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Oscillation criteria for second-order delay dynamic equations with a sub-linear neutral term on time scales

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Abstract. This paper addresses the oscillatory behavior of the solutions of second-order dynamic equations with a sublinear neutral term. Using Riccati transformation and comparison principles, we obtain new oscillation criteria. The obtained results essentially improve, complement, and simplify some of the previous ones in the literature. Some examples have been provided herein to illustrate our main results.

1. Introduction

The paper aims to study the oscillation problem of the class of second-order nonlinear dynamic equations with a sublinear neutral term

$$\left(r(t)z^{\Delta}(t)\right)^{\Delta} + q(t)x^{\beta}(\delta(t)) = 0, \quad t \geq t_0, \tag{1}$$

where $z(t) := x(t) + p(t)x^{\alpha}(\tau(t))$. Throughout, the following assumptions are satisfied

(H1) $0 < \alpha \leq 1, \beta$ are ratios of odd positive integers;

(H2) $r, p, q \in C_{rd}([t_0, \infty)_{\mathbb{T}}, [0, \infty))$ and $q(t)$ is not eventually zero for sufficiently large t and $\lim_{t\to\infty} p(t) = 0$, $R(t) = \int_{t_0}^{t} \dfrac{\Delta s}{r(s)}$;

(H3) $\tau,\ \delta \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{T}),\ \tau(t) \leq t,\ \delta(t) \leq t$ and $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \delta(t) = \infty$.

Herein, we consider

$$\int_{t_0}^{\infty} \frac{\Delta s}{r(s)} = \infty. \tag{2}$$

By a solution of (1), we mean a function $x \in C_{rd}[T_x, \infty)_{\mathbb{T}},\ T_x \in [t_0, \infty)_{\mathbb{T}}$ which has the property $r(z^{\Delta}) \in C^1_{rd}[T_x, \infty)_{\mathbb{T}}$ and satisfies (1) on $[T_x, \infty)_{\mathbb{T}}$. We consider only those solutions x of (1) which satisfy $\sup |x(t)| : t \in [T_x, \infty)_{\mathbb{T}} > 0$ for all $T \in [T_x, \infty)_{\mathbb{T}}$. We assume that (1) possesses such solutions. A solution of

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(1) is called oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is termed nonoscillatory. Dynamic equations on time scales have an enormous potential for applications fields such as in biology, engineering, economics, physics, neural networks, and social sciences [6].

Within the final two decades, there are numerous thinks about studying oscillatory behavior of solutions of nonlinear neutral delay dynamic equations, see [2–4, 14, 18, 19, 22] . However, we have relatively fewer results in the literature for dynamic equations with a sublinear neutral term. Dzurina *et al.*. [9] studied the oscillation of second-order difference equation with several sublinear neutral terms of the following form:

$$(a(t)z'(t))' + q(t)x^{\beta}(\sigma(t)) = 0, \quad t \geq t_0 > 0$$

where $m > 0$ is an integer, $z(t) = x(t) + \sum_{i=1}^{m} p_i(t)x^{\alpha_i}(\tau_i(t))$.

In [21] Sivaraj *et al.* established adequate conditions for the oscillation of all solutions of a nonlinear differential equations

$$\left(a(t)\left(x(t) + p(t)x^{\alpha}(\tau(t))\right)'\right)' + q(t)x^{\beta}(\sigma(t)) = 0, \quad t \geqslant t_0$$

where α and β are ratio of odd positive integers.

In [8], Dharuman *et al.*obtained the oscillation criteria of the solution of a particular case for (1) when $\mathbb{T} = \mathbb{Z}$

$$\Delta\left(a_n\Delta\left(x_n + p_n x_{n-k}^{\alpha}\right)\right) + q_n x_{n+1-l}^{\beta} = 0, n \geq n_0$$

where $0 < \alpha \leq 1, \beta$ are ratio of odd positive integers.

