



On characteristic functions of generalized resolvents generated by integral equations with operator measures

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Abstract. We consider a symmetric minimal relation L_0 generated by an integral equation with operators measures. We describe the generalized resolvents of L_0 using the characteristic function $M(\lambda)$ ($\lambda \in \mathbb{C}$), i.e., a function that has the property $(\operatorname{Im}\lambda)^{-1}\operatorname{Im}M(\lambda) \geq 0$. We obtain a necessary and sufficient condition for a holomorphic function $M(\lambda)$ to be a characteristic function of a generalized resolvent. We give a detailed example of finding the characteristic function.

1. Introduction

Generalized resolvents of symmetric operators were introduced by M.A. Naimark in 1940 (see, for example, [1]). In the article [24], A.V. Straus described the generalized resolvents of a symmetric operator generated by a formally self-adjoint differential expression of even order in the scalar case. In such a description, a function $M(\lambda)$ ($\lambda \in \mathbb{C}$) plays an essential role. This function has the property $(\operatorname{Im}\lambda)^{-1}\operatorname{Im}M(\lambda) \geq 0$. In [24], the function $M(\lambda)$ is called the characteristic function of the generalized resolvent. The characteristic function is used in solving many problems associated with the generalized resolvents (see [24]).

In [5], these results from [24] were extended to the operator case, and in [7], to the case of a differential-operator expression with a non-negative weight operator function. Further, the generalized resolvents of differential operators were studied in many works (a detailed bibliography is available, for example, in [22], [20]).

In this paper, we consider the integral equation

$$y(t) = x_0 - iJ \int_a^t d\mathbf{p}(s)y(s) - iJ \int_a^t d\mathbf{m}(s)f(s), \quad (1)$$

where y is an unknown function, $a \leq t \leq b$; J is an operator in a separable Hilbert space H , $J = J^*$, $J^2 = E$ (E is the identical operator); \mathbf{p} , \mathbf{m} are operator-valued measures defined on Borel sets $\Delta \subset [a, b]$ and taking values in the set of linear bounded operators acting in H ; $x_0 \in H$, $f \in L_2(H, d\mathbf{m}; a, b)$. We assume that the measures \mathbf{p} , \mathbf{m} have bounded variations and \mathbf{p} is self-adjoint, \mathbf{m} is non-negative.

If the measures \mathbf{p} , \mathbf{m} are absolutely continuous (i.e., $\mathbf{p}(\Delta) = \int_\Delta p(t)dt$, $\mathbf{m}(\Delta) = \int_\Delta m(t)dt$ for all Borel sets $\Delta \subset [a, b]$, where $p(t)$, $m(t)$ are bounded operators for fixed t and the functions $\|p(t)\|$, $\|m(t)\|$ belong

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to $L_1(a, b)$), then integral equation (1) is transformed to a differential equation with a non-negative weight operator function. Linear relations and operators generated by such differential equations were considered in many works (see [21], [6], [7], further detailed bibliography can be found, for example, in [20], [3]).

The study of integral equation (1) differs essentially from the study of differential equations by the presence of the following features: i) a representation of a solution of equation (1) using an evolutionary family of operators is possible if the measures \mathbf{p} , \mathbf{m} have not common single-point atoms (see [10]); ii) the Lagrange formula contains summands relating to single-point atoms of the measures \mathbf{p} , \mathbf{m} (see [11]).

We consider a symmetric minimal relation L_0 generated by equation (1). We obtain a form of generalized resolvents of L_0 . This form contains the characteristic function. The proposed article is a continuation of the works [15], [16]. In [15], continuously invertible relations $T(\lambda)$ with the property $L_0 - \lambda E \subset T(\lambda) \subset L_0^* - \lambda E$ are described. In this description, a function $M(\lambda)$ plays an important role. The function $M(\lambda)$ is determined by the relation $T(\lambda)$ and the choice of a system of functions v_k , where $v_k \in \ker(L_0^* - \lambda E)$ and the linear span of the set $\{v_k\}$ is dense in $\ker(L_0^* - \lambda E)$. In [16], the generalized resolvents of the relation L_0 are described using the function $M(\lambda)$ from [15]. However, this function $M(\lambda)$ is not characteristic (see the example in [16]).

In this article, we construct a system of functions ϑ_k different from the system $\{v_k\}$ in [15], [16], but having the same properties: $\vartheta_k \in \ker(L_0^* - \lambda E)$ and the linear span of the set $\{\vartheta_k\}$ is dense in $\ker(L_0^* - \lambda E)$. We describe generalized resolvents using the system of functions ϑ_k . The corresponding function $M(\lambda)$ is characteristic. We obtain a necessary and sufficient condition for a holomorphic function $M(\lambda)$ to be a characteristic function of a generalized resolvent. We give a detailed example of finding the characteristic function.

We note that this article partially corrects the errors made in the work [9]. Also note that some special cases of the describing of generalized resolvents and related problems for equation (1) are considered in [12], [13], [14].

2. Designations and preliminary assertions

Let \mathbf{B} be a Hilbert space. A linear relation T is understood as any linear manifold $T \subset \mathbf{B} \times \mathbf{B}$. The terminology on the linear relations can be found, for example, in [18], [22], [2]. In what follows we make use of the following notations: $\{\cdot, \cdot\}$ is an ordered pair; $\mathcal{D}(T)$ is the domain of T ; $\mathcal{R}(T)$ is the range of T ; $\ker T$ is a set of elements $x \in \mathbf{B}$ such that $\{x, 0\} \in T$; T^{-1} is the relation inverse for T , i.e., the relation formed by the pairs $\{x', x\}$, where $\{x, x'\} \in T$. A relation T is called surjective if $\mathcal{R}(T) = \mathbf{B}$. A relation T is called invertible or injective if $\ker T = \{0\}$ (i.e., the relation T^{-1} is an operator); it is called continuously invertible if it is closed, invertible, and surjective (i.e., T^{-1} is a bounded everywhere defined operator). A relation T^* is called adjoint for T if T^* consists of all pairs $\{y_1, y_2\}$ such that the equality $(x_2, y_1) = (x_1, y_2)$ holds for all pairs $\{x_1, x_2\} \in T$. A relation T is called symmetric if $T \subset T^*$ and self-adjoint if $T = T^*$.

It is known (see, for example, [19, ch.3], [18, ch.1]) that the graph of an operator $T: \mathcal{D}(T) \rightarrow \mathbf{B}$ is the set of pairs $\{x, Tx\} \in \mathbf{B} \times \mathbf{B}$, where $x \in \mathcal{D}(T) \subset \mathbf{B}$. Consequently, the linear operators can be treated as linear relations; this is why the notation $\{x_1, x_2\} \in T$ is used also for the operator T . Since all considered relations are linear, we shall often omit the word "linear".

Let T be a closed symmetric relation, $T \subset \mathbf{B} \times \mathbf{B}$, and let \tilde{T} be a self-adjoint extension of T to $\tilde{\mathbf{B}}$, where $\tilde{\mathbf{B}}$ is a Hilbert space, $\tilde{\mathbf{B}} \supset \mathbf{B}$, and scalar products coincide in \mathbf{B} and $\tilde{\mathbf{B}}$. By P denote an orthogonal projection of $\tilde{\mathbf{B}}$ onto \mathbf{B} . The function $\lambda \rightarrow R_\lambda$ defined by the formula $R_\lambda = P(\tilde{T} - \lambda E)^{-1}|_{\mathbf{B}}$, $\text{Im} \lambda \neq 0$, is called the generalized resolvent of the relation T (see, for example, [1, ch.9], [17]).

A.V. Straus (see [23]) obtained a formula for all generalized resolvents of a symmetric operator. It is shown in [17] that this formula remains true for symmetric relations also. By \mathfrak{N}_λ denote a defect subspace of the closed symmetric relation T , i.e., the orthogonal complement in \mathbf{B} of the range of the relation $T - \lambda E$. We fix some number λ_0 ($\text{Im} \lambda_0 \neq 0$). Let $\lambda \rightarrow \mathcal{F}(\lambda)$ be a holomorphic operator function, where $\mathcal{F}(\lambda): \mathfrak{N}_{\lambda_0} \rightarrow \mathfrak{N}_{\lambda_0}$ is a bounded operator, $\|\mathcal{F}(\lambda)\| \leq 1$, $\text{Im} \lambda \cdot \text{Im} \lambda_0 > 0$. Let $T_{\mathcal{F}(\lambda)}$ be the relation consisting of all pairs of the form $\{y_0 + \mathcal{F}(\lambda)z - z, y_1 + \lambda_0 \mathcal{F}(\lambda)z - \bar{\lambda}_0 z\}$, where $\{y_0, y_1\} \in T$, $z \in \mathfrak{N}_{\lambda_0}$. Then (see [23], [17]) the family of operators R_λ is a generalized resolvent of T if and only if R_λ can be represented in the form

$$R_\lambda = (T_{\mathcal{F}(\lambda)} - \lambda E)^{-1}, \quad \text{Im} \lambda \cdot \text{Im} \lambda_0 > 0. \tag{2}$$

In what follows, we use the following notions from measure theory. Let H be a separable Hilbert space with a scalar product (\cdot, \cdot) and a norm $\|\cdot\|$. By \mathcal{B} denote a set of Borel subsets $\Delta \subset [a, b]$. We consider a function $\Delta \rightarrow \mathbf{P}(\Delta)$ defined on \mathcal{B} and taking values in the set of linear bounded operators acting in H . The function \mathbf{P} is called an operator measure on $[a, b]$ (see, for example, [4, ch. 5]) if it is zero on the empty set and the equality $\mathbf{P}(\bigcup_{n=1}^{\infty} \Delta_n) = \sum_{n=1}^{\infty} \mathbf{P}(\Delta_n)$ holds for disjoint Borel sets Δ_n , where the series converges weakly. Further, we extend any measure \mathbf{P} on $[a, b]$ to a segment $[a, b_0]$ ($b_0 > b$) letting $\mathbf{P}(\Delta) = 0$ for each Borel sets $\Delta \subset (b, b_0]$.

By $\mathbf{V}_{\Delta}(\mathbf{P})$ we denote $\mathbf{V}_{\Delta}(\mathbf{P}) = \rho_{\mathbf{P}}(\Delta) = \sup \sum_n \|\mathbf{P}(\Delta_n)\|$, where the supremum is taken over finite sums of disjoint Borel sets $\Delta_n \subset \Delta$. The number $\mathbf{V}_{\Delta}(\mathbf{P})$ is called the variation of the measure \mathbf{P} on the Borel set Δ . Suppose that the measure \mathbf{P} has the bounded variation on $[a, b]$. Then for $\rho_{\mathbf{P}}$ -almost all $\xi \in [a, b]$ there exists an operator function $\xi \rightarrow \Psi_{\mathbf{P}}(\xi)$ such that $\Psi_{\mathbf{P}}$ possesses the values in the set of linear bounded operators acting in H , $\|\Psi_{\mathbf{P}}(\xi)\| = 1$, and the equality

$$\mathbf{P}(\Delta) = \int_{\Delta} \Psi_{\mathbf{P}}(s) d\rho_{\mathbf{P}} \tag{3}$$

holds for each set $\Delta \in \mathcal{B}$. The function $\Psi_{\mathbf{P}}$ is uniquely determined up to values on a set of zero $\rho_{\mathbf{P}}$ -measure. Integral (3) converges in the sense of usual operator norm ([4, ch. 5]).

Further, $\int_{t_0}^t$ stands for $\int_{[t_0, t]}$ if $t_0 < t$, for $-\int_{[t, t_0]}$ if $t_0 > t$, and for 0 if $t_0 = t$. This implies that $y(a) = x_0$ in equation (1). A function h is integrable with respect to the measure \mathbf{P} on a set $\Delta \in \mathcal{B}$ if there exists the Bochner integral $\int_{\Delta} \Psi_{\mathbf{P}}(t)h(t)d\rho_{\mathbf{P}} = \int_{\Delta}(d\mathbf{P})h(t)$. Then the function $y(t) = \int_{t_0}^t(d\mathbf{P})h(s)$ is continuous from the left.

By $\mathcal{S}_{\mathbf{P}}$ denote a set of single-point atoms of the measure \mathbf{P} (i.e., a set $t \in [a, b]$ such that $\mathbf{P}(\{t\}) \neq 0$). The set $\mathcal{S}_{\mathbf{P}}$ is at most countable. The measure \mathbf{P} is continuous if $\mathcal{S}_{\mathbf{P}} = \emptyset$, it is self-adjoint if $(\mathbf{P}(\Delta))^* = \mathbf{P}(\Delta)$ for each Borel set $\Delta \in \mathcal{B}$, it is non-negative if $(\mathbf{P}(\Delta)x, x) \geq 0$ for all Borel sets $\Delta \in \mathcal{B}$ and for all elements $x \in H$.

In addition, we use the following notation. We construct a continuous measure \mathbf{P}_0 from the measure \mathbf{P} in the following way. We set $\mathbf{P}_0(\{t_k\}) = 0$ for $t_k \in \mathcal{S}_{\mathbf{P}}$ and we set $\mathbf{P}_0(\Delta) = \mathbf{P}(\Delta)$ for all Borel sets such that $\Delta \cap \mathcal{S}_{\mathbf{P}} = \emptyset$. We denote $\widehat{\mathbf{P}} = \mathbf{P} - \mathbf{P}_0$. Then $\widehat{\mathbf{P}}(\{t_k\}) = \mathbf{P}(\{t_k\})$ for all $t_k \in \mathcal{S}_{\mathbf{P}}$ and $\widehat{\mathbf{P}}(\Delta) = 0$ for all Borel sets Δ such that $\Delta \cap \mathcal{S}_{\mathbf{P}} = \emptyset$.

In following Lemma 2.1, $\mathbf{p}_1, \mathbf{p}_2, \mathbf{q}$ are operator measures having bounded variations and taking values in the set of linear bounded operators acting in H . Suppose that the measure \mathbf{q} is self-adjoint and assume that these measures are extended on the segment $[a, b_0] \supset [a, b_0] \supset [a, b]$ in the manner described above.

Lemma 2.1. [11] *Let f, g be functions integrable on $[a, b_0]$ with respect to the measure \mathbf{q} and $y_0, z_0 \in H$. Then any functions*

$$y(t) = y_0 - iJ \int_{t_0}^t d\mathbf{p}_1(s)y(s) - iJ \int_{t_0}^t d\mathbf{q}(s)f(s), \quad z(t) = z_0 - iJ \int_{t_0}^t d\mathbf{p}_2(s)z(s) - iJ \int_{t_0}^t d\mathbf{q}(s)g(s) \quad (a \leq t_0 < b_0, \quad t_0 \leq t \leq b_0)$$

satisfy the following formula (analogous to the Lagrange one):

$$\begin{aligned} & \int_{c_1}^{c_2} (d\mathbf{q}(t)f(t), z(t)) - \int_{c_1}^{c_2} (y(t), d\mathbf{q}(t)g(t)) = (iJy(c_2), z(c_2)) - (iJy(c_1), z(c_1)) + \int_{c_1}^{c_2} (y(t), d\mathbf{p}_2(t)z(t)) - \\ & - \int_{c_1}^{c_2} (d\mathbf{p}_1(t)y(t), z(t)) - \sum_{t \in \mathcal{S}_{\mathbf{p}_1} \cap \mathcal{S}_{\mathbf{p}_2} \cap [c_1, c_2]} (iJ\mathbf{p}_1(\{t\})y(t), \mathbf{p}_2(\{t\})z(t)) - \sum_{t \in \mathcal{S}_{\mathbf{q}} \cap \mathcal{S}_{\mathbf{p}_2} \cap [c_1, c_2]} (iJ\mathbf{q}(\{t\})f(t), \mathbf{p}_2(\{t\})z(t)) - \\ & - \sum_{t \in \mathcal{S}_{\mathbf{p}_1} \cap \mathcal{S}_{\mathbf{q}} \cap [c_1, c_2]} (iJ\mathbf{p}_1(\{t\})y(t), \mathbf{q}(\{t\})g(t)) - \sum_{t \in \mathcal{S}_{\mathbf{q}} \cap [c_1, c_2]} (iJ\mathbf{q}(\{t\})f(t), \mathbf{q}(\{t\})g(t)), \quad t_0 \leq c_1 < c_2 \leq b_0. \end{aligned} \tag{4}$$

Further we assume that measures \mathbf{p}, \mathbf{m} have bounded variations and \mathbf{p} is self-adjoint, \mathbf{m} is non-negative. We consider equation (1), where $x_0 \in H$, f is integrable with respect to the measure \mathbf{m} on $[a, b]$, $a \leq t \leq b_0$. We replace \mathbf{p} by \mathbf{p}_0 and \mathbf{m} by \mathbf{m}_0 in (1). Then we obtain the equation

$$y(t) = x_0 - iJ \int_a^t d\mathbf{p}_0(s)y(s) - iJ \int_a^t d\mathbf{m}_0(s)f(s). \tag{5}$$

Equations (1), (5) have unique solutions (see [10]).

