following way*

$$\left(\frac{1-\beta}{1-\alpha}\right)^{\lambda}\left[(a\nabla_{\alpha}b)^{\lambda}-(a\sharp_{\alpha}b)^{\lambda}\right]\leq\left(a\nabla_{\beta}b\right)^{\lambda}-(a\sharp_{\beta}b)^{\lambda}, \tag{26}$$

where $0 \leq \alpha < \beta \leq 1$ *and* $\lambda \geq 1$. *Consequently, the first inequality in Corollary 3.6 present two refining term of* (25), *while the second inequality in Corollary 3.6 present two refining term of* (26). *Therefore, the Corollary 3.6 give a considerable refinement of Theorem 1.2. This is the main significance of our results. In the next sections, we present explicit examples of refined inequalities for both matrices.*

3.2. An improvement of Alzer-Fonseca-Kovačec's type inequalities

In this subsection, we present Alzer-Fonseca-Kovačec's type inequalities mentioned in the previous subsection for inequality (11). For this, we need the following Lemmas. The first one comes directly from Theorem 2.4.

Lemma 3.8. *Let* $a, b > 0$ *and* $0 < \alpha \leq \beta < \frac{1}{2}$. *Then*

$$\alpha^2 a+(1-\alpha)^2 b\geq\alpha^{\frac{\alpha}{\beta}}(a\sharp_{\alpha}b)+\frac{\alpha^2}{\beta}\left(a\nabla_{\beta}b-(a\sharp_{\beta}b)\right)+2\mathrm{r}_0\left(b\nabla(\alpha(a\sharp_{\beta}b))-\sqrt{\alpha}\left(a\sharp_{\frac{\beta}{2}}b\right)\right),$$

where $\mathrm{r}_0=\min\{\frac{\alpha}{\beta},1-\frac{\alpha}{\beta}\}$.

Lemma 3.9. *Let* $a, b > 0$, $0 < \alpha \leq \beta < \frac{1}{2}$ *and* $u=(u_1,u_2,u_3)$, $v=(v_1,v_2,v_3)\in\mathbb{R}^3$ *be two vectors with components*

$$u_1=\alpha^{\frac{\alpha}{\beta}}a\sharp_{\alpha}b,\ u_2=\frac{\alpha^2}{\beta}\left(a\nabla_{\beta}b\right),u_3=2\mathrm{r}_0\left(b\nabla\alpha(a\sharp_{\beta}b)\right),$$

and

$$v_1=\alpha^2a+(1-\alpha)^2b,\ v_2=\frac{\alpha^2}{\beta}(a\sharp_{\beta}b),\ v_3=2\mathrm{r}_0\sqrt{\alpha}a\sharp_{\frac{\beta}{2}}b,$$

where $\mathrm{r}_0=\min\{\frac{\alpha}{\beta},1-\frac{\alpha}{\beta}\}$. *Then, we have* $u\prec_w v$, *namely, the vectors* u^* *and* v^* *have components satisfying that*

$$u_1^*\leq v_1^*, \tag{27}$$
$$u_1^*+u_2^*\leq v_1^*+v_2^*, \tag{28}$$
$$u_1^*+u_2^*+u_3^*\leq v_1^*+v_2^*+v_3^*. \tag{29}$$

Proof. In order to prove (27) remark that v_1^* is exactly v_1. Indeed, on one hand we have $v_1 \geq v_2$, since

$$v_1-u_2=\left((1-\alpha)^2-\frac{\alpha^2(1-\beta)}{\beta}\right)b\geq\left(1-\frac{\alpha}{\beta}\right)b\geq 0,$$

and $u_2 \geq v_2$. On the other hand, we have

$$v_1-u_3=((1-\alpha)^2-\mathrm{r}_0)b+\alpha^2a-\mathrm{r}_0\alpha(a\sharp_{\beta}b)\geq 0. \tag{30}$$

In fact, first of all remark that

$$\left\{(\alpha,\beta)\in[0,1]^2\ :\ 0<\alpha\leq\beta<\frac{1}{2}\right\}=\left\{(\alpha,\beta)\in[0,1]^2\ :\ 0<\alpha\leq\frac{\beta}{2}\right\}\bigcup\left\{(\alpha,\beta)\in[0,1]^2\ :\ \frac{\beta}{2}\leq\alpha<\frac{1}{2}\right\}.$$