Recently, Soliman *et al.*[20] established the sufficient conditions for the oscillatory behavior of solutions of (1), under the conditions either

$$\int_{t_0}^{\infty} \frac{1}{r(s)}\Delta s = \infty, \quad \text{or} \quad \int_{t_0}^{\infty} \frac{1}{r(s)}\Delta s < \infty$$

Indeed, equation (1) has numerous applications in mathematical, theoretical, and chemical material science; for example, it appears in a variety of real-world issues such as in the study of p-Laplace equations non-Newtonian fluid theory, the turbulentflow of a polytrophic gas in a porous medium, and so on. As a result, there has been a lot of research activity concerning oscillatory behavior of various classes of differential equations. We refer the reader to [1, 5, 15–17]. See also [11] and [10] for models from mathematical biology formulated by PDE's.

This study aims to unify and continue further investigation of the oscillation criteria of (1). The obtained results are appropriate for non-neutral differential and difference equations ($p(t) = 0$). We present a more general study than those previously reported in the literature in a manner such that we cover all possible cases for (1) (Sublinear case, superlinear case and linear case).

2. Preliminary Lemmas

In this section, we state and demonstrate some preparatory lemmas that are crucial to prove the main results obtained herein.

Lemma 2.1. [12] *If a and b are nonnegative, then*

$$a^{\alpha}b^{1-\alpha} \leq \alpha a + (1-\alpha)b \quad \text{for } 0 < \alpha \leq 1, \tag{3}$$

where equality holds if and only if $a = b$.

Theorem 2.2. [7] *Assume that* $\nu : \mathbb{T} \to \mathbb{R}$ *is strictly increasing and* $\tilde{\mathbb{T}} := \nu(\mathbb{T})$ *is a time scale. Let* $y : \tilde{\mathbb{T}} \to \mathbb{R}$. *If* $y^{\tilde{\Delta}}(\nu(t))$ *and* $\nu^{\Delta}(t)$ *exist for* $t \in \mathbb{T}^{\kappa}$, *then*

$$(y \circ \nu)^{\Delta}(t) = y^{\tilde{\Delta}}(\nu(t))\nu^{\Delta}(t).$$

Where $\tilde{\Delta}$ *denotes to the derivative on* $\tilde{\mathbb{T}}$.

Lemma 2.3. *Let (H1)-(H3) and* (2) *hold. If* $x(t)$ *be an eventually positive solution of* (1), *then* $z(t)$ *satisfies*

(I) $z(t) > 0, z^{\Delta}(t) > 0$, *and* $\left(r(t)z^{\Delta}(t)\right)^{\Delta} < 0,\ \ t \geq t_1 \geq t_0$,

(II) $\frac{z(t)}{R(t)}$ *is decreasing for* $t \geq t_1$.

Proof. As x is an eventually positive solution of (1), then by (H2) and (H3) there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t) > 0$, $x(\delta(t)) > 0$ and $x(\tau(t)) > 0$ for all $t \in [t_0, \infty)_{\mathbb{T}}$. Now, from (1) we have

$$\left(r(t)z^{\Delta}(t)\right)^{\Delta} \leq -q(t)x^{\beta}(\delta(t)), \tag{4}$$

Hence $\left(r(t)z^{\Delta}(t)\right)$ is a nonincreasing function and is eventually of one sign. Claim that$z^{\Delta}(t) > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$. If not, then there exists $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that $z^{\Delta}(t) \leq 0$ for all $t_2 \in [t_1, \infty)_{\mathbb{T}}$. Since q not identically equal to zero eventually, we may assume that $z^{\Delta}(t) < 0$ for all $t \in [t_2, \infty)_{\mathbb{T}}$. From (4), we have

$$\left(r(t)z^{\Delta}(t)\right) \leq -c < 0, \quad \text{for all } t \in [t_2, \infty)_{\mathbb{T}}, \tag{5}$$

where $c := \left(r(t_2)z^{\Delta}(t_2)\right) > 0$, then

$$z^{\Delta}(t) \leq \frac{-c}{r(t)} \tag{6}$$

Integrate (6) on $[t_2, t) \subset [t_2, \infty)_{\mathbb{T}}$, we obtain

$$z(t) \leq z(t_2) - c \int_{t_2}^{t} \frac{\Delta s}{r(s)} \quad \text{for all } t \in [t_2, \infty)_{\mathbb{T}}. \tag{7}$$

Letting $t \to \infty$, then it follows from (2) that $\lim_{t\to\infty} z(t) = -\infty$, which is a inconsistency. Then

$$z(t) > 0, z^{\Delta}(t) > 0, \ \text{and} \ \left(r(t)z^{\Delta}(t)\right)^{\Delta} < 0, \ \ t \geq t_1 \geq t_0$$