By $W(t, \lambda)$ denote an operator solution of the equation

$$W(t, \lambda)x_0 = x_0 - iJ \int_a^t d\mathbf{p}_0(s)W(s, \lambda)x_0 - iJ\lambda \int_a^t d\mathbf{m}_0(s)W(s, \lambda)x_0, \tag{6}$$

where $x_0 \in H, \lambda \in \mathbb{C}$ (\mathbb{C} is the set of complex numbers). It follows from Lemma 2.1 that $W^*(t, \bar{\lambda})JW(t, \lambda) = J$. The functions $t \rightarrow W(t, \lambda)$ and $t \rightarrow W^{-1}(t, \lambda) = JW^*(t, \bar{\lambda})J$ are continuous with respect to the uniform operator topology. Consequently, there exist constants $\varepsilon_1 > 0, \varepsilon_2 > 0$ such that the inequality

$$\varepsilon_1 \|x\|^2 \leq \|W(t, \lambda)x\|^2 \leq \varepsilon_2 \|x\|^2 \tag{7}$$

holds for all $x \in H, t \in [a, b_0], \lambda \in C \subset \mathbb{C}$ (C is a compact set).

Lemma 2.2. [15]. *Suppose that a function f is integrable with respect to the measure \mathbf{m} . A function y is a solution of the equation*

$$y(t) = x_0 - iJ \int_a^t d\mathbf{p}_0(s)y(s) - iJ\lambda \int_a^t d\mathbf{m}_0(s)y(s) - iJ \int_a^t d\mathbf{m}(s)f(s), \quad x_0 \in H, \quad a \leq t \leq b_0,$$

if and only if y has the form

$$y(t) = W(t, \lambda)x_0 - W(t, \lambda)iJ \int_a^t W^*(\xi, \bar{\lambda})d\mathbf{m}(\xi)f(\xi).$$

3. Linear relations generated by the integral equation

This article substantially uses the results of [15], [16]. In this section, we provide definitions and statements from [15], [16] that are used in this article. Moreover, in this section, we construct a system of functions $\{\vartheta_k\}$ whose the linear span is dense in $\ker(L_{10}^* - \lambda E)$ (see below). This system is different from the system $\{v_k\}$ introduced in [15], [16]. In the next section, we use the system $\{\vartheta_k\}$ to find the characteristic function $M(\lambda)$.

Let \mathbf{m} is a non-negative operator measure defined on Borel sets $\Delta \subset [a, b]$ and taking values in the set of linear bounded operators acting in the space H . The measure \mathbf{m} is assumed to have a bounded variation on $[a, b]$. We introduce the quasi-scalar product $(x, y)_{\mathbf{m}} = \int_a^{b_0} ((d\mathbf{m})x(t), y(t))$ on a set of step-like functions with values in H defined on the segment $[a, b_0]$. Identifying with zero functions y obeying $(y, y)_{\mathbf{m}} = 0$ and making the completion, we arrive at the Hilbert space denoted by $L_2(H, d\mathbf{m}; a, b) = \mathfrak{H}$. The elements of \mathfrak{H} are the classes of functions identified with respect to the norm $\|y\|_{\mathbf{m}} = (y, y)_{\mathbf{m}}^{1/2}$. In order not to complicate the terminology, the class of functions with a representative y is indicated by the same symbol and we write $y \in \mathfrak{H}$. The equality of functions in \mathfrak{H} is understood as the equality for associated equivalence classes. We denote the scalar product in the space \mathfrak{H} by the symbols $(\cdot, \cdot)_{\mathfrak{H}}$ or $(\cdot, \cdot)_{\mathbf{m}}$. The scalar product generated by the measure \mathbf{m}_0 is denoted by $(\cdot, \cdot)_{\mathbf{m}_0}$.

Let us define a *minimal relation* L_0 in the following way. The relation L_0 consists of all pairs $\{\widetilde{y}, \widetilde{f}_0\} \in \mathfrak{H} \times \mathfrak{H}$ satisfying the condition: for each pair $\{\widetilde{y}, \widetilde{f}_0\}$ there exists a pair $\{y, f_0\}$ such that the pairs $\{\widetilde{y}, \widetilde{f}_0\}, \{y, f_0\}$ are identical in $\mathfrak{H} \times \mathfrak{H}$ and $\{y, f_0\}$ satisfies equation (1) and the equalities

$$y(a) = y(b_0) = y(\alpha) = 0, \quad \alpha \in \mathcal{S}_{\mathbf{p}}; \quad \mathbf{m}(\{\beta\})f_0(\beta) = 0, \quad \beta \in \mathcal{S}_{\mathbf{m}}. \tag{8}$$

Further, without loss of generality it can be assumed that if $\{y, f_0\} \in L_0$, then equalities (1), (8) hold for this pair. In general, the relation L_0 is not an operator since a function y can happen to be identified with zero in \mathfrak{H} , while f is non-zero. The relation L_0 is symmetric and closed. We note that if $y \in \mathcal{D}(L_0)$, then y is continuous and $y(b) = 0$ (see[14], [15]).

By $\overline{\mathcal{S}_{\mathbf{p}}}$ denote the closure of the set $\mathcal{S}_{\mathbf{p}}$. Let \mathcal{S}_0 be the set $t \in [a, b]$ such that $y(t) = 0$ for all $y \in \mathcal{D}(L_0)$. The set \mathcal{S}_0 is closed and $\overline{\mathcal{S}_{\mathbf{p}}} \cup \{a\} \cup \{b\} \subset \mathcal{S}_0$ (see[15]).

Lemma 3.1. [15]. *Suppose $\{y, f\} \in L_0$. Then $f(t) = 0$ for \mathbf{m} -almost all $t \in \mathcal{S}_0$.*

By \mathfrak{H}_0 (by \mathfrak{H}_1) denote a subspace of functions that vanish on $[a, b] \setminus \mathcal{S}_0$ (on \mathcal{S}_0 , respectively) with respect to the norm in \mathfrak{H} . The subspaces $\mathfrak{H}_0, \mathfrak{H}_1$ are orthogonal and $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_1$. We note that $\mathfrak{H}_0 = \{0\}$ if and only if $\mathbf{m}(\mathcal{S}_0) = 0$. We denote $L_{10} = L_0 \cap (\mathfrak{H}_1 \times \mathfrak{H}_1)$. It follows from Lemma 3.1 that $\mathcal{D}(L_{10}) \subset \mathfrak{H}_1, \mathcal{R}(L_{10}) \subset \mathfrak{H}_1$ and $L_0 \cap \mathfrak{H}_0 \times \mathfrak{H}_0 = \{0, 0\}$. Hence,

$$L_0^* = (\mathfrak{H}_0 \times \mathfrak{H}_0) \oplus L_{10}^*, \tag{9}$$

i.e., the relation L_0^* consists of all pairs $\{y, f\} \in \mathfrak{H}$ of the form $\{y, f\} = \{u, v\} + \{z, g\} = \{u + z, v + g\}$, where $u, v \in \mathfrak{H}_0, \{z, g\} \in L_{10}^*$.

Consequently, any generalized resolvent \widetilde{R}_λ of the relation L_0 has the form $\widetilde{R}_\lambda = R_{0\lambda} \oplus R_\lambda$, where R_λ is some generalized resolvent of L_{10} and $R_{0\lambda}$ is a generalized resolvent of the relation $\{0, 0\}$, i.e., $R_{0\lambda} = (T_{\mathcal{F}_0(\lambda)} - \lambda E)^{-1}$ (see (2)), $T_{\mathcal{F}_0(\lambda)}$ is the relation consisting of pairs of the form $\{\mathcal{F}_0(\lambda)z - z, \lambda_0 \mathcal{F}_0(\lambda)z - \bar{\lambda}_0 z\}$. Here $\mathcal{F}_0(\lambda): \mathfrak{H}_0 \rightarrow \mathfrak{H}_0$ is a bounded operator, $\|\mathcal{F}_0(\lambda)\| \leq 1, z \in \mathfrak{H}_0$, the operator function $\lambda \rightarrow \mathcal{F}_0(\lambda)$ is holomorphic, $\text{Im} \lambda \cdot \text{Im} \lambda_0 > 0$. Therefore, to find the characteristic function of the generalized resolvent, we consider the relation L_{10} .

The set $\mathcal{T}_p = (a, b) \setminus \mathcal{S}_0$ is open and it is the union of at most a countable number of disjoint open intervals \mathcal{J}_k , i.e., $\mathcal{T}_p = \bigcup_{k=1}^{\mathbb{k}_1} \mathcal{J}_k$ and $\mathcal{J}_k \cap \mathcal{J}_j = \emptyset$ for $k \neq j$, where \mathbb{k}_1 is a natural number (equal to the number of intervals if this number is finite) or the symbol ∞ (if the number of intervals is infinite). By \mathbb{J} denote the set of these intervals \mathcal{J}_k .

Remark 3.2. *The boundaries α_k, β_k of any interval $\mathcal{J}_k = (\alpha_k, \beta_k) \in \mathbb{J}$ belong to \mathcal{S}_0 . This follows from the closedness of \mathcal{S}_0 and the continuity of functions $y \in \mathcal{D}(L_0)$ (see [15]).*

Let $\mathfrak{X}_A = \mathfrak{X}_A(t)$ be an operator characteristic function of a set A , i.e., $\mathfrak{X}_A(t) = E$ if $t \in A$ and $\mathfrak{X}_A(t) = 0$ if $t \notin A$. We will often omit the argument t in the notation \mathfrak{X}_A . We denote

$$w_k(t, \lambda) = \mathfrak{X}_{[\alpha_k, \beta_k]} W(t, \lambda) W^{-1}(\alpha_k, \lambda), \tag{10}$$

where $(\alpha_k, \beta_k) = \mathcal{J}_k \in \mathbb{J}$. Then (see[15])

$$w_k^*(t, \bar{\lambda}) J w_k(t, \lambda) = J, \quad \alpha_k \leq t < \beta_k. \tag{11}$$

By \mathfrak{H}_{10} (by \mathfrak{H}_{11}) denote a subspace of functions that belong to \mathfrak{H}_1 and vanish on \mathcal{S}_m (on $[a, b] \setminus \mathcal{S}_m$, respectively) with respect to the norm in \mathfrak{H} . So, \mathfrak{H}_{10} (\mathfrak{H}_{11}) consists of functions of the form $\mathfrak{X}_{[a,b] \setminus (\mathcal{S}_0 \cup \mathcal{S}_m)} h$ (of the form $\mathfrak{X}_{\mathcal{S}_m \setminus \mathcal{S}_0} h$, respectively), where $h \in \mathfrak{H}$ is an arbitrary function. Therefore,

$$\mathfrak{H}_1 = \mathfrak{H}_{10} \oplus \mathfrak{H}_{11}, \quad \mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_{10} \oplus \mathfrak{H}_{11}.$$

Obviously, the space \mathfrak{H}_{11} is the closure in \mathfrak{H} of the linear span of functions that have the form $\mathfrak{X}_{\{\tau\}}(\cdot)x$, where $x \in H, \tau \in \mathcal{S}_m \setminus \mathcal{S}_0$. By (8), it follows that $\mathfrak{H}_{11} \subset \ker L_{10}^*$.

Let $u_k(\cdot, \lambda, \tau): H \rightarrow \mathfrak{H}_1$ be an operator acting by the formula

$$u_k(t, \lambda, \tau)x = -\mathfrak{X}_{[a,b] \setminus (\mathcal{S}_m \cup \mathcal{S}_0)} w_k(t, \lambda) i J \int_a^t w_k^*(s, \bar{\lambda}) d\mathbf{m}(s) \lambda \mathfrak{X}_{\{\tau\}}(s)x, \tag{12}$$

where $x \in H, \tau \in (\alpha_k, \beta_k) \cap \mathcal{S}_m, (\alpha_k, \beta_k) = \mathcal{J}_k \in \mathbb{J}$. Then (see[15]) for any $x \in H$ the function

$$u_k(\cdot, \lambda, \tau)x + \mathfrak{X}_{\{\tau\}}(\cdot)x \in \ker(L_{10}^* - \lambda E).$$

Lemma 3.3. [15]. *The linear span of functions of the form $\mathfrak{X}_{[a,b] \setminus (\mathcal{S}_m \cup \mathcal{S}_0)} w_k(\cdot, \lambda)x_0$ and $u_k(\cdot, \lambda, \tau)B_\tau x_j + \mathfrak{X}_{\{\tau\}}(\cdot)B_\tau x_j$ is dense in $\ker(L_{10}^* - \lambda E)$. Here $x_j, x_0 \in H; \tau \in (\alpha_k, \beta_k) \cap \mathcal{S}_m; B_\tau: H \rightarrow H$ is a bounded continuously invertible operator; $k = 1, \dots, \mathbb{k}_1$ if \mathbb{k}_1 is finite and $k \in \mathbb{N}$ if \mathbb{k}_1 is infinite (\mathbb{N} is the set of natural numbers).*

Let \mathbb{M} be a set consisting of intervals $\mathcal{J} \in \mathbb{J}$ and single-point sets $\{\tau\}$, where $\tau \in \mathcal{S}_m \setminus \mathcal{S}_0$. The set \mathbb{M} is at most countable. Let \mathbb{k} be the number of elements in \mathbb{M} . We arrange the elements of \mathbb{M} in the form of a finite or infinite sequence and denote these elements by \mathcal{E}_k , where k is any natural number if the number of elements in \mathbb{M} is infinite, and $1 \leq k \leq \mathbb{k}$ if the number of elements in \mathbb{M} is finite.