At this point we distinguish two situations.
If $\alpha \in \left(0, \frac{\beta}{2}\right]$, then $\mathrm{r}_0 = \frac{\alpha}{\beta}$, and so (30) becomes

$$\begin{aligned} v_1 - u_3 &= ((1-\alpha)^2 - \mathrm{r}_0)b + \alpha^2 a - \mathrm{r}_0\alpha(a\sharp_\beta b) \\ &= \left((1-\alpha)^2 - \frac{\alpha}{\beta}\right)b + \alpha^2 a - \frac{\alpha^2}{\beta} a\sharp_\beta b + \frac{\alpha^2}{\beta}\left(a\nabla_\beta b\right) - \frac{\alpha^2}{\beta}\left(a\nabla_\beta b\right) \\ &= \left((1-\alpha)^2 - \frac{\alpha^2}{\beta}(1-\beta) - \frac{\alpha}{\beta}\right)b + \frac{\alpha^2}{\beta}\left(a\nabla_\beta b - (a\sharp_\beta b)\right) \\ &\geq \left(1 - \frac{\alpha}{\beta} - \frac{\alpha}{\beta}\right)b + \frac{\alpha^2}{\beta}\left(a\nabla_\beta b - (a\sharp_\beta b)\right) \\ &= \left(\frac{\beta - 2\alpha}{\beta}\right)b + \frac{\alpha^2}{\beta}\left(a\nabla_\beta b - (a\sharp_\beta b)\right) \\ &\geq 0. \end{aligned}$$

If $\alpha \in \left[\frac{\beta}{2}, \frac{1}{2}\right)$, then $\mathrm{r}_0 = 1 - \frac{\alpha}{\beta}$. Therefore

$$\begin{aligned} v_1 - u_3 &= \left((1-\alpha)^2 - \left(1 - \frac{\alpha}{\beta}\right)\right)b + \alpha^2 a \\ &- \alpha\left(1 - \frac{\alpha}{\beta}\right)(a\nabla_\beta b) + \alpha\left(1 - \frac{\alpha}{\beta}\right)\left((a\nabla_\beta b) - (a\sharp_\beta b)\right) \\ &= \left((1-\alpha)^2 - \left(1 - \frac{\alpha}{\beta}\right) - \left(1 - \frac{\alpha}{\beta}\right)\alpha(1-\beta)\right)b \\ &+ (2\alpha - \beta)\alpha a + \alpha\left(1 - \frac{\alpha}{\beta}\right)\left((a\nabla_\beta b) - (a\sharp_\beta b)\right) \\ &= \left((1-\alpha)^2 - \frac{\alpha^2(1-\beta)}{\beta} - \left(1 - \frac{\alpha}{\beta}\right) + \frac{2\alpha^2(1-\beta)}{\beta} - \alpha(1-\beta)\right)b \\ &+ (2\alpha - \beta)\alpha a + \alpha\left(1 - \frac{\alpha}{\beta}\right)\left((a\nabla_\beta b) - (a\sharp_\beta b)\right) \\ &\geq \left(\frac{2\alpha^2(1-\beta)}{\beta} - \alpha(1-\beta)\right)b \\ &+ (2\alpha - \beta)\alpha a + \alpha\left(1 - \frac{\alpha}{\beta}\right)\left((a\nabla_\beta b) - (a\sharp_\beta b)\right) \\ &= \alpha\left(\frac{(1-\beta)(2\alpha - \beta)}{\beta}\right)b + (2\alpha - \beta)\alpha a + \alpha\left(1 - \frac{\alpha}{\beta}\right)\left((a\nabla_\beta b) - (a\sharp_\beta b)\right) \\ &\geq 0. \end{aligned}$$

This gives that $v_1 \geq v_3$, because $u_3 \geq v_3$ and $v_1 \geq u_3$. Hence, $v_1^* = v_1$. Moreover, by the previous note, we have $u_i \leq v_1$ for every $i = 1, 2, 3$. In particular, $u_1^* \leq v_1^*$.

The third inequality comes from Lemma 3.8 and we have

$$u_1 + u_2 + u_3 \leq v_1 + v_2 + v_3. \tag{31}$$

To prove the second inequality (28), the following inequalities should be shown.

$$\begin{aligned} u_1 + u_2 &\leq v_1 + v_2, & (32) \\ u_1 + u_3 &\leq v_1 + v_3, & (33) \\ u_2 + u_3 &\leq v_1 + v_2. & (34) \end{aligned}$$

