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Extensions of $\mathcal{G}$-outer inverses

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Abstract. Our first objective is to present equivalent conditions for the solvability of the system of matrix equations $ADA = A$, $DAB = B$ and $CAD = C$, where D is unknown, A, B, C are of appropriate dimensions, and to obtain its general solution in terms of appropriate inner inverses. Our leading idea is to find characterizations and representations of a subclass of inner inverses that satisfy some properties of outer inverses. A $\mathcal{G}$-(B, C) inverse of A is defined as a solution of this matrix system. In this way, $\mathcal{G}$-(B, C) inverses are defined and investigated as an extension of $\mathcal{G}$-outer inverses. One-sided versions of $\mathcal{G}$-(B, C) inverse are introduced as weaker kinds of $\mathcal{G}$-(B, C) inverses and generalizations of one-sided versions of $\mathcal{G}$-outer inverse. Applying the $\mathcal{G}$-(B, C) inverse and its one-sided versions, we propose three new partial orders on the set of complex matrices. These new partial orders extend the concepts of $\mathcal{G}$-outer (T, S)-partial order and one-sided $\mathcal{G}$-outer (T, S)-partial orders.

1. Introduction

Standardly, A^*, $\mathrm{rank}(A)$, $\mathcal{R}(A)$ and $\mathcal{N}(A)$, respectively, represent the conjugate transpose, rank, range (column space) and null space of $A \in \mathbb{C}^{m\times n}$, where $\mathbb{C}^{m\times n}$ is the set of all $m \times n$ complex matrices. If $A \in \mathbb{C}^{n\times n}$, the symbol $\mathrm{ind}(A)$ denotes its index, that is, the minimal integer $k \geq 0$ which satisfies the rank-invariant property $\mathrm{rank}(A^k) = \mathrm{rank}(A^{k+1})$.

The definitions of significant generalized inverses are firstly given. In the case if the matrix equation $XAX = X$ (or $AXA = A$) is satisfied, the matrix $X \in \mathbb{C}^{n\times m}$ is an outer (or inner) inverse of $A \in \mathbb{C}^{m\times n}$. The set of all outer (or inner) inverses of A is denoted by $A\{2\}$ (or $A\{1\}$). The uniquely determined outer inverse of A with prescribed range T and null space S is a matrix $X \in \mathbb{C}^{n\times m}$ (represented by $A^{(2)}_{T,S}$) for which

$$XAX = X, \quad \mathcal{R}(X) = T, \quad \mathcal{N}(X) = S,$$

where T is a subspace of $\mathbb{C}^n$ of dimension $s \leq r = \mathrm{rank}(A)$, and S is a subspace of $\mathbb{C}^m$ of dimension $m - s$. Note that $AT \oplus S = \mathbb{C}^m$ if and only if $A^{(2)}_{T,S}$ exists [3]. For a given $A \in \mathbb{C}^{m\times n}$ and appropriately chosen T, S, the notation $\mathbb{C}^{m\times n}_{T,S} = \{A \in \mathbb{C}^{m\times n} : AT \oplus S = \mathbb{C}^m\} \subseteq \mathbb{C}^{m\times n}$ will be used.

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Denote by $\mho_{H_1,\dots,H_k}$ the relation $\operatorname{rank}(H_1) = \cdots = \operatorname{rank}(H_k)$ between the matrices $H_1, \dots, H_k$ of appropriate order. For $A \in \mathbb{C}^{m\times n}$, $F \in \mathbb{C}^{n\times k}$ and $H \in \mathbb{C}^{l\times m}$, consider the sets

$$A\{2\}_{\mathcal{R}(F),*} = \{X \in \mathbb{C}^{n\times m} :\ XAX = X,\ \mathcal{R}(X) = \mathcal{R}(F)\};$$
$$A\{2\}_{*,\mathcal{N}(H)} = \{X \in \mathbb{C}^{n\times m} :\ XAX = X,\ \mathcal{N}(X) = \mathcal{N}(H)\}.$$

Evidently, the set $\{X \in \mathbb{C}^{n\times m} :\ XAX = X,\ \mathcal{R}(X) = \mathcal{R}(F), \mathcal{N}(X) = \mathcal{N}(H)\}$ contains only one element $A^{(2)}_{\mathcal{R}(F),\mathcal{N}(H)}$. If $A^{(2)}_{\mathcal{R}(F),\mathcal{N}(H)}$ is an inner inverse of A, it will be denoted by $A^{(1,2)}_{\mathcal{R}(F),\mathcal{N}(H)}$. The expressions $\complement_1 := F(HAF)^{(1)}H$ are important generalizations of $A^{(2)}_{\mathcal{R}(F),\mathcal{N}(H)}$, as presented in Proposition 1.1, summarized from [3] .

Proposition 1.1. (Urquhart formula). *For arbitrary $A \in \mathbb{C}^{m\times n}$, $F \in \mathbb{C}^{n\times k}$, $H \in \mathbb{C}^{l\times m}$ and a fixed but arbitrary element $(HAF)^{(1)} \in (HAF)\{1\}$, it follows*

$$\complement_1 := F(HAF)^{(1)}H = \begin{cases} A^{(2)}_{\mathcal{R}(F),*} \iff \mho_{HAF,F}; \\ A^{(2)}_{*,\mathcal{N}(H)} \iff \mho_{HAF,H}; \\ A^{(2)}_{\mathcal{R}(F),\mathcal{N}(H)} \iff \mho_{HAF,F,H}; \\ A^{(1,2)}_{\mathcal{R}(F),\mathcal{N}(H)} \iff \mho_{HAF,F,H,A}, \end{cases} \tag{1}$$

where $A^{(2)}_{\mathcal{R}(F),*} \in A\{2\}_{\mathcal{R}(F),*}$ and $A^{(2)}_{*,\mathcal{N}(H)} \in A\{2\}_{*,\mathcal{N}(H)}$.

According to Proposition 1.1, the expression $\complement_1 := F(HAF)^{(1)}H$ is an outer inverse $A^{(2)}_{\mathcal{R}(F),*}$ with known only range when $\mho_{FAH,F}$, an outer inverse $A^{(2)}_{*,\mathcal{N}(H)}$ with known only kernel when $\mho_{HAF,H}$, and becomes the outer inverse with defined both range and null space $A^{(2)}_{\mathcal{R}(F),\mathcal{N}(H)}$ if $\mho_{HAF,F}$ or $\mho_{HAF,F,H,A}$.

Recall that, for $A \in \mathbb{C}^{m\times n}$, the unique solver of the system: $AXA = A$, $XAX = X$, $(AX)^* = AX$ and $(XA)^* = XA$ is the Moore-Penrose inverse $A^\dagger$ of A [3, 26]. The system: $A^{k+1}X = A^k$, $XAX = X$ and $AX = XA$, where $A \in \mathbb{C}^{n\times n}$ and $k = \operatorname{ind}(A)$, has the unique solution called the Drazin inverse A^{D} of A [3, 26]. For recent results related to the Moore-Penrose and Drazin inverses see [1, 2, 4, 6–8, 13, 14, 23, 28, 29, 31, 32].

Several partial orders were presented and investigated in terms of generalized inverses [9, 15–17, 27, 30]. For $A, B \in \mathbb{C}^{m\times n}$, we say that A is below B with respect to the minus partial order (and denote as $A \leq^- B$) if there exists $A^- \in A\{1\}$ satisfying $AA^- = BA^-$ and $A^-A = A^-B$ [12].

A $\mathcal{G}$-Drazin inverse for a square matrix was defined by Wang and Liu in [25]. Let $A \in \mathbb{C}^{n\times n}$ and $k = \operatorname{ind}(A)$. A matrix $D \in \mathbb{C}^{n\times n}$ is a $\mathcal{G}$-Drazin inverse of A if it satisfies the system

$$ADA = A, \quad DA^{k+1} = A^k \quad \text{and} \quad A^{k+1}D = A^k. \tag{2}$$

Recall that (2) is equivalent to the system:

$$ADA = A, \quad DAA^{\mathrm{D}} = A^{\mathrm{D}} \quad \text{and} \quad A^{\mathrm{D}}AD = A^{\mathrm{D}}. \tag{3}$$

The symbol $A\{GD\}$ will stand for the set of all $\mathcal{G}$-Drazin inverses of A. More interesting results related to $\mathcal{G}$-Drazin inverses can be found in [5, 10, 11, 20].

A $\mathcal{G}$-outer inverse was introduced for a rectangular matrix in [18] extending the $\mathcal{G}$-Drazin inverse using outer inverse with prescribed range and null space. Precisely, $D \in \mathbb{C}^{n\times m}$ is defined as a $\mathcal{G}$-outer ($\mathcal{G}_O$-) (T, S)-inverse of $A \in \mathbb{C}^{m\times n}_{T,S}$ when it realizes

$$ADA = A, \qquad DAA^{(2)}_{T,S} = A^{(2)}_{T,S} \qquad \text{and} \qquad A^{(2)}_{T,S}AD = A^{(2)}_{T,S}. \tag{4}$$

We use the notation $A\{\mathcal{G}_O, T, S\}$ for the set of $\mathcal{G}_O$-(T, S)-inverses of A. Obviously, $A\{\mathcal{G}_O, T, S\} \subseteq A\{1\}$. Further results about $\mathcal{G}_O$ inverses can be found in [21].

The so-called $\mathcal{G}_O$-(T,S)-partial order was defined on $\mathbb{C}^{m\times n}_{T,S}$ [18] based on $\mathcal{G}$-outer (T,S)-inverses. For $A\in\mathbb{C}^{m\times n}_{T,S}$ and $B\in\mathbb{C}^{m\times n}$, A is below B with respect to the $\mathcal{G}_O$-(T,S)-partial order (denoted by $A\leq^{\mathcal{G}_O,T,S} B$) if there are $C_1,C_2\in A\{\mathcal{G}_O,T,S\}$ satisfying $AC_1=BC_1$ and $C_2A=C_2B$.

The left and right $\mathcal{G}_O$-inverses were introduced [22] as weakened versions of $\mathcal{G}_O$-inverses. A left $\mathcal{G}$-outer (l-$\mathcal{G}$-outer shortly) (T,S)-inverse of $A\in\mathbb{C}^{m\times n}_{T,S}$ is a matrix $D\in\mathbb{C}^{n\times m}$ such that

$$ADA=A \qquad \text{and} \qquad DAA^{(2)}_{T,S}=A^{(2)}_{T,S}.$$

A right $\mathcal{G}$-outer (r-$\mathcal{G}$-outer shortly) (T,S)-inverse of $A\in\mathbb{C}^{m\times n}_{T,S}$ is a matrix $D\in\mathbb{C}^{n\times m}$ satisfying

$$ADA=A \qquad \text{and} \qquad A^{(2)}_{T,S}AD=A^{(2)}_{T,S}.$$

The class of left (or right) $\mathcal{G}_O$-(T,S)-inverses of A is marked as $A\{l,\mathcal{G}_O,T,S\}$ ($A\{r,\mathcal{G}_O,T,S\}$). In the particular choice $m=n$ and $A^{(2)}_{T,S}=A^{\mathrm{D}}$, the left (resp. right) $\mathcal{G}_O$-(T,S)-inverse reduces to the left (resp. right) $\mathcal{G}$-Drazin inverse [22].

2. Preliminaries and motivation

Under the assumptions $C\in\mathbb{C}^{q\times m}$, $B\in\mathbb{C}^{n\times p}$, $A\in\mathbb{C}^{m\times n}_{\mathcal{R}(B),\mathcal{N}(C)}$ and $A\{\mathcal{G}_O,\mathcal{R}(B),\mathcal{N}(C)\}\neq\emptyset$, the general solution to the system with unknown matrix D (termed as $\mathcal{G}$-(B,C) system)

$$ADA=A,\quad DAB=B\quad\text{and}\quad CAD=C \tag{5}$$

was presented using a $\mathcal{G}$-outer $(\mathcal{R}(B),\mathcal{N}(C))$-inverse of A and corresponding inner inverses in [19]. Notice that, by [18, Theorem 2.2], the system (4) is equivalent to the system (5) when $A^{(2)}_{T,S}=A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$. Also, the system (3) is a particular case $B=C=A^{\mathrm{D}}$ of the system (5).

Motivated by recent investigations about $\mathcal{G}$-outer, l-$\mathcal{G}$-outer and r-$\mathcal{G}$-outer inverses, our research presents extensions of these inverses without the hypothesis related to the existence of corresponding outer inverse. Precisely, the following results are proved in this paper.

(1) The first aim of this research is to solve the system (5) for matrices $A\in\mathbb{C}^{m\times n}$, $C\in\mathbb{C}^{q\times m}$ and $B\in\mathbb{C}^{n\times p}$. Thus, we present necessary and sufficient conditions for the solvability of the system (5) without the assumptions $A\in\mathbb{C}^{m\times n}_{\mathcal{R}(B),\mathcal{N}(C)}$ and $A\{\mathcal{G}_O,\mathcal{R}(B),\mathcal{N}(C)\}\neq\emptyset$. In particular, the general solution to the system (5) is given in terms of inner inverses of A, AB and CA, omitting a $\mathcal{G}$-outer $(\mathcal{R}(B),\mathcal{N}(C))$-inverse of A.

