Filomat 37:23 (2023), 7849–7865
https://doi.org/10.2298/FIL2323849D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Solvability of (P, Q)-functional integral equations of fractional order using generalized Darbo's fixed point theorem

Anupam Das[a], Bipan Hazarika[b], Mohammad Mursaleen[c,d,*], Hemant Kumar Nashine[e,f], Vahid Parvaneh[g,*]

[a]*Department of Mathematics, Cotton University, Panbazar, Guwahati-781001, Assam, India*
[b]*Department of Mathematics, Gauhati University, Guwahati 781014, Assam, India*
[c]*Department of Medical Research, China Medical University Hospital, China Medical University (Taiwan), Taichung, Taiwan*
[d]*Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India*
[e]*Mathematics Division, School of Advanced Sciences and Languages,*
VIT Bhopal University, Bhopal-Indore Highway, Kothrikalan, Sehore, Madhya Pradesh - 466114, India
[f]*Department of Mathematics and Applied Mathematics,*
University of Johannesburg, Kingsway Campus, Auckland Park 2006, South Africa
[g]*Department of Mathematics, Gilan-E-Gharb Branch, Islamic Azad University, Gilan-E-Gharb, Iran*

Abstract.
In this article, we establish a generalized version of Darbo's fixed point theorem via some newly defined condensing operators and we define a new fractional integral using (P, Q)-calculus and study its properties. Finally, we apply this generalized Darbo's fixed point theorem to check the existence of a solution of (P, Q)-functional integral equations of fractional order in a Banach space. We explain the results with the help of simple examples.

1. Introduction

The measure of non-compactness which was first introduced by Kuratowski [14] plays a very important role in many branches of mathematics. There are several types of non-compactness measures in metric and topological spaces. For more information on the subject of measure of non-compactness, see [7]. Non-compactness measures are used in various types of integral and differential equations, see [7]. Arab et al. [6] proved the existence of solutions for infinite systems of integral equations that generate via two variables. In [10], the existence of solutions for singular integral equations was discussed using a measure of non-compactness.

The idea of Q-calculus was introduced by Jackson [11, 12]. Fractional q-difference concept was introduced by Agarwal [2] and Al-Salam [4]. In [13], the existence of solution of Q-integral equations of fractional order have been discussed.

2020 *Mathematics Subject Classification*. 34A34, 46B45, 47H10.
Keywords. Measure of noncompactness; Hausdorff measure of noncompactness; Integral equations; Darbo's fixed point theorem.
Received: 22 March 2022; Accepted: 10 April 2023
Communicated by Vladimir Rakočević
* Corresponding authors: Mohammad Mursaleen and Vahid Parvaneh
ORCID: https://orcid.org/0000-0002-0250-9172 (Hemant Kumar Nashine)
Email addresses: math.anupam@gmail.com (Anupam Das), bh_rgu@yahoo.co.in (Bipan Hazarika), mursaleenm@gmail.com (Mohammad Mursaleen), drhemantnashine@gmail.com, hemantkumar.nashine@vitbhopal.ac.in (Hemant Kumar Nashine), zam.dalahoo@gmail.com (Vahid Parvaneh)

In this article, we apply a generalized version of Darbo's theorem to study the solvability of the equation:

$$X(\theta) = \eta\left(\theta, X(\bar{a}(\theta)), \frac{\mathcal{F}(\theta, X(\bar{b}(\theta)))}{\Gamma_{P,Q}(\alpha)} \int_0^\theta (P\theta - Q\theta_1)_{P,Q}^{(\alpha-1)} \mathcal{U}(\theta_1, X(\theta_1)) d_{P,Q}\theta_1\right), \tag{1}$$

where $\theta \in I = [0,1],\ 0 < Q < P \leq 1,\ \mathcal{F}, \mathcal{U} : I \times \mathbb{R} \to \mathbb{R},\ \bar{a}, \bar{b} : I \to I,\ \eta : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\alpha > 1$.

2. Preliminaries

At first, we recall some facts about Q-calculus. For more details, we refer to [2, 5, 17]. Let $Q \in [1, \infty)$. For arbitrary $Ł \in \mathbb{R}$, the Q-real number $[Ł]_Q$ is defined by

$$[Ł]_Q = \frac{1 - Q^{Ł}}{1 - Q}.$$

The Q-shifted factorial of real number $Ł'$ is defined by

$$(Ł', Q)_0 = 1,\ (Ł', Q)_\ell = \prod_{i=0}^{\ell-1} \left(1 - Ł' Q^i\right),\ \ell = 1, 2, ..., \infty.$$

For $(Ł, Ł') \in \mathbb{R}^2$, the Q-analog of $(Ł - Ł')^\ell$ is defined by

$$(Ł - Ł')^{(0)} = 1,\ (Ł - Ł')^{(\ell)} = \prod_{i=0}^{\ell-1} \left(Ł - Ł' Q^i\right),\ \ell = 1, 2, ..., \infty.$$

For arbitrary $\beta \in \mathbb{R},\ (Ł, Ł') \in \mathbb{R}^2$ and $Ł \geq 0$,

$$(Ł - Ł')^{(\beta)} = Ł^\beta \prod_{i=0}^{\infty} \left(\frac{Ł - Ł' Q^i}{Ł - Ł' Q^{\beta+i}}\right).$$

For $Ł' = 0$, we have $Ł^{(\beta)} = Ł^\beta$.
The Q-gamma function is given by

$$\Gamma_Q(Ł) = \frac{(1 - Q)^{(Ł-1)}}{(1 - Q)^{Ł-1}},\ Ł \notin \{0, -1, -2, ...\}.$$

The (P, Q)-bracket or twin-basic number is defined by Sadjang [16] as follows. For arbitrary $Ł \in \mathbb{R}$, we have

$$[Ł]_{P,Q} = \frac{P^{Ł} - Q^{Ł}}{P - Q}.$$

For arbitrary $Ł, Ł' \in \mathbb{R}$, we define the (P, Q)-analog of $(Ł - Ł')^\ell$ as follows:

$$(Ł - Ł')_{P,Q}^{(0)} = 1,$$

$$(Ł - Ł')_{P,Q}^{(\ell)} = \prod_{i=0}^{\ell-1} \left(ŁP^i - Ł' Q^i\right), \ell = 1, 2, 3, \ldots$$

and for arbitrary $\beta \in \mathbb{R}$ and for arbitrary $Ł \geq 0$,

$$(Ł - Ł')_{P,Q}^{(\beta)} = Ł^\beta \prod_{i=0}^{\infty} \left(\frac{ŁP^i - Ł' Q^i}{ŁP^i - Ł' Q^{\beta+i}}\right).$$

For $Ł' = 0$, we have $(Ł - Ł')_{p,q}^{(\beta)} = Ł^\beta$.

Lemma 2.1. If $\beta > 0$ and $A \leq B \leq T$ then $(T-A)^{(\beta)}_{P,Q} \geq (T-B)^{(\beta)}_{P,Q}$.

Proof. We have to know that

$$T^{\beta}\prod_{i=0}^{\infty}\left(\frac{TP^i - AQ^i}{TP^i - AQ^{\beta+i}}\right) \geq T^{\beta}\prod_{i=0}^{\infty}\left(\frac{TP^i - BQ^i}{TP^i - BQ^{\beta+i}}\right).$$

For each $i \in \mathbb{N}_0$, we show that

$$\begin{aligned}
&\left(TP^i - AQ^i\right)\left(TP^i - BQ^{\beta+i}\right) \geq \left(TP^i - BQ^i\right)\left(TP^i - AQ^{\beta+i}\right)\\
&\Leftrightarrow BP^iQ^{\beta+i} + AP^iQ^i \geq AP^iQ^{\beta+i} + BP^iQ^i\\
&\Leftrightarrow A + BQ^{\beta} \geq B + AQ^{\beta}\\
&\Leftrightarrow B - A \leq Q^{\beta}(B-A).
\end{aligned}$$

For $A = B$, we have $B - A = Q^{\beta}(B-A)$ and for $A \neq B$, we have $Q^{\beta} \leq 1$. □

We define the (P,Q)-analogue of the Gamma function as follows:

$$\Gamma_{P,Q}(Ł) = \frac{(P-Q)^{(Ł-1)}_{P,Q}}{(P-Q)^{Ł-1}},\ Ł \notin \{0,-1,-2,...\}.$$

For $P = 1$, we can see that $\Gamma_{P,Q}$ reduces to Γ_Q. Clearly, we can see that

$$\Gamma_{P,Q}(Ł+1) \neq [Ł]_{P,Q}\Gamma_{P,Q}(Ł).$$

Only for $P = 1$ the equality holds.