To prove (II), since (I)holds, then for sufficiently large t_1

$$\begin{aligned} z(t) &\geq z(t_1) + \int_{t_1}^{t} \frac{r(s)z^{\Delta}(s)}{r(s)} \Delta s \\ &\geq r(t)R(t)z^{\Delta}(t). \end{aligned}$$

Moreover, using the previous inequality, we get

$$\left(\frac{z(t)}{R(t)}\right)^{\Delta} = \frac{z^{\Delta}(t)R(t) - r^{-1}(t)z(t)}{R(t)R(\sigma(t))} \leq \frac{r(t)z^{\Delta}(t)R(t) - z(t)}{r(t)R(t)R(\sigma(t))} \leq 0.$$

This implies that $\frac{z(t)}{R(t)}$ is decreasing for $t \geq t_1$, thereby completes the proof. □

Lemma 2.4. *Let (H1)-(H3) and* (2) *hold. Assume* $x(t)$ *be an eventually positive solution of* (1), *such that* $z(t)$ *satisfied (I) of Lemma 2.1. If*

$$\int_{t_0}^{\infty} \left(\frac{1}{r(u)} \int_{t_0}^{u} q(s)\Delta s\right) \Delta u = \infty. \tag{8}$$

Then $z(t) \to \infty$ *as* $t \to \infty$.

Proof. Assume that $x(t)$ be a nonoscillatory solution of (1) on $[t_0,\infty)_{\mathbb{T}}$ such that $x(t) > 0, x(\tau(t)) > 0$, and $x(\delta(t)) > 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$. We claim that (8) guarantees that $z(t) \to \infty$ as $t \to \infty$, since $z(t)$ is a positively increasing function; therefore there exists a constant $k > 0$ such that

$$z(t) \geq k > 0, \quad \text{for} \ \ t \geq t_1. \tag{9}$$

Besides, it follows from the definition of $z(t)$ that

$$\begin{aligned} x(t) =& z(t) - p(t)x^{\alpha}(\tau(t)) \\ \geq& z(t) - p(t)z^{\alpha}(\tau(t)) \end{aligned}$$

Applying (3) , we get

$$\begin{aligned} x(t) \geq& z(t) - p(t)\left(\alpha z(t) + (1-\alpha)\right) \\ \geq& z(t)\left(1 - \alpha p(t) - \frac{p(t)}{z(t)}(1-\alpha)\right). \end{aligned} \tag{10}$$

Substituting (9) into (10), we get

$$x(t) \geq k\left(1 - \alpha p(t) - \frac{p(t)}{z(t)}(1-\alpha)\right). \tag{11}$$

Considering (H2) , we obtain

$$x(t) \geq k > 0, \quad t \geq t_1. \tag{12}$$

Integrate (1) from t to t using (12), we obtain

$$z^{\Delta}(t) \geq \frac{k^{\beta}}{r(t)} \int_{t_1}^{t} q(s)\Delta s. \tag{13}$$

Integrate (13) from t_1 to t, we conclude

$$z(t) \geq z(t_1) + k^{\beta} \int_{t_1}^{t} \frac{1}{r(s)} \int_{t_1}^{t} q(s)\Delta s \Delta u. \tag{14}$$

In view of (8), we conclude that $z(t) \to \infty$ as $t \to \infty$, thereby the proof is complete. □

3. Main Results

In this section, we state and prove new oscillation criteria for Eq.(1), and we present oscillation criteria for the case when (1) is super-linear

Theorem 3.1. *Assume that $\beta > 1$, $(H1)-(H3)$ and (8) hold, and $\delta^{\Delta} > 0$. If there exists a function $\varphi(t) \in C^1_{rd}([t_0,\infty)_{\mathbb{T}},(0,\infty))$, such that for sufficiently large $t_2 \geq t_1$,*

$$\limsup_{t\to\infty} \left[\int_{t_0}^{t} \left(\xi^{\beta}\varphi(s)q(s) - \frac{(\varphi^{\Delta}(s))^2 r(\delta(\sigma(s)))}{4\varphi(s)\delta^{\Delta}(s)} \right) \Delta s \right] = \infty, \tag{15}$$

holds for $\xi \in (0,1)$, then (1) is oscillatory.