The first inequality (32) comes easily from Theorem 2.4 for $N = 0$. For the second inequality (33), since $v_2 \leq u_2$ together with (31), we get that

$$u_1 + u_3 \leq v_1 + v_3 - (u_2 - v_2) \leq v_1 + v_3.$$

Now, let us treat our last inequality (34). We discuss the following two cases.
If $\alpha \in \left(0, \frac{\beta}{2}\right]$, then $r_0 = \frac{\alpha}{\beta}$. We have

$$\begin{aligned} v_1 + v_2 - (u_2 + u_3) &= \alpha^2 a + (1-\alpha)^2 b + \frac{\alpha^2}{\beta}(a\sharp_\beta b) - \left(\frac{\alpha^2}{\beta}\left(a\nabla_\beta b\right) + 2r_0\left(b\nabla\alpha(a\sharp_\beta b)\right)\right) \\ &= \left((1-\alpha)^2 - \frac{\alpha^2(1-\beta)}{\beta} - \frac{\alpha}{\beta}\right)b \\ &\geq \left(1 - \frac{\alpha}{\beta} - \frac{\alpha}{\beta}\right)b \\ &= \left(\frac{\beta - 2\alpha}{\beta}\right)b \\ &\geq 0. \end{aligned}$$

If $\alpha \in \left[\frac{\beta}{2}, \frac{1}{2}\right)$, then $r_0 = 1 - \frac{\alpha}{\beta}$. Hence

$$\begin{aligned} v_1 + v_2 - (u_2 + u_3) &= \alpha^2 a + (1-\alpha)^2 b + \frac{\alpha^2}{\beta}(a\sharp_\beta b) \\ &- \left(\frac{\alpha^2}{\beta}(a\nabla_\beta b) + 2\left(1 - \frac{\alpha}{\beta}\right)(b\nabla\alpha(a\sharp_\beta b))\right) \\ &= \alpha\left(\frac{2\alpha - \beta}{\beta}\right)(a\sharp_\beta b) + \left((1-\alpha)^2 - \frac{\alpha^2(1-\beta)}{\beta} - \left(1 - \frac{\alpha}{\beta}\right)\right)b \\ &\geq 0. \end{aligned}$$

This completes the proof. □

In the following, we state our second main result of this section, which present a improvement of Alzer-Fonseca-Kovačec's type inequalities for inequality (11).

Theorem 3.10. *Let $a, b > 0$, $0 < \alpha < \beta \leq \frac{1}{2}$ and ϕ be a strictly increasing convex function defined on $(0, +\infty)$. Then*

$$\begin{aligned} \phi\left(\alpha^2 a + (1-\alpha)^2 b\right) &\geq \phi\left(\alpha^{\frac{\alpha}{\beta}} a\sharp_\alpha b\right) + \phi\left(\frac{\alpha^2}{\beta}(a\nabla_\beta b)\right) - \phi\left(\frac{\alpha^2}{\beta}(a\sharp_\beta b)\right) \\ &+ \phi\Big(2r_0(b\nabla\alpha(a\sharp_\beta b))\Big) - \phi\left(2r_0\sqrt{\alpha}\left(a\sharp_{\frac{\beta}{2}} b\right)\right). \end{aligned} \tag{35}$$

where $r_0 = \min\{\frac{\alpha}{\beta}, 1 - \frac{\alpha}{\beta}\}$.

Proof. Let $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ two vectors in $\mathbb{R}^3$ where the component are the same as Lemma 3.9. Since $u \prec_w v$, by applying Lemma 3.1, we get that

$$\phi(u_1) + \phi(u_2) + \phi(u_3) \leq \phi(v_1) + \phi(v_2) + \phi(v_3),$$

or equivalently,

$$\phi(v_1) \geq \phi(u_1) + \Big(\phi(u_2) - \phi(v_2)\Big) + \Big(\phi(u_3) - \phi(v_3)\Big)$$

This completes the proof. □

The following theorem presents the reversed version of the previous one.

Theorem 3.11. *Let $a, b > 0$, $\frac{1}{2} \leq \alpha < \beta \leq 1$ and ϕ be a strictly increasing convex function defined on $(0, +\infty)$. Then*

$$\begin{aligned}\phi\left(\alpha^2 a + (1-\alpha)^2 b\right) &\geq \phi\left(\left((1-\beta)^{\frac{1-\beta}{1-\alpha}}\right) a\sharp_\beta b\right) \\ &+ \phi\left(\frac{(1-\beta)^2}{1-\alpha}(a\nabla_\alpha b)\right) - \phi\left(\frac{(1-\beta)^2}{1-\alpha}(a\sharp_\alpha b)\right) \\ &+ \phi\Big(2\mathrm{R}_0(a\nabla(1-\beta)(a\sharp_\alpha b))\Big) - \phi\left(2\mathrm{R}_0\sqrt{(1-\beta)}\left(a\sharp_{\frac{1+\alpha}{2}} b\right)\right),\end{aligned}$$

where $\mathrm{R}_0 = \min\{\frac{1-\beta}{1-\alpha}, 1 - \frac{1-\beta}{1-\alpha}\}$.