To each element $\mathcal{E}_k \in \mathbb{M}$ assign the operator function ϑ_k in the following way. If \mathcal{E}_k is the interval, $\mathcal{E}_k = \mathcal{J}_k = (\alpha_k, \beta_k) \in \mathbb{J}$, then

$$\vartheta_k(t, \lambda) = \mathfrak{X}_{[\alpha_k, \beta_k] \setminus \mathcal{S}_m} w_k(t, \lambda) = \mathfrak{X}_{[a, b] \setminus \mathcal{S}_m} w_k(t, \lambda). \tag{13}$$

If \mathcal{E}_k is a single-point set, $\mathcal{E}_k = \{\tau_k\}$, then

$$\vartheta_k(t, \lambda) = \lambda^{-1} u_n(t, \lambda, \tau_k) + \lambda^{-1} \mathfrak{X}_{\{\tau_k\}}(t) \text{ if } \lambda \neq 0 \text{ and } \vartheta_k(t, 0) = \mathfrak{X}_{\{\tau_k\}}(t) \text{ if } \lambda = 0, \tag{14}$$

where $\tau_k \in (\mathcal{S}_m \setminus \mathcal{S}_0) \cap \mathcal{J}_n$, $\mathcal{J}_n = (\alpha_n, \beta_n) \in \mathbb{J}$, $n = 1, \dots, \mathbb{k}_1$ if \mathbb{k}_1 is finite and $n \in \mathbb{N}$ if \mathbb{k}_1 is infinite. In case (14), using (12), we get

$$\vartheta_k(t, \lambda)x = -\mathfrak{X}_{[a, b] \setminus (\mathcal{S}_m \cup \mathcal{S}_0)} w_n(t, \lambda) i \int_a^t w_n^*(s, \bar{\lambda}) d\mathbf{m}(s) \mathfrak{X}_{\{\tau_k\}}(s)x + \lambda^{-1} \mathfrak{X}_{\{\tau_k\}}(t)x \text{ if } \lambda \neq 0 \quad (x \in H). \tag{15}$$

Lemma 3.4. *The linear span of functions $t \rightarrow \vartheta_k(t, \lambda)\xi_j$ is dense in $\ker(L_{10}^* - \lambda E)$. (Here $\xi_j \in H$, $k \in \mathbb{N}$ if $\mathbb{k} = \infty$, and $1 \leq k \leq \mathbb{k}$ if \mathbb{k} is finite.)*

Proof. The required statement follows from Lemma 3.3 immediately. \square

Corollary 3.5. *A function $f \in \mathfrak{S}_1$ belongs to the range $\mathcal{R}(L_{10} - \lambda E)$ if and only if the equality $(f, \vartheta_k(\cdot, \bar{\lambda})\xi_j)_{\mathfrak{S}} = 0$ holds for all k and all $\xi_j \in H$. (Here $k \in \mathbb{N}$ if $\mathbb{k} = \infty$, and $1 \leq k \leq \mathbb{k}$ if \mathbb{k} is finite.)*

Proof. The proof follows from the equality $\mathcal{R}(L_{10} - \lambda E) \oplus \ker(L_{10}^* - \bar{\lambda} E) = \mathfrak{S}_1$ and Lemma 3.4. \square

Remark 3.6. *In [15], [16], to each element $\mathcal{E}_k \in \mathbb{M}$ was assigned the operator function v_k in the following way. If \mathcal{E}_k is the interval, $\mathcal{E}_k = \mathcal{J}_k = (\alpha_k, \beta_k) \in \mathbb{J}$, then $v_k(t, \lambda) = \vartheta_k(t, \lambda) = \mathfrak{X}_{[a, b] \setminus \mathcal{S}_m} w_k(t, \lambda)$. If \mathcal{E}_k is a single-point set, $\mathcal{E}_k = \{\tau_k\}$, then*

$$v_k(t, \lambda) = u_n(t, \lambda, \tau_k) \mathfrak{X}_{\{\tau_k\}}(t) w_n(\tau_k, \lambda) + \mathfrak{X}_{\{\tau_k\}}(t) w_n(\tau_k, \lambda), \tag{16}$$

where $\tau_k \in (\mathcal{S}_m \setminus \mathcal{S}_0) \cap \mathcal{J}_n$, $\mathcal{J}_n = (\alpha_n, \beta_n) \in \mathbb{J}$. The linear span of functions $t \rightarrow v_k(t, \lambda)\xi_j$ ($\xi_j \in H$) is dense in $\ker(L_{10}^* - \lambda E)$. There exist constants $\gamma_{1k} = \gamma_{1k}(\lambda)$, $\gamma_{2k} = \gamma_{2k}(\lambda)$ such that the inequality $\gamma_{1k} \|v_k(\cdot, 0)x\|_{\mathfrak{S}} \leq \|v_k(\cdot, \lambda)x\|_{\mathfrak{S}} \leq \gamma_{2k} \|v_k(\cdot, 0)x\|_{\mathfrak{S}}$ holds for all $x \in H$ (see [15]).

Lemma 3.7. *There exist constants $\gamma_1 = \gamma_{1k}(\lambda_1, \lambda_2)$, $\gamma_2 = \gamma_{2k}(\lambda_1, \lambda_2) > 0$ such that the inequality*

$$\gamma_{1k} \|\vartheta_k(\cdot, \lambda_1)x\|_{\mathfrak{S}} \leq \|\vartheta_k(\cdot, \lambda_2)x\|_{\mathfrak{S}} \leq \gamma_{2k} \|\vartheta_k(\cdot, \lambda_1)x\|_{\mathfrak{S}} \tag{17}$$

holds for all $x \in H$.

Proof. Using Lemma 2.2 and (6), we obtain

$$W(t, \lambda)x_0 = W(t, 0)x_0 - W(t, 0) i \int_a^t W^*(s, 0) d\mathbf{m}_0(s) \lambda W(s, \lambda)x_0, \quad x_0 \in H, \tag{18}$$

$$W(t, 0)x_0 = W(t, \lambda)x_0 + W(t, \lambda) i \int_a^t W^*(\xi, \bar{\lambda}) d\mathbf{m}_0(s) \lambda W(s, 0)x_0, \quad x_0 \in H. \tag{19}$$

Suppose that ϑ_k has form (13). Using (10), (18), (19), we get

$$\vartheta_k(t, \lambda)x_0 = \vartheta_k(t, 0)x_0 - \vartheta_k(t, 0) i \int_{\alpha_k}^t \vartheta_k^*(s, 0) d\mathbf{m}_0(s) \lambda \vartheta_k(s, \lambda)x_0, \quad x_0 \in H, \tag{20}$$

$$\vartheta_k(t, 0)x_0 = \vartheta_k(t, \lambda)x_0 + \vartheta_k(t, \lambda) i \int_{\alpha_k}^t \vartheta_k^*(\xi, \bar{\lambda}) d\mathbf{m}_0(s) \lambda \vartheta_k(s, 0)x_0, \quad x_0 \in H. \tag{21}$$

It follows from (3), (7), (20), (21) that there exist constants $\gamma_3 > 0, \gamma_4 > 0$ such that

$$\gamma_3 \|\Psi_{\mathbf{m}}^{1/2}(t)\vartheta_k(t, 0)x\| \leq \|\Psi_{\mathbf{m}}^{1/2}(t)\vartheta_k(t, \lambda)x\| \leq \gamma_4 \|\Psi_{\mathbf{m}}^{1/2}(t)\vartheta_k(t, 0)x\|$$

for all $x \in H$, all $t \in [a, b_0]$, and all $\lambda \in C \subset \mathbb{C}$ (C is a compact set). Equalities (3), (20), (21) imply (17) in the case when ϑ_k has form (13).

Suppose that ϑ_k has form (14). Using (14), (15), (12), we get

$$\|\vartheta_k(\cdot, \lambda)x\|_{\mathfrak{S}}^2 = \|u_n(\cdot, \lambda, \tau_k)\lambda^{-1}\mathfrak{X}_{\{\tau_k\}}(\cdot)x\|_{\mathfrak{S}}^2 + \|\lambda^{-1}\mathfrak{X}_{\{\tau_k\}}(\cdot)x\|_{\mathfrak{S}}^2 \geq \|\lambda^{-1}\mathfrak{X}_{\{\tau_k\}}(\cdot)x\|_{\mathfrak{S}}^2 \geq \gamma_5 \|\mathfrak{X}_{\{\tau_k\}}(\cdot)x\|_{\mathfrak{S}}^2, \quad \gamma_5 = \gamma_5(\lambda) > 0. \tag{22}$$

On the other hand, using (12), we obtain

$$\|\vartheta_k(\cdot, \lambda)x\|_{\mathfrak{S}} \leq \|u_n(\cdot, \lambda, \tau_k)\lambda^{-1}\mathfrak{X}_{\{\tau_k\}}(\cdot)x\|_{\mathfrak{S}} + \|\lambda^{-1}\mathfrak{X}_{\{\tau_k\}}(\cdot)x\|_{\mathfrak{S}} \leq \gamma_6 \|\mathfrak{X}_{\{\tau_k\}}(\cdot)x\|_{\mathfrak{S}}, \quad \gamma_6 = \gamma_6(\lambda) > 0, \tag{23}$$

for all $x \in H$, all $t \in [a, b_0]$, and all $\lambda \neq 0$. Inequalities (22), (23) imply (17) in the case when ϑ_k has form (14). The lemma is proved. \square

Lemma 3.8. *There exist constants $\delta_{1k} = \delta_{1k}(\lambda), \delta_{2k} = \delta_{2k}(\lambda) > 0$ such that the inequality*

$$\delta_{1k} \|\vartheta_k(\cdot, \lambda)x\|_{\mathfrak{S}} \leq \|v_k(\cdot, \lambda)x\|_{\mathfrak{S}} \leq \delta_{2k} \|\vartheta_k(\cdot, \lambda)x\|_{\mathfrak{S}} \tag{24}$$

holds for all $x \in H$.

Proof. According to Remark 3.6, $v_k = \vartheta_k$ if \mathcal{E}_k is the interval, $\mathcal{E}_k = \mathcal{J}_k = (\alpha_k, \beta_k) \in \mathbb{J}$. Obviously, inequality (24) holds in this case. Suppose that v_k, ϑ_k have form (16), (14), respectively. Using (16), (12), (7), (10), we obtain

$$\|v_k(\cdot, \lambda)x\|_{\mathfrak{S}}^2 = \|u_n(\cdot, \lambda, \tau_k)\mathfrak{X}_{\{\tau_k\}}(\cdot)w_n(\tau_k, \lambda)x\|_{\mathfrak{S}}^2 + \|\mathfrak{X}_{\{\tau_k\}}(\cdot)w_n(\tau_k, \lambda)x\|_{\mathfrak{S}}^2 \geq \delta_{3k} \|\mathfrak{X}_{\{\tau_k\}}(\cdot)x\|_{\mathfrak{S}}^2, \quad \delta_{3k} = \delta_{3k}(\lambda) > 0. \tag{25}$$

On the other hand, using (12), we get

$$\|v_k(\cdot, \lambda)x\|_{\mathfrak{S}} \leq \|u_n(\cdot, \lambda, \tau_k)\mathfrak{X}_{\{\tau_k\}}(\cdot)w_n(\tau_k, \lambda)x\|_{\mathfrak{S}} + \|\mathfrak{X}_{\{\tau_k\}}(\cdot)w_n(\tau_k, \lambda)x\|_{\mathfrak{S}} \leq \delta_{4k} \|\mathfrak{X}_{\{\tau_k\}}(\cdot)x\|_{\mathfrak{S}}, \quad \delta_{4k} = \delta_{4k}(\lambda) > 0, \tag{26}$$

for all $x \in H$. Inequalities (25), (26), and (22), (23) imply (24) in the case when v_k, ϑ_k have form (16), (14), respectively. The lemma is proved. \square

We fix some point $\lambda_0 \in \mathbb{C}$ and by $Q_{k,0}$ denote a set $x \in H$ such that the functions $t \rightarrow \vartheta_k(t, \lambda_0)x$ are identical with zero in \mathfrak{S} . We put $Q_k = H \ominus Q_{k,0}$. On the linear space Q_k we introduce a norm $\|\cdot\|_-$ by the equality

$$\|\xi_k\|_- = \|\vartheta_k(\cdot, \lambda_0)\xi_k\|_{\mathfrak{S}}, \quad \xi_k \in Q_k. \tag{27}$$

By Q_k^- denote the completion of Q_k with respect to norm (27). Using (7), (13), (14), we get the inequality $\|\xi_k\|_- \leq \gamma \|\xi_k\|$, where $\gamma > 0$. Hence the space Q_k^- can be treated as a space with a negative norm with respect to Q_k ([4, ch. 1], [18, ch.2]). By Q_k^+ denote the associated space with a positive norm. The definition of spaces with positive and negative norms implies that $Q_k^+ \subset Q_k \subset Q_k^-$. By $(\cdot, \cdot)_+$ and $\|\cdot\|_+$ we denote the scalar product and the norm in Q_k^+ , respectively.

Remark 3.9. *It follows from Remark 3.6, Lemmas 3.7, 3.8 that the set $Q_{k,0}$ will not change if the function $\vartheta_k(\cdot, \lambda_0)$ is replaced by $\vartheta_k(\cdot, \lambda)$ or by $v_k(\cdot, \lambda)$ in the definition of $Q_{k,0}$. Moreover, with such replacement, the space Q_k^- will not change in the following sense: the set Q_k^- will not change, and the norm in it will be replaced by the equivalent one. The similar statement holds for the space Q_k^+ .*

Suppose that a sequence $\{x_{kn}\}$, $x_{kn} \in Q_k$, converges in the space Q_k^- to $x_0 \in Q_k^-$ as $n \rightarrow \infty$. Then the sequence $\{\vartheta_k(\cdot, \lambda)x_{kn}\}$ is fundamental in \mathfrak{S}_1 . Therefore this sequence converges to some element in \mathfrak{S}_1 . By $\vartheta_k(\cdot, \lambda)x_0$ we denote this element.

Let $\widetilde{Q}_N^- = Q_1^- \times \dots \times Q_N^-$ ($\widetilde{Q}_N^+ = Q_1^+ \times \dots \times Q_N^+$) be the Cartesian product of the first N sets Q_k^- (Q_k^+ , respectively) and let $V_N(t, \lambda) = (\vartheta_1(t, \lambda), \dots, \vartheta_N(t, \lambda))$ be the operator one-row matrix. It is convenient to treat elements from \widetilde{Q}_N^- as one-column matrices, and to assume that $V_N(t, \lambda)\widetilde{\xi}_N = \sum_{k=1}^N \vartheta_k(t, \lambda)\xi_k$, where $\widetilde{\xi}_N = \text{col}(\xi_1, \dots, \xi_N) \in \widetilde{Q}_N^-$, $\xi_k \in Q_k^-$. Let $\ker_k(\lambda)$ be a linear space of functions $t \rightarrow \vartheta_k(t, \lambda)\xi_k$, $\xi_k \in Q_k^-$. The space $\ker_k(\lambda)$ is closed in \mathfrak{S} . We denote $\mathcal{K}_N(\lambda) = \ker_1(\lambda) + \dots + \ker_N(\lambda)$. Obviously, $\mathcal{K}_{N_1}(\lambda) \subset \mathcal{K}_{N_2}(\lambda)$ for $N_1 < N_2$. By $V_N(\lambda)$ denote the operator $\widetilde{\xi}_N \rightarrow V_N(\cdot, \lambda)\widetilde{\xi}_N$, where $\widetilde{\xi}_N \in \widetilde{Q}_N^-$. The operator $V_N(\lambda)$ maps continuously and one-to-one \widetilde{Q}_N^- onto $\mathcal{K}_N(\lambda) \subset \mathfrak{S}_1 \subset \mathfrak{S}$.

Let Q_-, Q_+, Q be linear spaces of sequences, respectively, $\widetilde{\eta} = \{\eta_k\}$, $\widetilde{\varphi} = \{\varphi_k\}$, $\widetilde{\xi} = \{\xi_k\}$, where $\eta_k \in Q_k^-$, $\varphi_k \in Q_k^+$, $\xi_k \in Q_k$; $k \in \mathbb{N}$ if $\mathbb{k} = \infty$, and $1 \leq k \leq \mathbb{k}$ if \mathbb{k} is finite; \mathbb{k} is the number of elements in \mathbb{M} . We assume that the series $\sum_{k=1}^{\infty} \|\eta_k\|_-^2$, $\sum_{k=1}^{\infty} \|\varphi_k\|_+^2$, $\sum_{k=1}^{\infty} \|\xi_k\|^2$ converge if $\mathbb{k} = \infty$. These spaces become Hilbert spaces if we introduce the scalar products by the formulas

$$(\widetilde{\eta}, \widetilde{\zeta})_- = \sum_{k=1}^{\mathbb{k}} (\eta_k, \zeta_k)_-, \quad \widetilde{\eta}, \widetilde{\zeta} \in Q_-; \quad (\widetilde{\varphi}, \widetilde{\psi})_+ = \sum_{k=1}^{\mathbb{k}} (\varphi_k, \psi_k)_+, \quad \widetilde{\varphi}, \widetilde{\psi} \in Q_+; \quad (\widetilde{\xi}, \widetilde{\sigma}) = \sum_{k=1}^{\mathbb{k}} (\xi_k, \sigma_k), \quad \widetilde{\xi}, \widetilde{\sigma} \in Q.$$

The spaces Q_+, Q_- can be treated as spaces with positive and negative norms with respect to Q ([4, ch. 1], [18, ch.2]). So, $Q_+ \subset Q \subset Q_-$ and $\gamma_1 \|\widetilde{\varphi}\|_- \leq \|\widetilde{\varphi}\| \leq \gamma_2 \|\widetilde{\varphi}\|_+$, where $\widetilde{\varphi} \in Q_+$, $\gamma_1, \gamma_2 > 0$. The "scalar product" $(\widetilde{\eta}, \widetilde{\varphi})$ is defined for all $\widetilde{\varphi} \in Q_+$, $\widetilde{\eta} \in Q_-$. If $\widetilde{\eta} \in Q$, then $(\widetilde{\eta}, \widetilde{\varphi})$ coincides with the scalar product in Q .