(2) A generalization of $\mathcal{G}$-outer inverse, called a $\mathcal{G}$-(B,C) inverse of A, is introduced as a solution to the system (5) in the case $A\in\mathbb{C}^{m\times n}$, $C\in\mathbb{C}^{q\times m}$ and $B\in\mathbb{C}^{n\times p}$. Replacing B and C in (5) with arbitrary $\complement\in\mathbb{C}^{n\times m}$, a $\mathcal{G}$-$\complement$ inverse of A is defined as a solution of the obtained system. Remark that for $\complement=A^{(2)}_{T,S}$, the $\mathcal{G}$-$\complement$ inverse is equal to the $\mathcal{G}_O$-(T,S)-inverse of A [18], and when $m=n$ and $\complement=A^{\mathrm{D}}$, the $\mathcal{G}$-$\complement$ inverse reduces to $\mathcal{G}$-Drazin inverse of A [25]. So, the $\mathcal{G}$-(B,C) inverse, as well as its particular case - the $\mathcal{G}$-$\complement$ inverse, present new classes of generalized inverses and extend the notion of the $\mathcal{G}$-outer inverse.

(3) The set of all $\mathcal{G}$-(B,C) inverses is described (and as consequences sets of $\mathcal{G}_O$-(T,S)-inverses and $\mathcal{G}$-Drazin inverses, respectively) based on adequate inner inverses. We prove that the $\mathcal{G}$-outer (T,S)-inverse of A exists if $A^{(2)}_{T,S}$ exists.

(4) By the Urquhart formula (1), the expression $\complement_1:=F(HAF)^{(1)}H$ is equal to outer inverses $A^{(2)}_{\mathcal{R}(F),*}$, $A^{(2)}_{*,\mathcal{N}(H)}$ and $A^{(2)}_{\mathcal{R}(F),\mathcal{N}(H)}$ under additional assumptions. This fact inspired us to investigate $\mathcal{G}$-$\complement_1$ inverses as new extensions of $\mathcal{G}$-outer inverses. Furthermore, for $(HAF)^{(2)}\in(HAF)\{2\}$ and $\complement_2:=F(HAF)^{(2)}H$, it is interesting to study $\mathcal{G}$-$\complement_2$ inverses as generalizations of $\mathcal{G}$-outer inverses.

(5) One-sided versions of $\mathcal{G}$-(B,C) inverse are defined as weaker kinds of $\mathcal{G}$-(B,C) inverses and investigated as new generalized inverses. Precisely, l-$\mathcal{G}$-B inverses and r-$\mathcal{G}$-C inverses are proposed as solutions of systems obtained omitting the third equation or the second equation in (5). Notice that one-sided versions of $\mathcal{G}$-(B,C) inverse present extensions of one-sided kinds of $\mathcal{G}$-outer inverse.

(6) Based on the $\mathcal{G}$-(B,C) inverse and its one-sided versions, we introduce three kinds of partial orders on the corresponding subsets of $\mathbb{C}^{m\times n}$. These new relations generalize the concepts of $\mathcal{G}$-outer (T,S)-partial order from [18] and one-sided $\mathcal{G}$-outer (T,S)-partial orders from [22].

A global content of involved sections and subsections is described now. After the survey of necessary facts in Section 1, some preliminary results and a detailed motivation are given in Section 2. Extensions of $\mathcal{G}$-outer inverses are proposed and considered in Section 3. Solvability of system (5) is considered in Subsection 3.1. Definitions and characterizations of $\mathcal{G}$-(B,C) inverse and $\mathcal{G}$-$\mathfrak{C}$ inverse are proved in Subsection 3.2. Subsection 3.3 investigates properties of $\mathcal{G}$-$\mathfrak{C}_1$ and $\mathcal{G}$-$\mathfrak{C}_2$ inverses. Extensions of one-sided $\mathcal{G}$-outer inverses are introduced and considered in Section 4. The sets of l-$\mathcal{G}$-B and r-$\mathcal{G}$-C inverses are studied in Subsection 4.1, while Subsection 4.2 gives characterizations of one-sided $\mathcal{G}$-$\mathfrak{C}_1$ and $\mathcal{G}$-$\mathfrak{C}_2$ inverses. Examples in the numerical and symbolic form are presented in Section 5. In Section 6, partial orders defined employing the $\mathcal{G}$-(B,C) inverse and its one-sided versions are introduced and investigated. Final remarks are stated in Section 7.

3. Extensions of $\mathcal{G}$-outer inverse

Solving the $\mathcal{G}$-(B,C) system (5) leads to the definition of new classes of generalized inverses which extend the notions of $\mathcal{G}$-outer and $\mathcal{G}$-Drazin inverses.

3.1. Solvability of $\mathcal{G}$-(B,C) system

In this subsection, our intent is to solve the system (5) for $A\in\mathbb{C}^{m\times n}$, $C\in\mathbb{C}^{q\times m}$ and $B\in\mathbb{C}^{n\times p}$, and without the conditions $A\in\mathbb{C}^{m\times n}_{\mathcal{R}(B),\mathcal{N}(C)}$ or $A\{\mathcal{G}_O,\mathcal{R}(B),\mathcal{N}(C)\}\neq\emptyset$.

The following result will be very useful in solving certain systems of matrix equations. In further, M^- is used to denote arbitrary $M^-\in M\{1\}$.

Lemma 3.1. [3, p. 52] *The matrix equation $AYB=C$ that involves arbitrary $A\in\mathbb{C}^{m\times n}$, $B\in\mathbb{C}^{p\times q}$ and $C\in\mathbb{C}^{m\times q}$, possesses a solution $Y\in\mathbb{C}^{n\times p}$ if and only if $AA^-CB^-B=C$ for some A^- and B^-. In addition, the general solution is $Y=A^-CB^-+Z-A^-AZBB^-$ for arbitrary $Z\in\mathbb{C}^{n\times p}$.*

A necessary and sufficient condition for the solvability of system (5) is established in Theorem 3.2 as well as the general solution of this system employing corresponding inner inverses. To simplify notation, the common situation $A\in\mathbb{C}^{m\times n}$, $B\in\mathbb{C}^{n\times p}$ and $C\in\mathbb{C}^{q\times m}$ will be denoted by $\{A,B,C\}\in\mathbb{C}^{[m,n,p,q]}$.

Theorem 3.2. *Let $\{A,B,C\}\in\mathbb{C}^{[m,n,p,q]}$. Then the $\mathcal{G}$-(B,C) system* (5) *has a solution if and only if*

$$B(I-(AB)^-AB)=0 \qquad \text{and} \qquad (I-CA(CA)^-)C=0, \tag{6}$$

for some inner inverses $(AB)^-$ and $(CA)^-$. Furthermore, the general solution to (5) *is equal to*

$$\begin{aligned} D \;=\; & A^-AA^-+(I-A^-A)B(AB)^-+(CA)^-C(I-AA^-) \\ & +K-(CA)^-CAK(I-AA^-)-(I-A^-A)KAB(AB)^--A^-AKAA^-, \end{aligned} \tag{7}$$

for arbitrary $K\in\mathbb{C}^{n\times m}$ and A^-.

Proof. If D is a solution to the system (5), it can be verified

$$B(AB)^-AB=D(AB(AB)^-AB)=DAB=B,$$
$$CA(CA)^-C=(CA(CA)^-CA)D=CAD=C,$$

which implies (6).

If (6) is satisfied, we conclude that $D = A^-AA^- + (I - A^-A)B(AB)^- + (CA)^-C(I - AA^-)$ is a solution to the system (5) by

$$ADA = AA^-AA^-A = A,$$
$$DAB = A^-AB + (I - A^-A)B(AB)^-AB = A^-AB + (I - A^-A)B = B,$$
$$CAD = CAA^- + CA(CA)^-C(I - AA^-) = CAA^- + C(I - AA^-) = C.$$

In order to obtain the general solution to the system (5), firstly, using Lemma 3.1, notice that the general solution to the equation $ADA = A$ is of the form

$$D = A^-AA^- + Z - A^-AZAA^-, \tag{8}$$

for arbitrary $Z \in \mathbb{C}^{n\times m}$. A substitution of (8) in $DAB = B$ leads to

$$(I - A^-A)ZAB = (I - A^-A)B. \tag{9}$$

By $I - A^-A \in (I - A^-A)\{1\}$, (6) and Lemma 3.1, the general solution to (9) is represented by

$$Z = (I - A^-A)B(AB)^- + Y - (I - A^-A)YAB(AB)^- \tag{10}$$

for arbitrary $Y \in \mathbb{C}^{n\times m}$. From $CAD = C$, (8) and (10), we obtain

$$CAY(I - AA^-) = C(I - AA^-). \tag{11}$$

According to $I - AA^- \in (I - AA^-)\{1\}$, (6) and Lemma 3.1, the general solution to (11) is expressed by

$$Y = (CA)^-C(I - AA^-) + K - (CA)^-CAK(I - AA^-), \tag{12}$$

for arbitrary $K \in \mathbb{C}^{n\times m}$. Applying (8), (10) and (12), we deduce that (7) is the general solution to the system (5). □

Computationally effective conditions of the constraints (6) follow results obtained in [24].

Corollary 3.3. [24] *Let* $\{A, B, C\} \in \mathbb{C}^{[m,n,p,q]}$.

(a) *The subsequent statements are mutually equivalent for arbitrary* $(AB)^-$*:*

(i) $B(I - (AB)^-AB) = 0$;

(ii) $\mathcal{N}(B) = \mathcal{N}(AB)$;

(iii) $\mho_{AB,B}$;

(iv) *there exists* $\Upsilon \in \mathbb{C}^{p\times m}$ *satisfying* $B\Upsilon AB = B$;

(v) *there exists* $A^{(2)}_{\mathcal{R}(B),*} = B(AB)^- = B\Upsilon$.

(b) *The subsequent statements are mutually equivalent for arbitrary* $(CA)^-$*:*

(i) $(I - CA(CA)^-)C = 0$;

(ii) $\mathcal{R}(C) = \mathcal{R}(CA)$;

(iii) $\mho_{CA,C}$;

(iv) *there exists* $\Psi \in \mathbb{C}^{n\times q}$ *satisfying* $CA\Psi C = C$;

(v) *there exists* $A^{(2)}_{*,\mathcal{N}(C)} = (CA)^-C = \Psi C$.

According to Corollary 3.3 and Theorem 3.2, a set of equivalent conditions for the solvability of system (5) can be obtained by corresponding conjunctions of equivalent conditions. Some of them are presented in Corollary 3.4.

Corollary 3.4. *Let $\{A, B, C\} \in \mathbb{C}^{[m,n,p,q]}$. Then the following statements are equivalent for some $(AB)^-$ and $(CA)^-$:*

(i) *the $\mathcal{G}$-(B, C) system* (5) *is solvable;*

(ii) $\mathcal{N}(B) = \mathcal{N}(AB)$ *and* $\mathcal{R}(C) = \mathcal{R}(CA)$;

(iii) $\mho_{AB,B}$ *and* $\mho_{CA,C}$;

(iv) *there exists* $\Upsilon \in \mathbb{C}^{p\times m}$ *satisfying* $B\Upsilon AB = B$ *and* $\Psi \in \mathbb{C}^{n\times q}$ *satisfying* $CA\Psi C = C$;

(v) *there exist* $A^{(2)}_{\mathcal{R}(B),*} = B(AB)^- = B\Upsilon$ *and* $A^{(2)}_{*,\mathcal{N}(C)} = (CA)^-C = \Psi C$.

In the particular case $\mathsf{C} = B = C$ in Theorem 3.2, we obtain the existence criterion for solvability and the general solution of the next system of matrix equations.

Corollary 3.5. *Let $A \in \mathbb{C}^{m\times n}$ and $\mathsf{C} \in \mathbb{C}^{n\times m}$. Then the system*

$$ADA = A, \quad DA\mathsf{C} = \mathsf{C} \quad \text{and} \quad \mathsf{C}AD = \mathsf{C} \tag{13}$$

has a solution if and only if

$$\mathsf{C}\left(I - (A\mathsf{C})^- A\mathsf{C}\right) = 0 \qquad \text{and} \qquad \left(I - \mathsf{C}A(\mathsf{C}A)^-\right)\mathsf{C} = 0,$$

for some $(A\mathsf{C})^-$ and $(\mathsf{C}A)^-$. Afterwards, an arbitrary solution D is of the form

$$\begin{aligned} D = {} & A^-AA^- + (I - A^-A)\mathsf{C}(A\mathsf{C})^- + (\mathsf{C}A)^-\mathsf{C}(I - AA^-) \\ & + K - (\mathsf{C}A)^-\mathsf{C}AK(I - AA^-) - (I - A^-A)KA\mathsf{C}(A\mathsf{C})^- - A^-AKAA^-, \end{aligned}$$

for arbitrary $K \in \mathbb{C}^{n\times m}$ and A^-.