Proof. Notice that, if $\frac{1}{2} \leq \alpha < \beta \leq 1$ then $0 < 1-\beta < 1-\alpha \leq \frac{1}{2}$. Hence, by changing respectively a, b, α and β by $b, a, 1-\beta$ and $1-\alpha$ in Theorem 3.10, we obtain the desired results. □

Now, by applying Theorems 3.10 and 3.11 on $\phi(x) = x^\lambda$ for $\lambda \geq 1$, we get the following result, refining the corresponding results in [5, 13, 16, 19].

Corollary 3.12. *Let $a, b > 0$, $0 < \alpha \leq \beta < 1$ and $\lambda > 1$.*

1. *If $0 < \alpha < \beta \leq \frac{1}{2}$ then*

$$\begin{aligned}\left(\alpha^2 a + (1-\alpha)^2 b\right)^\lambda &\geq \left(\alpha^{\frac{\alpha}{\beta}}\right)^\lambda (a\sharp_\alpha b)^\lambda + \left(\frac{\alpha^2}{\beta}\right)^\lambda \left(\left(a\nabla_\beta b\right)^\lambda - (a\sharp_\beta b)^\lambda\right) \\ &+ (2\mathrm{r}_0)^\lambda \left(\left(b\nabla\alpha(a\sharp_\beta b)\right)^\lambda - \left(\sqrt{\alpha}\left(a\sharp_{\frac{\beta}{2}} b\right)\right)^\lambda\right), \end{aligned} \tag{36}$$

   where $\mathrm{r}_0 = \min\{\frac{\alpha}{\beta}, 1 - \frac{\alpha}{\beta}\}$.

2. *If $\frac{1}{2} \leq \alpha < \beta \leq 1$ then*

$$\begin{aligned}\left(a\nabla_\beta b\right)^\lambda &\geq \left(1 + \lambda_2 \log^2\left(\frac{a}{b}\right)\right)^\lambda (a\sharp_\beta b)^\lambda + \left(\frac{1-\beta}{1-\alpha}\right)^\lambda \left((a\nabla_\alpha b)^\lambda - (a\sharp_\alpha b)^\lambda\right) \\ &+ (2\mathrm{R}_0)^\lambda \left((a\nabla(1-\beta)(a\sharp_\alpha b))^\lambda - \left(\sqrt{1-\beta}\, a\sharp_{\frac{1+\alpha}{2}} b\right)^\lambda\right), \end{aligned} \tag{37}$$

   where $\mathrm{R}_0 = \min\{\frac{1-\beta}{1-\alpha}, 1 - \frac{1-\beta}{1-\alpha}\}$.

4. Some refinements of matrix inequalities

In this section, we present some new matrix inequalities, that extend some known results in the literature. We point out that the results in this section are valid for the algebra $\mathcal{B}(\mathcal{H})$ instead of $\mathbf{M}_n$. However, our discussion will be limited to $\mathbf{M}_n$ only. A matrix $A \in \mathbf{M}_n$ is called Hermitian if $A = A^*$, where A^* is the adjoint of A. The notation $A \geq 0$ ($A > 0$) is used to mean that A is positive semi-definite (positive definite), if A and B are Hermitian and $A - B$ is positive semi-definite, then we write $A \geq B$. The set of all positive semi-definite matrices is denoted by $\mathbf{M}_n^+$ and the set of all definite matrices in $\mathbf{M}_n^+$ is denoted by $\mathbf{M}_n^{++}$.
For $A, B \in \mathbf{M}_n^{++}$, and $\alpha \in [0,1]$ the operators arithmetic, geometric, means are defined respectively, by $A\nabla_\alpha B := (1-\alpha)A + \alpha B$, $A\sharp_\alpha B := A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^\alpha A^{1/2}$. The famous Young's inequality for operators states that

$$A\sharp_\alpha B \leq A\nabla_\alpha B \quad \text{where} \quad \alpha \in [0,1].$$

For $x > 1$, the definition of $A\sharp_x B = A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^x A^{1/2}$ is still well defined. We start this section by the following Lemma quoted from [20, p. 3].

