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Existence of best proximity points for the sum of two operators

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Abstract. In this paper, we work on the existence of approximate solution for the fixed point equation $Rx + Qx = x$, when R is a weak contractive mapping and Q is a continuous self-map which is a new topic related to the background literature. Till now, no results were found to find the existence of approximate solution for such a kind of mappings in literature to the best of our knowledge. Here we prove the existence of best proximity point for the sum of a weak contractive non self map and a continuous self map. Also, we provide an example to support our main theorem. And also, we prove the existence of best proximity point for the sum of Geraghty- contraction and continuous map.

1. Introduction and preliminaries

Many problems in analysis may split in the form $J = R + Q$, where R is contraction in some sense and Q is a compact map, but J has neither of this properties. In such cases, fixed point results of self map namely Banach contraction principle, Brouwer's theorem, Shauder's theorem cannot be applied. So, it urge researchers to develop theory for such kind of situations.

Krasnoselskii encounters the problem called the existence of fixed points for sum of two operators while he studied a paper by Schauder on partial differential equations. He formulated that the inversion of perturbed differential operator gives rise to sum of contraction and compact mapping. This is the starting point for the fixed point problem for the sum of two operators. Krasnoselskii was the first researcher to prove existence of fixed point for sum of two operators. Later, it was attracted by many researchers because of its wide applications in nonlinear analysis, also these results are mainly applied to boundary value problems and for determining solutions of nonlinear integral equations. He proved the following result which combines both Banach contraction and Shauder's theorem.

Theorem 1.1. *[11] Let E be a closed, convex, and bounded subset of a Banach space X and let $R, Q : E \to X$ be two operators such that*

1. *R is a contraction;*
2. *Q is completely continuous; and*

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3. *$Rz + Qy \in E$, for all $z, y \in E$.*

Then the operator equation $Rx + Qx = x$ has a solution.

Later on, several improvements of the above fixed-point result have been made by many authors see [4, 6, 12–14, 16, 20]. In perturbation theory, the operator equation $Rx+Qx = x$ is considered as a perturbation of $Rx = x$ or vice versa. Hence, it is natural for us to look for the existence of solution of the above perturbed equation, while the unperturbed equation has a solution. But, the existence of fixed points for the sum of two operators $J = R + Q$ cannot be guaranteed even though both operators R, Q are self-map. To see this, consider the following simple example $R, Q : [0, 1] \to [0, 1]$ by $R(x) = 1$ and $Q(x) = x^2$. Then $R + Q : [0, 1] \to [1, 2]$, also $R + Q$ does not have fixed point. So, if condition 3. in Theorem 1. fails, then it may be impossible for us to find fixed point for the sum. Naturally, this motivates us to look for the approximate solution for the sum of two operators. In the past few years, researchers started concentrating more on existence of fixed points for non-self mappings R defined from a set C to a set D, where C and D are subsets of a metric space X. But it is not always possible to find an exact solution for the operator equation $Rx = x$ even though the map is a self-map, for example $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{x^2+x+1}{x+1}$ and so it urge us to look for an approximate solution. Subsequently, one targets to determine an element x that is in some sense close to Rx. In fact, best approximation theorems and best proximity point theorems are suitable to be explored in this direction. The best approximation theorem answers the question of existence of approximate solution in affirmative way, but such results may not afford an approximate solution that is optimal. On the other hand, best proximity point theorems deals with existence of an approximate solution which is optimal, that is there exists an element x such that $\rho(x, Rx)$ is minimum. Best proximity pair theorem analyzes the conditions under which, the optimization problem, $\min_{x\in C} \rho(x, Rx)$ has a solution. In this paper, we provide sufficient condition for the existence of best proximity point for the sum of two operators. The authors in [7] combined the aforementioned concepts in more general way for set valued and single valued mappings and to prove the existence of best proximity point results in the context of b-metric spaces. Endowing the concept of graph with b-metric space, they presented some best proximity point results. Pragadeeswarar et. all [15] proved the existence of a common best proximity point for a pair of multivalued non-self mappings in partially ordered metric spaces. Kostić et. all [10] introduced a new generalized distance, namely the w_0-distance, which is a special type of w-distance and using the concept of w_0 -distance, they generalized some recent best proximity point results involving simulation functions. Kadelburg and Radenović [8] showed how an easy lemma, proved recently by Abkar and Gabeleh, can be applied to obtain Geraghty-type results for best proximity points. They obtained the improvements of some recent best proximity point results. The next definition is well known in literature.