For $\mathsf{C} = A^*$ in Corollary 3.5, we get the solution of the next system of equations.

Corollary 3.6. *Let $A \in \mathbb{C}^{m\times n}$. Then the system*

$$ADA = A, \quad DAA^* = A^* \quad \text{and} \quad A^*AD = A^*$$

has a solution of the form

$$\begin{aligned} D = {} & A^-AA^- + (I - A^-A)A^*(AA^*)^- + (A^*A)^-A^*(I - AA^-) \\ & + K - (A^*A)^-A^*AK(I - AA^-) - (I - A^-A)KAA^*(AA^*)^- - A^-AKAA^-, \end{aligned}$$

for arbitrary $K \in \mathbb{C}^{n\times m}$ and A^-, and for some $(AA^*)^-$ and $(A^*A)^-$. In addition, $DAD = A^\dagger$.

Remark that, by Corollary 3.6, $A^\dagger$ is the unique solution to the system $DAD = D$, $ADA = A$, $DAA^* = A^*$ and $A^*AD = A^*$.

3.2. $\mathcal{G}$-(B,C) inverses

Considering the system (5) imposes us to define two new classes of generalized inverses. Precisely, we generalize the $\mathcal{G}$-outer and $\mathcal{G}$-Drazin inverses presenting broader classes of generalized inverses.

Definition 3.7. *Let $\{A,B,C\} \in \mathbb{C}^{[m,n,p,q]}$. A matrix $D \in \mathbb{C}^{n\times m}$ is:*

(i) *a $\mathcal{G}$-(B,C) inverse of A if it satisfies* (5);

(ii) *a $\mathcal{G}$-$\complement$ inverse of A if it satisfies* (13).

Theorem 3.2 implies that $\mathcal{G}$-(B,C) inverse is not uniquely determined. So, it is convenient to denote by $A\{\mathcal{G},B,C\}$ (or $A\{\mathcal{G},\complement\}$) the set of $\mathcal{G}$-(B,C) ($\mathcal{G}$-$\complement$) inverses of A. Evidently, $A\{\mathcal{G},B,C\} \subseteq A\{1\}$.

Applying Theorem 3.2, the set $A\{\mathcal{G},B,C\}$ can be described in terms of one parameter K or two parameters L and M.

Corollary 3.8. *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$ and $C \in \mathbb{C}^{q\times m}$. If* (6) *is satisfied, then*

$$\begin{aligned} A\{\mathcal{G},B,C\} &= \{A^-AA^- + (I-A^-A)B(AB)^- + (CA)^-C(I-AA^-) \\ &\quad + (I-(CA)^-CA)K(I-AA^-) - (I-A^-A)K(I-AB(AB)^-) \\ &\quad - (I-A^-A)K(I-AA^-) : K \in \mathbb{C}^{n\times m} \text{ is arbitrary}\} \qquad (14)\\ &= \{A^-AA^- + (I-A^-A)B(AB)^- + (CA)^-C(I-AA^-) \\ &\quad + (I-(CA)^-CA)L(I-AA^-) - (I-A^-A)M(I-AB(AB)^-) : \\ &\qquad L,M \in \mathbb{C}^{n\times m} \text{ are arbitrary}\}. \qquad (15) \end{aligned}$$

Proof. The equality (14) follows from (7) and direct calculations.

For $K = L(I-AA^-)+(I-A^-A)M$ in (14), where $L,M \in \mathbb{C}^{n\times m}$ are arbitrary, we observe that (15) is satisfied. Choosing $L = K$ and $M = (I-A^-A)K - K(I-AA^-)$ in (15), when $K \in \mathbb{C}^{n\times m}$ is arbitrary, we obtain (14). □

Corollary 3.9. *Let $\{A,B,C\} \in \mathbb{C}^{[m,n,p,q]}$. If* (6) *is satisfied, then*

$$\begin{aligned} A\{\mathcal{G},B,C\} &= \{A^-AA^- + (I-A^-A)A^{(2)}_{\mathcal{R}(B),*} + A^{(2)}_{*,\mathcal{N}(C)}(I-AA^-) \\ &\quad + \left(I - A^{(2)}_{*,\mathcal{N}(C)}A\right)K(I-AA^-) - (I-A^-A)K\left(I - AA^{(2)}_{\mathcal{R}(B),*}\right) \\ &\quad - (I-A^-A)K(I-AA^-) : \quad K \in \mathbb{C}^{n\times m} \text{ is arbitrary}\} \\ &= \{A^-AA^- + (I-A^-A)B\Upsilon + \Psi C(I-AA^-) \\ &\quad + (I-\Psi CA)K(I-AA^-) - (I-A^-A)K(I-AB\Upsilon) - (I-A^-A)K(I-AA^-) : \\ &\quad K \in \mathbb{C}^{n\times m} \text{ is arbitrary}, \Upsilon \in \mathbb{C}^{p\times m}, B\Upsilon AB = B,\ \Psi \in \mathbb{C}^{n\times q}, CA\Psi C = C\} \\ &= \{A^-AA^- + (I-A^-A)A^{(2)}_{\mathcal{R}(B),*} + A^{(2)}_{*,\mathcal{N}(C)}(I-AA^-) \\ &\quad + (I - A^{(2)}_{*,\mathcal{N}(C)}A)L(I-AA^-) - (I-A^-A)M\left(I - AA^{(2)}_{\mathcal{R}(B),*}\right) : \\ &\qquad L,M \in \mathbb{C}^{n\times m} \text{ are arbitrary}\}. \end{aligned}$$

Proof. Follows from the replacements $A^{(2)}_{\mathcal{R}(B),*} = B(AB)^- = B\Upsilon$ and $A^{(2)}_{*,\mathcal{N}(C)} = (CA)^-C = \Psi C$ in Corollary 3.8, which is ensured after the application of Corollary 3.4. □

Corollary 3.10. *Suppose that $\{A,B,C\} \in \mathbb{C}^{[m,n,p,q]}$ and* (6) *is satisfied. Then*

$$A\{\mathcal{G},B,C\} = A\{\mathcal{G},P,Q\},$$

where P and Q are arbitrary matrices satisfying $\mathcal{N}(P) = \mathcal{N}(AP)$, $\mathcal{R}(Q) = \mathcal{R}(QA)$, $\mathcal{R}(P) = \mathcal{R}(B)$ and $\mathcal{N}(Q) = \mathcal{N}(C)$.

Proof. On the basis of Corollary 3.4, it follows that the system $\mathcal{G}$-(P,Q) is solvable and the next identities hold:

$$\begin{aligned} P(AP)^- &= A^{(2)}_{\mathcal{R}(P),*} = A^{(2)}_{\mathcal{R}(B),*} = B(AB)^-; \\ (QA)^-Q &= A^{(2)}_{*,\mathcal{N}(Q)} = A^{(2)}_{*,\mathcal{N}(C)} = (CA)^-C. \end{aligned} \tag{16}$$

Further, Corollary 3.8 in common with (16) implies $A\{\mathcal{G},B,C\} = A\{\mathcal{G},P,Q\}$. □

Specializing B, C and $\complement$ in Definition 3.7, some important cases can be obtained.

(i) When $\complement = A^{(2)}_{T,S}$, the $\mathcal{G}$-$\complement$ inverse reduces to the $\mathcal{G}$-outer (T,S)-inverse of A [18], which implies $A\{\mathcal{G},A^{(2)}_{T,S}\} = A\{\mathcal{G}_O,T,S\}$.

(ii) Provided $m = n$ and $\complement = A^{\mathrm{D}}$, the $\mathcal{G}$-$\complement$ inverse coincides with the $\mathcal{G}$-Drazin inverse of A [25], which implies $A\{\mathcal{G},A^{\mathrm{D}}\} = A\{\mathcal{G},A^k\} = A\{\mathcal{G}_O,A^k,A^k\} = A\{\mathcal{G}_O,A^{\mathrm{D}},A^{\mathrm{D}}\}$, $k = \mathrm{ind}(A)$.

For $\complement = A^{(2)}_{T,S}$ or $\complement = A^{\mathrm{D}}$ in Corollary 3.5, we describe the sets $A\{\mathcal{G}_O,T,S\}$ and $A\{GD\}$ in terms of adequate inner inverses.

Corollary 3.11. *Suppose that $A \in \mathbb{C}^{m\times n}$.*

(i) *If $A^{(2)}_{T,S}$ exists, then*

$$\begin{aligned} A\{\mathcal{G}_O,T,S\} = \{&A^-AA^- + (I-A^-A)A^{(2)}_{T,S} + A^{(2)}_{T,S}(I-AA^-) \\ &+ K - A^{(2)}_{T,S}AK(I-AA^-) - (I-A^-A)KAA^{(2)}_{T,S} - A^-AKAA^- : \\ &K \in \mathbb{C}^{n\times m} \text{ is arbitrary}\}. \end{aligned}$$

(ii) *If $m = n$ and $\mathrm{ind}(A) = k$, then*

$$\begin{aligned} A\{GD\} &= \{A^-AA^- + (I-A^-A)A^{\mathrm{D}} + A^{\mathrm{D}}(I-AA^-) \\ &\quad + K - A^{\mathrm{D}}AK(I-AA^-) - (I-A^-A)KAA^{\mathrm{D}} - A^-AKAA^- : \\ &\quad K \in \mathbb{C}^{n\times m} \text{ is arbitrary}\} \\ &= \{A^-AA^- + (I-A^-A)A^k(A^{k+1})^- + (A^{k+1})^-A^k(I-AA^-) - A^-AKAA^- \\ &\quad + K - (A^{k+1})^-A^{k+1}K(I-AA^-) - (I-A^-A)KA^{k+1}(A^{k+1})^- : \\ &\quad K \in \mathbb{C}^{n\times m} \text{ is arbitrary}\}. \end{aligned}$$

Proof. (i) For $(AA^{(2)}_{T,S})^- = AA^{(2)}_{T,S} \in (AA^{(2)}_{T,S})\{1\}$ and $(A^{(2)}_{T,S}A)^- = A^{(2)}_{T,S}A \in (A^{(2)}_{T,S}A)\{1\}$, it follows

$$A^{(2)}_{T,S}\left(I - (AA^{(2)}_{T,S})^-AA^{(2)}_{T,S}\right) = A^{(2)}_{T,S}AA^{(2)}_{T,S}\left(I - (AA^{(2)}_{T,S})^-AA^{(2)}_{T,S}\right) = 0$$

and

$$\left(I - A^{(2)}_{T,S}A(A^{(2)}_{T,S}A)^-\right)A^{(2)}_{T,S} = \left(I - A^{(2)}_{T,S}A(A^{(2)}_{T,S}A)^-\right)A^{(2)}_{T,S}AA^{(2)}_{T,S} = 0.$$

The rest is clear by Corollary 3.5 when $\complement = A^{(2)}_{T,S}$.

(ii) In the cases $\complement = A^{\mathrm{D}}$ or $\complement = A^k$ in Corollary 3.5, we prove this part. □

Remark that, when $A^{(2)}_{T,S}$ exists, $A\{\mathcal{G}_O,T,S\} \neq \emptyset$, i.e., the $\mathcal{G}$-outer (T,S)-inverse of A exists.

Theorem 3.12 investigates a necessary and sufficient condition for the existence of a $\mathcal{G}$-(B,C) inverse.

Theorem 3.12. *For $\{A, B, C\} \in \mathbb{C}^{[m,n,p,q]}$ and $D \in \mathbb{C}^{n\times m}$, the following statements are equivalent:*

(i) $D \in A\{\mathcal{G}, B, C\}$;

(ii) $ADA = A$, $\mathcal{R}(B) \subseteq \mathcal{R}(DA)$ *and* $\mathcal{N}(AD) \subseteq \mathcal{N}(C)$.

Proof. (i) $\Rightarrow$ (ii): It is obvious.

(ii) $\Rightarrow$ (i): The conditions $ADA = A$, $\mathcal{R}(B) \subseteq \mathcal{R}(DA)$ and $\mathcal{N}(AD) \subseteq \mathcal{N}(C)$ imply $\mathcal{R}(B) \subseteq \mathcal{R}(DA) = \mathcal{N}(I - DA)$ and $\mathcal{R}(I - AD) = \mathcal{N}(AD) \subseteq \mathcal{N}(C)$. Therefore, $DAB = B$ and $CAD = C$, that is, D is a $\mathcal{G}$-(B, C) inverse of A. □

For arbitrary $\mathcal{G}$-(B, C) inverses D_1 and D_2 of A, Theorem 3.13 confirms that D_1AD_2 is a $\mathcal{G}$-(B, C) inverse of A.