Lemma 4.1. *Let $A \in \mathbf{M}_n$ be Hermitian. If f and g are both continuous real valued functions on an interval that contains the spectrum of A, with $f(t) \geq g(t)$ for $t \in Sp(A)$ (where $Sp(A)$ stands for the spectrum of A), then $f(A) \geq g(A)$.*

The following theorem presents the matrix version of Corollary 3.6.

Theorem 4.2. *Let $A, B \in \mathbf{M}_n^{++}$, $0 < \alpha \leq \beta < 1$ and $\lambda \geq 1$. Then we have*

$$\begin{aligned}
&\left(\frac{\alpha}{\beta}\right)^{\lambda}\Big(A\sharp_{\lambda}(A\nabla_{\beta}B) - A\sharp_{\lambda\beta}B\Big)\\
&+\Big(2\mathrm{r}_0\Big)^{\lambda}\Big(A\sharp_{\lambda}\Big(A\nabla(A\sharp_{\beta}B)\Big) - A\sharp_{\frac{\beta}{2}}B\Big) + \sqrt{\lambda_1}A^{-1}S(A\backslash B)A\sharp_{\lambda\alpha}B\\
&\leq A\sharp_{\lambda}(A\nabla_{\alpha}B) - A\sharp_{\lambda\alpha}B,
\end{aligned} \tag{38}$$

where $\mathrm{r}_0 = \min\{\frac{\alpha}{\beta}, 1-\frac{\alpha}{\beta}\}$, *and* $\lambda_1 = \min\left\{\frac{L\left(2\frac{\alpha}{\beta}\right)}{4}\beta^2, L\left(\frac{\alpha}{\beta}\right), L(\alpha)\right\}$. *And*

$$\begin{aligned}
&\left(\frac{1-\beta}{1-\alpha}\right)^{\lambda}\Big(A\sharp_{\lambda}(A\nabla_{\alpha}B) - A\sharp_{\lambda\alpha}B\Big)\\
&+\Big(2\mathrm{R}_0\Big)^{\lambda}\Big(A\sharp_{\lambda}\left(B\nabla(A\sharp_{\alpha}B)\right) - A\sharp_{\frac{1+\alpha}{2}}B\Big) + \sqrt{\lambda_2}A^{-1}S(A\backslash B)A\sharp_{\lambda\beta}B\\
&\leq A\sharp_{\lambda}(A\nabla_{\beta}B) - A\sharp_{\lambda\beta}B,
\end{aligned} \tag{39}$$

where $\mathrm{R}_0 = \min\{\frac{1-\beta}{1-\alpha}, 1-\frac{1-\beta}{1-\alpha}\}$. *and* $\lambda_2 = \min\left\{\frac{L\left(2\frac{1-\beta}{1-\alpha}\right)}{4}(1-\alpha)^2, L(\frac{1-\beta}{1-\alpha}), L(1-\beta)\right\}$.

Proof. Let $b = 1$ and $a = t$ in inequality (23), by using the following inequality

$$(1+\lambda_1\log^2(t))^{\lambda} \geq 1+\lambda_1\log^2(t), \quad t > 0$$

we get

$$\begin{aligned}
&\left(\frac{\alpha}{\beta}\right)^{\lambda}\Big(((1-\beta)+\beta t)^{\lambda} \quad - \quad t^{\beta\lambda}\Big)\\
&+(2\mathrm{r}_0)^{\lambda}\left[\left(\frac{1+t^{\beta}}{2}\right)^{\lambda} - t^{\frac{\lambda\beta}{2}}\right]\\
&\leq ((1-\alpha)+\alpha t)^{\lambda} - (1+\lambda_1\log^2(t))t^{\alpha\lambda}.
\end{aligned} \tag{40}$$

The matrix $X := A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ has a positive spectrum, then by Lemma 4.1 and inequality (40) we get

$$\begin{aligned}
&\left(\frac{\alpha}{\beta}\right)^{\lambda}\Big(((1-\beta)I+\beta X)^{\lambda} \quad - \quad X^{\beta\lambda}\Big)\\
&+(2\mathrm{r}_0)^{\lambda}\left[\left(\frac{I+X^{\beta}}{2}\right)^{\lambda} - X^{\frac{\lambda\beta}{2}}\right]\\
&\leq ((1-\alpha)I+\alpha X)^{\lambda} - X^{\alpha\lambda} - \lambda_1\log(X)\log(X)X^{\alpha\lambda}.
\end{aligned} \tag{41}$$