Let $\mathbf{f} : [0,\bar{a}] \to \mathbb{R}$ be a function where $\bar{a}$ is a nonnegative real number. Sadjang [16], defined the (P,Q)-integral of the function $\mathbf{f}$ as follows:

$$\int_0^{\theta}\mathbf{f}(Ł)d_{P,Q}Ł = (P-Q)\theta\sum_{\ell=0}^{\infty}\frac{Q^{\ell}}{P^{\ell+1}}\mathbf{f}(\frac{Q^{\ell}}{P^{\ell+1}}\theta),$$

where $\left|\frac{P}{Q}\right| > 1$ and $\theta \in [0,\bar{a}]$, provided that the sum converges absolutely. For $P = 1$, we get $\int_0^{\theta}\mathbf{f}(Ł)d_{P,Q}Ł = \int_0^{\theta}\mathbf{f}(Ł)d_QŁ$.

Lemma 2.2. Let $\mathbf{f} : [0,1] \to \mathbb{R}$ be a continuous function. Then

$$\left|\int_0^{\theta}\mathbf{f}(Ł)d_{P,Q}Ł\right| \leq \int_0^{\theta}|\mathbf{f}(Ł)|\,d_{P,Q}Ł$$

for all $\theta \in [0,1]$.

Proof. We have

$$\left|\int_0^\theta \mathbf{f}(Ł)d_{P,Q}Ł\right| = \left|(P-Q)\theta\sum_{\ell=0}^{\infty}\frac{Q^\ell}{P^{\ell+1}}\mathbf{f}(\frac{Q^\ell}{P^{\ell+1}}\theta)\right|, \left|\frac{P}{Q}\right| > 1$$
$$\leq (P-Q)\theta\sum_{\ell=0}^{\infty}\frac{Q^\ell}{P^{\ell+1}}\left|\mathbf{f}(\frac{Q^\ell}{P^{\ell+1}}\theta)\right|$$
$$= \int_0^\theta |\mathbf{f}(Ł)|\, d_{P,Q}Ł.$$

□

Remark 2.3. If $\mathbf{f}(Ł) = 1$ for all $Ł \in I = [0,1]$, then for any $\theta \in I$, we have

$$\int_0^\theta \mathbf{f}(Ł)d_{P,Q}Ł = \int_0^\theta d_{P,Q}Ł$$
$$= (P-Q)\theta\sum_{\ell=0}^{\infty}\frac{Q^\ell}{P^{\ell+1}}, \left|\frac{P}{Q}\right| > 1$$
$$= \left(\frac{P-Q}{P}\right)\theta\sum_{\ell=0}^{\infty}\left(\frac{Q}{P}\right)^\ell$$
$$= \left(\frac{P-Q}{P}\right)\theta\left(\frac{P}{P-Q}\right)$$
$$= \theta.$$

We introduce the fractional (P,Q)-integral of order $\alpha \geq 0$ of the function $\mathbf{f}$ which is given by

$$I^0_{P,Q}\mathbf{f}(\theta) = \mathbf{f}(\theta)$$

and

$$I^\alpha_{P,Q}\mathbf{f}(\theta) = \frac{1}{\Gamma_{P,Q}(\alpha)}\int_0^\theta (P\theta - Q\theta_1)^{(\alpha-1)}_{P,Q}\,\mathbf{f}(\theta_1)d_{P,Q}\theta_1,$$

where $\theta \in [0,1]$ and $\alpha > 1$. For $P = 1$, we get $I^\alpha_{P,Q}\mathbf{f}(\theta) = I^\alpha_Q\mathbf{f}(\theta)$.

Definition 2.4. *[8] A strongly continuous semigroup on E is a mapping $S : [0,\infty) \to \mathcal{L}(E)$ so that:*
(1) $S(0) = I_i$ and $S(t+s) = S(t)S(s)$ for all $t, s \geq 0$ where I_i is the identity mapping.
(2) $S(x)$ is continuous on $[0,\infty)$ for all $x \in E$ where E is a complex Banach space and $\mathcal{L}(E)$ is the Banach algebra of all continuous linear mappings defined on $\mathbb{B}$.

Let $f_1, f_2 \in C[0,1]$ and $k_1, k_2 \in \mathbb{R}$. Therefore

$$I^\alpha_{P,Q}\left[k_1f_1(\theta) + k_2f_2(\theta)\right]$$
$$= \frac{1}{\Gamma_{P,Q}(\alpha)}\int_0^\theta (P\theta - Q\theta_1)^{(\alpha-1)}_{P,Q}\left[k_1f_1(\theta_1) + k_2f_2(\theta_1)\right]d_{P,Q}\theta_1$$
$$= k_1I^\alpha_{P,Q}f_1(\theta) + k_2I^\alpha_{P,Q}f_2(\theta).$$

Hence, the operator $I^{\alpha}_{P,Q}$ is linear.
Again for $f_1(\theta), f_2(\theta) \geq 0$ we observe that

$$I^{\alpha}_{P,Q}[f_1(\theta)+f_2(\theta)] = I^{\alpha}_{P,Q}[f_1(\theta)] + I^{\alpha}_{P,Q}[f_2(\theta)] \neq I^{\alpha}_{P,Q}[f_1(\theta)]\, I^{\alpha}_{P,Q}[f_2(\theta)]$$

and $I^{\alpha}_{P,Q}[0] = 0 \neq I_i$.
Hence, we conclude that the operator $I^{\alpha}_{P,Q}$ is not an strongly continuous semigroup on $C([0,1])$.

Suppose that $\mathbf{E}$ be a real Banach space. Let $\bar{\mathbf{B}}(\mathbf{y_0}, \mathbf{d})$ be the closed ball in $\mathbf{E}$ with center $\mathbf{y_0}$ and radius $\mathbf{d}$. By $\bar{Ł}$ and ConvŁ we denote the closure and the convex closure of Ł. Moreover, let $\mathbf{NB_E}$ be the family of all nonempty and bounded subsets of $\mathbf{E}$ and $\mathbf{RC_E}$ be its subfamily consisting of all relatively compact sets.

The following definition of a measure of noncompactness has been presented in [7].

Definition 2.5. $\mu : \mathbf{NB_E} \to [0,\infty)$ is called a measure of noncompactness if:

(i) $\mu(Ł) = 0$ implies that Ł is precompact for all $Ł \in \mathbf{NB_E}$,
(ii) the family $\ker \mu = \{Ł \in \mathbf{NB_E} : \mu(Ł) = 0\}$ is nonempty and $\ker \mu \subset \mathbf{RC_E}$,
(iii) $Ł \subset Ł' \implies \mu(Ł) \leq \mu(Ł')$,
(iv) $\mu\left(\bar{Ł}\right) = \mu(Ł)$,
(v) $\mu(\text{Conv}Ł) = \mu(Ł)$,
(vi) $\mu(\lambda Ł + (1-\lambda) Ł') \leq \lambda\mu(Ł) + (1-\lambda)\mu(Ł')$ for all $\lambda \in [0,1]$,
(vii) $\bigcap_{\mathbf{n}=1}^{\infty} Ł_{\mathbf{n}} \neq \emptyset$ whenever $Ł_{\mathbf{n}} \in \mathbf{NB_E}$, $Ł_{\mathbf{n}} = \bar{Ł}_{\mathbf{n}}$, $Ł_{\mathbf{n+1}} \subseteq Ł_{\mathbf{n}}$ for all $\mathbf{n} = 1,2,3,\ldots$ and $\lim_{\mathbf{n}\to\infty} \mu(Ł_{\mathbf{n}}) = 0$.

The family $\ker \mu$ is said to be the *kernel of measure* μ.

A measure μ is called sublinear if:

(1) $\mu(\lambda Ł) = |\lambda|\, \mu(Ł)$ for all $\lambda \in \mathbb{R}$,
(2) $\mu(Ł + Ł') \leq \mu(Ł) + \mu(Ł')$.