Let $\mathcal{M} \subset Q_-$ be a set of sequences vanishing starting from a certain number (its own for each sequence). The set \mathcal{M} is dense in the space Q_- . The operator $V_N(\lambda)$ is the restriction of $V_{N+1}(\lambda)$ to \widetilde{Q}_N^- . By $V'(\lambda)$ denote an operator in \mathcal{M} such that $V'(\lambda)\widetilde{\eta} = V_N(\lambda)\widetilde{\eta}_N$ for all $N \in \mathbb{N}$, where $\widetilde{\eta} = (\widetilde{\eta}_N, 0, \dots)$, $\widetilde{\eta}_N \in \widetilde{Q}_N^-$. The operator $V'(\lambda)$ admits an extension by continuity to the space Q_- . By $V(\lambda)$ denote the extended operator. This operator maps continuously and one-to-one Q_- onto $\ker(L_{10}^* - \lambda E) \subset \mathfrak{S}_1 \subset \mathfrak{S}$. Moreover, we denote $\widetilde{V}(t, \lambda)\widetilde{\eta} = (V(\lambda)\widetilde{\eta})(t)$, where $\widetilde{\eta} = \{\eta_k\} \in Q_-$.

Remark 3.10. The operator $V(\lambda)$ is constructed by the functions ϑ_k (13), (14) in the same way that an operator $V(\lambda)$ is constructed by the functions v_k in [15] (see Remark 3.6). The operators $V(\lambda)$, $V(\lambda)$ have the same properties formulated in Lemma 3.11.

Lemma 3.11. The operator $V(\lambda)$ maps Q_- onto $\ker(L_{10}^* - \lambda E)$ continuously and one to one. A function z belongs to $\ker(L_{10}^* - \lambda E)$ if and only if there exists an element $\widetilde{\eta} = \{\eta_k\} \in Q_-$ such that $z(t) = (V(\lambda)\widetilde{\eta})(t) = \widetilde{V}(t, \lambda)\widetilde{\eta}$. The adjoint operator $V^*(\lambda)$ maps \mathfrak{S} onto Q_+ continuously, and acts by the formula

$$V^*(\lambda)f = \int_a^{b_0} \widetilde{V}^*(t, \lambda) d\mathbf{m}(t) f(t), \tag{28}$$

and $\ker V^*(\lambda) = \mathfrak{S}_0 \oplus \mathcal{R}(L_{10} - \bar{\lambda}E)$. Moreover, $V^*(\lambda)$ maps $\ker(L_{10}^* - \lambda E)$ onto Q_+ one to one.

Proof. The proof repeats verbatim the corresponding proof from [15] with the change of $V(\lambda)$ to $V(\lambda)$. \square

4. The description of generalized resolvents

Following Theorems 4.1 and 4.2 are proved in [15] for functions v_k (see Remark 3.6). For functions ϑ_k (13), (14), the proof is the same. We have changed some designations from [15] to shorten the record.

Theorem 4.1. A pair $\{\widetilde{y}, \widetilde{f}\} \in \mathfrak{S} \times \mathfrak{S}$ belongs to $L_0^* - \lambda E$ if and only if there exist a pair $\{\widehat{y}, \widehat{f}\} \in \mathfrak{S} \times \mathfrak{S}$, functions $y_0, y'_0 \in \mathfrak{S}_0, y, f \in \mathfrak{S}_1$ and an element $\widetilde{\eta} \in \mathcal{Q}_-$ such that the pairs $\{\widetilde{y}, \widetilde{f}\}, \{\widehat{y}, \widehat{f}\}$ are identical in $\mathfrak{S} \times \mathfrak{S}$ and the equalities

$$\begin{aligned} \widehat{y} &= y_0 + y, \widehat{f} = y'_0 + f, \\ y(t) &= \widetilde{V}(t, \lambda)\widetilde{\eta} - \sum_{k=1}^{\mathbb{k}_1} \mathfrak{X}_{[\alpha_k, \beta_k) \setminus \mathcal{S}_m} w_k(t, \lambda) i \int_{\alpha_k}^t w_k^*(s, \bar{\lambda}) d\mathbf{m}(s) f(s) \end{aligned} \tag{29}$$

hold, where the series in (29) converges in $\mathfrak{S}_1, \mathbb{k}_1$ is the number of intervals $\mathcal{J}_k \in \mathbb{J}$.

Theorem 4.2. Let $T(\lambda)$ be a linear relation such that $L_{10} - \lambda E \subset T(\lambda) \subset L_{10}^* - \lambda E$ and $\lambda \neq 0$. The relation $T(\lambda)$ is continuously invertible in the space \mathfrak{S}_1 if and only if there exists a bounded operator $M(\lambda) : \mathcal{Q}_+ \rightarrow \mathcal{Q}_-$ such that the following equality holds for any pair $\{y, f\} \in T(\lambda)$ and for any λ

$$\begin{aligned} y(t) = y(t, \lambda) &= \int_a^b \widetilde{V}(t, \lambda) M(\lambda) \widetilde{V}^*(s, \bar{\lambda}) d\mathbf{m}(s) f(s) + \\ &+ 2^{-1} \sum_{k=1}^{\mathbb{k}_1} \int_a^b \mathfrak{X}_{[\alpha_k, \beta_k) \setminus \mathcal{S}_m}(t) w_k(t, \lambda) \operatorname{sgn}(s - t) i \int_{\alpha_k}^s w_k^*(s, \bar{\lambda}) d\mathbf{m}(s) \mathfrak{X}_{[\alpha_k, \beta_k) \setminus \mathcal{S}_m}(s) f(s) - \lambda^{-1} \sum_{k=1}^{\mathbb{k}_1} \mathfrak{X}_{\mathcal{S}_m \cap (\alpha_k, \beta_k)}(t) f(t). \end{aligned} \tag{30}$$

Lemma 4.3. In Theorem 4.2, the function $\lambda \rightarrow T^{-1}(\lambda)f$ is holomorphic for any $f \in \mathfrak{S}_1$ at a point λ_1 ($\lambda_1 \neq 0$) if and only if the function $\lambda \rightarrow M(\lambda)\widetilde{x}$ is holomorphic for any element $\widetilde{x} \in \mathcal{Q}_+$ at the same point λ_1 .

Proof. First we assume that the function $\lambda \rightarrow M(\lambda)\widetilde{x}$ is holomorphic. Then all functions on the right side of equality (30) are holomorphic. Consequently, the function $y = T^{-1}(\lambda)f$ is holomorphic for any $f \in \mathfrak{S}_1$.

Now suppose that $y = T^{-1}(\lambda)f$ is a holomorphic function for any $f \in \mathfrak{S}_1$. Let us prove that the function $\lambda \rightarrow M(\lambda)\widetilde{x}$ is holomorphic for any $\widetilde{x} \in \mathcal{Q}_+$. We use the notation from Theorem 4.2. Moreover, we denote $S(\lambda) = M(\lambda)\mathbf{V}^*(\bar{\lambda})$. According to Lemma 3.11, $\mathbf{V}^*(\bar{\lambda})f$ is defined by equality (28). It follows from the holomorphicity of the function $\lambda \rightarrow T^{-1}(\lambda)$ that the function $\lambda \rightarrow \mathbf{V}(\lambda)S(\lambda)f$ is holomorphic. Using Lemma 3.11, we obtain that the function $\lambda \rightarrow S(\lambda)f$ is holomorphic. Now the holomorphicity of the function $\lambda \rightarrow M(\lambda)$ follows from Lemma 4.4 that is formulated after the proof of this lemma. In Lemma 4.4 it should be taken that $\mathcal{B}_1 = \mathfrak{S}_1, \mathcal{B}_2 = \mathcal{Q}_+, \mathcal{B}_3 = \mathcal{Q}_-, T_1(\lambda) = \mathbf{V}^*(\bar{\lambda}), T_2(\lambda) = M(\lambda), T_3(\lambda) = S(\lambda)$. The lemma is proved. \square

Lemma 4.4. [8]. Let $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ be Banach spaces. Suppose bounded operators $T_3(\lambda) : \mathcal{B}_1 \rightarrow \mathcal{B}_3, T_1(\lambda) : \mathcal{B}_1 \rightarrow \mathcal{B}_2, T_2(\lambda) : \mathcal{B}_2 \rightarrow \mathcal{B}_3$ satisfy the equality $T_3(\lambda) = T_2(\lambda)T_1(\lambda)$ for every fixed λ belonging to some neighborhood of a point λ_1 and suppose the range of operator $T_1(\lambda_1)$ coincides with \mathcal{B}_2 . If the functions $T_1(\lambda), T_3(\lambda)$ are strongly differentiable at the point λ_1 , then the function $T_2(\lambda)$ is are strongly differentiable at λ_1 .

It follows from Lemma 3.11 that for any element $\widetilde{x} \in \mathcal{Q}_+$ there exists a function $f \in \mathfrak{S}_1$ such that $\mathbf{V}^*(\bar{\lambda})f = \widetilde{x}$. We denote

$$z(t) = z(t, f, \lambda) = \widetilde{V}(t, \lambda) M(\lambda) \widetilde{x} - \sum_{n=1}^{\mathbb{k}_1} 2^{-1} \mathfrak{X}_{[\alpha_n, \beta_n) \setminus \mathcal{S}_m}(t) w_n(t, \lambda) i \int_{\alpha_n}^{\beta_n} w_n^*(s, \bar{\lambda}) d\mathbf{m}_0(s) f(s), \tag{31}$$

where $\widetilde{x} = \mathbf{V}^*(\bar{\lambda})f$. It follows from Lemmas 3.4, 3.11 that $z \in \ker(L_{10}^* - \lambda E)$.

Theorem 4.5. Suppose that the relation $T(\lambda)$ satisfy the conditions of Theorem 4.2, and $T(\lambda)$ is continuously invertible, and $R(\lambda) = T^{-1}(\lambda)$, and $\operatorname{Im} \lambda \neq 0$. Then the equality

$$(\operatorname{Im} \lambda)^{-1} \operatorname{Im}(M(\lambda)\widetilde{x}, \widetilde{x}) - (z, z)_m = (\operatorname{Im} \lambda)^{-1} \operatorname{Im}(R(\lambda)f, f)_m - (R(\lambda)f, R(\lambda)f)_m \tag{32}$$

holds for all $f \in \mathfrak{S}_1$ and $\widetilde{x} = \mathbf{V}^*(\bar{\lambda})f \in \mathcal{Q}_+$.

Proof. Using (30), we get

$$\begin{aligned}
 y(t) &= (R(\lambda)f)(t) = \widetilde{V}(t, \lambda)M(\lambda) \int_a^b \widetilde{V}^*(s, \bar{\lambda})d\mathbf{m}(s)f(s) + \\
 &+ \sum_{n=1}^{k_1} \left(-2^{-1} \mathfrak{X}_{[\alpha_n, \beta_n] \setminus \mathcal{S}_m}(t)w_n(t, \lambda) iJ \int_{\alpha_n}^t w_n^*(s, \bar{\lambda})d\mathbf{m}_0(s)f(s) + 2^{-1} \mathfrak{X}_{[\alpha_n, \beta_n] \setminus \mathcal{S}_m}(t)w_n(t, \lambda) iJ \int_t^{\beta_n} w_n^*(s, \bar{\lambda})d\mathbf{m}_0(s)f(s) \right) - \\
 &- \lambda^{-1} \sum_{n=1}^{k_1} \mathfrak{X}_{\mathcal{S}_m \cap (\alpha_n, \beta_n)}(t)f(t). \quad (33)
 \end{aligned}$$

By standard transformations, equality (33) is reduced to the form

$$\begin{aligned}
 y(t) &= \widetilde{V}(t, \lambda)M(\lambda) \int_a^b \widetilde{V}^*(s, \bar{\lambda})d\mathbf{m}(s)f(s) - \sum_{n=1}^{k_1} \mathfrak{X}_{[\alpha_n, \beta_n] \setminus \mathcal{S}_m}(t)w_n(t, \lambda) iJ \int_{\alpha_n}^t w_n^*(s, \bar{\lambda})d\mathbf{m}_0(s)f(s) + \\
 &+ \sum_{n=1}^{k_1} 2^{-1} \mathfrak{X}_{[\alpha_n, \beta_n] \setminus \mathcal{S}_m}(t)w_n(t, \lambda) iJ \int_{\alpha_n}^{\beta_n} w_n^*(s, \bar{\lambda})d\mathbf{m}_0(s)f(s) - \lambda^{-1} \sum_{n=1}^{k_1} \mathfrak{X}_{\mathcal{S}_m \cap (\alpha_n, \beta_n)}(t)f(t). \quad (34)
 \end{aligned}$$

We fix an interval $(\alpha_n, \beta_n) \in \mathbb{J}$ and denote $f_n = \mathfrak{X}_{[\alpha_n, \beta_n]}f$. Let $\mathbb{M}_n \subset \mathbb{M}$ be a set consisting of the interval $(\alpha_n, \beta_n) \in \mathbb{J}$ and single-point sets $\{\tau\}$, where $\tau \in \mathcal{S}_m \cap (\alpha_n, \beta_n)$. Suppose \bar{k}_n is a natural number equal to the number of elements in \mathbb{M}_n if this number is finite or \bar{k}_n is the symbol ∞ if the number of elements in \mathbb{M}_n is infinite.

We arrange the elements of \mathbb{M}_n in the form of a finite or infinite sequence and denote these elements by \mathcal{E}_{nk} , where k is any natural number if the number of elements in \mathbb{M}_n is infinite, and $1 \leq k \leq \bar{k}_n$ if the number of elements in \mathbb{M}_n is finite; \mathcal{E}_{n1} is the interval (α_n, β_n) , \mathcal{E}_{nk} ($k \geq 2$) is the single-point set $\{\tau_k\}$, $\tau_k \in \mathcal{S}_m \cap (\alpha_n, \beta_n)$.