Theorem 3.13. *For $\{A, B, C\} \in \mathbb{C}^{[m,n,p,q]}$, the next inclusions hold:*
$A\{\mathcal{G}, B, C\} \cdot A \cdot A\{\mathcal{G}, B, C\} \subseteq A\{\mathcal{G}, B, C\}$.

Proof. Let $D_1, D_2 \in A\{\mathcal{G}, B, C\}$ and $D = D_1AD_2$. Then

$$\begin{aligned} ADA &= (AD_1A)D_2A = AD_2A = A, \\ DAB &= D_1A(D_2AB) = D_1AB = B, \\ CAD &= (CAD_1)AD_2 = CAD_2 = C, \end{aligned}$$

which is equivalent with $D \in A\{\mathcal{G}, B, C\}$. □

One characterization of $\mathcal{G}$-(B, C) invertible matrices can be given in terms of appropriate idempotents.

Theorem 3.14. *For $\{A, B, C\} \in \mathbb{C}^{[m,n,p,q]}$, there is equivalence between the subsequent assertions:*

(i) $A\{\mathcal{G}, B, C\} \neq \emptyset$;

(ii) *there are two idempotents $T_1 \in \mathbb{C}^{m\times m}$ and $T_2 \in \mathbb{C}^{n\times n}$ such that*

$$\mathcal{R}(T_1) = \mathcal{R}(A), \quad \mathcal{N}(T_2) = \mathcal{N}(A), \quad \mathcal{R}(B) \subseteq \mathcal{R}(T_2) \quad \text{and} \quad \mathcal{N}(T_1) \subseteq \mathcal{N}(C).$$

Subsequently, $T_2A^- T_1 \in A\{\mathcal{G}, B, C\}$ for arbitrary A^-, which implies

$$T_2 \cdot A\{1\} \cdot T_1 \subseteq A\{\mathcal{G}, B, C\}.$$

Proof. (i) $\Rightarrow$ (ii): For $D \in A\{\mathcal{G}, B, C\}$, if $T_1 = AD$ and $T_2 = DA$, we observe by $ADA = A$, that $T_1 = T_1^2$, $T_2 = T_2^2$, $\mathcal{R}(T_1) = \mathcal{R}(A)$, and $\mathcal{N}(T_2) = \mathcal{N}(A)$. Since $DAB = B$ and $CAD = C$, we deduce $\mathcal{R}(B) \subseteq \mathcal{R}(T_2)$ and $\mathcal{N}(T_1) \subseteq \mathcal{N}(C)$.

(ii) $\Rightarrow$ (i): Note that, for $A^- \in A\{1\}$, the assumption $\mathcal{R}(T_1) = \mathcal{R}(A)$ gives $A = T_1A$ and $T_1 = AA^-T_1$. From $\mathcal{N}(A) = \mathcal{N}(T_2)$, it follows $A = AT_2$ and $T_2 = T_2A^-A$. The conditions $\mathcal{R}(B) \subseteq \mathcal{R}(T_2)$ and $\mathcal{N}(T_1) \subseteq \mathcal{N}(C)$ imply $B = T_2B$ and $C = CT_1$. If we take $D = T_2A^-T_1$, notice that

$$\begin{aligned} ADA &= (AT_2)A^-(T_1A) = AA^-A = A, \\ DAB &= T_2A^-(T_1A)B = (T_2A^-A)B = T_2B = B, \\ CAD &= C(AT_2)A^-T_1 = C(AA^-T_1) = CT_1 = C. \end{aligned}$$

Thus, $D \in A\{\mathcal{G}, B, C\}$. □

Taking $\mathfrak{C} = B = C$ in Theorem 3.12, Theorem 3.13 and Theorem 3.14, we obtain corresponding results for $\mathcal{G}$-$\mathfrak{C}$ inverse, which reveal appropriate results from [18] in the complex matrix case.

3.3. $\mathcal{G}$-$\complement_1$ and $\mathcal{G}$-$\complement_2$ inverses

In this subsection, let $A \in \mathbb{C}^{m\times n}$, $F \in \mathbb{C}^{n\times k}$, $H \in \mathbb{C}^{l\times m}$ and let $(HAF)^{(1)} \in (HAF)\{1\}$ and $(HAF)^{(2)} \in (HAF)\{2\}$ be fixed but arbitrary. If we state $\complement_1 := F(HAF)^{(1)}H$ or $\complement_2 := F(HAF)^{(2)}H$ instead of $\complement$ in Definition 3.7(ii) of the $\mathcal{G}$-$\complement$ inverse, we obtain $\mathcal{G}$-$\complement_1$ and $\mathcal{G}$-$\complement_2$ inverses.

Under extra assumptions, by Urquhart formula (1), the expression $\complement_1 := F(HAF)^{(1)}H$ is equal to $A^{(2)}_{\mathcal{R}(F),*}$, $A^{(2)}_{*,\mathcal{N}(H)}$ or $A^{(2)}_{\mathcal{R}(F),\mathcal{N}(H)}$. According to this fact, $\mathcal{G}$-$\complement_1$ inverses can be considered as new extensions of $\mathcal{G}$-outer inverses.

Theorem 3.15. *The set of $\mathcal{G}$-$\complement_1$ inverses of A satisfies*

$$A\{\mathcal{G},\complement_1\} = \begin{cases} A\{\mathcal{G}, A^{(2)}_{\mathcal{R}(F),*}\} = A\{\mathcal{G}, F, \complement_1\}, & \mho_{HAF,F}; \\ A\{\mathcal{G}, A^{(2)}_{*,\mathcal{N}(H)}\} = A\{\mathcal{G}, \complement_1, H\}, & \mho_{HAF,H}; \\ A\{\mathcal{G}_O, \mathcal{R}(F), \mathcal{N}(H)\} = A\{\mathcal{G}, F, H\}, & \mho_{HAF,F,H}. \end{cases}$$

Proof. By Proposition 1.1, an arbitrary $D \in A\{\mathcal{G},\complement_1\}$ satisfies

(i) $\mho_{HAF,F} \Longrightarrow D \in A\{\mathcal{G}, A^{(2)}_{\mathcal{R}(F),*}\}$;

(ii) $\mho_{HAF,H} \Longrightarrow D \in A\{\mathcal{G}, A^{(2)}_{*,\mathcal{N}(H)}\}$;

(iii) $\mho_{HAF,F,H} \Longrightarrow D \in A\{\mathcal{G}_O, \mathcal{R}(F), \mathcal{N}(H)\}$,

which confirms mentioned equalities. According to [18, Theorem 2.2], note that $\mho_{HAF,F,H}$ gives $A\{\mathcal{G}, F, H\} = A\{\mathcal{G}_O, \mathcal{R}(F), \mathcal{N}(H)\}$. $\square$

As a consequence of Theorem 3.15, we obtain the next result about $\mathcal{G}$-$\complement_2$ inverses.

Corollary 3.16. *If P and Q be suitable Hermitian idempotents satisfying $(HAF)^{(2)} = Q(PHAFQ)^\dagger P$, then*

$$A\{\mathcal{G},\complement_2\} = \begin{cases} A\{\mathcal{G}, A^{(2)}_{\mathcal{R}(FQ),*}\} = A\{\mathcal{G}, FQ, \complement_1\}, & \mho_{PHAFQ,FQ}; \\ A\{\mathcal{G}, A^{(2)}_{*,\mathcal{N}(PH)}\} = A\{\mathcal{G}, \complement_1, PH\}, & \mho_{PHAFQ,PH}; \\ A\{\mathcal{G}_O, \mathcal{R}(FQ), \mathcal{N}(PH)\} = A\{\mathcal{G}, FQ, PH\}, & \mho_{PHAFQ,FQ,PH}. \end{cases}$$

Proof. Notice that, by [3], $Z \in (HAF)\{2\}$ if and only if $Z = (PHAFQ)^\dagger$, where P and Q are suitable Hermitian idempotents. In this case, $(HAF)^{(2)} = (PHAFQ)^\dagger = Q(PHAFQ)^\dagger = (PHAFQ)^\dagger P$. An application of Theorem 3.15 implies that an arbitrary $\mathcal{G}$-$\complement_2$ inverse D of A satisfies

(i) $\mho_{PHAFQ,FQ} \Longrightarrow D \in A\{\mathcal{G}, A^{(2)}_{\mathcal{R}(FQ),*}\}$;

(ii) $\mho_{PHAFQ,PH} \Longrightarrow D \in A\{\mathcal{G}, A^{(2)}_{*,\mathcal{N}(PH)}\}$;

(iii) $\mho_{PHAFQ,FQ,PH} \Longrightarrow D \in A\{\mathcal{G}_O, \mathcal{R}(FQ), \mathcal{N}(PH)\}$.

The proof follows immediately. $\square$

Because $\complement_2 := F(HAF)^{(2)}H$ is an outer inverse of A, we have the following characterizations of $\mathcal{G}$-$\complement_2$ inverses.

Theorem 3.17. *For $D \in \mathbb{C}^{n\times m}$, the following statements are equivalent:*

(i) $D \in A\{\mathcal{G},\complement_2\}$;

(ii) $ADA = A$ *and* $DA\complement_2 = \complement_2 AD$;

(iii) $ADA = A$, $DA\complement_2 A = \complement_2 A$ *and* $A\complement_2 = A\complement_2 AD$.

Proof. (i) ⇒ (ii): This implication is clear by the definition of a $\mathcal{G}$-$\complement_2$ inverse.

(ii) ⇒ (iii): Since $ADA = A$ and $DA\complement_2 = \complement_2 AD$, then $DA\complement_2 A = \complement_2(ADA) = \complement_2 A$ and $A\complement_2 AD = (ADA)\complement_2 = A\complement_2$.

(iii) ⇒ (i): Assume that $ADA = A$, $DA\complement_2 A = \complement_2 A$ and $A\complement_2 = A\complement_2 AD$. Because $\complement_2 \in A\{2\}$, we get

$$\complement_2 = (\complement_2 A)\complement_2 = DA(\complement_2 A\complement_2) = DA\complement_2$$

and

$$\complement_2 = \complement_2(A\complement_2) = (\complement_2 A\complement_2)AD = \complement_2 AD.$$

Hence, $D \in A\{\mathcal{G}, \complement_2\}$. □

4. Extensions of one-sided $\mathcal{G}$-outer inverses

Considering weaker versions of the system (5), we introduce one-sided kinds of $\mathcal{G}$-(B,C) inverses and generalize l-$\mathcal{G}$-outer and r-$\mathcal{G}$-outer inverses.

4.1. L- $\mathcal{G}$-B and r- $\mathcal{G}$-C inverses

By omitting the second or third equation involved in the system (5), we present two classes of novel generalized inverses that represent weaker versions of $\mathcal{G}$-(B,C) invertibility. By Corollary 3.3, Theorem 4.1 can be verified by a similar procedure as in the verification of Theorem 3.2.

Theorem 4.1. *Then the following statements hold for* $\{A, B, C\} \in \mathbb{C}^{[m,n,p,q]}$*:*

(a) *The system (termed as L-$\mathcal{G}$-B system)*

$$ADA = A \quad \text{and} \quad DAB = B \tag{17}$$

has a solution if and only if one of the following two conditions is satisfied:

$$\begin{aligned} &\text{(i)} \quad B(I - (AB)^- AB) = 0, \text{ for some } (AB)^- \\ &\text{(ii)} \quad B\Upsilon AB = B,\ \Upsilon \in \mathbb{C}^{p\times m}. \end{aligned} \tag{18}$$

Furthermore, the general solution to (17) *is equal to*

$$\begin{aligned} D &= A^- AA^- + (I - A^- A)B(AB)^- + Y - (I - A^- A)YAB(AB)^- - A^- AYAA^- \\ &= A^- AA^- + (I - A^- A)B\Upsilon + Y - (I - A^- A)YA\Upsilon - A^- AYAA^-,\ B\Upsilon AB = B,\ \Upsilon \in \mathbb{C}^{p\times m} \end{aligned}$$

for arbitrary $Y \in \mathbb{C}^{n\times m}$ *and* A^-.

(b) *The system (termed as R-$\mathcal{G}$-C system)*

$$ADA = A \quad \text{and} \quad CAD = C \tag{19}$$

has a solution if and only if one of the following two conditions is satisfied:

$$\begin{aligned} &\text{(i)} \quad (I - CA(CA)^-)C = 0, \text{ for some } (CA)^- \\ &\text{(ii)} \quad CA\Psi C = C,\ \Psi \in \mathbb{C}^{n\times q}. \end{aligned} \tag{20}$$

Furthermore, the general solution is equal to

$$\begin{aligned} D &= A^- AA^- + (CA)^- C(I - AA^-) + Y - (CA)^- CAY(I - AA^-) - A^- AYAA^- \\ &= A^- AA^- + \Psi C(I - AA^-) + Y - \Psi CAY(I - AA^-) - A^- AYAA^-,\ CA\Psi C = C,\ \Psi \in \mathbb{C}^{n\times q} \end{aligned}$$

for arbitrary $Y \in \mathbb{C}^{n\times m}$ *and* A^-.