A sublinear measure of noncompactness μ so that

$$\mu(Ł \cup Ł') = \max\{\mu(Ł), \mu(Ł')\}$$

and $\ker \mu = \mathbf{RC_E}$ is said to be regular.
For a bounded subset Q of a metric space Ł,

$$\alpha(Q) = \inf\left\{\delta > 0 : Q = \bigcup_{\mathbf{i}=1}^{\mathbf{n}} Q_{\mathbf{i}},\ \text{diam}(Q_i) \leq \delta \ \textit{for}\ 1 \leq \mathbf{i} \leq \mathbf{n} \leq \infty\right\},$$

is the Kuratowski measure of noncompactness of Q where $\text{diam}(Q_{\mathbf{i}})$ denotes the diameter of the set $Q_{\mathbf{i}}$, that is,

$$\text{diam}(Q_{\mathbf{i}}) = \sup\{\mathbf{d}(\mathbf{x}, \mathbf{y}) : \mathbf{x}, \mathbf{y} \in Q_{\mathbf{i}}\},$$

and

$$\chi(Q) = \inf\{\epsilon > 0 : Q \text{ has a finite } \epsilon-\text{net in } Ł\},$$

is the Hausdorff measure of noncompactness for Q.
Recall the following fixed point theorems:

Theorem 2.6. *[1, Schauder fixed-point theorem] Let* $\mathbb{E}$ *be a Banach space and* $\aleph(\neq \emptyset) \subseteq \mathbb{E}$ *be closed and convex. Then any* $\Delta : \aleph \to \aleph$ *which is continuous and compact, admits at least one fixed point.*

Theorem 2.7. *[9, Darbo fixed-point theorem] Let $\mathbb{E}$ be a Banach space and $\aleph \subseteq \mathbb{E}$ be nonempty, bounded, closed and convex (NBCC) and μ is a measure of noncompactness defined in $\mathbb{E}$. Also, let $\Delta : \aleph \to \aleph$ be continuous and there exists a constant $0 \leq \tau < 1$ with*

$$\mu(\Delta\Pi) \leq \tau \cdot \mu(\Pi),\ \Pi \subseteq \aleph.$$

Then Δ has a fixed point.

In this section, we establish a generalization of Darbo's fixed point theorem with the help of following concepts:

Definition 2.8. *[15] Let functions $\wp_1, \wp_2 : \mathbb{R}_+ \to \mathbb{R}$ be given. The pair $(\wp_1, \wp_2)$ is called a pair of shifting distance functions(SDF) if:*
(1) $\wp_1(l) \leq \wp_2(m)$, then $l \leq m$, for all $l, m \in \mathbb{R}_+$,
(2) for all $l_k, m_k \in \mathbb{R}_+$ with $\lim_{k\to\infty} l_k = \lim_{k\to\infty} m_k = w$, if $\wp_1(l_k) \leq \wp_2(m_k)$ for all k, then $w = 0$.

Following examples of $\wp$ represents a pair $(\wp_1, \wp_2)$ of a SDF.
(1) $\wp_1(\xi) = \ln\left(\frac{1+2\xi}{2}\right)$ and $\wp_2(\xi) = \ln\left(\frac{1+\xi}{2}\right)$.
(2) $\wp_1(\xi) = \xi$ and $\wp_2(\xi) = \lambda\xi,\ \lambda \in [0, 1)$.

Definition 2.9. *Let $\mathbb{F}$ be the family of all continuous and nondecreasing maps $F : \mathbb{R}_+^3 \to \mathbb{R}_+$ with:*

(1) $\max\{a, b, c\} \leq F(a, b, c)$ *for all* $a, b, c \geq 0$,
(2) $F(a, 0, 0) = a$ *for all* $a \geq 0$.

For example, $F : \mathbb{R}_+^3 \to \mathbb{R}_+$ defined by $F(a, b, c) = a + b + c$ is an element of $\mathbb{F}$.

3. New results

From now on, let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be nondecreasing and continuous with $\phi(t) = 0$ iff $t = 0$ and $\phi(t) < t$ for all $t > 0$.

Theorem 3.1. *Let $\mathbb{E}$ be a Banach space, $\mathbb{C} \subseteq \mathbb{E}$ be a NBCC and $T : \mathbb{C} \to \mathbb{C}$ be a continuous function with*

$$\wp_1\left[F\left(\mu(T\text{Ł}), \gamma_1\left(\mu(T\text{Ł})\right), \gamma_2\left(\mu(T\text{Ł})\right)\right)\right] \leq \wp_2\left[\phi\left\{F\left(\mu(\text{Ł}), \gamma_1\left(\mu(\text{Ł})\right), \gamma_2\left(\mu(\text{Ł})\right)\right)\right\}\right], \tag{2}$$

for all Ł$(\neq \emptyset) \subseteq \mathbb{C}$; $F \in \mathbb{F}$; $\wp_1, \wp_2 \in \wp$ and $\gamma_1, \gamma_2 : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous nondecreasing functions where μ is an arbitrary MNC. Then T admits a fixed point in $\mathbb{C}$.

Proof. Define a sequence $(\mathbb{C}_s)$, where $\mathbb{C}_1 = \mathbb{C}$ and $\mathbb{C}_{s+1} = \overline{Conv}(T\mathbb{C}_s)$, for all $s \geq 1$. Also, $T\mathbb{C}_1 = T\mathbb{C} \subseteq \mathbb{C} = \mathbb{C}_1$, $\mathbb{C}_2 = \overline{Conv}(T\mathbb{C}_1) \subseteq \mathbb{C} = \mathbb{C}_1$. Similarly, $\mathbb{C}_1 \supseteq \mathbb{C}_2 \supseteq \mathbb{C}_3 \supseteq \ldots \supseteq \mathbb{C}_s \supseteq \mathbb{C}_{s+1} \supseteq \ldots$.
If $s_0 \in \mathbb{N}$ with $\mu(\mathbb{C}_{s_0}) = 0$, then $\mathbb{C}_{s_0}$ is compact. So, applying Theorem 2.6 we observed that T admits a fixed point.
Let $\mu(\mathbb{C}_s) > 0$ for all $s \geq 0$. By (2) we have

$$\begin{aligned}
&\wp_1\left[F\left(\mu(\mathbb{C}_{s+1}), \gamma_1\left(\mu(\mathbb{C}_{s+1})\right), \gamma_2\left(\mu(\mathbb{C}_{s+1})\right)\right)\right] \\
&= \wp_1\left[F\left(\mu(\overline{Conv}(T\mathbb{C}_s)), \gamma_1\left(\mu(\overline{Conv}(T\mathbb{C}_s))\right), \gamma_2\left(\mu(\overline{Conv}(T\mathbb{C}_s))\right)\right)\right] \\
&= \wp_1\left[F\left(\mu(T\mathbb{C}_s), \gamma_1\left(\mu(T\mathbb{C}_s)\right), \gamma_2\left(\mu(T\mathbb{C}_s)\right)\right)\right] \\
&\leq \wp_2\left[\phi\left\{F\left(\mu(\mathbb{C}_s), \gamma_1\left(\mu(\mathbb{C}_s)\right), \gamma_2\left(\mu(\mathbb{C}_s)\right)\right)\right\}\right],
\end{aligned}$$

which gives

$$\begin{aligned}&F\left(\mu(\mathbb{C}_{s+1}),\gamma_1\left(\mu(\mathbb{C}_{s+1})\right),\gamma_2\left(\mu(\mathbb{C}_{s+1})\right)\right)\\ &\le \phi\left\{F\left(\mu(\mathbb{C}_s),\gamma_1\left(\mu(\mathbb{C}_s)\right),\gamma_2\left(\mu(\mathbb{C}_s)\right)\right)\right\}\\ &< F\left(\mu(\mathbb{C}_s),\gamma_1\left(\mu(\mathbb{C}_s)\right),\gamma_2\left(\mu(\mathbb{C}_s)\right)\right).\end{aligned}$$

Clearly, the sequence $\{F\left(\mu(T\mathbb{C}_s),\gamma_1\left(\mu(T\mathbb{C}_s)\right),\gamma_2\left(\mu(T\mathbb{C}_s)\right)\right)\}_{s=1}^{\infty}$ is positive and decreasing. So, we can find a $d \ge 0$ such that

$$\lim_{s\to\infty} F\left(\mu(T\mathbb{C}_s),\gamma_1\left(\mu(\mathbb{C}_s)\right),\gamma_2\left(\mu(T\mathbb{C}_s)\right)\right) = d.$$

If $d = 0$, then the result is obvious.

If possible, assume that $d > 0$.

As $s \to \infty$, then we get $d < \phi(d)$ which is a contradiction. Hence, $\lim_{s\to\infty} F\left(\mu(\mathbb{C}_s),\gamma_1\left(\mu(\mathbb{C}_s)\right),\gamma_2\left(\mu(\mathbb{C}_s)\right)\right) = 0$, i.e., $d = 0$ which gives

$$F\left(\lim_{s\to\infty}\mu(\mathbb{C}_s),\lim_{s\to\infty}\gamma_1\left(\mu(\mathbb{C}_s)\right),\lim_{s\to\infty}\gamma_2\left(\mu(\mathbb{C}_s)\right)\right) = 0.$$

By using the property of F we get $\lim_{s\to\infty}\mu\left(\mathbb{C}_s\right) = 0$.