Finally, multiply the inequality (41) by $A^{\frac{1}{2}}$ on the left and right hand sides we get

$$\begin{aligned}
&\left(\frac{\alpha}{\beta}\right)^{\lambda}\Big(A\sharp_{\lambda}(A\nabla_{\beta}B) - A\sharp_{\lambda\beta}B\Big)\\
&+\Big(2\mathrm{r}_0\Big)^{\lambda}\Big(A\sharp_{\lambda}\Big(A\nabla(A\sharp_{\beta}B)\Big) - A\sharp_{\frac{\beta}{2}}B\Big) + \sqrt{\lambda_1}A^{-1}S(A\backslash B)A\sharp_{\lambda\alpha}B\\
&\leq A\sharp_{\lambda}(A\nabla_{\alpha}B) - A\sharp_{\lambda\alpha}B,
\end{aligned} \tag{42}$$

Using the same technique, we can obtain the other inequality. This completes the proof. □

The following theorem presents the matrix version of Corollary 3.12.

Theorem 4.3. *Let* $A, B \in \mathbf{M}_n^{++}$, $0 < \alpha \leq \beta < 1$ *and* $\lambda \geq 1$. *Then we have*

1. *If* $0 < \alpha < \beta \leq \frac{1}{2}$ *then*

$$\begin{aligned}&\Big(\frac{\alpha^2}{\beta}\Big)^{\lambda}\Big(A\sharp_{\lambda}(A\nabla_{\beta}B) - A\sharp_{\lambda\beta}B\Big)\\ &+\Big(2\mathrm{r}_0\Big)^{\lambda}\Big(A\sharp_{\lambda}\Big(A\nabla\alpha(A\sharp_{\beta}B)\Big) - \sqrt{\alpha}A\sharp_{\frac{\beta}{2}}B\Big)\\ &\leq A\sharp_{\lambda}((1-\alpha)^2A + \alpha^2B) - A\sharp_{\lambda\alpha}B,\end{aligned} \tag{43}$$

where $\mathrm{r}_0 = \min\{\frac{\alpha}{\beta}, 1 - \frac{\alpha}{\beta}\}$.

1. *If* $\dfrac{1}{2} \leq \alpha < \beta \leq 1$ *then*

$$\begin{aligned}&\Big(\frac{(1-\beta)^2}{1-\alpha}\Big)^{\lambda}\Big(A\sharp_{\lambda}(A\nabla_{\alpha}B) - A\sharp_{\lambda\alpha}B\Big)\\ &+\Big(2\mathrm{R}_0\Big)^{\lambda}\Big(A\sharp_{\lambda}\left(B\nabla(1-\beta)(A\sharp_{\alpha}B)\right) - \sqrt{1-\beta}A\sharp_{\frac{1+\alpha}{2}}B\Big)\\ &\leq A\sharp_{\lambda}((1-\beta)^2A + \beta^2B) - A\sharp_{\lambda\beta}B,\end{aligned} \tag{44}$$

where $\mathrm{R}_0 = \min\{\frac{1-\beta}{1-\alpha}, 1 - \frac{1-\beta}{1-\alpha}\}$.

Proof. Let $b = 1$ and $a = t$ in the first inequality of Corollary (3.12), then

$$\begin{aligned}&\Big(\frac{\alpha^2}{\beta}\Big)^{\lambda}\Big(((1-\beta)+\beta t)^{\lambda} \quad - \quad t^{\beta\lambda}\Big)\\ &+(2\mathrm{r}_0)^{\lambda}\left[\left(\frac{1+\alpha t^{\beta}}{2}\right)^{\lambda} - \sqrt{\alpha}t^{\frac{\lambda\beta}{2}}\right]\\ &\leq ((1-\alpha)^2 + \alpha^2 t)^{\lambda} - t^{\alpha\lambda}.\end{aligned} \tag{45}$$

By applying Lemma 4.1 and inequality (45) to $X := A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ we get

$$\begin{aligned}&\Big(\frac{\alpha^2}{\beta}\Big)^{\lambda}\Big(((1-\beta)I+\beta X)^{\lambda} \quad - \quad X^{\beta\lambda}\Big)\\ &+(2\mathrm{r}_0)^{\lambda}\left[\left(\frac{I+X^{\beta}}{2}\right)^{\lambda} - \sqrt{\alpha}X^{\frac{\lambda\beta}{2}}\right]\\ &\leq ((1-\alpha)^2 I + \alpha^2 X)^{\lambda} - X^{\alpha\lambda}.\end{aligned} \tag{46}$$