Proof. Assume that $x(t)$ be a nonoscillatory solution of (1) on $[t_0,\infty)_{\mathbb{T}}$ such that $x(t)>0$, $x(\tau(t))>0$, and $x(\delta(t))>0$ for $t\in[t_1,\infty)_{\mathbb{T}}$. From properties of $p(t)$ and $z(t)$, one can see for $\epsilon\in(0,1)$ that

$$\alpha p(t)+\frac{p(t)}{z(t)}(1-\alpha)<\epsilon \tag{16}$$

This combined with (10) provides

$$x(t)\geq \xi z(t) \tag{17}$$

where $\xi=(1-\epsilon)\in(0,1)$. Substituting (17) into (1), we have

$$\left(r(t)z^{\Delta}(t)\right)^{\Delta}+q(t)\xi^{\beta}z^{\beta}(\delta(t))\leq 0. \tag{18}$$

Using (8) it follows that $z(t)\to\infty$ as $t\to\infty$ and for $\beta>1$, we have

$$z^{\beta}(\delta(t))\geq z(\delta(t)).$$

This combined with (18) provides

$$\left(r(t)z^{\Delta}(t)\right)^{\Delta}+q(t)\xi^{\beta}z(\delta(t))\leq 0,\quad t\geq t_2. \tag{19}$$

Defining the function

$$\omega(t)=\varphi(t)\frac{r(t)z^{\Delta}(t)}{z(\delta(t))},\quad t\geq t_2. \tag{20}$$

It is clear that $w(t)>0$ for $t\geq t_2$ and

$$\begin{aligned}\omega^{\Delta}(t)=&\varphi^{\Delta}(t)\frac{r(\sigma(t))z^{\Delta}(\sigma(t))}{z(\delta(\sigma(t)))}+\varphi(t)\left(\frac{r(t)z^{\Delta}(t)}{z(\delta(t))}\right)^{\Delta}\\ =&\varphi^{\Delta}(t)\frac{r(\sigma(t))z^{\Delta}(\sigma(t))}{z(\delta(\sigma(t)))}+\varphi(t)\frac{[r(t)z^{\Delta}(t)]^{\Delta}}{z(\delta(t))}+\varphi(t)r(\sigma(t))z^{\Delta}(\sigma(t))\left(\frac{1}{z(\delta(t))}\right)^{\Delta}\\ \leq&-\xi^{\beta}\varphi(t)q(t)+\frac{\varphi^{\Delta}(t)}{\varphi(\sigma(t))}\omega(\sigma(t))-\frac{\varphi(t)r(\sigma(t))z^{\Delta}(\sigma(t))z^{\Delta}(\delta(t))\delta^{\Delta}(t)}{z(\delta(t))z(\delta(\sigma(t)))}.\end{aligned} \tag{21}$$

Since $z^{\Delta}(t)>0$, $\delta^{\Delta}(t)>0$ and $(r(t)z^{\Delta}(t))$ is non increasing, we get

$$\begin{aligned}\omega^{\Delta}(t)\leq&-\xi^{\beta}\varphi(t)q(t)+\frac{\varphi^{\Delta}(t)}{\varphi(\sigma(t))}\omega(\sigma(t))-\frac{\varphi(t)[r(\sigma(t))z^{\Delta}(\sigma(t))]^{2}\delta^{\Delta}(t)}{r(\delta(\sigma(t)))z^{2}(\delta(\sigma(t)))}\\ \leq&-\xi^{\beta}\varphi(t)q(t)+\frac{\varphi^{\Delta}(t)}{\varphi(\sigma(t))}\omega(\sigma(t))-\frac{\varphi(t)\delta^{\Delta}(t)}{\varphi^{2}(\sigma(t))r(\delta(\sigma(t)))}\omega^{2}(\sigma(t))\end{aligned} \tag{22}$$