We know that $\mathbb{C}_s \supseteq \mathbb{C}_{s+1}$ and by Definition 2.5 we get $\mathbb{C}_\infty = \bigcap_{s=1}^{\infty}\mathbb{C}_s \subseteq \mathbb{C}$ is nonempty, closed and convex. Also, $\mathbb{C}_\infty$ is invariant under F. Thus, Theorem 2.6 implies that F has a fixed point in $\mathbb{C}_\infty \subseteq \mathbb{C}$. □

Theorem 3.2. *Let $\mathbb{E}$ be a Banach space, $\mathbb{C} \subseteq \mathbb{E}$ be a NBCC and $T : \mathbb{C} \to \mathbb{C}$ be a continuous function such that*

$$\wp_1\left[\mu(T\text{Ł}) + \gamma_1\left(\mu(T\text{Ł})\right) + \gamma_2\left(\mu(T\text{Ł})\right)\right] \le \wp_2\left[\phi\left\{\mu(\text{Ł}) + \gamma_1\left(\mu(\text{Ł})\right) + \gamma_2\left(\mu(\text{Ł})\right)\right\}\right], \tag{3}$$

for all $\text{Ł}(\neq \emptyset) \subseteq \mathbb{C}$; $\wp_1, \wp_2 \in \wp$ where $\gamma_1, \gamma_2 : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous nondecreasing functions and μ is an arbitrary MNC. Then T admits a fixed point in $\mathbb{C}$.

Proof. The result can be obtained by taking $F(a, b, c) = a + b + c$ in Theorem 3.1. □

Corollary 3.3. *Let $\mathbb{E}$ be a Banach space, $\mathbb{C} \subseteq \mathbb{E}$ be a NBCC and $T : \mathbb{C} \to \mathbb{C}$ be a continuous function with*

$$\wp_1\left[\mu(T\text{Ł})\right] \le \wp_2\left[\phi\left\{\mu(\text{Ł})\right\}\right], \tag{4}$$

for all $\text{Ł}(\neq \emptyset) \subseteq \mathbb{C}$ and $\wp_1, \wp_2 \in \wp$ where μ is an arbitrary MNC. Then T admits a fixed point in $\mathbb{C}$.

Proof. Let $\gamma_1(t) = \gamma_2(t) = 0$ for all $t \ge 0$ in Theorem 3.2. □

Corollary 3.4. *Let $\mathbb{E}$ be a Banach space, $\mathbb{C} \subseteq \mathbb{E}$ be a NBCC and $T : \mathbb{C} \to \mathbb{C}$ be a continuous function with*

$$\mu(T\text{Ł}) \le \phi\left\{\mu(\text{Ł})\right\}, \tag{5}$$

for all $\text{Ł}(\neq \emptyset) \subseteq \mathbb{C}$ where μ is an arbitrary MNC. Then T admits a fixed point in $\mathbb{C}$.

Proof. Let $\wp_1(t) = \wp_2(t) = t$ for all $t \ge 0$ in Corollary 3.3. □

Remark 3.5. *For $\phi(t) = kt$ where $k \in [0, 1)$ and $t \in \mathbb{R}_+$ in Corollary 3.4 we obtain the Darbo's fixed point theorem.*

4. Application

In this section, we establish the existence of solution of the equation (1) in the space $\mathbf{E} = C(I)$, where $C(I)$ is the set of real and continuous functions defined on the compact set I. We also know that $\mathbf{E}$ is a Banach space with respect to the norm

$$\| Ł \| = \max\{|Ł(\theta)| : \theta \in I\}, \ Ł \in \mathbf{E}.$$

Let $\mathbf{M} \in \mathbf{NB_E}$. For $(Ł, \mathbf{r}) \in \mathbf{M} \times (0,\infty)$, we denote by $\omega(Ł, \mathbf{r})$ the modulus of continuity of Ł, i.e.,

$$\omega(Ł, \mathbf{r}) = \sup\{|Ł(\theta) - Ł(\theta_1)| : \theta, \theta_1 \in I, |\theta - \theta_1| \le \mathbf{r}\}.$$

Further we define

$$\omega(\mathbf{M}, \mathbf{r}) = \sup\{\omega(Ł, \mathbf{r}) : Ł \in \mathbf{M}\}.$$

Define the mapping $\mu : \mathbf{NB_E} \to [0,\infty)$ by

$$\mu(\mathbf{M}) = \lim_{\mathbf{r}\to 0^+} \omega(\mathbf{M}, \mathbf{r}), \mathbf{M} \in \mathbf{NB_E}.$$

Then μ is a measure of non-compactness in $\mathbf{E}$ (see[7]).

Let us define the operator $\mathcal{T}$ on $\mathbf{E}$ by

$$(\mathcal{T}Ł)(\theta) = \eta\left(\theta, Ł(\bar{a}(\theta)), \frac{\mathcal{F}(\theta, Ł(\bar{b}(\theta)))}{\Gamma_{P,Q}(\alpha)} \int_0^\theta (P\theta - Q\theta_1)_{P,Q}^{(\alpha-1)} \mathcal{U}(\theta_1, Ł(\theta_1)) d_{P,Q}\theta_1\right),$$

where $Ł \in \mathbf{E}$ and $\theta \in I$.

We consider the following assumptions:

(1) The functions $\mathcal{F}, \mathcal{U} : I \times \mathbb{R} \to \mathbb{R}; \bar{a}, \bar{b} : I \to I$ and $\eta : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous.

(2) There exists a constant $\mathcal{D}_\eta > 0$ and non-decreasing function $\psi_\eta : [0,\infty) \to [0,\infty)$ such that

$$\left|\eta(\theta, Ł, Ł') - \eta(\theta, Z, W)\right| \le \psi_\eta(|Ł - Z|) + \mathcal{D}_\eta |Ł' - W|$$

for all $\theta \in I$ and for all $Ł, Ł', Z, W \in \mathbb{R}$.

(3) There exists a constant $\mathcal{D}_\mathcal{F} > 0$ such that

$$|\mathcal{F}(\theta, Ł) - \mathcal{F}(\theta, Ł')| \le \mathcal{D}_\mathcal{F} |Ł - Ł'|$$

for all $\theta \in I$ and for all $Ł, Ł' \in \mathbb{R}$.

(4) There exists a non-decreasing and continuous function $\psi_\mathcal{U} : [0,\infty) \to [0,\infty)$ such that

$$|\mathcal{U}(\theta, Ł) - \mathcal{U}(\theta, Ł')| \le \psi_\mathcal{U}(|Ł - Ł'|),$$

where $\theta \in I$ and $Ł, Ł' \in \mathbb{R}$. Also, $\psi_\mathcal{U}(\theta) < \theta, \theta > 0$ and $\mathcal{U}(\theta, 0) = 0$ for all $\theta \in I$.

(5) There exists $\mathbf{r}_0 > 0$ such that

$$\psi_\eta(\mathbf{r}_0) + \frac{\mathcal{D}_\eta \psi_\mathcal{U}(\mathbf{r}_0)}{\left|\Gamma_{P,Q}(\alpha)\right|}\left(\mathcal{D}_\mathcal{F}\mathbf{r}_0 + \hat{\mathcal{F}}\right) + \hat{\eta} \le \mathbf{r}_0,$$

where $\hat{\eta} = \max\left\{\left|\eta(\theta, 0, 0)\right| : \theta \in I\right\}$ and $\hat{\mathcal{F}} = \max\{|\mathcal{F}(\theta, 0)| : \theta \in I\}$.

(6) The function $\psi_\eta : [0,\infty) \to [0,\infty)$ is continuous so that $\psi_\eta(\theta) < \hat{L}\theta$ for all $\theta > 0$ where $\hat{L} > 0$ is a constant.

(7) The function $\bar{a} : I \to I$ satisfies

$$|\bar{a}(\theta) - \bar{a}(\theta_1)| \leq \psi_{\bar{a}}(|\theta - \theta_1|)$$

for all $\theta, \theta_1 \in I$ and $\psi_{\bar{a}} : [0, \infty) \to [0, \infty)$ is non-decreasing and $\lim_{\theta \to 0^+} \psi_{\bar{a}}(\theta) = 0$.

(8) The function $\bar{b} : I \to I$ satisfies

$$\left|\bar{b}(\theta) - \bar{b}(\theta_1)\right| \leq \psi_{\bar{b}}(|\theta - \theta_1|)$$

for all $\theta, \theta_1 \in I$ and $\psi_{\bar{b}} : [0, \infty) \to [0, \infty)$ is non-decreasing and $\lim_{\theta \to 0^+} \psi_{\bar{b}}(\theta) = 0$.