To each element $\mathcal{E}_{nk} \in \mathbb{M}_n$ assign the operator function ϑ_{nk} in the following way (see also (10), (13), (14)). If $k = 1$, then

$$\vartheta_{n1}(t, \lambda) = \mathfrak{X}_{[a, b] \setminus \mathcal{S}_m} w_n(t, \lambda) = \mathfrak{X}_{[\alpha_n, \beta_n] \setminus \mathcal{S}_m} w_n(t, \lambda), \quad w_n(t, \lambda) = \mathfrak{X}_{[\alpha_n, \beta_n]} W(t, \lambda) W^{-1}(\alpha_n, \lambda); \quad (35)$$

if $k \geq 2$, then

$$\begin{aligned}
 \vartheta_{nk}(t, \lambda) &= \lambda^{-1} u_n(t, \lambda, \tau_k) + \lambda^{-1} \mathfrak{X}_{\{\tau_k\}}(t) = -\mathfrak{X}_{[\alpha_n, \beta_n] \setminus \mathcal{S}_m} w_n(t, \lambda) iJ \int_{\alpha_n}^t w_n^*(s, \bar{\lambda})d\mathbf{m}(s) \mathfrak{X}_{\{\tau_k\}}(s) + \lambda^{-1} \mathfrak{X}_{\{\tau_k\}}(t) = \\
 &= \begin{cases} 0 & \text{for } t < \tau_k, \\ \lambda^{-1} \mathfrak{X}_{\{\tau_k\}}(t) & \text{for } t = \tau_k, \\ -\mathfrak{X}_{[\alpha_n, \beta_n] \setminus \mathcal{S}_m} w_n(t, \lambda) iJ \mathfrak{X}_{(\tau_k, b]} \mathfrak{X}_{[\alpha_n, \beta_n] \setminus \mathcal{S}_m} w_n^*(t, \bar{\lambda}) \mathbf{m}(\{\tau_k\}) & \text{for } t > \tau_k. \end{cases} \quad (36)
 \end{aligned}$$

Using (35), (36), we get

$$\vartheta_{n1}^*(t, \bar{\lambda}) = \mathfrak{X}_{[\alpha_n, \beta_n] \setminus \mathcal{S}_m} w_n^*(t, \bar{\lambda}); \quad \vartheta_{nk}^*(t, \bar{\lambda}) = \begin{cases} 0 & \text{for } t < \tau_k, \\ \lambda^{-1} \mathfrak{X}_{\{\tau_k\}}(t) & \text{for } t = \tau_k, \\ \mathbf{m}(\{\tau_k\}) w_n(\tau_k, \lambda) iJ \mathfrak{X}_{(\tau_k, b]} \mathfrak{X}_{[\alpha_n, \beta_n] \setminus \mathcal{S}_m} w_n^*(t, \bar{\lambda}) & \text{for } t > \tau_k. \end{cases}$$

We denote $\mathbf{V}^*(\bar{\lambda})f_n = \bar{x}_n$ and

$$x_{n1} = \int_{\alpha_n}^{\beta_n} \vartheta_{n1}^*(s, \bar{\lambda})d\mathbf{m}(s)f_n(s) = \int_{\alpha_n}^{\beta_n} w_n^*(s, \bar{\lambda})d\mathbf{m}_0(s)f_n(s), \quad (37)$$

$$\begin{aligned}
 x_{nk} &= \int_{\alpha_n}^{\beta_n} \vartheta_{nk}^*(s, \bar{\lambda})d\mathbf{m}(s)f_n(s) = \\
 &= \lambda^{-1} \int_{\{\tau_k\}} d\mathbf{m}(s) \mathfrak{X}_{\{\tau_k\}}(s) f(\tau_k) + \mathbf{m}(\{\tau_k\}) w_n(\tau_k, \lambda) iJ \int_{\tau_k}^{\beta_n} w_n^*(s, \bar{\lambda})d\mathbf{m}_0(s)f_n(s), \quad k \geq 2. \quad (38)
 \end{aligned}$$

It follows from (28) and Lemma 3.11 that $\tilde{x}_n \in \mathcal{Q}_+ \subset \mathcal{Q} \subset \mathcal{Q}_-$ is a sequence $\tilde{x}_n = \{x_j\}$ with elements x_{n1}, x_{nk} and zeros. Suppose $p_j : \mathcal{Q}_- \rightarrow \mathcal{Q}_-$ is an operator defined by the formula $p_j \tilde{\eta} = \eta_j$, where $\tilde{\eta} = \{\eta_j\} \in \mathcal{Q}_-$. Let $x_{j_{n1}} = x_{n1}, x_{j_{nk}} = x_{nk}$. Then by $p_{j_{n1}}$ (by $p_{j_{nk}}$ for $k \geq 2$) we denote the operator such that

$$p_{j_{n1}} \tilde{\eta} = \eta_{n1} \quad (p_{j_{nk}} \tilde{\eta} = \eta_{nk} \text{ for } k \geq 2, \text{ respectively}). \tag{39}$$

Hence $p_{j_{n1}} \tilde{x}_n = x_{n1}$ and $p_{j_{nk}} \tilde{x}_n = x_{nk}$ for $k \geq 2$. Let $\mu_{n1} = p_{j_{n1}} M(\lambda) \tilde{x}_n$ and $\mu_{nk} = p_{j_{nk}} M(\lambda) \tilde{x}_n$ for $k \geq 2$. So, $M(\lambda) \tilde{x}_n, \tilde{x}_n = \sum_{k=1}^{\mathbb{k}_n} (\mu_{nk}, x_{nk})$.

Let $y_n = R(\lambda) f_n$. It follows from (34) that the equality

$$y_n(t) = \tilde{V}(t, \lambda) M(\lambda) \tilde{x}_n - \mathfrak{X}_{[\alpha_n, \beta_n] \setminus \mathcal{S}_m}(t) w_n(t, \lambda) iJ \int_{\alpha_n}^t w_n^*(s, \bar{\lambda}) d\mathbf{m}_0(s) f_n(s) + 2^{-1} \mathfrak{X}_{[\alpha_n, \beta_n] \setminus \mathcal{S}_m}(t) w_n(t, \lambda) iJ \int_{\alpha_n}^{\beta_n} w_n^*(s, \bar{\lambda}) d\mathbf{m}_0(s) f_n(s) - \lambda^{-1} \mathfrak{X}_{\mathcal{S}_m \cap (\alpha_n, \beta_n)}(t) f_n(t) \tag{40}$$

holds. Using (35), (36), (40), and the equality

$$\tilde{V}(t, \lambda) M(\lambda) \tilde{x}_n = \mathfrak{X}_{[a, b] \setminus (\mathcal{S}_0 \cup [\alpha_n, \beta_n])}(t) \tilde{V}(t, \lambda) M(\lambda) \tilde{x}_n + \mathfrak{X}_{[\alpha_n, \beta_n]}(t) \tilde{V}(t, \lambda) M(\lambda) \tilde{x}_n, \tag{41}$$

we get

$$y_n(t) = \mathfrak{X}_{[a, b] \setminus (\mathcal{S}_0 \cup [\alpha_n, \beta_n])}(t) \tilde{V}(t, \lambda) M(\lambda) \tilde{x}_n + \mathfrak{X}_{[\alpha_n, \beta_n] \setminus \mathcal{S}_m}(t) w_n(t, \lambda) \mu_{n1} + \sum_{k=2}^{\mathbb{k}_n} \left(-\mathfrak{X}_{[\alpha_n, \beta_n] \setminus \mathcal{S}_m}(t) w_n(t, \lambda) iJ \int_{\alpha_n}^t w_n^*(s, \bar{\lambda}) d\mathbf{m}(s) \mathfrak{X}_{\{\tau_k\}}(s) \mu_{nk} + \lambda^{-1} \mathfrak{X}_{\{\tau_k\}}(t) \mu_{nk} \right) - \mathfrak{X}_{[\alpha_n, \beta_n] \setminus \mathcal{S}_m}(t) w_n(t, \lambda) iJ \int_{\alpha_n}^t w_n^*(s, \bar{\lambda}) d\mathbf{m}_0(s) f_n(s) + 2^{-1} \mathfrak{X}_{[\alpha_n, \beta_n] \setminus \mathcal{S}_m}(t) w_n(t, \lambda) iJ x_{n1} - \sum_{k=2}^{\mathbb{k}_n} \lambda^{-1} \mathfrak{X}_{\{\tau_k\}}(t) f_n(t), \tag{42}$$

where $\tau_k \in \mathcal{S}_m \cap (\alpha_n, \beta_n)$. We denote

$$z_n(t) = \tilde{V}(t, \lambda) M(\lambda) \tilde{x}_n - 2^{-1} \mathfrak{X}_{[\alpha_n, \beta_n] \setminus \mathcal{S}_m}(t) w_n(t, \lambda) iJ \int_{\alpha_n}^{\beta_n} w_n^*(s, \bar{\lambda}) d\mathbf{m}_0(s) f_n(s).$$

By (31), so that $z(t) = \sum_{n=1}^{\mathbb{k}_1} z_n(t)$. It follows from (37), (41) that

$$z_n(t) = \mathfrak{X}_{[a, b] \setminus (\mathcal{S}_0 \cup [\alpha_n, \beta_n])}(t) \tilde{V}(t, \lambda) M(\lambda) \tilde{x}_n + \mathfrak{X}_{[\alpha_n, \beta_n] \setminus \mathcal{S}_m}(t) w_n(t, \lambda) \mu_{n1} + \sum_{k=2}^{\mathbb{k}_n} \left(-\mathfrak{X}_{[\alpha_n, \beta_n] \setminus \mathcal{S}_m}(t) w_n(t, \lambda) iJ \int_{\alpha_n}^t w_n^*(s, \bar{\lambda}) d\mathbf{m}(s) \mathfrak{X}_{\{\tau_k\}}(s) \mu_{nk} + \lambda^{-1} \mathfrak{X}_{\{\tau_k\}}(t) \mu_{nk} \right) - 2^{-1} \mathfrak{X}_{[\alpha_n, \beta_n] \setminus \mathcal{S}_m}(t) w_n(t, \lambda) iJ x_{n1}, \tag{43}$$

where $\tau_k \in \mathcal{S}_m \cap (\alpha_n, \beta_n)$. We note that $z_n \in \ker(L_{10}^* - \lambda E)$.

Using (42), (43), we obtain

$$y_n(t) - z_n(t) = -\mathfrak{X}_{[\alpha_n, \beta_n] \setminus \mathcal{S}_m}(t) w_n(t, \lambda) iJ \int_{\alpha_n}^t w_n^*(s, \bar{\lambda}) d\mathbf{m}_0(s) f_n(s) + \mathfrak{X}_{[\alpha_n, \beta_n] \setminus \mathcal{S}_m}(t) w_n(t, \lambda) iJ x_{n1} - \sum_{k=2}^{\mathbb{k}_n} \lambda^{-1} \mathfrak{X}_{\{\tau_k\}}(t) f_n(t);$$

$$y_n(t) + z_n(t) = 2\mathfrak{X}_{[a, b] \setminus (\mathcal{S}_0 \cup [\alpha_n, \beta_n])}(t) \tilde{V}(t, \lambda) M(\lambda) \tilde{x}_n + 2\mathfrak{X}_{[\alpha_n, \beta_n] \setminus \mathcal{S}_m}(t) w_n(t, \lambda) \mu_{n1} + 2 \sum_{k=2}^{\mathbb{k}_n} \left(-\mathfrak{X}_{[\alpha_n, \beta_n] \setminus \mathcal{S}_m}(t) w_n(t, \lambda) iJ \int_{\alpha_n}^t w_n^*(s, \bar{\lambda}) d\mathbf{m}(s) \mathfrak{X}_{\{\tau_k\} \cap (\alpha_n, \beta_n)}(s) \mu_{nk} + \lambda^{-1} \mathfrak{X}_{\{\tau_k\} \cap (\alpha_n, \beta_n)}(t) \mu_{nk} \right) - \mathfrak{X}_{[\alpha_n, \beta_n] \setminus \mathcal{S}_m}(t) w_n(t, \lambda) iJ \int_{\alpha_n}^t w_n^*(s, \bar{\lambda}) d\mathbf{m}_0(s) f_n(s) - \sum_{k=2}^{\mathbb{k}_n} \lambda^{-1} \mathfrak{X}_{\{\tau_k\} \cap (\alpha_n, \beta_n)}(t) f_n(t).$$

We decompose the functions $y_n - z_n, y_n + z_n$ into terms to which Lagrange formula (4) is applicable. Let us introduce the following designations:

$$\begin{aligned} \varphi_1(t) &= -\mathfrak{X}_{\{\alpha_n, \beta_n\} \setminus S_m}(t)w_n(t, \lambda) iJ \int_{\alpha_n}^t w_n^*(s, \bar{\lambda}) d\mathbf{m}_0(s) f_n(s) + \mathfrak{X}_{\{\alpha_n, \beta_n\} \setminus S_m}(t)w_n(t, \lambda) iJ x_{n1}; \\ \varphi_2(t) &= -\sum_{k=2}^{\bar{\mathbb{K}}_n} \lambda^{-1} \mathfrak{X}_{\{\tau_k\} \cap (\alpha_n, \beta_n)}(t) f_n(t); \end{aligned} \tag{44}$$

$$\begin{aligned} \psi_1(t) &= 2\mathfrak{X}_{\{\alpha_n, \beta_n\} \setminus S_m}(t)w_n(t, \lambda) \mu_{n1} - \mathfrak{X}_{\{\alpha_n, \beta_n\} \setminus S_m}(t)w_n(t, \lambda) iJ \int_{\alpha_n}^t w_n^*(s, \bar{\lambda}) d\mathbf{m}_0(s) f_n(s); \\ \psi_2(t) &= 2\lambda^{-1} \sum_{k=2}^{\bar{\mathbb{K}}_n} \mathfrak{X}_{\{\tau_k\} \cap (\alpha_n, \beta_n)}(t) \mu_{nk} - \lambda^{-1} \sum_{k=2}^{\bar{\mathbb{K}}_n} \mathfrak{X}_{\{\tau_k\} \cap (\alpha_n, \beta_n)}(t) f_n(\tau_k); \end{aligned} \tag{45}$$

$$\psi_3(t) = -2 \sum_{k=2}^{\bar{\mathbb{K}}_n} \mathfrak{X}_{\{\alpha_n, \beta_n\} \setminus S_m} w_n(t, \lambda) iJ \int_{\alpha_n}^t w_n^*(s, \bar{\lambda}) d\mathbf{m}(s) \mathfrak{X}_{\{\tau_k\} \cap (\alpha_n, \beta_n)}(s) \mu_{nk} = \sum_{k=2}^{\bar{\mathbb{K}}_n} \psi_{3k}(t),$$

where

$$\psi_{3k}(t) = -2\mathfrak{X}_{\{\alpha_n, \beta_n\} \setminus S_m} w_n(t, \lambda) iJ \int_{\alpha_n}^t w_n^*(s, \bar{\lambda}) d\mathbf{m}(s) \mathfrak{X}_{\{\tau_k\}}(s) \mu_{nk} = \begin{cases} 0 & \text{for } t \leq \tau_k, \\ -2\mathfrak{X}_{\{\alpha_n, \beta_n\} \setminus S_m} w_n(t, \lambda) iJ w_n^*(\tau_k, \bar{\lambda}) \mathbf{m}(\{\tau_k\}) \mu_{nk} & \text{for } t > \tau_k. \end{cases}$$

Then

$$y_n - z_n = \varphi_1 + \varphi_2, \quad y_n + z_n = \psi_1 + \psi_2 + \psi_3 + 2\mathfrak{X}_{[a,b] \setminus (S_0 \cup \{\alpha_n, \beta_n\})}(t) \tilde{V}(t, \lambda) M(\lambda) \tilde{x}_n. \tag{46}$$

Since $z_n \in \ker(L_{10}^* - \lambda E), \{y_n, f_n\} \in L_{10}^* - \lambda E$, it follows that $\{y_n - z_n, f_n\} \in L_{10}^* - \lambda E, \{y_n + z_n, f_n\} \in L_{10}^* - \lambda E$. Theorem 4.1 implies that $\{\varphi_1, \mathfrak{X}_{\{\alpha_n, \beta_n\} \setminus S_m} f_n\} \in L_{10}^* - \lambda E, \{\psi_1, \mathfrak{X}_{\{\alpha_n, \beta_n\} \setminus S_m} f_n\} \in L_{10}^* - \lambda E$. By (8), so that $\varphi_2 \in \ker L_{10}^*, \psi_2 \in \ker L_{10}^*$. Therefore,

$$\begin{aligned} \{y_n - z_n, \lambda(y_n - z_n) + f_n\} &\in L_{10}^*, \quad \{y_n + z_n, \lambda(y_n + z_n) + f_n\} \in L_{10}^*, \\ \{\varphi_1, \lambda\varphi_1 + \mathfrak{X}_{\{\alpha_n, \beta_n\} \setminus S_m} f_n\} &\in L_{10}^*, \quad \{\psi_1, \lambda\psi_1 + \mathfrak{X}_{\{\alpha_n, \beta_n\} \setminus S_m} f_n\} \in L_{10}^*. \end{aligned}$$

We denote

$$u_- = \lambda(y_n - z_n) + f_n, \quad u_+ = \lambda(y_n + z_n) + f_n. \tag{47}$$

Then $\{y_n - z_n, u_-\} \in L_{10}^*, \{y_n + z_n, u_+\} \in L_{10}^*$. Moreover, $(\lambda\varphi_1 + \mathfrak{X}_{\{\alpha_n, \beta_n\} \setminus S_m} f_n, \psi_2)_{\mathbf{m}} = 0, (\varphi_2, \lambda\psi_1 + \mathfrak{X}_{\{\alpha_n, \beta_n\} \setminus S_m} f_n)_{\mathbf{m}} = 0$.