So, multiply the inequality (46) by $A^{\frac{1}{2}}$ on the left and right hand sides we find

$$\begin{aligned}&\Big(\frac{\alpha^2}{\beta}\Big)^{\lambda}\Big(A\sharp_{\lambda}(A\nabla_{\beta}B) - A\sharp_{\lambda\beta}B\Big)\\ &+\Big(2\mathrm{r}_0\Big)^{\lambda}\Big(A\sharp_{\lambda}\Big(A\nabla\alpha(A\sharp_{\beta}B)\Big) - \sqrt{\alpha}A\sharp_{\frac{\beta}{2}}B\Big)\\ &\leq A\sharp_{\lambda}((1-\alpha)^2A + \alpha^2B) - A\sharp_{\lambda\alpha}B.\end{aligned} \tag{47}$$

Using the same technique we obtain the other inequality. This completes the proof. □

5. Matrix norm inequalities

The singular values of a matrix $A \in \mathbf{M}_n$ are the eigenvalues of the positive semi-definite matrix $|A| := (A^*A)^{1/2}$, denoted by $s_j(A)$ for $j = 1, 2, 3, \ldots, n$; arranged in a non-increasing order. A unitarily invariant norm $|||\cdot|||$ on $\mathbf{M}_n$ is a matrix norm that satisfies the invariance property: $|||UAV||| = |||A|||$ for every $A \in \mathbf{M}_n$ and for all unitary matrices $U, V \in \mathbf{M}_n$. The trace norm is given by $\|A\|_1 := \mathrm{tr}|A| = \sum_{j=1}^{n} s_j(A)$, where tr is the usual trace. This norm is unitarily invariant. An important example of unitarily invariant norm is the Hilbert-Schmidt norm $\|\cdot\|_2$ defined by

$$\|A\|_2 := \mathrm{tr}(AA^*)^{\frac{1}{2}} = \Big(\sum_{i,j} |a_{i,j}|^2\Big)^{\frac{1}{2}}, \qquad \Big(A = (a_{i,j})\Big).$$

Let $A, B \in \mathbf{M}_n^{++}$, $X \in \mathbf{M}_n \setminus \{0\}$. The classical Young inequality $a\sharp_\alpha b \le a\nabla_\alpha b$ has been extended to matrices as follows

$$|||A^\alpha XB^{1-\alpha}||| \le \alpha|||AX||| + (1-\alpha)|||XB|||, \ 0 \le \alpha \le 1. \tag{48}$$

Now, we use Corollary 3.6 to obtain the following refined version of the corresponding Young's inequality (48) for norms.

Theorem 5.1. *Let $A, B \in \mathbf{M}_n^{++}$, $X \in \mathbf{M}_n \setminus \{0\}$, $0 < \alpha \le \beta < 1$ and $\lambda \ge 1$. Then we have*

$$\begin{aligned}
&(|||AX|||\nabla_\alpha|||XB|||)^\lambda - \left(1 + \lambda_1 \log^2\left(\frac{|||AX|||}{|||XB|||}\right)\right)^\lambda (|||A^\alpha XB^{1-\alpha}|||)^\lambda \\
&\ge \left(\frac{\alpha}{\beta}\right)^\lambda \Big((|||AX|||\nabla_\beta|||XB|||)^\lambda - (|||AX|||\sharp_\beta|||XB|||)^\lambda\Big) \\
&+ (2\mathrm{r}_0)^\lambda \left(\Big(|||XB|||\nabla(|||AX|||\sharp_\beta|||XB|||)\Big)^\lambda - \Big(|||AX|||\sharp_{\frac{\beta}{2}}|||XB|||\Big)^\lambda\right),
\end{aligned} \tag{49}$$

where $\mathrm{r}_0 = \min\{\frac{\alpha}{\beta}, 1 - \frac{\alpha}{\beta}\}$ *and* $\lambda_1 = \min\left\{\frac{L\left(2\frac{\alpha}{\beta}\right)}{4}\beta^2, L(\frac{\alpha}{\beta}), L(\alpha)\right\}$. *And*

$$\begin{aligned}
&(|||AX|||\nabla_\beta|||XB|||)^\lambda - \left(1 + \lambda_2 \log^2\left(\frac{|||AX|||}{|||XB|||}\right)\right)(|||A^\beta XB^{1-\beta}|||)^\lambda \\
&\ge \left(\frac{1-\beta}{1-\alpha}\right)^\lambda \Big((|||AX|||\nabla_\alpha|||XB|||)^\lambda - (|||AX|||\sharp_\alpha|||XB|||)^\lambda\Big) \\
&+ (2\mathrm{r}_0)^\lambda \left((|||AX|||\nabla(|||AX|||\sharp_\alpha|||XB|||))^\lambda - \Big(|||AX|||\sharp_{\frac{1+\alpha}{2}}|||XB|||\Big)^\lambda\right),
\end{aligned} \tag{50}$$

where $\mathrm{R}_0 = \min\{\frac{1-\beta}{1-\alpha}, 1 - \frac{1-\beta}{1-\alpha}\}$ *and* $\lambda_2 = \min\left\{\frac{L\left(2\frac{1-\beta}{1-\alpha}\right)}{4}(1-\alpha)^2, L(\frac{1-\beta}{1-\alpha}), L(1-\beta)\right\}$.