(9) We suppose that $0 < \psi_{\mathcal{U}}(\mathbf{r}_0) < \frac{|\Gamma_{P,Q}(\alpha)|}{\mathcal{D}_{\mathcal{F}} \mathcal{D}_\eta}$ and $\frac{\mathcal{D}_\eta}{|\Gamma_{P,Q}(\alpha)|}\left(\mathcal{D}_{\mathcal{F}} \mathbf{r}_0 + \hat{\mathcal{F}}\right) < 1$. Also,

$$\hat{L} + \mathcal{L} + \mathcal{N} < 1,$$

where $\mathcal{L} = \frac{\mathcal{D}_\eta(\mathcal{D}_{\mathcal{F}} \mathbf{r}_0 + \hat{\mathcal{F}})}{|\Gamma_{P,Q}(\alpha)|}$ and $\mathcal{N} = \frac{\mathcal{D}_\eta \mathcal{D}_{\mathcal{F}} \psi_{\mathcal{U}}(\mathbf{r}_0)}{|\Gamma_{P,Q}(\alpha)|}$.

Let the closed ball with center 0 and radius $\mathbf{r}_0$ be denoted by $\bar{\mathbf{B}}(0, \mathbf{r}_0) = \{Ł \in \mathbf{E} : \| Ł \| \leq \mathbf{r}_0\}$.

Theorem 4.1. Under the hypothesis (1)-(9), equation (1) has at least one solution in $\mathbf{E} = C(I)$.

Proof. As $\theta_1 \in [0, 1] = I$ so $\theta_1 \geq 1$. Also $\alpha - 1 > 0$. By applying Lemma 2.1 we have

$$(P\theta - Q\theta_1)_{P,Q}^{(\alpha-1)} \leq (P\theta - 0)_{P,Q}^{(\alpha-1)} = (P\theta)_{P,Q}^{(\alpha-1)}$$

i.e

$$(P\theta - Q\theta_1)_{P,Q}^{(\alpha-1)} \leq P^{\alpha-1}\theta^{\alpha-1}.$$

Since $P \leq 1$ therefore $(P\theta - Q\theta_1)_{P,Q}^{(\alpha-1)} \leq \theta^{\alpha-1}$.

Let $Ł \in \bar{\mathbf{B}}(0, \mathbf{r}_0)$. By using assumptions (1)-(9), for all $\theta \in I$, we have

$$\begin{aligned}
&|(\mathcal{T}Ł)(\theta)| \\
&\leq \left|\eta\left(\theta, Ł(\bar{a}(\theta)), \frac{\mathcal{F}(\theta, Ł(\bar{b}(\theta)))}{\Gamma_{P,Q}(\alpha)} \int_0^\theta (P\theta - Q\theta_1)_{P,Q}^{(\alpha-1)} \mathcal{U}(\theta_1, Ł(\theta_1)) d_{P,Q}\theta_1\right) - \eta(\theta, 0, 0)\right| + |\eta(\theta, 0, 0)| \\
&\leq \psi_\eta\left(|Ł(\bar{a}(\theta))|\right) + \mathcal{D}_\eta \frac{\left|\mathcal{F}(\theta, Ł(\bar{b}(\theta)))\right|}{\left|\Gamma_{P,Q}(\alpha)\right|} \int_0^\theta \left|(P\theta - Q\theta_1)_{P,Q}^{(\alpha-1)}\right| |\mathcal{U}(\theta_1, Ł(\theta_1))| \, d_{P,Q}\theta_1 + \hat{\eta} \\
&\leq \psi_\eta\left(\| Ł \|\right) + \frac{\mathcal{D}_\eta}{\left|\Gamma_{P,Q}(\alpha)\right|}\left\{\left|\mathcal{F}(\theta, Ł(\bar{b}(\theta))) - \mathcal{F}(\theta, 0)\right| + |\mathcal{F}(\theta, 0)|\right\} \int_0^\theta \theta^{\alpha-1} |\mathcal{U}(\theta_1, Ł(\theta_1))| \, d_{P,Q}\theta_1 + \hat{\eta} \\
&\leq \psi_\eta\left(\mathbf{r}_0\right) + \frac{\mathcal{D}_\eta}{\left|\Gamma_{P,Q}(\alpha)\right|}\left(\mathcal{D}_{\mathcal{F}} \| Ł \| + \hat{\mathcal{F}}\right) \int_0^\theta \theta^{\alpha-1} \psi_{\mathcal{U}}(\| Ł \|) d_{P,Q}\theta_1 + \hat{\eta} \\
&\leq \psi_\eta\left(\mathbf{r}_0\right) + \frac{\mathcal{D}_\eta \psi_{\mathcal{U}}(\mathbf{r}_0)}{\left|\Gamma_{P,Q}(\alpha)\right|}\left(\mathcal{D}_{\mathcal{F}} \mathbf{r}_0 + \hat{\mathcal{F}}\right) \int_0^\theta \theta^{\alpha-1} d_{P,Q}\theta_1 + \hat{\eta} \\
&= \psi_\eta\left(\mathbf{r}_0\right) + \frac{\mathcal{D}_\eta \psi_{\mathcal{U}}(\mathbf{r}_0)}{\left|\Gamma_{P,Q}(\alpha)\right|}\left(\mathcal{D}_{\mathcal{F}} \mathbf{r}_0 + \hat{\mathcal{F}}\right) \theta^\alpha + \hat{\eta} \\
&\leq \mathbf{r}_0,
\end{aligned}$$

i.e., $(\mathcal{T}Ł)(\theta) \in \bar{\mathbf{B}}(0, \mathbf{r}_0)$. Thus, $\mathcal{T}$ maps $\bar{\mathbf{B}}(0, \mathbf{r}_0)$ into itself.

We have to show that $\mathcal{T}$ is continuous on $\bar{\mathbf{B}}(0, \mathbf{r}_0)$. Let us define the operators λ_1, λ_2 and λ_3 on $\mathbf{E}$ by

$$(\lambda_1 Ł)(\theta) = \theta$$

$$(\lambda_2 Ł)(\theta) = Ł(\bar{a}(\theta))$$

and

$$(\lambda_3 Ł)(\theta) = \mathcal{F}(\theta, Ł(\bar{b}(\theta)))$$

for all $\theta \in I$ and $Ł \in \mathbf{E}$. It is obvious that λ_1 is continuous.
For all $Ł, Ł' \in \mathbf{E}$ we have

$$|(\lambda_2 Ł)(\theta) - (\lambda_2 Ł')(\theta)| = |Ł(\bar{a}(\theta)) - Ł'(\bar{a}(\theta))| \leq \| Ł - Ł' \|,$$

for all $\theta \in I$ which gives $\| \lambda_2 Ł - \lambda_2 Ł' \| \leq \| Ł - Ł' \|$. Therefore, λ_2 is uniformly continuous on $\mathbf{E}$.
Similarly, we can show that $\| \lambda_3 Ł - \lambda_3 Ł' \| \leq \mathcal{D}_{\mathcal{F}} \| Ł - Ł' \|$ for all $Ł, Ł' \in \mathbf{E}$. Therefore, λ_3 is also uniformly continuous on $\mathbf{E}$.
To prove that $\mathcal{T}$ is continuous on $\bar{\mathbf{B}}(0, \mathbf{r}_0)$, for this we show that

$$(\mathcal{H}Ł)(\theta) = \int_0^\theta (P\theta - Q\theta_1)_{P,Q}^{(\alpha-1)} \mathcal{U}(\theta_1, Ł(\theta_1)) d_{P,Q}\theta_1$$

is continuous on $\bar{\mathbf{B}}(0, \mathbf{r}_0)$. Let $\epsilon > 0$ and $Ł, Ł' \in \bar{\mathbf{B}}(0, \mathbf{r}_0)$ such that $\| Ł - Ł' \| < \epsilon$. For all $\theta \in I$ we have

$$(\mathcal{H}Ł)(\theta) - (\mathcal{H}Ł')(\theta) = \int_0^\theta (P\theta - Q\theta_1)_{P,Q}^{(\alpha-1)} \{\mathcal{U}(\theta_1, Ł(\theta_1)) - \mathcal{U}(\theta_1, Ł'(\theta_1))\} d_{P,Q}\theta_1.$$

Let $\mathcal{U}_{\mathbf{r}_0}(\epsilon) = \sup \{|\mathcal{U}(\theta, Ł) - \mathcal{U}(\theta, Ł')| : \theta \in I; Ł, Ł' \in \bar{\mathbf{B}}(0, \mathbf{r}_0); \| Ł - Ł' \| < \epsilon\}$.
Therefore,

$$\begin{aligned}(\mathcal{H}Ł)(\theta) - (\mathcal{H}Ł')(\theta) &\leq \mathcal{U}_{\mathbf{r}_0}(\epsilon) \int_0^\theta (P\theta - Q\theta_1)_{P,Q}^{(\alpha-1)} d_{P,Q}\theta_1 \\ &\leq \mathcal{U}_{\mathbf{r}_0}(\epsilon) \int_0^\theta \theta^{\alpha-1} d_{P,Q}\theta_1 \\ &= \theta^\alpha \mathcal{U}_{\mathbf{r}_0}(\epsilon) \\ &\leq \mathcal{U}_{\mathbf{r}_0}(\epsilon).\end{aligned}$$