Completing squares, we get

$$\begin{aligned}\omega^{\Delta}(t)\leq&-\xi^{\beta}\varphi(t)q(t)\\ &-\frac{\varphi(t)\delta^{\Delta}(t)}{\varphi^{2}(\sigma(t))r(\delta(\sigma(t)))}\left[\left(\omega(\sigma(t))-\frac{\varphi^{\Delta}(t)r(\delta(\sigma(t)))\varphi(\sigma(t))}{2\varphi(t)\delta^{\Delta}(t)}\right)^{2}-\left(\frac{\varphi^{\Delta}(t)r(\delta(\sigma(t)))\varphi(\sigma(t))}{2\varphi(t)\delta^{\Delta}(t)}\right)^{2}\right]\\ \leq&-\xi^{\beta}\varphi(t)q(t)+\frac{(\varphi^{\Delta}(t))^{2}r(\delta(\sigma(t)))}{4\varphi(t)\delta^{\Delta}(t)}.\end{aligned} \tag{23}$$

Integrating (23) from t_2 to t, we get

$$\int_{t_2}^{t}\left(\xi^{\beta}\varphi(s)q(s)-\frac{(\varphi^{\Delta}(s))^{2}r(\delta(\sigma(s)))}{4\varphi(s)\delta^{\Delta}(s)}\right)\Delta s\leq\omega(t_2).$$

Taking $\limsup_{t\to\infty}$, we get a inconsistency with (15). This completes the proof. □

The following oscillation results cover the case when (1) is linear.
From Theorem3.1, one can immediately obtain the following oscillation results when $\beta = 1$.

Theorem 3.2. *Assume that* $\beta = 1$, $(H1) - (H3)$ *and* (8) *hold, and* $\delta^{\Delta} > 0$. *If there exists a function* $\varphi(t) \in C^1_{rd}([t_0, \infty)_{\mathbb{T}}, (0, \infty))$, *such that for sufficiently large* $t_2 \geq t_1$,

$$\limsup_{t\to\infty} \left[\int_{t_2}^{t} \left(\xi\varphi(s)q(s) - \frac{(\varphi^{\Delta}(s))^2 r(\delta(\sigma(s)))}{4\varphi(s)\delta^{\Delta}(s)} \right) \Delta s \right] = \infty, \tag{24}$$

holds for $\xi \in (0, 1)$, *then* (1) *is oscillatory.*

In the following, we use the comparison theorem to establish new oscillation criteria for the linear case of (1).

Theorem 3.3. *Assume that for sufficient large* $t \in [t_0, \infty)_{\mathbb{T}}$, (2) *holds if the first-order dynamic equation*

$$u^{\Delta}(t) + \xi q(t)R(\delta(t))u(\delta(t)) = 0, \tag{25}$$

oscillatory, then (1) *is also oscillatory.*

Proof. Suppose that (1) has a nonoscillatory solution $x(t)$ on $[t_0, \infty)_{\mathbb{T}}$. Without loss of generality, we may assume that there exists a $t_1 \geq t_0$ such that $x(t) > 0, x(\tau(t)) > 0$, and $x(\delta(t)) > 0$, for $t \geq t_1$. From (1) and condition (2) we obtain the following for $t \geq t_1$,

$$z(\delta(t)) = z(t_1) + \int_{t_1}^{\delta(t)} \frac{r(s)z^{\Delta}(s)}{r(s)} \Delta s \geq r(\delta(t))z^{\Delta}(\delta(t))R(\delta(t)). \tag{26}$$

As the same Proceeding in the proof of Theorem 3.1, we get

$$\left(r(t)z^{\Delta}(t)\right)^{\Delta} + q(t)\xi z(\delta(t)) \leq 0, \quad t \geq t_2. \tag{27}$$

This combined with (26) provides

$$\left(r(t)z^{\Delta}(t)\right)^{\Delta} + \xi q(t)R(\delta(t))r(\delta(t))z^{\Delta}(\delta(t)) \leq 0, \quad t \geq t_2. \tag{28}$$

Defining $u(t) := r(t)z^{\Delta}(t) > 0$ 0 and combining this with (28) provides

$$u^{\Delta}(t) + \xi q(t)R(\delta(t))u(\delta(t)) \leq 0, \quad t \geq t_2. \tag{29}$$

where $u := r(t)z^{\Delta}(t)$, is a positive solution of the first order delay dynamic inequality (29). By [[13] ,Theorem 3.1], equation (25) also presents a nonoscillatory solution. This contradiction proves that (1) is oscillatory. □

In view of Theorem3.3 and [Theorem 1, Theorem 2, [13]].