Definition 1.2. *Let (C, D) be a pair of nonempty subsets of a metric space X. Then a point $e \in C$ is said to be best proximity point of a mapping $R : C \to D$, if it satisfies the following condition:*

$$\rho(e, Re) = \rho(C, D).$$

In [1], Alber and Guerre introduced the notion called weakly contractive self-map and proved the existence of fixed point for weakly contractive self-map in Hilbert space.

Definition 1.3. *[1] Let X be a metric space and C be a nonempty subset of X. A map $R : C \to C$ is said to be a weakly contractive self-map if $\rho(Ru, Rv) \leq \rho(u, v) - \psi(\rho(u, v))$, for all $u, v \in C$, where $\psi : [0, +\infty) \to [0, +\infty)$ is a continuous and nondecreasing function such that ψ is positive on $(0, +\infty)$, $\psi(0) = 0$ and $\lim_{t\to+\infty} \psi(t) = +\infty$. If C is bounded, then the infinity limit condition can be omitted.*

Later on, in [17] Rhoades extended the result of Alber and Guerre to arbitrary complete metric space. Further, in [19] V. Sankar Raj has defined the notion called weakly contractive mapping for non self mapping and generalizes the fixed point theorem of Rhoades for weak contractive self maps to non self maps.

Definition 1.4. *[19]Let C, D be nonempty subsets of a metric space X . A map $R : C \to D$ is said to be a weakly contractive mapping if $\rho(Ru, Rv) \leq \rho(u, v) - \psi(\rho(u, v))$, for all $u, v \in C$, where $\psi : [0, +\infty) \to [0, +\infty)$ is a continuous and nondecreasing function such that ψ is positive on $(0, +\infty)$, $\psi(0) = 0$ and $\lim_{t\to+\infty} \psi(t) = +\infty$. If C is bounded, then the infinity condition can be omitted.*

In recent years, researchers started to concentrate on existence of best proximity points for different types of contraction mapping. one such is Gregarty type contraction mapping. In [3] J. Caballero proved existence of best proximity point for Gregarty type contraction map.

Definition 1.5. *Let C, D be any two nonempty subsets of a metric space (X, d). A mapping $R : C \to D$ is said to be a Geraghty-contraction if there exists $\alpha \in \mathcal{F}$ satisfying $d(Ru, Rv) \leq \alpha(d(u, v)).d(u, v)$ for all $u, v \in C$. Here, $\mathcal{F}$ is a collection of functions $\alpha : [0, +\infty) \to [0, 1)$ with $\alpha(s_n) \to 1-$ implies $s_n \to 0+$.*

The following few notations were used frequently in this paper,

$$\rho(C, D) := \inf\{\rho(u, v) : u \in C \text{ and } v \in D\},$$

$$C_0 = \{u \in C : \rho(u, v) = \rho(C, D) \text{ for some } v \in D\},$$

$$D_0 = \{v \in D : \rho(u, v) = \rho(C, D) \text{ for some } u \in C\}.$$

In [18], the sufficient conditions for the the non-emptiness of C_0 and D_0 were discussed. V. Sankar Raj in [19] has introduced new notion called P - property.

Definition 1.6. *([19]). Let (C, D) be a pair of non-empty subsets of a metric space X with $C_0 \neq \emptyset$. Then the pair (C, D) is said to have the P-property if and only if*

$$\left.\begin{array}{l}\rho(u_1, v_1) = \rho(C, D)\\ \rho(u_2, v_2) = \rho(C, D)\end{array}\right\} \textit{ implies } \rho(u_1, u_2) = \rho(v_1, v_2)$$

where $u_1, u_2 \in C_0$ and $v_1, v_2 \in D_0$.

Definition 1.7. *[14] The set B is said to be approximatively compact with respect to A if every sequence (y_n) of B satisfying the condition that $\rho(x, y_n) \to \rho(x, B)$ for some x in A has a convergent subsequence.*

It is obvious that any compact set is approximatively compact, and that any set is approximatively compact with respect to itself. Further, it is given in [14] that, if C is compact and D is approximatively compact with respect to C, then it is ensured that C_0 and D_0 are non- empty.

Definition 1.8. *Let X, Y be any two topological spaces. Let $F : X \to 2^Y$ be any multivalued mapping. Then we say F is an upper semicontinuous mapping if $F^{-1}(E)$ is a closed subset of X whenever E is closed in Y.*

Lemma 1.9. *[2] If C is any nonempty compact subset of a metric space (X, ρ). Then the metric projection map defined as $P_C(u) := \{v \in C : \rho(u, v) = \rho(u, C)\}$ becomes an upper semi continuous map.*

2. Main results

In [5], Ky Fan discussed existence of fixed point for an upper semi continuous mapping on a topological vector space. In our result we used Banach space version of that theorem. The following is the Banach space version of Ky Fan's theorem.