We note that for all functions $g, h \in \mathfrak{S}$ the equalities

$$(\mathfrak{X}_{[a,b] \setminus S_m} g, \mathfrak{X}_{S_m} h)_{\mathbf{m}} = 0; \quad (\mathfrak{X}_{[a,b] \setminus S_m} g, h)_{\mathbf{m}} = (\mathfrak{X}_{[a,b] \setminus S_m} g, \mathfrak{X}_{[a,b] \setminus S_m} h)_{\mathbf{m}} = (g, h)_{\mathbf{m}_0}; \quad (\mathfrak{X}_{S_m} g, h)_{\mathbf{m}} = (\mathfrak{X}_{S_m} g, \mathfrak{X}_{S_m} h)_{\mathbf{m}}$$

hold. Moreover,

$$(\mathfrak{X}_{[a,b] \setminus (S_0 \cup \{\alpha_n, \beta_n\})} \tilde{V}(\cdot, \lambda) M(\lambda) \tilde{x}_n, y_n - z_n)_{\mathbf{m}} = 0, \quad (\mathfrak{X}_{[a,b] \setminus (S_0 \cup \{\alpha_n, \beta_n\})} \tilde{V}(\cdot, \lambda) M(\lambda) \tilde{x}_n, f_n)_{\mathbf{m}} = 0.$$

Therefore by (46), (47), we get

$$\begin{aligned} (u_-, y_n + z_n)_{\mathbf{m}} - (y_n - z_n, u_+)_{\mathbf{m}} &= (\lambda(\varphi_1 + \varphi_2) + f_n, \psi_1 + \psi_2 + \psi_3)_{\mathbf{m}} - (\varphi_1 + \varphi_2, \lambda(\psi_1 + \psi_2 + \psi_3) + f_n)_{\mathbf{m}} = \\ &= (\lambda\varphi_1, \psi_1)_{\mathbf{m}_0} + (f_n, \psi_1)_{\mathbf{m}_0} + (\lambda\varphi_2, \psi_2)_{\mathbf{m}} + (f_n, \psi_2)_{\mathbf{m}} + (\lambda\varphi_1, \psi_3)_{\mathbf{m}_0} + (f_n, \psi_3)_{\mathbf{m}_0} - \\ &\quad - (\varphi_1, \lambda\psi_1)_{\mathbf{m}_0} - (\varphi_1, \lambda\psi_3)_{\mathbf{m}_0} - (\varphi_1, f_n)_{\mathbf{m}_0} - (\varphi_2, \lambda\psi_2)_{\mathbf{m}} - (\varphi_2, f_n)_{\mathbf{m}}. \end{aligned} \tag{48}$$

It follows from (44), (45) that

$$(\lambda\varphi_2, \psi_2)_m = \sum_{k=2}^{\bar{k}_n} (-\mathfrak{X}_{\{\tau_k\} \cap (\alpha_n, \beta_n)} f_n(\tau_k), \lambda^{-1} \mathfrak{X}_{\{\tau_k\} \cap (\alpha_n, \beta_n)} (2\mu_{nk} - f_n(\tau_k)))_m; \tag{49}$$

$$(f_n, \psi_2)_m = \sum_{k=2}^{\bar{k}_n} (\mathfrak{X}_{\{\tau_k\} \cap (\alpha_n, \beta_n)} f_n(\tau_k), \lambda^{-1} \mathfrak{X}_{\{\tau_k\} \cap (\alpha_n, \beta_n)} (2\mu_{nk} - f_n(\tau_k)))_m; \tag{50}$$

$$-(\varphi_2, \lambda\psi_2)_m = \sum_{k=2}^{\bar{k}_n} (\lambda^{-1} \mathfrak{X}_{\{\tau_k\} \cap (\alpha_n, \beta_n)} f_n(\tau_k), \mathfrak{X}_{\{\tau_k\} \cap (\alpha_n, \beta_n)} (2\mu_{nk} - f_n(\tau_k)))_m; \tag{51}$$

$$-(\varphi_2, f_n)_m = \sum_{k=2}^{\bar{k}_n} (\lambda^{-1} \mathfrak{X}_{\{\tau_k\} \cap (\alpha_n, \beta_n)} f_n(\tau_k), \mathfrak{X}_{\{\tau_k\} \cap (\alpha_n, \beta_n)} f_n(\tau_k))_m. \tag{52}$$

Summing (49), (50), (51), (52), we get

$$\begin{aligned} ((\lambda\varphi_2, \psi_2)_m + (f_n, \psi_2)_m - (\varphi_2, \lambda\psi_2)_m - (\varphi_2, f_n)_m) = \\ = 2 \sum_{k=2}^{\bar{k}_n} (\lambda^{-1} \mathfrak{X}_{\{\tau_k\}} f_n(\tau_k), \mathfrak{X}_{\{\tau_k\}} \mu_{nk})_m = 2 \sum_{k=2}^{\bar{k}_n} (\lambda^{-1} f_n(\tau_k), \mathbf{m}(\{\tau_k\}) \mu_{nk}), \quad \tau_k \in (\alpha_n, \beta_n). \end{aligned} \tag{53}$$

Taking into account (53), we continue equality (48)

$$\begin{aligned} (u_-, y + z)_m - (y - z, u_+)_m = [(\lambda\varphi_1 + f_n, \psi_1)_{m_0} - (\varphi_1, \lambda\psi_1 + f_n)_{m_0}] + [(\lambda\varphi_1 + f_n, \psi_3)_{m_0} - (\varphi_1, \lambda\psi_3)_{m_0}] + \\ + 2 \sum_{k=2}^{\bar{k}_2} (\lambda^{-1} f_n(\tau_k), \mathbf{m}(\{\tau_k\}) \mu_{nk}), \quad \tau_k \in (\alpha_n, \beta_n). \end{aligned} \tag{54}$$

We denote

$$\tilde{\varphi}_1(t) = w_n(t, \lambda) iJ x_{n1} - w_n(t, \lambda) iJ \int_{\alpha_n}^t w_n^*(s, \bar{\lambda}) d\mathbf{m}_0(s) f_n(s); \tag{55}$$

$$\tilde{\psi}_1(t) = 2w_n(t, \lambda) \mu_{n1} - w_n(t, \lambda) iJ \int_{\alpha_n}^t w_n^*(s, \bar{\lambda}) d\mathbf{m}_0(s) f_n(s); \tag{56}$$

$$\tilde{\psi}_{3k}(t) = \begin{cases} 0 & \text{for } t < \tau_k, \\ -2w_n(t, \lambda) iJ w_n^*(\tau_k, \bar{\lambda}) \mathbf{m}(\{\tau_k\}) \mu_{nk} & \text{for } t \geq \tau_k \end{cases}; \quad \tilde{\psi}_3(t) = \sum_{k=2}^{\bar{k}_n} \tilde{\psi}_{3k}(t),$$

where $\tau_k \in \mathcal{S}_m \cap (\alpha_n, \beta_n)$. Then

$$\tilde{\varphi}_1(t) - \varphi_1(t) = \mathfrak{X}_{[\alpha_n, \beta_n] \cap \mathcal{S}_m}(t) \tilde{\varphi}_1(t), \quad \tilde{\psi}_1(t) - \psi_1(t) = \mathfrak{X}_{[\alpha_n, \beta_n] \cap \mathcal{S}_m}(t) \tilde{\psi}_1(t), \quad \tilde{\psi}_{3k}(t) - \psi_{3k}(t) = \mathfrak{X}_{\{\tau_k\}}(t) \tilde{\psi}_{3k}(t).$$

This implies that

$$(\varphi_1, \psi_1)_{m_0} = (\tilde{\varphi}_1, \tilde{\psi}_1)_{m_0}, \quad (\varphi_1, f_n)_{m_0} = (\tilde{\varphi}_1, f_n)_{m_0}, \quad (f_n, \psi_1)_{m_0} = (f_n, \tilde{\psi}_1)_{m_0}, \quad (\varphi_1, \psi_{3k})_{m_0} = (\tilde{\varphi}_1, \tilde{\psi}_{3k})_{m_0}. \tag{57}$$

By (37), it follows that $\lim_{t \rightarrow \beta_n-0} \tilde{\varphi}_1(t) = \tilde{\varphi}_1(\beta_n) = 0$. Using (57), we continue equality (54)

$$\begin{aligned} (u_-, y_n + z_n)_m - (y_n - z_n, u_+)_m = [(\lambda\tilde{\varphi}_1 + f_n, \tilde{\psi}_1)_{m_0} - (\tilde{\varphi}_1, \lambda\tilde{\psi}_1 + f_n)_{m_0}] + \\ + \sum_{k=2}^{\bar{k}_n} [(\lambda\tilde{\varphi}_1 + f_n, \tilde{\psi}_{3k})_{m_0} - (\tilde{\varphi}_1, \lambda\tilde{\psi}_{3k})_{m_0}] + 2\lambda^{-1} \sum_{k=2}^{\bar{k}_n} (f_n(\tau_k), \mathbf{m}(\{\tau_k\}) \mu_{nk}), \quad \tau_k \in \mathcal{S}_m \cap (\alpha_n, \beta_n). \end{aligned} \tag{58}$$

Let us transform the formula $(\lambda\tilde{\varphi}_1 + f_n, \tilde{\psi}_1)_{\mathbf{m}_0} - (\tilde{\varphi}_1, \lambda\tilde{\psi}_1 + f_n)_{\mathbf{m}_0}$. It follows from Lemma 2.2 and equalities (55), (56) that

$$\tilde{\varphi}_1(t) = iJx_{n1} - iJ \int_{\alpha_n}^t d\mathbf{p}_0(s)\tilde{\varphi}_1(s) - iJ\lambda \int_{\alpha_n}^t d\mathbf{m}_0(s)\tilde{\varphi}_1(s) - iJ \int_{\alpha_n}^t d\mathbf{m}_0(s)f_n(s), \tag{59}$$

$$\tilde{\psi}_1(t) = 2\mu_{n1} - iJ \int_{\alpha_n}^t d\mathbf{p}_0(s)\tilde{\psi}_1(s) - iJ\lambda \int_{\alpha_n}^t d\mathbf{m}_0(s)\tilde{\psi}_1(s) - iJ \int_{\alpha_n}^t d\mathbf{m}_0(s)f_n(s). \tag{60}$$

We denote $\tilde{\Phi}_1 = \lambda\tilde{\varphi}_1 + f_n, \tilde{\Psi}_1 = \lambda\tilde{\psi}_1 + f_n$. Then

$$(\lambda\tilde{\varphi}_1 + f_n, \tilde{\psi}_1)_{\mathbf{m}_0} - (\tilde{\varphi}_1, \lambda\tilde{\psi}_1 + f_n)_{\mathbf{m}_0} = (\tilde{\Phi}_1, \tilde{\psi}_1)_{\mathbf{m}_0} - (\tilde{\varphi}_1, \tilde{\Psi}_1)_{\mathbf{m}_0}.$$

Using (59), (60), we obtain that the pairs $\{\tilde{\varphi}_1, \tilde{\Phi}_1\}, \{\tilde{\psi}_1, \tilde{\Psi}_1\}$ satisfy equations (61), (62), respectively (see below)

$$\tilde{\varphi}_1(t) = iJx_{n1} - iJ \int_{\alpha_n}^t d\mathbf{p}_0(s)\tilde{\varphi}_1(s) - iJ \int_{\alpha_n}^t d\mathbf{m}_0(s)\tilde{\Phi}_1(s), \tag{61}$$

$$\tilde{\psi}_1(t) = 2\mu_{n1} - iJ \int_{\alpha_n}^t d\mathbf{p}_0(s)\tilde{\psi}_1(s) - iJ \int_{\alpha_n}^t d\mathbf{m}_0(s)\tilde{\Psi}_1(s). \tag{62}$$

Therefore we can apply Lagrange formula (4) to the functions $\tilde{\varphi}_1, \tilde{\Phi}_1, \tilde{\psi}_1, \tilde{\Psi}_1$ for $c_1 = \alpha_n, c_2 = \beta_n, \mathbf{q} = \mathbf{m}_0, \mathbf{p}_1 = \mathbf{p}_2 = \mathbf{p}_0$. Using (4) and the equality $\lim_{t \rightarrow \beta_n - 0} \tilde{\varphi}_1(t) = \tilde{\varphi}_1(\beta_n) = 0$, we get

$$\begin{aligned} (\lambda\tilde{\varphi}_1 + f_n, \tilde{\psi}_1)_{\mathbf{m}_0} - (\tilde{\varphi}_1, \lambda\tilde{\psi}_1 + f_n)_{\mathbf{m}_0} &= \int_{\alpha_n}^{\beta_n} (d\mathbf{m}_0(t)\tilde{\Phi}_1(t), \tilde{\psi}_1(t)) - \int_{\alpha_n}^{\beta_n} (d\mathbf{m}_0(t)\tilde{\varphi}_1(t), \tilde{\Psi}_1(t)) = \\ &= (iJ\tilde{\varphi}_1(\beta_n), \tilde{\psi}_1(\beta_n)) - (iJ\tilde{\varphi}_1(\alpha_n), \tilde{\psi}_1(\alpha_n)) = -(iJiJx_{n1}, 2\mu_{n1}) = 2(x_{n1}, \mu_{n1}). \end{aligned} \tag{63}$$

Now let us transform the formula $(\lambda\tilde{\varphi}_1 + f_n, \tilde{\psi}_{3k})_{\mathbf{m}_0} - (\tilde{\varphi}_1, \lambda\tilde{\psi}_{3k})_{\mathbf{m}_0}$. For this we denote

$$\tilde{\psi}_{4k}(t) = -2^{-1}\tilde{\psi}_{3k}(t) = \begin{cases} 0 & \text{for } t < \tau_k, \\ w_n(t, \lambda)iJw_n^*(\tau_k, \bar{\lambda})\mathbf{m}(\{\tau_k\})\mu_{nk} & \text{for } t \geq \tau_k, \end{cases} \tag{64}$$

where $\tau_k \in \mathcal{S}_m \cap (\alpha_n, \beta_n)$. It follows from Lemma 2.2, equalities (55), (64), and (10), (11) that the equality

$$\tilde{\psi}_{4k}(t) = iJ\mathbf{m}(\{\tau_k\})\mu_{nk} - iJ \int_{\tau_k}^t d\mathbf{p}_0(s)\tilde{\psi}_{4k}(s) - iJ\lambda \int_{\tau_k}^t d\mathbf{m}_0(s)\tilde{\psi}_{4k}(s)$$

holds. We denote $\tilde{\Psi}_{4k} = \lambda\tilde{\psi}_{4k}$. Then the pair $\{\tilde{\psi}_{4k}, \tilde{\Psi}_{4k}\}$ satisfies the equation

$$\tilde{\psi}_{4k}(t) = iJ\mathbf{m}(\{\tau_k\})\mu_{nk} - iJ \int_{\tau_k}^t d\mathbf{p}_0(s)\tilde{\psi}_{4k}(s) - iJ \int_{\tau_k}^t d\mathbf{m}_0(s)\tilde{\Psi}_{4k}(s).$$

We can apply Lagrange formula (4) to the functions $\tilde{\varphi}_1, \tilde{\Phi}_1, \tilde{\psi}_{4k}, \tilde{\Psi}_{4k}$ for $c_1 = \tau_k, c_2 = \beta_n, \mathbf{q} = \mathbf{m}_0, \mathbf{p}_1 = \mathbf{p}_2 = \mathbf{p}_0$. We first note that equalities (55), (37) imply

$$\begin{aligned} \tilde{\varphi}_1(\tau_k) &= -w_n(\tau_k, \lambda)iJ \int_{\alpha_n}^{\tau_k} w_n^*(s, \bar{\lambda})d\mathbf{m}_0(s)f_n(s) + \\ &\quad + w_n(\tau_k, \lambda)iJ \int_{\alpha_n}^{\beta_n} w_n^*(s, \bar{\lambda})d\mathbf{m}_0(s)f_n(s) = w_n(\tau_k, \lambda)iJ \int_{\tau_k}^{\beta_n} w_n^*(s, \bar{\lambda})d\mathbf{m}_0(s)f_n(s). \end{aligned}$$