Corollary 3.4. *Let* $\delta^{\sigma}(t) \leq t$ *and* $\delta^{\Delta}(t) \geq 0$, *if*

$$\limsup_{t\to\infty} \left(\frac{\int_{\delta(t)}^{\sigma(t)} \xi q(s)R(\delta(s))\Delta s}{1 - [1 - \xi\mu(\delta(t))q(\delta(t))R(\delta(\delta(t)))]\xi\mu(\sigma(t))q(\sigma(t))R(\delta(\sigma(t)))} \right) > 1, \tag{30}$$

holds for $\xi \in (0, 1)$, *then every solution of* (1) *is oscillatory.*

The following theorem presents oscillation criteria when (1) is sub-linear.

Theorem 3.5. *Assume that* $0 < \beta < 1$, $(H1) - (H3)$ *and* (8) *hold, and* $\delta^{\Delta} > 0$. *If there exists a function* $\varphi(t) \in C^1_{rd}([t_0, \infty)_{\mathbb{T}}, (0, \infty))$, *such that for sufficiently large* $t_2 \geq t_1$,

$$\limsup_{t\to\infty} \left[\int_{t_2}^{t} \left(\xi^{\beta} R^{\beta-1}(\delta(s)) K^{\beta-1} q(s)\varphi(s) - \frac{(\varphi^{\Delta}(s))^2 r(\delta(\sigma(s)))}{4\varphi(s)\delta^{\Delta}(s)} \right) \Delta s \right] \Delta s = \infty, \tag{31}$$

holds for $\xi \in (0, 1)$, *then* (1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1) on $[t_0, \infty)_{\mathbb{T}}$ such that $x(t) > 0, x(\tau(t)) > 0$, and $x(\delta(t)) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. Using the proof of Theorem 3.1. We rewrite (18) as

$$\left(r(t)z^{\Delta}(t)\right)^{\Delta} + q(t)\xi^{\beta} R^{\beta-1}(\delta(t)) \frac{z^{\beta-1}(\delta(t))}{R^{\beta-1}(\delta(t))} z(\delta(t)) \leq 0, \quad t \geq t_2 \geq t_1. \tag{32}$$

As $\frac{z(t)}{R(t)}$ is decreasing, there exists a constant $K > 0$ such that

$$\frac{z(t)}{R(t)} \leq K, \quad t \geq t_2 \geq t_1. \tag{33}$$

Using (33) and $\beta < 1$, then (32) takes the form

$$\left(r(t)z^{\Delta}(t)\right)^{\Delta} + q(t)\xi^{\beta} R^{\beta-1}(\delta(t)) K^{\beta-1} z(\delta(t)) \leq 0, \quad t \geq t_2 \geq t_1. \tag{34}$$

Use the definition of $\omega(t)$ in (20). Using the proof of Theorem 3.1, we get

$$\omega^{\Delta}(t) \leq -\xi^{\beta} R^{\beta-1}(\delta(t)) K^{\beta-1} q(t)\varphi(t) + \frac{(\varphi^{\Delta}(t))^2 r(\delta(\sigma(t)))}{4\varphi(t)\delta^{\Delta}(t)}.$$

Integrating (**??**) from t_2 to t, we get

$$\int_{t_2}^{t} \left(\xi^{\beta} R^{\beta-1}(\delta(s)) K^{\beta-1} q(s)\varphi(s) - \frac{(\varphi^{\Delta}(s))^2 r(\delta(\sigma(s)))}{4\varphi(s)\delta^{\Delta}(s)} \right) \Delta s \leq \omega(t_2).$$

Considering $\limsup_{t\to\infty}$, we obtain a contradiction with (31) . This completes the proof . □

Now, we use comparison theorem to establish an oscillation criteria in the case of $0 < \beta < 1$.