Proof. Using the first inequality in Corollary 3.6 with $a = |||AX|||$, $b = |||XB|||$ and noting that the unitarily invariant norm $|||.|||$ satisfying (see, [2–4])

$$|||A^\alpha XB^{1-\alpha}||| \le |||AX|||^\alpha|||XB|||^{1-\alpha}.$$

Hence, we obtain the desired results. □

The famous Young inequality for trace can be stated as follows:

$$\mathrm{tr}(A^\alpha B^{1-\alpha}) \le \mathrm{tr}\left((1-\alpha)A + \alpha B\right), \ 0 \le \alpha \le 1. \tag{51}$$

We have the following theorem which present two refining term of the corresponding Young's inequality (51) for trace of matrices.

Corollary 5.2. *Let* $A, B \in \mathbf{M}_n^+$, $0 < \alpha \leq \beta < 1$ *and* $\lambda \geq 1$. *Then we have*

$$\begin{aligned} \operatorname{tr}^{\lambda}(A\nabla_{\alpha}B) &\geq \left(1+\lambda_1 \log^2\left(\frac{\operatorname{tr}(A)}{\operatorname{tr}(B)}\right)\right)^{\lambda} \operatorname{tr}^{\lambda}\left(A^{\alpha}B^{1-\alpha}\right) \\ &+ \left(\frac{\alpha}{\beta}\right)^{\lambda}\left(\operatorname{tr}^{\lambda}\left(A\nabla_{\beta}B\right) - (\operatorname{tr}(A)\sharp_{\beta}\operatorname{tr}(B))^{\lambda}\right) \\ &+ (2\mathrm{r}_0)^{\lambda}\left(\left(\operatorname{tr}(B)\nabla(\operatorname{tr}(A)\sharp_{\beta}\operatorname{tr}(B))\right)^{\lambda} - \left(\operatorname{tr}(A)\sharp_{\frac{\beta}{2}}\operatorname{tr}(B)\right)^{\lambda}\right), \end{aligned} \tag{52}$$

where $\mathrm{r}_0 = \min\{\frac{\alpha}{\beta}, 1-\frac{\alpha}{\beta}\}$ *and* $\lambda_1 = \min\left\{\frac{L\left(2\frac{\alpha}{\beta}\right)}{4}\beta^2, L\left(\frac{\alpha}{\beta}\right), L(\alpha)\right\}$. *And*

$$\begin{aligned} \operatorname{tr}^{\lambda}\left(A\nabla_{\beta}B\right) &\geq \left(1+\lambda_2 \log^2\left(\frac{\operatorname{tr}(A)}{\operatorname{tr}(B)}\right)\right)^{\lambda} \operatorname{tr}^{\lambda}\left(A^{\beta}B^{1-\beta}\right) \\ &+ \left(\frac{1-\beta}{1-\alpha}\right)^{\lambda}\left(\operatorname{tr}^{\lambda}(A\nabla_{\alpha}B) - (\operatorname{tr}(A)\sharp_{\alpha}\operatorname{tr}(B))^{\lambda}\right) \\ &+ (2\mathrm{r}_0)^{\lambda}\left((\operatorname{tr}(A)\nabla(\operatorname{tr}(A)\sharp_{\alpha}\operatorname{tr}(B)))^{\lambda} - \left(\operatorname{tr}(A)\sharp_{\frac{1+\alpha}{2}}\operatorname{tr}(B)\right)^{\lambda}\right), \end{aligned} \tag{53}$$

where $\mathrm{R}_0 = \min\{\frac{1-\beta}{1-\alpha}, 1-\frac{1-\beta}{1-\alpha}\}$ *and* $\lambda_2 = \min\left\{\frac{L\left(2\frac{1-\beta}{1-\alpha}\right)}{4}(1-\alpha)^2, L(\frac{1-\beta}{1-\alpha}), L(1-\beta)\right\}$.

Acknowledgement

We would like to extend our sincere thanks to the anonymous referee who carefully reviewed our manuscript and provided valuable feedback and suggestions.

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