Theorem 2.1. *[5] Let C be any non empty compact, convex subsets of a Banach space X. Let $Cl(C)$ be the collection of all closed convex subsets of C. Then for any upper semi-continuous function $R : C \to Cl(C)$, there exists a point $u_0 \in C$ such that $u_0 \in R(u_0)$.*

In [19], V. Sankar Raj proved the following theorem.

Theorem 2.2. *[19] Let (C, D) be a pair of nonempty closed subsets of a complete metric space (X, ρ) such that C_0 is nonempty. Let $R : C \to D$ be a weakly contractive mapping. Assume that the pair (C, D) has the P-property. If $R(C_0) \subseteq D_0$, then there exists a unique $x^{\hat{a}} \in C$ such that $\rho(x^{\hat{a}}, Rx^{\hat{a}}) = \rho(C, D)$.*

If $\psi(t) = (1 - k)t$ for some $k \in (0, 1)$ and $t \geq 0$ in the above theorem, then R becomes contractive mapping.

2.1. Best proximity point theorems for sum of two operators

Now, we are going to explore our main theorem and provide an example to support our main result.

Theorem 2.3. *Let C, D be non empty closed subsets of a Banach space X and also, (C, D) satisfies the P- property. Let $R : C \to D$ be a weakly contractive mapping with $R(C_0) \subseteq D_0$ and $Q : C \to C$ be a continuous mapping with conditions that $Rx + Qy \in D$ for all $x \in C, y \in C_0$. Also, assume that C_0 is nonempty compact convex subset with $R(C_0) + Qx \subseteq D_0$ for all $x \in C_0$. Then there exists $x \in C$ such that $\rho(x, Rx + Qx) = \rho(C, D)$.*

Proof. For each $y \in C_0$, define the map $\tilde{R_y} : C \to D$, by $\tilde{R_y}(x) = Rx + Qy$. For all $x_1, x_2 \in C$, we have

$$||\tilde{R_y}(x_1) - \tilde{R_y}(x_2)|| = ||Rx_1 + Qy - Rx_2 - Qy|| = ||Rx_1 - Rx_2||. \tag{1}$$

It is clear from (1), that the map $\tilde{R_y}$ is a weakly contractive map with $\tilde{R_y}(C_0) \subseteq D_0$ for all $y \in C_0$. Thus, for each $y \in C_0$, there exists $p_y \in C$ such that $\rho(p_y, \tilde{R}(p_y)) = \rho(C, D)$.

Now, define $H : C_0 \to D_0$ by $H(x) = \tilde{R_x}(p_x)$, where $\rho(p_x, \tilde{R_x}(p_x)) = \rho(C, D)$. By the P - property, we have

$$\left.\begin{aligned} \rho(p_u, \tilde{R_u}(p_u)) = \rho(C, D) \\ \rho(p_v, \tilde{R_v}(p_v)) = \rho(C, D) \end{aligned}\right\} \text{ implies } \rho(p_u, p_v) = \rho(\tilde{R_u}(p_u), \tilde{R_v}(p_v)). \tag{2}$$

Thus, for all $u, v \in C_0$

$$\begin{aligned} ||Hu - Hv|| &= ||\tilde{R_u}(p_u) - \tilde{R_v}(p_v)|| \\ &= ||p_u - p_v|| \quad (\because \text{ by (2) }). \end{aligned} \tag{3}$$

Also,

$$\begin{aligned} ||Hu - Hv|| &= ||\tilde{R_u}(p_u) - \tilde{R_v}(p_v)|| \\ &= ||R(p_u) + Qu - R(p_v) - Qv|| \\ &\leq ||R(p_u) - R(p_v)|| + ||Qu - Qv|| \\ &\leq ||p_u - p_v|| - \psi(||p_u - p_v||) + ||Qu - Qv||. \end{aligned} \tag{4}$$

Combining (3) and (4), we have

$$\psi(||Hu - Hv||) \leq ||Qu - Qv||. \tag{5}$$

To see H is continuous, take (x_n) in C_0 such that $x_n \to x$. Since Q is continuous on C. Restriction of Q onto C_0 is also continuous and hence $Qx_n \to Qx$.

$$\begin{aligned} \lim_{n\to+\infty} \psi(||Hx_n - Hx||) &\leq \lim_{n\to+\infty} ||Qx_n - Qx|| \\ &= 0. \end{aligned} \tag{6}$$