Theorem 3.6. *Assume that for sufficient large* $t \in [t_0, \infty)_{\mathbb{T}}$, (2) *holds if the first order dynamic equation*

$$u^{\Delta}(t) + q(t)\xi^{\beta} R^{\beta}(\delta(t)) K^{\beta-1} u(\delta(t)) = 0, \tag{35}$$

oscillatory, then (1) *is also oscillatory.*

Proof. Suppose that (1) has a nonoscillatory solution $x(t)$ on $[t_0, \infty)_{\mathbb{T}}$. Without loss of generality, we may assume that there exists a $t_1 \geq t_0$ such that $x(t) > 0, x(\tau(t)) > 0$, and $x(\delta(t)) > 0$, for $t \geq t_1$. From (1) and condition (2) we obtain the following for $t \geq t_1$,

$$z(\delta(t)) = z(t_1) + \int_{t_1}^{\delta(t)} \frac{r(s)z^{\Delta}(s)}{r(s)} \Delta s \geq r(\delta(t)) z^{\Delta}(\delta(t)) R(\delta(t)). \tag{36}$$

Using the proof of Theorem 3.5, we get

$$\left(r(t)z^{\Delta}(t)\right)^{\Delta} + q(t)\xi^{\beta} R^{\beta-1}(\delta(t)) K^{\beta-1} z(\delta(t)) \leq 0, \quad t \geq t_2. \tag{37}$$

Combining this with (36) provides

$$\left(r(t)z^{\Delta}(t)\right)^{\Delta} + q(t)\xi^{\beta}R^{\beta}(\delta(t))K^{\beta-1}r(\delta(t))z^{\Delta}(\delta(t)) \le 0, \quad t \ge t_2. \tag{38}$$

Defining $u(t) := r(t)z^{\Delta}(t) > 0$, and combining this with (38) provides

$$u^{\Delta}(t) + q(t)\xi^{\beta}R^{\beta}(\delta(t))K^{\beta-1}u(\delta(t)) \le 0, \quad t \ge t_2. \tag{39}$$

where $u(t) := r(t)z^{\Delta}(t)$, is a positive solution of the first order delay dynamic inequality (39). By [[13] ,Theorem 3.1], equation (35) also presents a nonoscillatory solution. This contradiction proves that (1) is oscillatory. □

Example 3.7. *Assume $\mathbb{T} = \mathbb{R}$. Consider the second-order neutral differential equation*

$$\left(t\left(x(t) + \frac{1}{t}x^{1/3}(\eta t)\right)'\right)' + \gamma x^3(\lambda t) = 0, \quad t \ge 1. \tag{40}$$

Where $\eta, \lambda \in (0,1]$. Here $\alpha = 1/3, \beta = 3$, $p(t) = \frac{1}{t}$, $q(t) = \gamma$, $\tau(t) = \eta t$, $\delta(t) = \lambda t$ and $r(t) = t$. It is clear that conditions (2) and (8) are satisfied. By using Theorem 3.1 with $\varphi(t) = t$, we have

$$\begin{aligned}\limsup_{t\to\infty}\left[\int_{t_0}^{t}\left(\xi^{\beta}\varphi(s)q(s) - \frac{(\varphi^{\Delta}(s))^2 r(\delta(\sigma(s)))}{4\varphi(s)\delta^{\Delta}(s)}\right)\Delta s\right] &= \limsup_{t\to\infty}\left[\int_{t_0}^{t}\left(\xi^3\gamma s - \frac{1}{4}\right)ds\right] \\ &= \infty.\end{aligned}$$

For $\xi \in (0,1)$, it is clear that (40) satisfied the condition of Theorem 3.1, then equation (40) is oscillatory for all $\gamma > 0$.

Example 3.8. *Assume that $\mathbb{T} = \mathbb{R}$. Consider the second-order neutral differential equation*

$$\left(x(t) + \frac{p_0}{t^{1-\alpha}}x^{\alpha}(\tau(t))\right)'' + \frac{q_0}{t^{\beta+1}}x^{\beta}(\lambda t) = 0, \quad t > 0. \tag{41}$$

Where $0 < \alpha \le 1$, $p(t) = \frac{p_0}{t^{1-\alpha}}$, $q(t) = \frac{q_0}{t^{\beta+1}}$, $\tau(t) \le t$, $\delta(t) = \lambda t$ and $r(t) = 1$. Here, we have three possible cases $0 < \beta < 1$, $\beta = 1$ and $\beta > 1$. In the case of $\beta = 1$, Eq.(41) takes the from

$$\left(x(t) + \frac{p_0}{t^{1-\alpha}}x^{\alpha}(\tau(t))\right)'' + \frac{q_0}{t^2}x(\lambda t) = 0$$