Definition 4.2. *The following definitions are valid for* $\{A, B, C\} \in \mathbb{C}^{[m,n,p,q]}$*:*

(a) *a matrix* $D \in \mathbb{C}^{n\times m}$ *satisfying* (17) *is a left* $\mathcal{G}$*-B (l-*$\mathcal{G}$*-B) inverse of A;*

(b) *a matrix* $D \in \mathbb{C}^{n\times m}$*satisfying* (19) *is a right* $\mathcal{G}$*-C (r-*$\mathcal{G}$*-C) inverse of A.*

We will write $A\{l, \mathcal{G}, B\}$ (or $A\{r, \mathcal{G}, C\}$), respectively, as labels for the set of l-$\mathcal{G}$-B (r-$\mathcal{G}$-C) inverses of A. Observe the inclusions $A\{\mathcal{G}, B, C\} \subseteq A\{l, \mathcal{G}, B\}$, $A\{\mathcal{G}, B, C\} \subseteq A\{r, \mathcal{G}, C\}$ and the equality $A\{\mathcal{G}, B, C\} = A\{l, \mathcal{G}, B\} \cap A\{r, \mathcal{G}, C\}$.

For $B = A^{(2)}_{T,S}$ (or $B = A^{\mathrm{D}}$), a l-$\mathcal{G}$-B inverse of A reduces to the l-$\mathcal{G}$-outer (T, S)-inverse (l-$\mathcal{G}$-Drazin inverse) of A. Analogously, for $C = A^{(2)}_{T,S}$ (or $C = A^{\mathrm{D}}$), a r-$\mathcal{G}$-C inverse of A reduces to the r-$\mathcal{G}$-outer (T, S)-inverse (r-$\mathcal{G}$-Drazin inverse) of A.

Some set identities can be verified as in Corollary 3.10.

Corollary 4.3. *Suppose that* $\{A, B, C\} \in \mathbb{C}^{[m,n,p,q]}$.

(a) *If* (18) *is satisfied, we have*

$$A\{l, \mathcal{G}, B\} = A\{l, \mathcal{G}, P\},$$

where P is an arbitrary matrix satisfying $\mathcal{N}(P) = \mathcal{N}(AP)$ *and* $\mathcal{R}(P) = \mathcal{R}(B)$.

(b) *If* (20) *is satisfied, we have*

$$A\{r, \mathcal{G}, C\} = A\{r, \mathcal{G}, Q\},$$

where Q is an arbitrary matrix satisfying $\mathcal{R}(Q) = \mathcal{R}(QA)$ *and* $\mathcal{N}(Q) = \mathcal{N}(C)$.

Applying Theorem 4.1, the sets of one-sided $\mathcal{G}$-outer as well as $\mathcal{G}$-Drazin inverses are characterized as in Corollary 3.11.

Corollary 4.4. *Let* $A \in \mathbb{C}^{m\times n}$.

(i) *If* $A^{(2)}_{T,S}$ *exists, then*

$$\begin{aligned} A\{l, \mathcal{G}_O, T, S\} = \{&A^-AA^- + (I - A^-A)A^{(2)}_{T,S} + Y - (I - A^-A)YAA^{(2)}_{T,S} - A^-AYAA^- :\\ &Y \in \mathbb{C}^{n\times m} \text{ is arbitrary}\} \end{aligned}$$

and

$$\begin{aligned} A\{r, \mathcal{G}_O, T, S\} = \{&A^-AA^- + A^{(2)}_{T,S}(I - AA^-) + Y - A^{(2)}_{T,S}AY(I - AA^-) - A^-AYAA^- :\\ &Y \in \mathbb{C}^{n\times m} \text{ is arbitrary}\} \end{aligned}$$

for $A^- \in A\{1\}$.

(ii) *If* $m = n$ *and* $\mathrm{ind}(A) = k$, *then*

$$\begin{aligned} A\{l, GD\} &= \{A^-AA^- + (I - A^-A)A^{\mathrm{D}} + Y - (I - A^-A)YAA^{\mathrm{D}} - A^-AKAA^- :\\ &\quad Y \in \mathbb{C}^{n\times m} \text{ is arbitrary}\}\\ &= \{A^-AA^- + (I - A^-A)A^k(A^{k+1})^- + Y - (I - A^-A)YA^{k+1}(A^{k+1})^- - A^-AYAA^- :\\ &\quad Y \in \mathbb{C}^{n\times m} \text{ is arbitrary}\} \end{aligned}$$

and

$$\begin{aligned} A\{r, GD\} &= \{A^-AA^- + A^{\mathrm{D}}(I - AA^-) + Y - A^{\mathrm{D}}AY(I - AA^-) - A^-AYAA^- :\\ &\quad Y \in \mathbb{C}^{n\times m} \text{ is arbitrary}\}\\ &= \{A^-AA^- + (A^{k+1})^-A^k(I - AA^-) + Y - (A^{k+1})^-A^{k+1}Y(I - AA^-) - A^-AYAA^- :\\ &\quad Y \in \mathbb{C}^{n\times m} \text{ is arbitrary}\} \end{aligned}$$

for $A^- \in A\{1\}$ *and* $(A^{k+1})^- \in (A^{k+1})\{1\}$.

Theorem 4.5 gives the subsequent characterizations of l-$\mathcal{G}$-B and r-$\mathcal{G}$-C inverses, as a continuation of Theorem 3.12.

Theorem 4.5. *Let* $\{A, B, C\} \in \mathbb{C}^{[m,n,p,q]}$ *and* $D \in \mathbb{C}^{n\times m}$.
(a) *The next statements are equivalent:*

(i) $D \in A\{l, \mathcal{G}, B\}$;

(ii) $ADA = A$ *and* $\mathcal{R}(B) \subseteq \mathcal{R}(DA)$.

(b) *The next statements are mutually equivalent:*

(i) $D \in A\{r, \mathcal{G}, C\}$;

(ii) $ADA = A$ *and* $\mathcal{N}(AD) \subseteq \mathcal{N}(C)$.

In Theorem 4.6, we check some additional inclusions for the l-$\mathcal{G}$-B and r-$\mathcal{G}$-C inverses as in Theorem 3.13.

Theorem 4.6. *Let* $\{A, B, C\} \in \mathbb{C}^{[m,n,p,q]}$. *The next inclusions hold:*

$$A\{l, \mathcal{G}, B\} \cdot A \cdot A\{l, \mathcal{G}, B\} \subseteq A\{l, \mathcal{G}, B\} \quad \textit{and} \quad A\{r, \mathcal{G}, C\} \cdot A \cdot A\{r, \mathcal{G}, C\} \subseteq A\{r, \mathcal{G}, C\}.$$

Identities included in Corollary 4.7 follow immediately from corresponding definitions, Theorem 3.13 and Theorem 4.6.

Corollary 4.7. *The next identities hold for* $\{A, B, C\} \in \mathbb{C}^{[m,n,p,q]}$:
(a) $A \cdot A\{\mathcal{G}, B, C\} \cdot A = A \cdot A\{l, \mathcal{G}, B\} \cdot A = A \cdot A\{r, \mathcal{G}, C\} \cdot A = A$;
(b) $$\begin{aligned} A\{\mathcal{G}, B, C\} \cdot AB &= A\{l, \mathcal{G}, B\} \cdot AB \\ &= A\{\mathcal{G}, B, C\} \cdot A \cdot A\{\mathcal{G}, B, C\} \cdot AB = A\{l, \mathcal{G}, B\} \cdot A \cdot A\{l, \mathcal{G}, B\} \cdot AB \\ &= B; \end{aligned}$$
(c) $$\begin{aligned} CA \cdot A\{\mathcal{G}, B, C\} &= CA \cdot A\{r, \mathcal{G}, C\} \\ &= CA \cdot A\{\mathcal{G}, B, C\} \cdot A \cdot A\{\mathcal{G}, B, C\} = CA \cdot A\{r, \mathcal{G}, C\} \cdot A \cdot A\{r, \mathcal{G}, C\} \\ &= C. \end{aligned}$$

Some characterizations for $A\{l, \mathcal{G}, B\}$ and $A\{r, \mathcal{G}, C\}$ that involve idempotents are introduced in Theorem 4.8.

Theorem 4.8. *Consider* $\{A, B, C\} \in \mathbb{C}^{[m,n,p,q]}$.
(a) *The next claims are mutually equivalent:*

(i) $A\{l, \mathcal{G}, B\} \neq \emptyset$;

(ii) *there are idempotents* $T_1 \in \mathbb{C}^{m\times m}$ *and* $T_2 \in \mathbb{C}^{n\times n}$ *satisfying*

$$\mathcal{R}(A) = \mathcal{R}(T_1), \quad \mathcal{N}(A) = \mathcal{N}(T_2) \quad \textit{and} \quad \mathcal{R}(B) \subseteq \mathcal{R}(T_2).$$

Moreover, for arbitrary $A^- \in A\{1\}$, *it follows* $T_2 A^- T_1 \in A\{l, \mathcal{G}, B\}$, *which implies*

$$T_2 \cdot A\{1\} \cdot T_1 \subseteq A\{l, \mathcal{G}, B\}.$$

(b) *The subsequent claims are mutually equivalent:*

(i) $A\{r, \mathcal{G}, C\} \neq \emptyset$;

(ii) *there are idempotents* $T_1 \in \mathbb{C}^{m\times m}$ *and* $T_2 \in \mathbb{C}^{n\times n}$ *satisfying*

$$\mathcal{R}(A) = \mathcal{R}(T_1), \quad \mathcal{N}(A) = \mathcal{N}(T_2) \quad \textit{and} \quad \mathcal{N}(T_1) \subseteq \mathcal{N}(C).$$

Furthermore, for any $A^- \in A\{1\}$*, we have* $T_2A^-T_1 \in A\{r, \mathcal{G}, C\}$*, which implies*

$$T_2 \cdot A\{1\} \cdot T_1 \subseteq A\{r, \mathcal{G}, C\}.$$

Theorem 4.9. *For* $\{A, B, C\} \in \mathbb{C}^{[m,n,p,q]}$*, there is equivalence between the subsequent assertions:*

(i) $D_1 \in A\{l, \mathcal{G}, B\}$*,* $D_2 \in A\{r, \mathcal{G}, C\}$ *and* $D_1AD_2 = D_2AD_1$*;*

(ii) $D_1, D_2 \in A\{\mathcal{G}, B, C\}$*,* $AD_2 = AD_1$ *and* $D_1A = D_2A$*.*

Proof. (i) ⇒ (ii): Multiplying the equality $D_1AD_2 = D_2AD_1$ by A from the left (or right) hand side, we get $AD_2 = AD_1$ ($D_1A = D_2A$). Since $B = (D_1A)B = D_2AB$ and $C = CAD_2 = CAD_1$, we deduce $D_1, D_2 \in A\{\mathcal{G}, B, C\}$.

(ii) ⇒ (i): It is clear by $D_1(AD_2) = (D_1A)D_1 = D_2AD_1$. □

4.2. One-sided $\mathcal{G}$-$\mathfrak{C}_1$ and $\mathcal{G}$-$\mathfrak{C}_2$ inverses

Consider $A \in \mathbb{C}^{m\times n}$, $F \in \mathbb{C}^{n\times k}$, $H \in \mathbb{C}^{l\times m}$ and fixed but arbitrary $(HAF)^{(1)} \in (HAF)\{1\}$, and $(HAF)^{(2)} \in (HAF)\{2\}$ in this subsection. It is interesting to state $\mathfrak{C}_1 := F(HAF)^{(1)}H$ or $\mathfrak{C}_2 := F(HAF)^{(2)}H$ instead of B and C in the definition of the l-$\mathcal{G}$-B inverse and r-$\mathcal{G}$-C inverse and characterize one-sided $\mathcal{G}$-$\mathfrak{C}_1$ and $\mathcal{G}$-$\mathfrak{C}_2$ inverses.

Applying Urquhart formula (1), l- and r-$\mathcal{G}$-$\mathfrak{C}_1$ inverses are additional extensions of l- and r-$\mathcal{G}$-outer inverses.