Since ψ is continuous, we have

$$\lim_{n\to+\infty} \psi(||Hx_n - Hx||) = \psi(\lim_{n\to+\infty} ||Hx_n - Hx||) = 0.$$

By the property of ψ, we have $\lim_{n\to+\infty} ||Hx_n - Hx|| = 0$. This proves that H is continuous map. Now, consider the metric projection, $P_{C_0} : X \to 2^{C_0}$ where $P_{C_0}(x) = \{u \in C_0 : ||u - x|| = \rho(x, C_0)\}$. Since C_0 is nonempty compact convex set, $P_{C_0}(x)$ is a nonempty closed, convex subset. Hence by lemma 1.9, P_{C_0} is an upper semicontinuous map. Now, consider the composition $F := P_{C_0} \circ H$. Then F is a convex-valued multivalued mapping. By applying Theorem 2.1 for F, we get $\bar{x} \in F(\bar{x})$. That is, $\bar{x} \in P_{C_0}(H\bar{x})$, also $H\bar{x} \in D_0$, there exists $a \in C_0$ such that $||H\bar{x} - a|| = \rho(C, D)$. But, $\rho(H\bar{x}, C_0) = ||\bar{x} - H\bar{x}|| \leq ||H\bar{x} - a|| = \rho(C, D)$. This implies that, $||\bar{x} - H(\bar{x})|| = \rho(C, D)$. we have $\rho(p_{\bar{x}}, \tilde{R}_{p_{\bar{x}}}(p_{\bar{x}})) = \rho(C, D)$. Hence by P- property,

$$||p_{\bar{x}} - \bar{x}|| = ||H(\bar{x}) - \tilde{R}_{p_{\bar{x}}}(p_{\bar{x}})|| = 0.$$

That is, $\rho(\bar{x}, R\bar{x} + Q\bar{x}) = \rho(C, D)$. □

Example 2.4. *Consider $X = \mathbb{R}^2$ with the usual metric. Choose $C = \{(x_1, x_2) : x_1 \in [-1, 1], x_2 \in [-1, 1]\}$ and $D = \{(2, x_2) : x_2 \in [-\frac{3}{2}, \frac{3}{2}]\}$. Then C and D are non empty closed subsets of X and $C_0 = \{(x_1, x_2) : x_1 = 1, x_2 \in [-1, 1]\}$ and $D_0 = \{(2, x_2) : x_2 \in [-1, 1]\}$.It is clear that C_0 is compact convex subset. Note that $\rho(C, D) = 1$. Let $R : C \to D$ be a mapping defined as $R(x_1, x_2) = (2, \frac{x_2}{2})$ and $Q : C \to C$ be a mapping defined as $Q(x_1, x_2) = (0, x_2)$. It is clear that P is a non self contraction and hence it is a weak contractive non self mapping. Also, it is easy to verify that (C, D) satisfies the P - property. It is also evident that C, D, R and Q satisfy all the conditions of Theorem 4. Note that $x = (1, 0)$ is the best proximity point of $R + Q$, that is, $\rho(x, Rx + Qx) = 1 = \rho(C, D)$ where $x = (1, 0)$.*

In case of self mapping, that is, if $C = D$, then Theorem 4. reduces to the following.

Corollary 2.5. *Let C be a non empty compact subset of a Banach space X. Let $R : C \to C$ be a weakly contractive mapping and $Q : C \to C$ be a continuous mapping satisfying $Rx + Qy \in C$ for all $x, y \in C$. Then there exists $x \in C$ such that $\rho(x, Rx + Qx) = 0$, that is $Rx + Qx = x$ has a solution in C.*

Now, we are going to work on another important contraction mapping called Gregarty type contraction mapping.

Theorem 2.6. *[3] Let (C, D) be a pair of nonempty closed subsets of a complete metric space X such that C_0 is nonempty. Let $R : C \to D$ be a Geraghty-contraction satisfying $R(C_0) \subseteq D_0$. Also, if (C, D) has P-property, then there exists $p \in C$ such that $d(p, Rp) = d(C, D)$.*

Using the above result we are going to prove existence of best proximity point for sum of two operators as follows:

Theorem 2.7. *Let C, D be non empty closed subsets of a Banach space X and also, (C, D) satisfies the P- property. Let $R : C \to D$ be a Gregarty contraction mapping with $R(C_0) \subseteq D_0$ and $Q : C \to C$ be a continuous mapping with conditions that $Rx + Qy \in D$ for all $x \in C, y \in C_0$. Also, assume that C_0 is nonempty compact convex subset with $R(C_0) + Qx \subseteq D_0$ for all $x \in C_0$. Then there exists $x \in C$ such that $\rho(x, Rx + Qx) = \rho(C, D)$.*