Now using (4) and the equality $\lim_{t \rightarrow \beta_n - 0} \tilde{\varphi}_1(t) = \tilde{\varphi}_1(\beta_n) = 0$, we get

$$\begin{aligned} & (\lambda \tilde{\varphi}_1 + f_n, \tilde{\psi}_{3k})_{\mathbf{m}_0} - (\tilde{\varphi}_1, \lambda \tilde{\psi}_{3k})_{\mathbf{m}_0} = -2[(\tilde{\Phi}_1, \tilde{\psi}_{4k})_{\mathbf{m}_0} - (\tilde{\varphi}_1, \tilde{\Psi}_{4k})_{\mathbf{m}_0}] = \\ & = -2 \left[\int_{\tau_k}^{\beta_n} (d\mathbf{m}_0(t) \tilde{\Phi}_1(t), \tilde{\psi}_{4k}(t)) - \int_{\tau_k}^{\beta_n} (d\mathbf{m}_0(t) \tilde{\varphi}_1(t), \tilde{\Psi}_{4k}(t)) \right] = -2[(iJ\tilde{\varphi}_1(\beta_n), \tilde{\psi}_{4k}(\beta_n)) - (iJ\tilde{\varphi}_1(\tau_k), \tilde{\psi}_{4k}(\tau_k))] = \\ & = -2[-iJw_n(\tau_k, \lambda) iJ \int_{\tau_k}^{\beta_n} w_n^*(s, \bar{\lambda}) d\mathbf{m}_0(s) f_n(s), iJ\mathbf{m}(\{\tau_k\})\mu_{nk}] = \\ & = 2(w_n(\tau_k, \lambda) iJ \int_{\tau_k}^{\beta_n} w_n^*(s, \bar{\lambda}) d\mathbf{m}_0(s) f_n(s), \mathbf{m}(\{\tau_k\})\mu_{nk}). \end{aligned} \tag{65}$$

Using (37), (38), (63), (65), and the equality $\mathbf{m}(\{\tau_k\}) = \mathbf{m}^*(\{\tau_k\})$, we continue equality (58):

$$\begin{aligned} & (u_-, y_n + z_n)_{\mathbf{m}} - (y_n - z_n, u_+)_{\mathbf{m}} = 2(x_{n1}, \mu_{n1}) + \\ & + 2 \sum_{k=2}^{\tilde{k}_n} \left((\lambda^{-1} f_n(\tau_k), \mathbf{m}(\{\tau_k\})\mu_{nk}) + (w_n(\tau_k, \lambda) iJ \int_{\tau_k}^{\beta_n} w_n^*(s, \bar{\lambda}) d\mathbf{m}_0(s) f_n(s), \mathbf{m}(\{\tau_k\})\mu_{nk}) \right) = 2(\tilde{x}_n, M(\lambda)\tilde{x}_n). \end{aligned} \tag{66}$$

On the other hand, using (47), we get

$$\begin{aligned} & (f_n, y_n + z_n)_{\mathfrak{S}} - (y_n - z_n, f_n)_{\mathfrak{S}} = (u_- - \lambda(y_n - z_n), y_n + z_n)_{\mathfrak{S}} - (y_n - z_n, u_+ - \lambda(y_n + z_n))_{\mathfrak{S}} = \\ & = (u_-, y_n + z_n)_{\mathfrak{S}} - (y_n - z_n, u_+)_{\mathfrak{S}} - (\lambda y_n, y_n)_{\mathfrak{S}} + (\lambda z_n, y_n)_{\mathfrak{S}} - (\lambda y_n, z_n)_{\mathfrak{S}} + \\ & + (\lambda z_n, z_n)_{\mathfrak{S}} + (y_n, \lambda y_n)_{\mathfrak{S}} - (z_n, \lambda y_n)_{\mathfrak{S}} + (y_n, \lambda z_n)_{\mathfrak{S}} - (z_n, \lambda z_n)_{\mathfrak{S}}. \end{aligned} \tag{67}$$

Combining (66) and (67), we obtain

$$\begin{aligned} & (f_n, y_n + z_n)_{\mathfrak{S}} - (y_n - z_n, f_n)_{\mathfrak{S}} = 2(\tilde{x}_n, M(\lambda)\tilde{x}_n) - (\lambda y_n, y_n)_{\mathfrak{S}} + (\lambda z_n, y_n)_{\mathfrak{S}} - \\ & - (\lambda y_n, z_n)_{\mathfrak{S}} + (\lambda z_n, z_n)_{\mathfrak{S}} + (y_n, \lambda y_n)_{\mathfrak{S}} - (z_n, \lambda y_n)_{\mathfrak{S}} + (y_n, \lambda z_n)_{\mathfrak{S}} - (z_n, \lambda z_n)_{\mathfrak{S}}. \end{aligned}$$

This implies

$$\text{Im}(f_n, y_n)_{\mathfrak{S}} = \text{Im}(\tilde{x}_n, M(\lambda)\tilde{x}_n) - (\text{Im}\lambda)[(y_n, y_n)_{\mathfrak{S}} - (z_n, z_n)_{\mathfrak{S}}]. \tag{68}$$

Using (68), we get

$$\text{Im}(f, y)_{\mathfrak{S}} = \text{Im}(\tilde{x}, M(\lambda)\tilde{x}) - (\text{Im}\lambda)[(y, y)_{\mathfrak{S}} - (z, z)_{\mathfrak{S}}].$$

Since $y = R(\lambda)f$, we obtain

$$\text{Im}(M(\lambda)\tilde{x}, \tilde{x}) - (\text{Im}\lambda)(z, z)_{\mathbf{m}} = (R(\lambda)f, f)_{\mathbf{m}} - (\text{Im}\lambda)(R(\lambda)f, R(\lambda)f)_{\mathbf{m}}. \tag{69}$$

Equality (69) implies (32). The theorem is proved. \square

Corollary 4.6. *If*

$$(\text{Im}\lambda)^{-1} \text{Im}(R(\lambda)f, f)_{\mathbf{m}} - (R(\lambda)f, R(\lambda)f)_{\mathbf{m}} \geq 0 \tag{70}$$

for all $f \in \mathfrak{S}$, then

$$(\text{Im}\lambda)^{-1} \text{Im}(M(\lambda)\tilde{x}, \tilde{x}) \geq 0 \tag{71}$$

for all $\tilde{x} \in Q_+$.

Theorem 4.7. Let $R(\lambda)$ ($\text{Im}\lambda \neq 0$) be a generalized resolvent of the relation L_{10} and $y = R(\lambda)f$. Then y has the form (30), where $M(\lambda) : \mathcal{Q}_+ \rightarrow \mathcal{Q}_-$ is the bounded operator such that $M(\bar{\lambda}) = M^*(\lambda)$, $\text{Im}\lambda \neq 0$. The function $\lambda \rightarrow M(\lambda)\tilde{x}$ is holomorphic for every $\tilde{x} \in \mathcal{Q}_+$ on the half-planes $\text{Im}\lambda \neq 0$ and inequality (71) holds for every λ ($\text{Im}\lambda \neq 0$) and for every $\tilde{x} \in \mathcal{Q}_+$.

Proof. Equality (30) follows from (2) and Theorem 4.2. The equality $R^*(\lambda) = R(\bar{\lambda})$ implies $M(\bar{\lambda}) = M^*(\lambda)$. The generalized resolvent is holomorphic. Then it follows from Lemma 4.3 that the function $M(\lambda)$ is holomorphic. Inequality (70) holds for the generalized resolvent $R(\lambda)$ (see, for example, [17]). Using Corollary 4.6, we obtain that inequality (71) holds for the function $M(\lambda)$. The theorem is proved. \square

Theorem 4.8. Suppose that the relation $T(\lambda)$ satisfies the conditions of Theorem 4.2, and $T(\lambda)$ is continuously invertible, and $R(\lambda) = T^{-1}(\lambda)$, and $M(\lambda)$ is the operator from Theorem 4.2. Assume that the function $\lambda \rightarrow M(\lambda)$ is holomorphic on the half-planes $\text{Im}\lambda \neq 0$ and $M(\lambda) = M^*(\bar{\lambda})$. Then the function $\lambda \rightarrow R(\lambda)$ is a generalized resolvent of the relation L_{10} if and only if the inequality

$$(\text{Im}\lambda)^{-1}\text{Im}M(\lambda) - \left(M^*(\lambda)\mathbf{V}^*(\lambda) + 2^{-1}i \sum_{n=1}^{k_1} p_{j_{n1}}^* J \mathfrak{X}_{[\alpha_n, \beta_n) \setminus \mathcal{S}_m} w_n^*(\cdot, \lambda) \right) \left(\mathbf{V}(\lambda)M(\lambda) - 2^{-1}i \sum_{n=1}^{k_1} \mathfrak{X}_{[\alpha_n, \beta_n) \setminus \mathcal{S}_m} w_n(\cdot, \lambda) J p_{j_{n1}} \right) \geq 0 \quad (72)$$

holds, i.e., the operator on the left side of inequality (72) is non-negative.

Proof. By \mathcal{A} denote the operator on the left side (72). Then $(\mathcal{A}\tilde{x}, \tilde{x}) = (\text{Im}\lambda)^{-1}\text{Im}(M(\lambda)\tilde{x}, \tilde{x}) - (z, z)_m$, where $\tilde{x} = \mathbf{V}^*(\bar{\lambda})f \in \mathcal{Q}_+ \subset \mathcal{Q}_-$, $f \in \mathfrak{H}_1$, z is defined by equality (31), $p_{j_{n1}}$ is the operator such that $p_{j_{n1}}\tilde{x} = x_{n1}$ (see (39)), and x_{n1} is defined by equality (37). Using (32), we get

$$(\mathcal{A}\tilde{x}, \tilde{x}) = (\text{Im}\lambda)^{-1}\text{Im}(R(\lambda)f, f)_m - (R(\lambda)f, R(\lambda)f)_m.$$

Thus, inequality (72) is equivalent to the following

$$(\text{Im}\lambda)^{-1}\text{Im}(R(\lambda)f, f)_m - (R(\lambda)f, R(\lambda)f)_m \geq 0. \quad (73)$$

It follows from Lemma 4.3 that the function $\lambda \rightarrow R(\lambda)$ is holomorphic on the half-planes $\text{Im}\lambda \neq 0$. The equality $M(\lambda) = M^*(\bar{\lambda})$ is equivalent to the equality $R(\lambda) = R^*(\bar{\lambda})$. Now the desired assertion follows from [17], where it was established that a function $\lambda \rightarrow R(\lambda)$ is a generalized resolvent if and only if this function is holomorphic on the half-planes $\text{Im}\lambda \neq 0$, $R(\lambda) = R^*(\bar{\lambda})$, and inequality (73) holds. The theorem is proved. \square

5. The example

We consider equation (1) on a segment $[0, b]$ and assume that $H = \mathbb{C}$, $J = E = 1$, $\mathbf{p} = 0$, $\mathbf{m} = \mathbf{m}_0 + \widehat{\mathbf{m}}$, where \mathbf{m}_0 is the usual Lebesgue measure (i.e., $\mathbf{m}([\alpha, \beta]) = \beta - \alpha$ for $0 \leq \alpha < \beta \leq b$; we write ds instead of $d\mathbf{m}_0(s)$), $0 < \tau < b$, $\widehat{\mathbf{m}}(\{\tau\}) = 1$ and $\widehat{\mathbf{m}}(\Delta) = 0$ for all Borel sets such that $\tau \notin \Delta$. So, $\mathcal{S}_m = \{\tau\}$. Thus equation (1) has the form

$$y(t) = x_0 - i \int_0^t d\mathbf{m}(s)f(s). \quad (74)$$

In this example, we consider the minimal operator L_0 generated by equation (74). We find a function $M(\lambda)$ corresponding to the resolvent of a self-adjoint extension of the operator L_0 and use the system of functions $\{\vartheta_k\}$ (13), (14) to find $M(\lambda)$. This function $M(\lambda)$ has the property $(\text{Im}\lambda)^{-1}\text{Im}M(\lambda) \geq 0$. In the example from [16], a function $M(\lambda)$ was found using the system of functions $\{v_k\}$ (see (16) and Remark 3.6). This function from [16] does not have the property $(\text{Im}\lambda)^{-1}\text{Im}M(\lambda) \geq 0$ (see [16]).

It follows from the definition of L_0 and (8), (74) that L_0 is an operator and if $y = L_0 f$, then

$$y(t) = -i \int_0^t f(s) ds, \quad y(b) = 0, \quad f(\tau) = 0 \Leftrightarrow y'(t) = -if(t), \quad y(0) = y(b) = 0, \quad f(\tau) = 0.$$

Since $S_0 = \{0, b\}$ and $\mathbf{m}(S_0) = 0$, we have $\mathfrak{S}_0 = \{0\}$ and $L_{10}^* = L_0^*$ in equality (9).

Equation (6) (for $x_0 = 1$) takes the form

$$W(t, \lambda) = 1 - i\lambda \int_0^t W(s, \lambda) ds, \quad \lambda \in \mathbb{C}.$$

Therefore, $W(t, \lambda) = e^{-i\lambda t}$. Obviously, if $\lambda = 0$, then $W(t, 0) = 1$. The number of intervals $\mathcal{J}_k \in \mathbb{J}$ is $\mathbb{k}_1 = 1$. We write $w(t, \lambda)$ instead of $w_1(t, \lambda)$. Then $w(t, \lambda) = \mathfrak{X}_{[0,b]} W(t, \lambda)$. Without loss of generality it can be assumed that $w(t, \lambda) = W(t, \lambda) = e^{-i\lambda t}$.

The set \mathbb{M} consists of the interval $(0, b)$ and the single-point set $\{\tau\}$. Hence the number of elements of \mathbb{M} is $\mathbb{k} = 2$. By (27), it follows that $Q_{10} = Q_{20} = \{0\}$ and $Q_1 = Q_1^- = Q_1^+ = Q_2 = Q_2^- = Q_2^+ = H = \mathbb{C}$. Therefore, $Q = Q_- = Q_+ = \mathbb{C}^2$.

The domain $\mathcal{D}(L_0)$ of L_0 is dense in $\mathfrak{H} = L_2(\mathbb{C}, d\mathbf{m}; 0, b)$. This yields that L_0^* is an operator. A function y belongs to $\mathcal{D}(L_0^*)$ if and only if there exist numbers $\eta_1, \eta_2 \in \mathbb{C}$ such that

$$y(t) = \mathfrak{X}_{[0,b] \setminus \{\tau\}}(t)\eta_1 + \mathfrak{X}_{\{\tau\}}(t)\eta_2 - \mathfrak{X}_{[0,b] \setminus \{\tau\}}(t)i \int_0^t d\mathbf{m}(s)u(s), \tag{75}$$

where $u = L_0^* y$ (see [16]).

With each pair $\{y, u\} \in L_0^*$ we associate a pair of boundary values $\{Y, Y'\} \in \mathbb{C} \times \mathbb{C}$ by the formulas

$$Y = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} - 2^{-1}i \begin{pmatrix} \int_0^b d\mathbf{m}(s)u(s) \\ \int_0^b d\mathbf{m}(s)u(s) \end{pmatrix} + 2^{-1}i \begin{pmatrix} 0 \\ u(\tau) \end{pmatrix} + i \begin{pmatrix} 0 \\ \int_0^\tau d\mathbf{m}(s)u(s) \end{pmatrix}, \quad Y' = \begin{pmatrix} \int_0^b u(s) ds \\ u(\tau) \end{pmatrix}, \tag{76}$$

where y has form (75), $u = L_0^* y$. Let \mathcal{L} be a restriction of L_0^* to a set of functions satisfying the condition $Y = 0$. Then \mathcal{L} is the self-adjoint operator and $L_0 \subset \mathcal{L} \subset L_0^*$ (see [16]). Let us find the function

$$M(\lambda) = \begin{pmatrix} M_{11}(\lambda) & M_{12}(\lambda) \\ M_{21}(\lambda) & M_{22}(\lambda) \end{pmatrix} : \mathbb{C}^2 \rightarrow \mathbb{C}^2 \tag{77}$$

that corresponds to the resolvent $R_\lambda = (\mathcal{L} - \lambda E)^{-1}$.