Applying corollary 3.4 and take in your account that for $\mathbb{T} = \mathbb{R}$; $\sigma(t) = t$ and $\mu(t) = 0$. It follows that condition (30) becomes

$$\begin{aligned}\limsup_{t\to\infty}\left(\int_{\delta(t)}^{\sigma(t)}\xi q(s)R(\delta(s))ds\right) &= \limsup_{t\to\infty}\left(\int_{\lambda t}^{t}\frac{\xi q_0}{s^2}(\lambda s)ds\right) \\ &= \xi q_0\left(\ln(t) - \ln(\lambda t)\right) \\ &= \xi q_0 \ln\left(\frac{1}{\lambda}\right)\end{aligned}$$

Hence , for $\beta = 1$ Eq.(41) is oscillatory for $q_0 > \frac{1}{\xi\ln\left(\frac{1}{\lambda}\right)}$.

Consider $\beta > 1$, it is clear that conditions (2) and (8) are satisfied. By using Theorem 3.1 with $\varphi(t) = t^{\beta}$, we have

$$\begin{aligned}\limsup_{t\to\infty}\left[\int_{t_0}^{t}\left(\xi^{\beta}\varphi(s)q(s) - \frac{(\varphi^{\Delta}(s))^2 r(\delta(\sigma(s)))}{4\varphi(s)\delta^{\Delta}(s)}\right)\Delta s\right] &= \limsup_{t\to\infty}\left[\int_{t_0}^{t}\left(\frac{\xi^{\beta}}{s} - \frac{\beta^2 s^{2\beta-2}}{4s^{\beta}\lambda}\right)ds\right] \\ &= \infty.\end{aligned}$$

It is clear that (41) *satisfied the condition of Corollary 3.1, then equation* (41) *is oscillatory.*

Finally consider $0 < \beta < 1$. *Applying Theorem 3.5, we have* $R(t) = t$ *and*

$$\limsup_{t\to\infty}\left[\int_{t_0}^{t}\left(\xi^\beta\lambda^{\beta-1}(s)^{\beta-1}K^{\beta-1}\frac{q_0}{s^{\beta+1}}(s)-\frac{1}{4\lambda s}\right)ds\right]=\left(\xi^\beta\lambda^{\beta-1}K^{\beta-1}q_0-\frac{1}{4\lambda}\right)\ln(t)$$

Hence (41) *is oscillatory when* $q_0 > \frac{1}{4\xi^\beta\lambda^\beta K^{\beta-1}}$.

As a special case of (41), *consider* $\beta = \alpha = 1/3$ *and* $\tau(t) = \delta(t) = 1/2$; *Eq.*(41) *takes the form*

$$\left(x(t)+\frac{1}{t}x^{1/3}\left(\frac{t}{2}\right)\right)''+\frac{q_0}{t^{4/3}}x^{1/3}\left(\frac{t}{2}\right)=0 \tag{42}$$

It follows that (42) *oscillates when* $q_0 > \frac{K^{2/3}}{2^{5/3}\xi^{1/3}}$. *By choosing small values of K and* $\xi = 0.9$, *the criteria for oscillation of this equation consistent with the results of [20]*

Example 3.9. [8] *Assume* $\mathbb{T} = \mathbb{Z}$. *Consider the second order neutral difference equation*

$$\Delta\left((n+1)\Delta\left(x_n+\frac{1}{n}x_{n-2}^{1/3}\right)\right)+\left(4n+10+\frac{2n+1}{n(n+1)}\right)x_{n-3}^3=0,\quad n\geq 1. \tag{43}$$

Here $\alpha = 1/3, \beta = 3, r_n = n+1, p_n = \frac{1}{n}, q_n = 4n+10+\frac{2n+1}{n(n+1)}, \tau_n = n-2, \delta_n = n-3,$. *It is clear that conditions* (2) *and* (8) *are satisfied. By using Theorem 3.1; take* $\varphi_n = n$, *then condition* (15) *that becomes*

$$\limsup_{n\to\infty}\left[\sum_{s=1}^{n}\xi^3\left(4s+10+\frac{2s+1}{s(s+1)}\right)\right]=\infty.$$

Therefore, it is Clear that (43) *satisfied the condition of Theorem 3.1, then equation* (43) *is oscillatory.*

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