Proof. For each $y \in C_0$, define the map $\tilde{R_y} : C \to D$, by $\tilde{R_y}(x) = Rx + Qy$. For all $x_1, x_2 \in C$, we have

$$||\tilde{R_y}(x_1) - \tilde{R_y}(x_2)|| = ||Rx_1 + Qy - Rx_2 - Qy|| = ||Rx_1 - Rx_2||. \tag{7}$$

It is clear from (7), that the map $\tilde{R_y}$ is a Gregarty contraction map with $\tilde{R_y}(C_0) \subseteq D_0$ for all $y \in C_0$. Thus, for each $y \in C_0$, there exists $p_y \in C$ such that $\rho(p_y, \tilde{R}(p_y)) = \rho(C, D)$.

Now, define $H : C_0 \to D_0$ by $H(x) = \tilde{R_x}(p_x)$, where $\rho(p_x, \tilde{R_x}(p_x)) = \rho(C, D)$. By the P - property, we have

$$\left.\begin{aligned} \rho(p_u, \tilde{R_u}(p_u)) = \rho(C, D) \\ \rho(p_v, \tilde{R_v}(p_v)) = \rho(C, D) \end{aligned}\right\} \text{ implies } \rho(p_u, p_v) = \rho(\tilde{R_u}(p_u), \tilde{R_v}(p_v)). \tag{8}$$

Thus, for all $u, v \in C_0$

$$\begin{aligned} ||Hu - Hv|| &= ||\tilde{R_u}(p_u) - \tilde{R_v}(p_v)|| \\ &= ||p_u - p_v|| \quad (\because \text{ by (8) }). \end{aligned} \tag{9}$$

Also,

$$\begin{aligned} ||Hu - Hv|| &= ||\tilde{R_u}(p_u) - \tilde{R_v}(p_v)|| \\ &= ||R(p_u) + Qu - R(p_v) - Qv|| \\ &\leq ||R(p_u) - R(p_v)|| + ||Qu - Qv|| \\ &\leq \alpha(||p_u - p_v||)(||p_u - p_v||) + ||Qu - Qv||. \end{aligned} \tag{10}$$

Combining (9) and (10), we have

$$(1 - \alpha(||Hu - Hv||))||Hu - Hv|| \leq ||Qu - Qv||. \tag{11}$$

To see H is continuous, take (a_n) in C_0 such that $a_n \to a$. Since Q is continuous on C. Restriction of Q onto C_0 is also continuous and hence $Qa_n \to Qa$.

$$0 \leq \lim_{n\to+\infty}(1-\alpha(\|Ha_n - Ha\|))\|Ha_n - Ha\| \leq \lim_{n\to+\infty}\|Qa_n - Qa\| = 0.$$

By Sandwich theorem for sequences, we get

$$\lim_{n\to+\infty}(1-\alpha(\|Ha_n - Ha\|))\|Ha_n - Ha\| = 0,$$

which implies

$$\lim_{n\to+\infty}\|Ha_n - Ha\| = 0.$$

This proves that H is continuous map. Now, consider the metric projection, $P_{C_0} : X \to 2^{C_0}$ where $P_{C_0}(x) = \{u \in C_0 : \|u - x\| = \rho(x, C_0)\}$. Since C_0 is nonempty compact convex set, $P_{C_0}(x)$ is a nonempty closed, convex subset. Hence by lemma 1.9, P_{C_0} is an upper semicontinuous map. Now, consider the composition $F := P_{C_0} \circ H$. Then F is a convex-valued multivalued mapping. By applying Theorem 2.1 for F, we get $\bar{x} \in F(\bar{x})$. That is, $\bar{x} \in P_{C_0}(H\bar{x})$, also $H\bar{x} \in D_0$, there exists $a \in C_0$ such that $\|H\bar{x} - a\| = \rho(C, D)$. But, $\rho(H\bar{x}, C_0) = \|\bar{x} - H\bar{x}\| \leq \|H\bar{x} - a\| = \rho(C, D)$. This implies that, $\|\bar{x} - H(\bar{x})\| = \rho(C, D)$. we have $\rho(p_{\bar{x}}, \tilde{R}_{p_{\bar{x}}}(p_{\bar{x}})) = \rho(C, D)$. Hence by P- property,

$$\|p_{\bar{x}} - \bar{x}\| = \|H(\bar{x}) - \tilde{R}_{p_{\bar{x}}}(p_{\bar{x}})\| = 0.$$

That is, $\rho(\bar{x}, R\bar{x} + Q\bar{x}) = \rho(C, D)$. □

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