So, we have

$$\| \mathcal{H}Ł - \mathcal{H}Ł' \| \leq \mathcal{U}_{\mathbf{r}_0}(\epsilon).$$

Using the uniform continuity of $\mathcal{U}$ on the compact set $I \times [\mathbf{r}_0, \mathbf{r}_0]$ we get

$$\lim_{\epsilon \to 0^+} \mathcal{U}_{\mathbf{r}_0}(\epsilon) = 0.$$

Thus, $\mathcal{H}$ is continuous. So, we can conclude that $\mathcal{T}$ is also continuous. Let $\mathbf{d} > 0$ and $\theta_1, \theta_2 \in I$ such that $|\theta_1 - \theta_2| \leq \mathbf{d}$. We also assume that $\theta_1 \geq \theta_2$. Now

$$\begin{aligned}
&|(\mathcal{T}Ł)(\theta_1) - (\mathcal{T}Ł)(\theta_2)| \\
&= \left|\eta\left(\theta_1, Ł(\bar{a}(\theta_1)), \frac{\mathcal{F}(\theta_1, Ł(\bar{b}(\theta_1)))}{\Gamma_{P,Q}(\alpha)} \int_0^{\theta_1} (P\theta_1 - Q\theta_3)_{P,Q}^{(\alpha-1)} \mathcal{U}(\theta_3, Ł(\theta_3)) d_{P,Q}\theta_3\right)\right. \\
&\left. - \eta\left(\theta_2, Ł(\bar{a}(\theta_2)), \frac{\mathcal{F}(\theta_2, Ł(\bar{b}(\theta_2)))}{\Gamma_{P,Q}(\alpha)} \int_0^{\theta_2} (P\theta_2 - Q\theta_3)_{P,Q}^{(\alpha-1)} \mathcal{U}(\theta_3, Ł(\theta_3)) d_{P,Q}\theta_3\right)\right| \\
&\leq \left|\eta\left(\theta_1, Ł(\bar{a}(\theta_1)), \frac{\mathcal{F}(\theta_1, Ł(\bar{b}(\theta_1)))}{\Gamma_{P,Q}(\alpha)} \int_0^{\theta_1} (P\theta_1 - Q\theta_3)_{P,Q}^{(\alpha-1)} \mathcal{U}(\theta_3, Ł(\theta_3)) d_{P,Q}\theta_3\right)\right. \\
&\left. - \eta\left(\theta_2, Ł(\bar{a}(\theta_1)), \frac{\mathcal{F}(\theta_1, Ł(\bar{b}(\theta_1)))}{\Gamma_{P,Q}(\alpha)} \int_0^{\theta_1} (P\theta_1 - Q\theta_3)_{P,Q}^{(\alpha-1)} \mathcal{U}(\theta_3, Ł(\theta_3)) d_{P,Q}\theta_3\right)\right| \\
&+ \left|\eta\left(\theta_2, Ł(\bar{a}(\theta_1)), \frac{\mathcal{F}(\theta_1, Ł(\bar{b}(\theta_1)))}{\Gamma_{P,Q}(\alpha)} \int_0^{\theta_1} (P\theta_1 - Q\theta_3)_{P,Q}^{(\alpha-1)} \mathcal{U}(\theta_3, Ł(\theta_3)) d_{P,Q}\theta_3\right)\right. \\
&\left. - \eta\left(\theta_2, Ł(\bar{a}(\theta_2)), \frac{\mathcal{F}(\theta_2, Ł(\bar{b}(\theta_2)))}{\Gamma_{P,Q}(\alpha)} \int_0^{\theta_2} (P\theta_2 - Q\theta_3)_{P,Q}^{(\alpha-1)} \mathcal{U}(\theta_3, Ł(\theta_3)) d_{P,Q}\theta_3\right)\right| \\
&= I_1 + I_2.
\end{aligned}$$

Also,

$$\begin{aligned}
&\left|\frac{\mathcal{F}(\theta_1, Ł(\bar{b}(\theta_1)))}{\Gamma_{P,Q}(\alpha)} \int_0^{\theta_1} (P\theta_1 - Q\theta_3)_{P,Q}^{(\alpha-1)} \mathcal{U}(\theta_3, Ł(\theta_3)) d_{P,Q}\theta_3\right| \\
&\leq \frac{\left|\mathcal{F}(\theta_1, Ł(\bar{b}(\theta_1)))\right|}{\left|\Gamma_{P,Q}(\alpha)\right|} \int_0^{\theta_1} \left|(P\theta_1 - Q\theta_3)_{P,Q}^{(\alpha-1)}\right| |\mathcal{U}(\theta_3, Ł(\theta_3))| \, d_{P,Q}\theta_3 \\
&\leq \frac{\left|\mathcal{F}(\theta_1, Ł(\bar{b}(\theta_1))) - \mathcal{F}(\theta_1, 0)\right| + |\mathcal{F}(\theta_1, 0)|}{\left|\Gamma_{P,Q}(\alpha)\right|} \int_0^{\theta_1} \left|(P\theta_1 - Q\theta_3)_{P,Q}^{(\alpha-1)}\right| \psi_{\mathcal{U}}(|Ł(\theta_3)|) d_{P,Q}\theta_3 \\
&\leq \frac{\left(\mathcal{D}_{\mathcal{F}} \left|Ł(\bar{b}(\theta_1))\right| + \hat{\mathcal{F}}\right) \psi_{\mathcal{U}}(\| Ł \|)}{\left|\Gamma_{P,Q}(\alpha)\right|} \int_0^{\theta_1} \left|(P\theta_1 - Q\theta_3)_{P,Q}^{(\alpha-1)}\right| d_{P,Q}\theta_3 \\
&\leq \frac{\left(\mathcal{D}_{\mathcal{F}} \| Ł \| + \hat{\mathcal{F}}\right) \psi_{\mathcal{U}}(\| Ł \|)}{\left|\Gamma_{P,Q}(\alpha)\right|} \theta_1^{\alpha} \\
&\leq \frac{\left(\mathcal{D}_{\mathcal{F}} \mathbf{r}_0 + \hat{\mathcal{F}}\right) \psi_{\mathcal{U}}(\mathbf{r}_0)}{\left|\Gamma_{P,Q}(\alpha)\right|} = \hat{\mathcal{D}}.
\end{aligned}$$

Set

$$\mathcal{D}(\eta, \mathbf{d}) = \sup\left\{\left|\eta(\theta, Ł, Ł') - \eta(\theta_1, Ł, Ł')\right| : \theta, \theta_1 \in I; |\theta - \theta_1| < \mathbf{d}, Ł \in [-\mathbf{r}_0, \mathbf{r}_0], Ł' \in [-\hat{\mathcal{D}}, \hat{\mathcal{D}}]\right\}.$$

Therefore, $I_1 \leq \mathcal{D}(\eta, \mathbf{d})$. Again

$$I_2 \leq \psi_\eta\left(\left|Ł(\bar{a}(\theta_1)) - Ł(\bar{a}(\theta_2))\right|\right)$$
$$+ \frac{\mathcal{D}_\eta}{\left|\Gamma_{P,Q}(\alpha)\right|}\left|\mathcal{F}(\theta_1, Ł(\bar{b}(\theta_1)))\int_0^{\theta_1}(P\theta_1 - Q\theta_3)_{P,Q}^{(\alpha-1)}\,\mathcal{U}(\theta_3, Ł(\theta_3))d_{P,Q}\theta_3\right.$$
$$\left.- \mathcal{F}(\theta_2, Ł(\bar{b}(\theta_2)))\int_0^{\theta_2}(P\theta_2 - Q\theta_3)_{P,Q}^{(\alpha-1)}\,\mathcal{U}(\theta_3, Ł(\theta_3))d_{P,Q}\theta_3\right|.$$

We have

$$|Ł(\bar{a}(\theta_1)) - Ł(\bar{a}(\theta_2))| \leq \omega(Ł \; 0 \; \bar{a}, \mathbf{d})$$

which gives

$$\psi_\eta\left(|Ł(\bar{a}(\theta_1)) - Ł(\bar{a}(\theta_2))|\right) \leq \psi_\eta\left(\omega(Ł \; 0 \; \bar{a}, \mathbf{d})\right).$$