Theorem 4.10. (a) *Arbitrary element* $D \in A\{l, \mathcal{G}, \mathfrak{C}_1\}$ *satisfies:*

(i) $\mho_{HAF,F} \Longrightarrow D \in A\{l, \mathcal{G}, A^{(2)}_{\mathcal{R}(F),*}\}$*;*

(ii) $\mho_{HAF,H} \Longrightarrow D \in A\{l, \mathcal{G}, A^{(2)}_{*,\mathcal{N}(H)}\}$*;*

(iii) $\mho_{HAF,F,H} \Longrightarrow D \in A\{l, \mathcal{G}_O, \mathcal{R}(F), \mathcal{N}(H)\}$*.*

(b) *Arbitrary element* $D \in A\{r, \mathcal{G}, \mathfrak{C}_1\}$ *satisfies:*

(i) $\mho_{HAF,F} \Longrightarrow D \in A\{r, \mathcal{G}, A^{(2)}_{\mathcal{R}(F),*}\}$*;*

(ii) $\mho_{HAF,H} \Longrightarrow D \in A\{r, \mathcal{G}, A^{(2)}_{*,\mathcal{N}(H)}\}$*;*

(iii) $\mho_{HAF,F,H} \Longrightarrow D \in A\{r, \mathcal{G}_O, \mathcal{R}(F), \mathcal{N}(H)\}$*.*

Corollary 4.11. *The following set equalities are valid:*

(a) $$A\{l, \mathcal{G}, \mathfrak{C}_1\} = \begin{cases} A\{l, \mathcal{G}, A^{(2)}_{\mathcal{R}(F),*}\} = A\{l, \mathcal{G}, F\}, & \mho_{HAF,F}; \\ A\{l, \mathcal{G}, A^{(2)}_{*,\mathcal{N}(H)}\}, & \mho_{HAF,H}; \\ A\{l, \mathcal{G}_O, \mathcal{R}(F), \mathcal{N}(H)\}, & \mho_{HAF,F,H}. \end{cases}$$

(b) $$A\{r, \mathcal{G}, \mathfrak{C}_1\} = \begin{cases} A\{r, \mathcal{G}, A^{(2)}_{\mathcal{R}(F),*}\}, & \mho_{HAF,F}; \\ A\{r, \mathcal{G}, A^{(2)}_{*,\mathcal{N}(H)}\} = A\{r, \mathcal{G}, H\}, & \mho_{HAF,H}; \\ A\{r, \mathcal{G}_O, \mathcal{R}(F), \mathcal{N}(H)\}, & \mho_{HAF,F,H}. \end{cases}$$

Corollary 4.12. *If P and Q be suitable Hermitian idempotents satisfying $(HAF)^{(2)} = Q(PHAFQ)^{\dagger}P$, then it follows:*

$$\text{(a) } A\{l,\mathcal{G},\complement_2\} = \begin{cases} A\{l,\mathcal{G},A^{(2)}_{\mathcal{R}(FQ),*}\} = A\{l,\mathcal{G},FQ\}, & \mho_{PHAFQ,FQ};\\ A\{l,\mathcal{G},A^{(2)}_{*,\mathcal{N}(PH)}\}, & \mho_{PHAFQ,PH};\\ A\{l,\mathcal{G}_O,\mathcal{R}(FQ),\mathcal{N}(PH)\}, & \mho_{PHAFQ,FQ,PH}.\end{cases}$$

$$\text{(b) } A\{r,\mathcal{G},\complement_2\} = \begin{cases} A\{r,\mathcal{G},A^{(2)}_{\mathcal{R}(FQ),*}\}, & \mho_{PHAFQ,FQ};\\ A\{r,\mathcal{G},A^{(2)}_{*,\mathcal{N}(PH)}\} = A\{r,\mathcal{G},PH\}, & \mho_{PHAFQ,PH};\\ A\{r,\mathcal{G}_O,\mathcal{R}(FQ),\mathcal{N}(PH)\}, & \mho_{PHAFQ,FQ,PH}.\end{cases}$$

Proof. (a) Similarly as in Corollary 3.16, we verify using Theorem 4.10 that an arbitrary element $D \in A\{l,G,\complement_2\}$ satisfies:

(i) $\mho_{PHAFQ,FQ} \Longrightarrow D \in A\{l,\mathcal{G},A^{(2)}_{\mathcal{R}(FQ),*}\}$;

(ii) $\mho_{PHAFQ,PH} \Longrightarrow D \in A\{l,\mathcal{G},A^{(2)}_{*,\mathcal{N}(PH)}\}$;

(iii) $\mho_{PHAFQ,FQ,PH} \Longrightarrow D \in A\{l,\mathcal{G}_O,\mathcal{R}(FQ),\mathcal{N}(PH)\}$.

(b) Arbitrary element $D \in A\{r,\mathcal{G},\complement_2\}$ satisfies:

(i) $\mho_{PHAFQ,FQ} \Longrightarrow D \in A\{r,\mathcal{G},A^{(2)}_{\mathcal{R}(FQ),*}\}$;

(ii) $\mho_{PHAFQ,PH} \Longrightarrow D \in A\{r,\mathcal{G},A^{(2)}_{*,\mathcal{N}(PH)}\}$;

(iii) $\mho_{PHAFQ,FQ,PH} \Longrightarrow D \in A\{r,\mathcal{G}_O,\mathcal{R}(FQ),\mathcal{N}(PH)\}$.

The rest of the proof is evident. □

The next characterizations of l- and r-$\mathcal{G}$-$\complement_2$ inverses are proposed in Theorem 3.17.

Theorem 4.13. *Let $D \in \mathbb{C}^{n\times m}$.*

(a) *Then the next claims are equivalent:*

(i) *$D \in A\{l,\mathcal{G},\complement_2\}$;*

(ii) *$ADA = A$ and $DA\complement_2A = \complement_2A$.*

(b) *Then the next claims are equivalent:*

(i) *$D \in A\{r,\mathcal{G},\complement_2\}$;*

(ii) *$ADA = A$ and $A\complement_2 = A\complement_2AD$.*

5. Examples in numerical and symbolic form

Example 5.1. *Let*

$$A = \begin{bmatrix} a & b\\ 0 & 0\end{bmatrix},\quad F = \begin{bmatrix} 1 & 0\\ 1 & 0\end{bmatrix},\quad H = \begin{bmatrix} 1 & 0\\ 0 & 0\end{bmatrix} \in \mathbb{C}^{2\times 2},$$

where $a, b \neq 0$ are unassigned variables. Unknown matrix D is of the general form

$$D = \begin{bmatrix} d_{1,1} & d_{1,2}\\ d_{2,1} & d_{2,2}\end{bmatrix},\ d_{1,2}, d_{1,2}, d_{2,1}, d_{2,2} \in \mathbb{C}.$$

Since F and H satisfy $\operatorname{rank}(AF) = \operatorname{rank}(F) = \operatorname{rank}(HA) = \operatorname{rank}(H) = 1$, *Corollary 3.3 implies solvability of the system* (5). *The form of general solution to the system*

$$ADA = A,\ \ DAF = F,\ \ HAD = H$$

gives

$$A\{\mathcal{G}, F, H\} = \left\{ \begin{bmatrix} \frac{1}{a+b} & 0 \\ \frac{1}{a+b} & -\frac{ad_{1,2}}{b} \end{bmatrix} : \ d_{1,2} \in \mathbb{C} \right\}.$$

Further, the general solution to

$$ADA = A,\ DAF = F$$

gives

$$A\{l, \mathcal{G}, F\} = \left\{ \begin{bmatrix} \frac{1}{a+b} & d_{1,2} \\ \frac{1}{a+b} & d_{2,2} \end{bmatrix} : \ d_{1,1}, d_{2,2} \in \mathbb{C} \right\}.$$

Finally, the general solution to

$$ADA = A,\ \ HAD = H$$

is equal to

$$A\{r, \mathcal{G}, H\} = \left\{ \begin{bmatrix} d_{1,1} & d_{1,2} \\ \frac{1-ad_{1,1}}{b} & -\frac{ad_{1,2}}{b} \end{bmatrix} : \ d_{1,1}, d_{1,2} \in \mathbb{C} \right\}.$$

Further, for

$$\mathbb{C}_1 = F(HAF)^{\dagger} H = \begin{bmatrix} \frac{1}{a+b} & 0 \\ \frac{1}{a+b} & 0 \end{bmatrix},$$

the general solution to

$$ADA = A,\ \ DA\mathbb{C}_1 = \mathbb{C}_1,\ \ \mathbb{C}_1 AD = \mathbb{C}_1$$

gives $A\{\mathcal{G}, \mathbb{C}_1\} = A\{\mathcal{G}, F, H\} = A\{\mathcal{G}_O, \mathcal{R}(F), \mathcal{N}(H)\}$. *In this case,* $\operatorname{rank}(HAF) = \operatorname{rank}(H) = \operatorname{rank}(F) = \operatorname{rank}(A) = 1$, *so that* $\mathbb{C}_1 = A^{(1,2)}_{\mathcal{R}(F),\mathcal{N}(H)}$, *which is a verification of Theorem 3.15, part (iii). Further, the general solution to*

$$ADA = A,\ \ DA\mathbb{C}_1 = \mathbb{C}_1$$

gives $A\{l, \mathcal{G}, \mathbb{C}_1\} = A\{l, \mathcal{G}, F\} = A\{l, \mathcal{G}_O, \mathcal{R}(F), \mathcal{N}(H)\}$, *which is a confirmation of Theorem 4.10, part (a)(iii). Finally, the general solution to*

$$ADA = A,\ \ \mathbb{C}_1 AD = \mathbb{C}_1$$

gives $A\{r, \mathcal{G}, \mathbb{C}_1\} = A\{r, \mathcal{G}, H\} = A\{r, \mathcal{G}_O, \mathcal{R}(F), \mathcal{N}(H)\}$, *which is a confirmation of Theorem 4.10, part (b)(iii) and Corollary 4.3.*

Example 5.2. *Consider*

$$A = \begin{bmatrix} 8 & 8 & 4 \\ 12 & 12 & 2 \\ 8 & 8 & 4 \\ 8 & 8 & 2 \end{bmatrix}, \quad F = \begin{bmatrix} 12 & 13 & 17 \\ 12 & 13 & 17 \\ 18 & 20 & 25 \end{bmatrix}, \quad H = \begin{bmatrix} 12 & 15 & 6 & 9 \\ 8 & 2 & 0 & 8 \end{bmatrix}.$$

Let us mention that $\operatorname{rank}(HAF) = \operatorname{rank}(H) = \operatorname{rank}(F) = \operatorname{rank}(A) = 2$, *so that* $\mathbb{C}_1 = A^{(1,2)}_{\mathcal{R}(F),\mathcal{N}(H)}$. *Unknown matrix D is in symbolic form*

$$D = \begin{bmatrix} d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} \\ d_{2,1} & d_{2,2} & d_{2,3} & d_{2,4} \\ d_{3,1} & d_{3,2} & d_{3,3} & d_{3,4} \end{bmatrix}, \ d_{i,j} \in \mathbb{C}.$$

The general solution to

$$ADA = A,\ DAF = F,\ HAD = H$$

defines the set

$$A\{\mathcal{G},F,H\}=\left\{\begin{bmatrix} d_{1,1} & d_{1,2} & -d_{1,1}+\frac{d_{1,2}}{2}-\frac{1}{16} & \frac{1}{8}-2d_{1,2} \\ -d_{1,1}-\frac{1}{7} & \frac{45}{196}-d_{1,2} & d_{1,1}-\frac{d_{1,2}}{2}+\frac{153}{784} & 2d_{1,2}-\frac{131}{392} \\ \frac{4}{7} & -\frac{31}{49} & -\frac{19}{49} & \frac{75}{98} \end{bmatrix} :\ d_{1,1},d_{1,2}\in\mathbb{C}\right\}.$$

Further, the general solution to the matrix system

$$ADA=A,\ DAF=F$$

is equal to

$$A\{l,\mathcal{G},F\}=\left\{\begin{bmatrix} d_{1,1} & d_{1,2} & -d_{1,1}+\frac{d_{1,2}}{2}-\frac{1}{16} & \frac{1}{8}-2d_{1,2} \\ d_{2,1} & d_{2,2} & -d_{2,1}+\frac{d_{2,2}}{2}-\frac{1}{16} & \frac{1}{8}-2d_{2,2} \\ d_{3,1} & d_{3,2} & -d_{3,1}+\frac{d_{3,2}}{2}+\frac{1}{2} & -2d_{3,2}-\frac{1}{2} \end{bmatrix} :\ d_{i,j}\in\mathbb{C}\right\}$$

and the general solution to

$$ADA=A,\ HAD=H$$

is

$$A\{r,\mathcal{G},H\}=\left\{\begin{bmatrix} d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} \\ -d_{1,1}-\frac{1}{7} & \frac{45}{196}-d_{1,2} & \frac{13}{98}-d_{1,3} & -d_{1,4}-\frac{41}{196} \\ \frac{4}{7} & -\frac{31}{49} & -\frac{19}{49} & \frac{75}{98} \end{bmatrix} :\ d_{i,j}\in\mathbb{C}\right\}.$$

For

$$\mathsf{C}_1=F(HAF)^{\dagger}H=\begin{bmatrix} -\frac{1}{14} & \frac{45}{392} & \frac{13}{196} & -\frac{41}{392} \\ -\frac{1}{14} & \frac{45}{392} & \frac{13}{196} & -\frac{41}{392} \\ \frac{4}{7} & -\frac{31}{49} & -\frac{19}{49} & \frac{75}{98} \end{bmatrix},$$

symbolic calculation gives

$$\begin{aligned} A\{\mathcal{G},\mathsf{C}_1\}&=A\{\mathcal{G},F,H\}=A\{\mathcal{G}_O,\mathcal{R}(F),\mathcal{N}(H)\},\\ A\{l,\mathcal{G},\mathsf{C}_1\}&=A\{l,\mathcal{G},F\}=A\{l,\mathcal{G}_O,\mathcal{R}(F),\mathcal{N}(H)\},\\ A\{r,\mathcal{G},\mathsf{C}_1\}&=A\{r,\mathcal{G},H\}=A\{r,\mathcal{G}_O,\mathcal{R}(F),\mathcal{N}(H)\}, \end{aligned}$$

which is a verification of Theorem 3.15, part (iii) and Theorem 4.10, parts (a)(iii) and (b)(iii) and Corollary 4.3.