Using (13), (14), and the equality $\mathbf{m}(\{\tau\}) = 1$, we get

$$\mathfrak{D}_1(t, \lambda) = \mathfrak{X}_{[0,b] \setminus \{\tau\}} w(t, \lambda) = \mathfrak{X}_{[0,b] \setminus \{\tau\}} e^{-i\lambda t} = \begin{cases} e^{-i\lambda t} & \text{for } t \neq \tau, \\ 0 & \text{for } t = \tau \end{cases}; \tag{78}$$

$$\begin{aligned} \mathfrak{D}_2(t, \lambda) &= u_1(t, \lambda, \tau)\lambda^{-1} + \mathfrak{X}_{\{\tau\}}(t)\lambda^{-1} = \\ &= -\mathfrak{X}_{[0,b] \setminus \{\tau\}} e^{-i\lambda t} i \int_0^t e^{i\lambda s} d\mathbf{m}(s) \lambda \mathfrak{X}_{\{\tau\}}(s)\lambda^{-1} ds + \mathfrak{X}_{\{\tau\}}(t)\lambda^{-1} = \begin{cases} 0 & \text{for } t < \tau, \\ \lambda^{-1} & \text{for } t = \tau, \\ -ie^{-i\lambda t} e^{i\lambda \tau} & \text{for } t > \tau; \lambda \neq 0. \end{cases} \end{aligned} \tag{79}$$

Therefore,

$$\mathfrak{D}_1^*(t, \bar{\lambda}) = \mathfrak{X}_{[0,b] \setminus \{\tau\}} e^{i\lambda t} = \begin{cases} e^{i\lambda t} & \text{for } t \neq \tau, \\ 0 & \text{for } t = \tau \end{cases}; \quad \mathfrak{D}_2^*(t, \bar{\lambda}) = \begin{cases} 0 & \text{for } t < \tau, \\ \lambda^{-1} & \text{for } t = \tau, \\ ie^{-i\lambda \tau} \mathfrak{X}_{(\tau,b]} e^{i\lambda t} & \text{for } t > \tau; \lambda \neq 0. \end{cases}$$

We denote

$$x_1 = x_1(f, \lambda) = \int_0^b \vartheta_1^*(s, \bar{\lambda}) d\mathbf{m}(s) f(s) = \int_0^b e^{i\lambda s} f(s) ds, \tag{80}$$

$$x_2 = x_2(f, \lambda) = \int_0^b \vartheta_2^*(s, \bar{\lambda}) d\mathbf{m}(s) f(s) = \lambda^{-1} f(\tau) + \int_\tau^b i e^{i\lambda s} e^{-i\lambda \tau} f(s) ds. \tag{81}$$

Using (34), we get

$$y(t) = \vartheta_1(t, \lambda)(M_{11}(\lambda)x_1 + M_{12}(\lambda)x_2) + \vartheta_2(t, \lambda)(M_{21}(\lambda)x_1 + M_{22}(\lambda)x_2) - \mathfrak{X}_{[0,b] \setminus \{\tau\}} e^{-i\lambda t} i \int_0^t e^{i\lambda s} f(s) ds + 2^{-1} \mathfrak{X}_{[0,b] \setminus \{\tau\}} e^{-i\lambda t} i \int_0^b e^{i\lambda s} f(s) ds - \lambda^{-1} \mathfrak{X}_{\{\tau\}}(t) f(\tau). \tag{82}$$

By (78), (79), it follows that

$$\vartheta_1(t, \lambda)(M_{11}(\lambda)x_1 + M_{12}(\lambda)x_2) = \mathfrak{X}_{[0,b] \setminus \{\tau\}} e^{-i\lambda t} (M_{11}(\lambda)x_1 + M_{12}(\lambda)x_2), \tag{83}$$

$$\vartheta_2(t, \lambda)(M_{21}(\lambda)x_1 + M_{22}(\lambda)x_2) = \begin{cases} 0 & \text{for } t < \tau, \\ \lambda^{-1}(M_{21}(\lambda)x_1 + M_{22}(\lambda)x_2) & \text{for } t = \tau, \\ -ie^{-i\lambda t} e^{i\lambda \tau} (M_{21}(\lambda)x_1 + M_{22}(\lambda)x_2) & \text{for } t > \tau; \lambda \neq 0. \end{cases} \tag{84}$$

To find $M_{12}(\lambda)$, $M_{22}(\lambda)$, we take the function $f_1(t) = \mathfrak{X}_{\{\tau\}}(t)$, i.e., $f_1(t) = 1$ if $t = \tau$ and $f_1(t) = 0$ if $t \neq \tau$ (by $\{Y_1, Y_1'\}$ denote the corresponding pair of boundary values). It follows from (80), (81) that $x_1 = 0$, $x_2 = \lambda^{-1}$. We denote $y_1 = R_\lambda f_1$. Using (82), (83), (84), we obtain

$$y_1(t) = \vartheta_1(t, \lambda)M_{12}(\lambda)\lambda^{-1} + \vartheta_2(t, \lambda)M_{22}(\lambda)\lambda^{-1} - \lambda^{-1} \mathfrak{X}_{\{\tau\}}(t). \tag{85}$$

We denote $u_1 = L_0^* y_1 = \lambda y_1 + f_1$. Then using (85), (78), (79), we get

$$u_1(t) = \vartheta_1(t, \lambda)M_{12}(\lambda) + \vartheta_2(t, \lambda)M_{22}(\lambda) = \mathfrak{X}_{[0,b] \setminus \{\tau\}}(t) e^{-i\lambda t} M_{12}(\lambda) + \mathfrak{X}_{\{\tau\}}(t) \lambda^{-1} M_{22}(\lambda) - \mathfrak{X}_{(\tau,b]}(t) i e^{-i\lambda t} e^{i\lambda \tau} M_{22}(\lambda). \tag{86}$$

Taking into account (75), (85), we obtain

$$y_1(t) = \mathfrak{X}_{[0,b] \setminus \{\tau\}}(t) M_{12}(\lambda) \lambda^{-1} + \mathfrak{X}_{\{\tau\}}(t) \lambda^{-1} (M_{22}(\lambda) \lambda^{-1} - 1) - \mathfrak{X}_{[0,b] \setminus \{\tau\}}(t) i \int_0^t d\mathbf{m}(s) u_1(s). \tag{87}$$

Using (86), by direct calculations, we get

$$\int_0^b d\mathbf{m}(s) u_1(s) = \lambda^{-1} (i(e^{-i\lambda b} - 1)M_{12}(\lambda) + e^{-i\lambda b} e^{i\lambda \tau} M_{22}(\lambda)); \quad \int_0^\tau d\mathbf{m}(s) u_1(s) = i\lambda^{-1} (e^{-i\lambda \tau} - 1)M_{12}(\lambda). \tag{88}$$

By (76), (87), (88), so that

$$Y_1 = \left(\begin{matrix} M_{12}(\lambda)\lambda^{-1} \\ \lambda^{-1}(M_{22}(\lambda)\lambda^{-1} - 1) \end{matrix} \right) - 2^{-1} i \left(\begin{matrix} \lambda^{-1}(i(e^{-i\lambda b} - 1)M_{12}(\lambda) + e^{-i\lambda b} e^{i\lambda \tau} M_{22}(\lambda)) \\ \lambda^{-1}(i(e^{-i\lambda b} - 1)M_{12}(\lambda) + e^{-i\lambda b} e^{i\lambda \tau} M_{22}(\lambda)) \end{matrix} \right) + 2^{-1} i \left(\begin{matrix} 0 \\ \lambda^{-1} M_{22}(\lambda) \end{matrix} \right) + i \left(\begin{matrix} 0 \\ i\lambda^{-1}(e^{-i\lambda \tau} - 1)M_{12}(\lambda) \end{matrix} \right).$$

The equality $Y_1 = 0$ is equivalent to two equalities

$$\begin{cases} M_{12}(\lambda)\lambda^{-1} + 2^{-1}\lambda^{-1}((e^{-i\lambda b} - 1)M_{12}(\lambda) - ie^{-i\lambda b}e^{i\lambda \tau}M_{22}(\lambda)) = 0, \\ \lambda^{-1}(M_{22}(\lambda)\lambda^{-1} - 1) + 2^{-1}\lambda^{-1}((e^{-i\lambda b} - 1)M_{12}(\lambda) - ie^{-i\lambda b}e^{i\lambda \tau}M_{22}(\lambda)) + 2^{-1}\lambda^{-1}iM_{22}(\lambda) - \lambda^{-1}(e^{i\lambda \tau} - 1)M_{12}(\lambda) = 0. \end{cases}$$

Solving this system of equations, we get

$$M_{12}(\lambda) = i \frac{2\lambda e^{i\lambda\tau} e^{-i\lambda b}}{e^{-i\lambda b}(2 - i\lambda) + (2 + i\lambda)} ; \quad M_{22}(\lambda) = \frac{2\lambda(e^{-i\lambda b} + 1)}{e^{-i\lambda b}(2 - i\lambda) + (2 + i\lambda)} . \tag{89}$$

To find $M_{11}(\lambda), M_{21}(\lambda)$, we take the function $f_2(t) = \mathfrak{X}_{[0,\tau)}(t)$, i.e., $f_2(t) = 1$ if $t < \tau$ and $f_2(t) = 0$ if $t \geq \tau$ (by $\{Y_2, Y'_2\}$ denote the corresponding pair of boundary values). It follows from (80), (81) that $x_1 = i\lambda^{-1}(1 - e^{i\lambda\tau})$, $x_2 = 0$. We denote $y_2 = R_\lambda f_2$. Using (82), (83), (84), we obtain

$$y_2(t) = \vartheta_1(t, \lambda)M_{11}(\lambda)x_1 + \vartheta_2(t, \lambda)M_{21}(\lambda)x_1 - \mathfrak{X}_{[0,b]\setminus\{\tau\}} e^{-i\lambda t} i \int_0^t e^{i\lambda s} f_2(s) ds + 2^{-1} \mathfrak{X}_{[0,b]\setminus\{\tau\}} e^{-i\lambda t} i \int_0^b e^{i\lambda s} f_2(s) ds. \tag{90}$$

The equality $f_2(t) = \mathfrak{X}_{[0,\tau)}(t)$ implies

$$-\mathfrak{X}_{[0,b]\setminus\{\tau\}} e^{-i\lambda t} i \int_0^t e^{i\lambda s} f_2(s) ds = \begin{cases} \lambda^{-1}(e^{-i\lambda t} - 1) & \text{for } t < \tau, \\ 0 & \text{for } t = \tau, \\ \lambda^{-1}e^{-i\lambda t}(1 - e^{i\lambda\tau}) & \text{for } t > \tau; \end{cases} \tag{91}$$

$$2^{-1} \mathfrak{X}_{[0,b]\setminus\{\tau\}} e^{-i\lambda t} i \int_0^b e^{i\lambda s} f_2(s) ds = \begin{cases} 2^{-1}ie^{-i\lambda t}x_1 & \text{for } t \neq \tau, \\ 0 & \text{for } t = \tau. \end{cases} \tag{92}$$

By (75), (90), (91), (92), it follows that

$$y_2(t) = \mathfrak{X}_{[0,b]\setminus\{\tau\}}(M_{11}(\lambda)x_1 + 2^{-1}ix_1) + \mathfrak{X}_{\{\tau\}}(t)\lambda^{-1}M_{21}(\lambda)x_1 - \mathfrak{X}_{[0,b]\setminus\{\tau\}}(t)i \int_0^t d\mathbf{m}(s)u_2(s), \tag{93}$$

where $u_2 = L_0^* y_2 = \lambda y_2 + f_2$. Equalities (78), (79), (90), (91), (92) imply that $u_2(t) = u_{21}(t) + u_{22}(t) + u_{23}(t) + u_{24}(t)$, where

$$u_{21}(t) = \mathfrak{X}_{[0,b]\setminus\{\tau\}} \lambda e^{-i\lambda t} M_{11}(\lambda)x_1; \quad u_{22}(t) = \begin{cases} 0 & \text{for } t < \tau, \\ M_{21}(\lambda)x_1 & \text{for } t = \tau, \\ -\lambda i e^{i\lambda\tau} e^{-i\lambda t} M_{21}(\lambda)x_1 & \text{for } t > \tau; \end{cases} \tag{94}$$

$$u_{23}(t) = \begin{cases} e^{-i\lambda t} & \text{for } t < \tau, \\ 0 & \text{for } t = \tau, \\ e^{-i\lambda t}(1 - e^{i\lambda\tau}) & \text{for } t > \tau; \end{cases} \quad u_{24}(t) = \begin{cases} 2^{-1}\lambda i e^{-i\lambda t} x_1 & \text{for } t \neq \tau, \\ 0 & \text{for } t = \tau. \end{cases} . \tag{95}$$

Using (94), (95), and the equality $x_1 = i\lambda^{-1}(1 - e^{i\lambda\tau})$, by direct calculations, we obtain

$$\int_0^b d\mathbf{m}(s)u_2(s) = i(e^{-i\lambda b} - 1)M_{11}(\lambda)x_1 + e^{i\lambda\tau} e^{-i\lambda b} M_{21}(\lambda)x_1 + 2^{-1}(e^{-i\lambda b} + 1)x_1, \tag{96}$$

$$\int_0^\tau d\mathbf{m}(s)u_2(s) = i(e^{-i\lambda\tau} - 1)M_{11}(\lambda)x_1 + 2^{-1}(e^{-i\lambda\tau} + 1)x_1. \tag{97}$$

By (76), (93), so that

$$Y_2 = \begin{pmatrix} M_{11}(\lambda)x_1 + 2^{-1}ix_1 \\ \lambda^{-1}M_{21}(\lambda)x_1 \end{pmatrix} - 2^{-1}i \begin{pmatrix} \int_0^b d\mathbf{m}(s)u_2(s) \\ \int_0^b d\mathbf{m}(s)u_2(s) \end{pmatrix} + 2^{-1}i \begin{pmatrix} 0 \\ M_{21}(\lambda)x_1 \end{pmatrix} + i \begin{pmatrix} 0 \\ \int_0^\tau d\mathbf{m}(s)u_2(s) \end{pmatrix},$$

where the integrals $\int_0^b d\mathbf{m}(s)u_2(s), \int_0^\tau d\mathbf{m}(s)u_2(s)$ are calculated by formulas (96), (97), respectively.

The equality $Y_2 = 0$ is equivalent to two equalities

$$\begin{cases} M_{11}(\lambda)x_1 + 2^{-1}ix_1 - 2^{-1}i \int_0^b d\mathbf{m}(s)u_2(s) = 0, \\ \lambda^{-1}M_{21}(\lambda)x_1 - 2^{-1}i \int_0^b d\mathbf{m}(s)u_2(s) + 2^{-1}iM_{21}(\lambda)x_1 + i \int_0^\tau d\mathbf{m}(s)u_2(s) = 0. \end{cases}$$

Solving this system of equations, we obtain

$$M_{11}(\lambda) = i \frac{(2 - i\lambda)e^{-i\lambda b} - (2 + i\lambda)}{2((2 - i\lambda)e^{-i\lambda b} + (2 + i\lambda))}; \quad M_{21}(\lambda) = i \frac{-2\lambda e^{-i\lambda\tau}}{(2 - i\lambda)e^{-i\lambda b} + (2 + i\lambda)}. \quad (98)$$

Thus the matrix $M(\lambda)$ (77) is calculated by equalities (98), (89).

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