Now, we have

$$\left|\mathcal{F}(\theta_1, Ł(\bar{b}(\theta_1)))\int_0^{\theta_1}(P\theta_1 - Q\theta_3)_{P,Q}^{(\alpha-1)}\,\mathcal{U}(\theta_3, Ł(\theta_3))d_{P,Q}\theta_3\right.$$
$$\left.- \mathcal{F}(\theta_2, Ł(\bar{b}(\theta_2)))\int_0^{\theta_2}(P\theta_2 - Q\theta_3)_{P,Q}^{(\alpha-1)}\,\mathcal{U}(\theta_3, Ł(\theta_3))d_{P,Q}\theta_3\right|$$
$$\leq \left|\mathcal{F}(\theta_1, Ł(\bar{b}(\theta_1)))\int_0^{\theta_1}(P\theta_1 - Q\theta_3)_{P,Q}^{(\alpha-1)}\,\mathcal{U}(\theta_3, Ł(\theta_3))d_{P,Q}\theta_3\right.$$
$$\left.- \mathcal{F}(\theta_2, Ł(\bar{b}(\theta_2)))\int_0^{\theta_1}(P\theta_1 - Q\theta_3)_{P,Q}^{(\alpha-1)}\,\mathcal{U}(\theta_3, Ł(\theta_3))d_{P,Q}\theta_3\right|$$
$$+ \left|\mathcal{F}(\theta_2, Ł(\bar{b}(\theta_2)))\int_0^{\theta_1}(P\theta_1 - Q\theta_3)_{P,Q}^{(\alpha-1)}\,\mathcal{U}(\theta_3, Ł(\theta_3))d_{P,Q}\theta_3\right.$$
$$\left.- \mathcal{F}(\theta_2, Ł(\bar{b}(\theta_2)))\int_0^{\theta_2}(P\theta_2 - Q\theta_3)_{P,Q}^{(\alpha-1)}\,\mathcal{U}(\theta_3, Ł(\theta_3))d_{P,Q}\theta_3\right|$$
$$\leq \left|\mathcal{F}(\theta_1, Ł(\bar{b}(\theta_1))) - \mathcal{F}(\theta_2, Ł(\bar{b}(\theta_2)))\right|\int_0^{\theta_1}\left|(P\theta_1 - Q\theta_3)_{P,Q}^{(\alpha-1)}\right||\mathcal{U}(\theta_3, Ł(\theta_3))|\,d_{p,q}\theta_3$$
$$+ \left|\mathcal{F}(\theta_2, Ł(\bar{b}(\theta_2)))\right|\left|\int_0^{\theta_1}(P\theta_1 - Q\theta_3)_{P,Q}^{(\alpha-1)}\,\mathcal{U}(\theta_3, Ł(\theta_3))d_{P,Q}\theta_3 - \int_0^{\theta_2}(P\theta_2 - Q\theta_3)_{P,Q}^{(\alpha-1)}\,\mathcal{U}(\theta_3, Ł(\theta_3))d_{P,Q}\theta_3\right|$$

$$\leq \left|\mathcal{F}(\theta_1, Ł(\bar{b}(\theta_1))) - \mathcal{F}(\theta_2, Ł(\bar{b}(\theta_2)))\right| \psi_{\mathcal{U}}(\| Ł \|)$$
$$+ \left|\mathcal{F}(\theta_2, Ł(\bar{b}(\theta_2)))\right| \left| \int_0^{\theta_1} (P\theta_1 - Q\theta_3)_{P,Q}^{(\alpha-1)} \mathcal{U}(\theta_3, Ł(\theta_3)) d_{P,Q}\theta_3 - \int_0^{\theta_2} (P\theta_2 - Q\theta_3)_{P,Q}^{(\alpha-1)} \mathcal{U}(\theta_3, Ł(\theta_3)) d_{P,Q}\theta_3 \right|$$
$$= I_3 + I_4.$$

We define

$$\omega_{\mathcal{F}}(\mathbf{r}_0, \mathbf{d}) = \sup \{|\mathcal{F}(\theta_1, Ł) - \mathcal{F}(\theta_2, Ł)| : \theta_1, \theta_2 \in I, |\theta_1 - \theta_2| \leq \mathbf{d}, Ł \in [-\mathbf{r}_0, \mathbf{r}_0]\}.$$

Then

$$I_3 \leq \psi_{\mathcal{U}}(\| Ł \|) \left|\mathcal{F}(\theta_1, Ł(\bar{b}(\theta_1))) - \mathcal{F}(\theta_1, Ł(\bar{b}(\theta_2)))\right|$$
$$+ \psi_{\mathcal{U}}(\| Ł \|) \left|\mathcal{F}(\theta_1, Ł(\bar{b}(\theta_2))) - \mathcal{F}(\theta_2, Ł(\bar{b}(\theta_2)))\right|$$
$$\leq \psi_{\mathcal{U}}(\mathbf{r}_0) \left[\mathcal{D}_{\mathcal{F}} \left|Ł(\bar{b}(\theta_1)) - Ł(\bar{b}(\theta_2))\right|\right] + \psi_{\mathcal{U}}(\mathbf{r}_0)\omega_{\mathcal{F}}(\mathbf{r}_0, \mathbf{d})$$
$$\leq \psi_{\mathcal{U}}(\mathbf{r}_0) \left[\mathcal{D}_{\mathcal{F}}\omega(Ł \; o \; \bar{b}, \mathbf{d}) + \omega_{\mathcal{F}}(\mathbf{r}_0, \mathbf{d})\right].$$

We have

$$\left|\mathcal{F}(\theta_2, Ł(\bar{b}(\theta_2)))\right| \leq \left|\mathcal{F}(\theta_2, Ł(\bar{b}(\theta_2))) - \mathcal{F}(\theta_2, 0)\right| + |\mathcal{F}(\theta_2, 0)|$$
$$\leq \mathcal{D}_{\mathcal{F}} \left|Ł(\bar{b}(\theta_2))\right| + \hat{\mathcal{F}}$$
$$\leq \mathcal{D}_{\mathcal{F}}\mathbf{r}_0 + \hat{\mathcal{F}}.$$

Also, we have

$$\int_0^{\theta_1} (P\theta_1 - Q\theta_3)_{P,Q}^{(\alpha-1)} \mathcal{U}(\theta_3, Ł(\theta_3)) d_{P,Q}\theta_3$$
$$= (P-Q)\theta_1 \sum_{n=0}^{\infty} \frac{Q^n}{P^{n+1}} \left(P\theta_1 - \frac{Q^{n+1}}{P^{n+1}}\theta_1\right)_{P,Q}^{(\alpha-1)} \mathcal{U}\left(\frac{Q^n\theta_1}{P^{n+1}}, Ł\left(\frac{Q^n\theta_1}{P^{n+1}}\right)\right)$$

and

$$\left(P\theta_1 - \frac{Q^{n+1}}{P^{n+1}}\theta_1\right)_{P,Q}^{(\alpha-1)} = \theta_1^{\alpha-1}\left(P - \frac{Q^{n+1}}{P^{n+1}}\right)_{P,Q}^{(\alpha-1)}.$$

Therefore,

$$\int_0^{\theta_1} (P\theta_1 - Q\theta_3)_{P,Q}^{(\alpha-1)} \mathcal{U}(\theta_3, Ł(\theta_3)) d_{P,Q}\theta_3$$
$$= (P-Q)\theta_1^{\alpha} \sum_{n=0}^{\infty} \frac{Q^n}{P^{n+1}} \left(P - \frac{Q^{n+1}}{P^{n+1}}\right)_{P,Q}^{(\alpha-1)} \mathcal{U}\left(\frac{Q^n\theta_1}{P^{n+1}}, Ł\left(\frac{Q^n\theta_1}{P^{n+1}}\right)\right)$$

and

$$\left|\int_0^{\theta_1}(P\theta_1-Q\theta_3)_{P,Q}^{(\alpha-1)}\,\mathcal{U}(\theta_3,Ł(\theta_3))d_{P,Q}\theta_3-\int_0^{\theta_2}(P\theta_2-Q\theta_3)_{P,Q}^{(\alpha-1)}\,\mathcal{U}(\theta_3,Ł(\theta_3))d_{P,Q}\theta_3\right|$$
$$\leq (P-Q)\sum_{n=0}^{\infty}\frac{Q^n}{P^{n+1}}\left|\left(P-\frac{Q^{n+1}}{P^{n+1}}\right)_{P,Q}^{(\alpha-1)}\right|\left|\theta_1^{\alpha}\mathcal{U}\left(\frac{Q^n\theta_1}{P^{n+1}},Ł\left(\frac{Q^n\theta_1}{P^{n+1}}\right)\right)-\theta_2^{\alpha}\mathcal{U}\left(\frac{Q^n\theta_2}{P^{n+1}},Ł\left(\frac{Q^n\theta_2}{P^{n+1}}\right)\right)\right|.$$