Example 5.3. *Let*

$$A=\begin{bmatrix} 2 & 4 & 4 & 2 \\ 4 & 6 & 8 & 0 \\ 0 & 4 & 4 & 0 \\ 1 & 6 & 6 & 1 \\ 6 & 10 & 12 & 2 \end{bmatrix},\quad F=\begin{bmatrix} 2 & 2 & 0 \\ 3 & 3 & 2 \\ 0 & 0 & 0 \\ 3 & 3 & 2 \end{bmatrix},\quad H=\begin{bmatrix} 3 & 0 & 6 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Unknown matrix D is of the general form

$$D=\begin{bmatrix} d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & d_{1,5} & d_{1,6} \\ d_{2,1} & d_{2,2} & d_{2,3} & d_{2,4} & d_{2,5} & d_{2,6} \\ d_{3,1} & d_{3,2} & d_{3,3} & d_{3,4} & d_{3,5} & d_{3,6} \\ d_{4,1} & d_{4,2} & d_{4,3} & d_{4,4} & d_{4,5} & d_{4,6} \end{bmatrix},\ d_{i,j}\in\mathbb{C}.$$

Matrices F and H satisfy $\operatorname{rank}(AB)=\operatorname{rank}(B)$ *and* $\operatorname{rank}(CA)=\operatorname{rank}(C)$. *So, according to Corollary 3.3, the system* (5) *is solvable. The general solution to*

$$ADA=A,\ DAF=F,\ HAD=H,$$

corresponding to (5)*, is given as*

$$A\{\mathcal{G},F,H\}=\left\{\begin{bmatrix} d_{1,1} & d_{1,2} & d_{1,3} & \frac{1}{30}\left(-8d_{1,1}-22d_{1,2}+4d_{1,3}+7\right) & \\ d_{2,1} & d_{2,2} & 2\left(-2d_{1,1}+2d_{1,2}+d_{1,3}+d_{2,1}-d_{2,2}\right) & \frac{1}{5}\left(16d_{1,1}-16d_{1,2}-8d_{1,3}-10d_{2,1}+10d_{2,2}+1\right) & \frac{1}{30}\left(-16d_{1,1}+16d_{1,2}+8d_{1,3}-30d_{2,2}-1\right) \\ d_{3,1} & d_{3,2} & 4d_{1,1}-4d_{1,2}-2d_{1,3}+2d_{3,1}-2d_{3,2}+\frac{1}{4} & \frac{1}{5}\left(-16d_{1,1}+16d_{1,2}+8d_{1,3}-10d_{3,1}+10d_{3,2}-1\right) & \frac{1}{30}\left(16d_{1,1}-16d_{1,2}-8d_{1,3}-30d_{3,2}+1\right) \\ -d_{1,1}-6d_{2,1}-6d_{3,1}+\frac{1}{3} & -d_{1,2}-6\left(d_{2,2}+d_{3,2}\right) & -d_{1,3}-12d_{2,1}+12d_{2,2}-12d_{3,1}+12d_{3,2}-\frac{5}{6} & \frac{2d_{1,1}}{5}-\frac{2d_{1,2}}{5}+\frac{4d_{1,3}}{5}+12d_{2,1}-12d_{2,2}+12d_{3,1}-12d_{3,2}+\frac{11}{15} & \frac{1}{30}\left(8d_{1,1}+22d_{1,2}-4d_{1,3}+180d_{2,2}+180d_{3,2}-7\right) \end{bmatrix}\right\}.$$

Simplification in symbolic computation gives

$$A\cdot A\{\mathcal{G},F,H\}\cdot A-A=A\{\mathcal{G},F,H\}\cdot AF-F=HA\cdot A\{\mathcal{G},F,H\}-H=\{0\}.$$

Further, the general solution to

$$ADA=A,\ DAF=F$$

is given by

$$A\{l,\mathcal{G},F\}=\left\{\begin{bmatrix} d_{1,1} & d_{1,2} & d_{1,3} & -\frac{2}{5}\left(d_{1,1}-d_{1,2}+2d_{1,3}+1\right) & \frac{1}{30}\left(-8d_{1,1}-22d_{1,2}+4d_{1,3}+7\right) \\ d_{2,1} & d_{2,2} & 2\left(-2d_{1,1}+2d_{1,2}+d_{1,3}+d_{2,1}-d_{2,2}\right) & \frac{1}{5}\left(16d_{1,1}-16d_{1,2}-8d_{1,3}-10d_{2,1}+10d_{2,2}+1\right) & \frac{1}{30}\left(-16d_{1,1}+16d_{1,2}+8d_{1,3}-30d_{2,2}-1\right) \\ d_{3,1} & d_{3,2} & 4d_{1,1}-4d_{1,2}-2d_{1,3}+2d_{3,1}-2d_{3,2}+\frac{1}{4} & \frac{1}{5}\left(-16d_{1,1}+16d_{1,2}+8d_{1,3}-10d_{3,1}+10d_{3,2}-1\right) & \frac{1}{30}\left(16d_{1,1}-16d_{1,2}-8d_{1,3}-30d_{3,2}+1\right) \\ d_{4,1} & d_{4,2} & 2d_{1,1}-2d_{1,2}-d_{1,3}+2d_{4,1}-2d_{4,2}-\frac{3}{2} & \frac{1}{5}\left(-8d_{1,1}+8d_{1,2}+4d_{1,3}-10d_{4,1}+10d_{4,2}+7\right) & \frac{1}{30}\left(8d_{1,1}-8d_{1,2}-4d_{1,3}-30d_{4,2}-7\right) \end{bmatrix}\right\}$$

and the general solution to

$$ADA=A,\ HAD=H$$

is

$$A\{r,\mathcal{G},H\}=\left\{\begin{bmatrix} d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & d_{1,5} \\ d_{2,1} & d_{2,2} & 2\left(-2d_{1,1}+2d_{1,2}+d_{1,3}+d_{2,1}-d_{2,2}\right) & 4d_{1,1}-4d_{1,2}+2d_{1,4}-2d_{2,1}+2d_{2,2}+1 & 2d_{1,2}+2d_{1,5}-d_{2,2}-\frac{1}{2} \\ d_{3,1} & d_{3,2} & 4d_{1,1}-4d_{1,2}-2d_{1,3}+2d_{3,1}-2d_{3,2}+\frac{1}{4} & -4d_{1,1}+4d_{1,2}-2d_{1,4}-2d_{3,1}+2d_{3,2}-1 & -2d_{1,2}-2d_{1,5}-d_{3,2}+\frac{1}{2} \\ -d_{1,1}-6d_{2,1}-6d_{3,1}+\frac{1}{3} & -d_{1,2}-6\left(d_{2,2}+d_{3,2}\right) & -d_{1,3}-12d_{2,1}+12d_{2,2}-12d_{3,1}+12d_{3,2}-\frac{5}{6} & -d_{1,4}+12d_{2,1}-12d_{2,2}+12d_{3,1}-12d_{3,2}+\frac{1}{3} & 6\left(d_{2,2}+d_{3,2}\right)-d_{1,5} \end{bmatrix}\right\}.$$

Further,

$$\mathsf{C}_1=F(HAF)^{\dagger}H=\begin{bmatrix} \frac{46}{1881} & 0 & \frac{92}{1881} & \frac{46}{1881} & 0 \\ \frac{83}{1881} & 0 & \frac{166}{1881} & \frac{83}{1881} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{83}{1881} & 0 & \frac{166}{1881} & \frac{83}{1881} & 0 \end{bmatrix}.$$

Ranks relevant in view of Proposition 1.1 satisfy $\operatorname{rank}(HAF)=1=\operatorname{rank}(H)<2=\operatorname{rank}(F)<3=\operatorname{rank}(A)$*, so that* $\mathsf{C}_1=A^{(2)}_{*,\mathcal{N}(H)}$*. The general solution to the matrix system*

$$ADA=A,\ \ DA\mathsf{C}_1=\mathsf{C}_1,\ \ \mathsf{C}_1AD=\mathsf{C}_1$$

is equal to

$$A\{\mathcal{G},\mathsf{C}_1\}=\left\{\begin{bmatrix} d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & \frac{-590d_{1,1}-682d_{1,2}-332d_{1,3}-627d_{1,4}+46}{1272} \\ d_{2,1} & d_{2,2} & 2\left(-2d_{1,1}+2d_{1,2}+d_{1,3}+d_{2,1}-d_{2,2}\right) & 4d_{1,1}-4d_{1,2}+2d_{1,4}-2d_{2,1}+2d_{2,2}+1 & \frac{1}{636}\left(-590d_{1,1}+590d_{1,2}-332d_{1,3}-627d_{1,4}-636d_{2,2}-272\right) \\ d_{3,1} & d_{3,2} & 4d_{1,1}-4d_{1,2}-2d_{1,3}+2d_{3,1}-2d_{3,2}+\frac{1}{4} & -4d_{1,1}+4d_{1,2}-2d_{1,4}-2d_{3,1}+2d_{3,2}-1 & \frac{1}{636}\left(590d_{1,1}-590d_{1,2}+332d_{1,3}+627d_{1,4}-636d_{3,2}+272\right) \\ -d_{1,1}-6d_{2,1}-6d_{3,1}+\frac{1}{3} & -d_{1,2}-6\left(d_{2,2}+d_{3,2}\right) & -d_{1,3}-12d_{2,1}+12d_{2,2}-12d_{3,1}+12d_{3,2}-\frac{5}{6} & -d_{1,4}+12d_{2,1}-12d_{2,2}+12d_{3,1}-12d_{3,2}+\frac{1}{3} & \frac{590d_{1,1}+682d_{1,2}+332d_{1,3}+627d_{1,4}+7632d_{2,2}+7632d_{3,2}-46}{1272} \end{bmatrix}\right\}.$$

Further, the general solution to

$$ADA=A,\ \ DA\mathsf{C}_1=\mathsf{C}_1$$

gives

$$A\{l,\mathcal{G},\mathsf{C}_1\}=\left\{\begin{bmatrix} d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & d_{1,5} \\ d_{2,1} & d_{2,2} & 2\left(-2d_{1,1}+2d_{1,2}+d_{1,3}+d_{2,1}-d_{2,2}\right) & 4d_{1,1}-4d_{1,2}+2d_{1,4}-2d_{2,1}+2d_{2,2}+1 & 2d_{1,2}+2d_{1,5}-d_{2,2}-\frac{1}{2} \\ d_{3,1} & d_{3,2} & 4d_{1,1}-4d_{1,2}-2d_{1,3}+2d_{3,1}-2d_{3,2}+\frac{1}{4} & -4d_{1,1}+4d_{1,2}-2d_{1,4}-2d_{3,1}+2d_{3,2}-1 & -2d_{1,2}-2d_{1,5}-d_{3,2}+\frac{1}{2} \\ -d_{1,1}-6d_{2,1}-6d_{3,1}+\frac{1}{3} & -d_{1,2}-6\left(d_{2,2}+d_{3,2}\right) & -d_{1,3}-12d_{2,1}+12d_{2,2}-12d_{3,1}+12d_{3,2}-\frac{5}{6} & -d_{1,4}+12d_{2,1}-12d_{2,2}+12d_{3,1}-12d_{3,2}+\frac{1}{3} & 6\left(d_{2,2}+d_{3,2}\right)-d_{1,5} \end{bmatrix}\right\}.$$

Finally, the general solution to

$$ADA=A,\ \ \mathsf{C}_1AD=\mathsf{C}_1$$

gives

$$A\{r,\mathcal{G},\mathsf{C}_1\}=A\{r,\mathcal{G},H\}=A\{r,\mathcal{G},A^{(2)}_{*,\mathcal{N}(H)}\},$$

which confirms part (ii) of Theorem 4.10 because of $\mho_{HAF,H}$ *and Corollary 4.3.*

6. $\mathcal{G}$-(B,C) partial orders

In this section, suppose that $A\in\mathbb{C}^{m\times n}$, $B\in\mathbb{C}^{n\times p}$ and $C\in\mathbb{C}^{q\times m}$. Based on the $\mathcal{G}$-(B,C) inverse, l-$\mathcal{G}$-B inverse and r-$\mathcal{G}$-C inverse, we present three new binary relations on $\mathbb{C}^{m\times n}$. We can observe that these new relations generalized the concepts of $\mathcal{G}$-outer (T,S)-partial order proposed in [18] and l- and r-$\mathcal{G}$-outer (T,S)-partial orders given in [22].

Definition 6.1. *Let* $E\in\mathbb{C}^{m\times n}$. *Then*

(a) *A is below E with respect to the* $\mathcal{G}$-(B,C) *relation (marked as* $A\leq^{\mathcal{G},B,C}E$*) if there is* $D_1,D_2\in A\{\mathcal{G},B,C\}$ *such that*

$$AD_1=ED_1 \qquad \text{and} \qquad D_2A=D_2E;$$

(b) *A is below E with respect to the l-*$\mathcal{G}$*-B relation (marked as* $A\leq^{l,\mathcal{G},B}E$*) if there is* $D_1,D_2\in A\{l,\mathcal{G},B\}$ *such that*

$$AD_1=ED_1 \qquad \text{and} \qquad D_2A=D_2E;$$

(c) *A is below E with respect to the r-*$\mathcal{G}$*-C relation (marked as* $A\leq^{r,\mathcal{G},C}E$*) if there is* $D_1,D_2\in A\{r,\mathcal{G},C\}$ *such that*

$$AD_1=ED_1 \qquad \text{and} \qquad D_2A=D_2E.$$

Several necessary and sufficient conditions to fulfilling the relation $A\leq^{\mathcal{G},B,C}E$ are developed.

Theorem 6.2. *For $E \in \mathbb{C}^{m\times n}$, the next statements are equivalent one to another:*

(i) $A \leq^{\mathcal{G},B,C} E$;

(ii) *there is $D \in A\{\mathcal{G}, B, C\}$ such that*

$$AD = ED \qquad \text{and} \qquad DA = DE;$$

(iii) *there is $D \in A\{\mathcal{G}, B, C\}$ such that*

$$ADE = A = EDA;$$

(iv) *there are idempotents $T_1 \in \mathbb{C}^{m\times m}$ and $T_2 \in \mathbb{C}^{n\times n}$ satisfying*

$$\mathcal{R}(T_1) = \mathcal{R}(A),\ \mathcal{N}(T_2) = \mathcal{N}(A),\ \mathcal{R}(B) \subseteq \mathcal{R}(T_2),\ \mathcal{N}(T_1) \subseteq \mathcal{N}(C) \text{ and } T_1E = A = ET_2.$$

Proof. (i) ⇒ (ii): Observe that $A \leq^{\mathcal{G},B,C} E$ implies the existence of $D_1, D_2 \in A\{\mathcal{G}, B, C\}$ such that $AD_1 = ED_1$ and $D_2A = D_2E$. By Theorem 3.13, $D = D_1AD_2 \in A\{\mathcal{G}, B, C\}$. We observe that $AD = (AD_1)AD_2 = E(D_1AD_2) = ED$ and in a same way $DA = DE$.

(ii) ⇒ (iii): For $D \in A\{\mathcal{G}, B, C\}$, the equalities $AD = ED$ and $DA = DE$ yield $A = (AD)A = EDA$ and $A = A(DA) = ADE$.

(iii) ⇒ (i): If $ADE = A = EDA$ for $D \in A\{\mathcal{G}, B, C\}$, it follows $D' = DAD \in A\{\mathcal{G}, B, C\}$. Also,

$$AD' = (ADA)D = AD = E(DAD) = ED'$$

and similarly $D'A = D'E$.

(ii) ⇒ (iv): Assume that $AD = ED$ and $DA = DE$ for some $D \in A\{\mathcal{G}, B, C\}$. Using Theorem 3.14 for $T_1 = AD$ and $T_2 = DA$, one can see $\mathcal{R}(T_1) = \mathcal{R}(A)$, $\mathcal{N}(T_2) = \mathcal{N}(A)$, $\mathcal{R}(B) \subseteq \mathcal{R}(T_2)$, and $\mathcal{N}(T_1) \subseteq \mathcal{N}(C)$. Further, $T_1E = A(DE) = ADA = A = E(DA) = ET_2$.

(iv) ⇒ (ii): According to Theorem 3.14, we obtain $D = T_2A^- T_1 \in A\{\mathcal{G}, B, C\}$, where $A^- \in A\{1\}$. Therefore, $AD = (AT_2)A^-T_1 = AA^-T_1 = E(T_2A^- T_1) = ED$ and analogously $DA = DE$. □

Similarly as in Theorem 6.2, we characterize l-$\mathcal{G}$-B and r-$\mathcal{G}$-C relations.

Corollary 6.3. *Let $E \in \mathbb{C}^{m\times n}$.*

(a) *The subsequent claims are mutually equivalent:*

(i) $A \leq^{l,\mathcal{G},B} E$;

(ii) *there is $D \in A\{l, \mathcal{G}, B\}$ satisfying*

$$AD = ED \qquad \text{and} \qquad DA = DE;$$

(iii) *there is $D \in A\{l, \mathcal{G}, B\}$ satisfying*

$$A = ADE = EDA;$$

(iv) *there are idempotents $T_1 \in \mathbb{C}^{m\times m}$ and $T_2 \in \mathbb{C}^{n\times n}$ satisfying*

$$\mathcal{R}(A) = \mathcal{R}(T_1), \quad \mathcal{N}(A) = \mathcal{N}(T_2), \quad \mathcal{R}(B) \subseteq \mathcal{R}(T_2) \quad \text{and} \quad T_1E = A = ET_2.$$

(b) *The subsequent claims are mutually equivalent:*

(i) $A \leq^{r,\mathcal{G},C} E$;

(ii) *there is $D \in A\{r, \mathcal{G}, C\}$ satisfying*

$$AD = ED \qquad \text{and} \qquad DA = DE;$$

(iii) *there is* $D \in A\{r, \mathcal{G}, C\}$ *satisfying*

$$A = ADE = EDA;$$

(iv) *there are idempotents* $T_1 \in \mathbb{C}^{m\times m}$ *and* $T_2 \in \mathbb{C}^{n\times n}$ *satisfying*

$$\mathcal{R}(A) = \mathcal{R}(T_1), \quad \mathcal{N}(A) = \mathcal{N}(T_2), \quad \mathcal{N}(T_1) \subseteq \mathcal{N}(C) \quad \textit{and} \quad T_1E = A = ET_2.$$

Under the hypothesis $A \leq^{\mathcal{G},B,C} E$, we verify that arbitrary $\mathcal{G}$-(B,C) inverse of E is $\mathcal{G}$-(B,C) inverse of A. The analog results hold for l- and r-$\mathcal{G}$-(B,C) inverses.

Theorem 6.4. *The following implications are valid for* $E \in \mathbb{C}^{m\times n}$*:*

(a) $A \leq^{\mathcal{G},B,C} E \Longrightarrow E\{\mathcal{G}, B, C\} \subseteq A\{\mathcal{G}, B, C\}$;

(b) $A \leq^{l,\mathcal{G},B} E \Longrightarrow E\{l, \mathcal{G}, B\} \subseteq A\{l, \mathcal{G}, B\}$;

(c) $A \leq^{r,\mathcal{G},C} E \Longrightarrow E\{r, \mathcal{G}, C\} \subseteq A\{r, \mathcal{G}, C\}$.

Proof. (a) The hypothesis $A \leq^{\mathcal{G},B,C} E$ and Theorem 6.2 imply the existence of $D \in A\{\mathcal{G}, B, C\}$ satisfying $AD = ED$ and $DA = DE$. If $F \in E\{\mathcal{G}, B, C\}$, then

$$EFE = E, \quad FEB = B \quad \text{and} \quad CEF = C,$$

which give

$$\begin{aligned} AFA &= A(DA)F(AD)A = AD(EFE)DA = A(DE)DA = ADADA = A, \\ FAB &= F(AD)AB = FE(DAB) = FEB = B \\ CAF &= CA(DA)F = (CAD)EF = CEF = C. \end{aligned}$$

Thus, $F \in A\{\mathcal{G}, B, C\}$, which yields $B\{\mathcal{G}, B, C\} \subseteq A\{\mathcal{G}, B, C\}$.

Similarly, we can verify the parts (b) and (c). □

In the next theorem, we check that partial orders on $\mathbb{C}^{m\times n}$ are relations $\mathcal{G}$-(B,C), l-$\mathcal{G}$-B and r-$\mathcal{G}$-C.

Theorem 6.5. *The* $\mathcal{G}$-(B,C), *l*-$\mathcal{G}$-B *and r*-$\mathcal{G}$-C *relations are partial orders, respectively, on* $\{A \in \mathbb{C}^{m\times n} : A\{\mathcal{G}, B, C\} \neq \emptyset\}$, $\{A \in \mathbb{C}^{m\times n} : A\{l, \mathcal{G}, B\} \neq \emptyset\}$ *and* $\{A \in \mathbb{C}^{m\times n} : A\{r, \mathcal{G}, C\} \neq \emptyset\}$.

Proof. We will prove that the $\mathcal{G}$-(B,C) is a partial order, because the rest follows in a similar way. It is clear that $\leq^{\mathcal{G},B,C}$ is reflexive.

If $A, E \in \mathbb{C}^{m\times n}$ satisfy $A \leq^{\mathcal{G},B,C} E$ and $E \leq^{\mathcal{G},B,C} A$, then $A \leq^{-} E$ and $E \leq^{-} A$. Because the relation $\leq^{-}$ is antisymmetric, we deduce that $A = B$ and so $\leq^{\mathcal{G},B,C}$ is antisymmetric.

To prove that $\leq^{\mathcal{G},B,C}$ is transitive, assume the existence of $A, E, G \in \mathbb{C}^{m\times n}$ satisfying $A \leq^{\mathcal{G},B,C} E$ and $E \leq^{\mathcal{G},B,C} G$. Applying Theorem 6.2, there are $D \in A\{\mathcal{G}, B, C\}$ and $F \in E\{\mathcal{G}, B, C\}$ satisfying $AD = ED$, $DA = DE$, $EF = GF$ and $FE = FG$. According to Theorem 6.4, it follows $F \in A\{\mathcal{G}, B, C\}$. Now, by

$$\begin{aligned} A &= A(DA) = ADE = ADE(FE) = (ADE)FG = AFG, \\ A &= (AD)A = EDA = (EF)EDA = GF(EDA) = GFA \end{aligned}$$

and Theorem 6.2(iii), we conclude that $A \leq^{\mathcal{G},B,C} G$. □

7. Conclusion

The system of matrix equations (5) involving $A \in \mathbb{C}^{m\times n}$, $C \in \mathbb{C}^{q\times m}$ and $B \in \mathbb{C}^{n\times p}$ is investigated. This system is an atypical combination of the equation $AXA = A$ corresponding to inner inverses and the equations $XAB = B$, $CAX =$C used in the characterization of outer inverses. Firstly, equivalent conditions for solvability of system (5) are given. The general solution of (5) is determined in terms of inner inverses of A, AB, and CA. So, we omit the conditions $A \in \mathbb{C}^{m\times n}_{\mathcal{R}(B),\mathcal{N}(C)}$ and $A\{\mathcal{G}_O, \mathcal{R}(B), \mathcal{N}(C)\} \neq \emptyset$ which appeared in [19], and prove that the hypothesis $A\{\mathcal{G}_O, \mathcal{R}(B), \mathcal{N}(C)\} \neq \emptyset$ is satisfied when $A \in \mathbb{C}^{m\times n}_{\mathcal{R}(B),\mathcal{N}(C)}$ holds.

Our contributions are some extensions of $\mathcal{G}$-outer and $\mathcal{G}$-Drazin inverses which are proposed as solutions of the system (5) and its particular cases. Precisely, we define and characterize a $\mathcal{G}$-(B, C) and $\mathcal{G}$-$\complement$ inverses as new classes of generalized inverses. Using Urquhart formula, for $\complement_1 := F(HAF)^{(1)}H$ and $\complement_2 := F(HAF)^{(2)}H$, $\mathcal{G}$-$\complement_1$ and $\mathcal{G}$-$\complement_2$ inverses are also new generalizations of $\mathcal{G}$-outer inverses.

One-sided versions of $\mathcal{G}$-(B, C) inverse are introduced as weaker versions of $\mathcal{G}$-(B, C) inverses and extensions of one-sided kinds of $\mathcal{G}$-outer inverse from [22]. We also characterize one-sided $\mathcal{G}$-$\complement_1$ and $\mathcal{G}$-$\complement_2$ inverses.

Tree new kinds of partial orders on the adequate subsets of $\mathbb{C}^{m\times n}$ are presented using the $\mathcal{G}$-(B, C) inverse and its one-sided versions. Notice that these partial orders generalize the concepts of $\mathcal{G}$-outer (T, S)-partial order [18] and one-sided $\mathcal{G}$-outer (T, S)-partial orders [22].

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