We have

$$\left|\left(P-\frac{Q^{n+1}}{P^{n+1}}\right)_{P,Q}^{(\alpha-1)}\right|\leq 1$$

and since $\theta_1\leq 1$ therefore

$$\begin{aligned}
&\left|\theta_1^{\alpha}\mathcal{U}\left(\frac{Q^n\theta_1}{P^{n+1}},Ł\left(\frac{Q^n\theta_1}{P^{n+1}}\right)\right)-\theta_2^{\alpha}\mathcal{U}\left(\frac{Q^n\theta_2}{P^{n+1}},Ł\left(\frac{Q^n\theta_2}{P^{n+1}}\right)\right)\right|\\
&\leq \theta_1^{\alpha}\left|\mathcal{U}\left(\frac{Q^n\theta_1}{P^{n+1}},Ł\left(\frac{Q^n\theta_1}{P^{n+1}}\right)\right)-\mathcal{U}\left(\frac{Q^n\theta_1}{P^{n+1}},Ł\left(\frac{Q^n\theta_2}{P^{n+1}}\right)\right)\right|\\
&+\left|\theta_1^{\alpha}\mathcal{U}\left(\frac{Q^n\theta_1}{P^{n+1}},Ł\left(\frac{Q^n\theta_2}{P^{n+1}}\right)\right)-\theta_2^{\alpha}\mathcal{U}\left(\frac{Q^n\theta_2}{P^{n+1}},Ł\left(\frac{Q^n\theta_2}{P^{n+1}}\right)\right)\right|\\
&\leq \psi_{\mathcal{U}}\left(\left|Ł\left(\frac{Q^n\theta_1}{P^{n+1}}\right)-Ł\left(\frac{Q^n\theta_2}{P^{n+1}}\right)\right|\right)+\mathcal{A}_{\mathbf{d}}\\
&\leq \psi_{\mathcal{U}}\left(\omega(Ł,\mathbf{d})\right)+\mathcal{A}_{\mathbf{d}},
\end{aligned}$$

where

$$\mathcal{A}_{\mathbf{d}}=\sup\left\{\left|\theta_1^{\alpha}\mathcal{U}(\theta_4,Ł)-\theta_2^{\alpha}\mathcal{U}(\theta_5,Ł)\right|:\theta_1,\theta_2,\theta_4,\theta_5\in I,|\theta_1-\theta_2|\leq\mathbf{d},|\theta_4-\theta_5|\leq\mathbf{d},Ł\in[-\mathbf{r}_0,\mathbf{r}_0]\right\}.$$

Again

$$\left|\int_0^{\theta_1}(P\theta_1-Q\theta_3)_{P,Q}^{(\alpha-1)}\,\mathcal{U}(\theta_3,Ł(\theta_3))d_{P,Q}\theta_3-\int_0^{\theta_2}(P\theta_2-Q\theta_3)_{P,Q}^{(\alpha-1)}\,\mathcal{U}(\theta_3,Ł(\theta_3))d_{P,Q}\theta_3\right|$$
$$\begin{aligned}
&\leq\left(\psi_{\mathcal{U}}\left(\omega(Ł,\mathbf{d})\right)+\mathcal{A}_{\mathbf{d}}\right)(P-Q)\sum_{n=0}^{\infty}\frac{Q^n}{P^{n+1}}\\
&\leq \psi_{\mathcal{U}}\left(\omega(Ł,\mathbf{d})\right)+\mathcal{A}_{\mathbf{d}}.
\end{aligned}$$

Therefore

$$I_4\leq\left(\mathcal{D}_{\mathcal{F}}\mathbf{r}_0+\hat{\mathcal{F}}\right)\left(\psi_{\mathcal{U}}\left(\omega(Ł,\mathbf{d})\right)+\mathcal{A}_{\mathbf{d}}\right).$$

Using the above inequalities we get

$$I_2\leq\left\{\psi_{\mathcal{U}}(\mathbf{r}_0)\left(\mathcal{D}_{\mathcal{F}}\omega(Ł\ 0\ \bar{b},\mathbf{d})+\omega_{\mathcal{F}}(\mathbf{r}_0,\mathbf{d})\right)+\left(\mathcal{D}_{\mathcal{F}}\mathbf{r}_0+\hat{\mathcal{F}}\right)\left(\psi_{\mathcal{U}}\left(\omega(Ł,\mathbf{d})\right)+\mathcal{A}_{\mathbf{d}}\right)\right\}\frac{\mathcal{D}_{\eta}}{\left|\Gamma_{P,Q}(\alpha)\right|}+\psi_{\eta}\left(\omega(Ł\ o\ \bar{a},\mathbf{d})\right).$$

Now, from assumption (7) we have

$$\begin{aligned}
\omega(Ł\ 0\ \bar{a},\mathbf{d})&=\sup\left\{|Ł(\bar{a}(\theta))-Ł(\bar{a}(\theta_1))|:\theta,\theta_1\in I,|\theta-\theta_1|\leq\mathbf{d}\right\}\\
&\leq\sup\left\{\left|Ł(\hat{\theta})-Ł(\hat{\theta}_1)\right|:\hat{\theta},\hat{\theta}_1\in I,\left|\hat{\theta}-\hat{\theta}_1\right|\leq\psi_{\bar{a}}(\mathbf{d})\right\}\\
&=\omega(Ł,\psi_{\bar{a}}(\mathbf{d})).
\end{aligned}$$

Similarly, from assumption (8) we have

$$\omega(\text{Ł } 0\ \bar{b}, \mathbf{d}) \leq \omega(\text{Ł}, \psi_{\bar{b}}(\mathbf{d})).$$

Then

$$I_2 \leq \left\{\psi_{\mathcal{U}}(\mathbf{r}_0)\left(\mathcal{D}_{\mathcal{F}}\omega(\text{Ł}, \psi_{\bar{b}}(\mathbf{d})) + \omega_{\mathcal{F}}(\mathbf{r}_0, \mathbf{d})\right) + \left(\mathcal{D}_{\mathcal{F}}\mathbf{r}_0 + \hat{\mathcal{F}}\right)\left(\psi_{\mathcal{U}}\left(\omega(\text{Ł}, \mathbf{d})\right) + \mathcal{A}_{\mathbf{d}}\right)\right\} \frac{\mathcal{D}_\eta}{\left|\Gamma_{P,Q}(\alpha)\right|} + \psi_\eta\left(\omega(\text{Ł}, \psi_{\bar{a}}(\mathbf{d})\right).$$

Therefore

$$\begin{aligned}\omega(\mathcal{T}\text{Ł}, \mathbf{d}) &\leq \mathcal{D}(\eta, \mathbf{d}) + \psi_\eta\left(\omega(\text{Ł}, \psi_{\bar{a}}(\mathbf{d}))\right) \\ &\quad + \frac{\mathcal{D}_\eta}{\left|\Gamma_{P,Q}(\alpha)\right|}\left\{\psi_{\mathcal{U}}(\mathbf{r}_0)\left(\mathcal{D}_{\mathcal{F}}\omega(\text{Ł}, \psi_{\bar{b}}(\mathbf{d})) + \omega_{\mathcal{F}}(\mathbf{r}_0, \mathbf{d})\right) + \left(D_{\mathcal{F}}\mathbf{r}_0 + \hat{\mathcal{F}}\right)\left(\psi_{\mathcal{U}}\left(\omega(\text{Ł}, \mathbf{d})\right) + \mathcal{A}_{\mathbf{d}}\right)\right\}.\end{aligned}$$

Let $\mathbf{M}$ be a non-empty subset of $\bar{\mathbf{B}}(0, \mathbf{r}_0)$. Then we have

$$\begin{aligned}\omega(\mathcal{T}\mathbf{M}, \mathbf{d}) &\leq \mathcal{D}(\eta, \mathbf{d}) + \psi_\eta\left(\omega(\mathbf{M}, \psi_{\bar{a}}(\mathbf{d}))\right) \\ &\quad + \frac{\mathcal{D}_\eta}{\left|\Gamma_{P,Q}(\alpha)\right|}\left\{\psi_{\mathcal{U}}(\mathbf{r}_0)\left(\mathcal{D}_{\mathcal{F}}\omega(\mathbf{M}, \psi_{\bar{b}}(\mathbf{d})) + \omega_{\mathcal{F}}(\mathbf{r}_0, \mathbf{d})\right) + \left(\mathcal{D}_{\mathcal{F}}\mathbf{r}_0 + \hat{\mathcal{F}}\right)\left(\psi_{\mathcal{U}}\left(\omega(\mathbf{M}, \mathbf{d})\right) + \mathcal{A}_{\mathbf{d}}\right)\right\}.\end{aligned}$$

We have

$$\lim_{\theta\to 0^+}\psi_{\bar{a}}(\theta) = \lim_{\theta\to 0^+}\psi_{\bar{b}}(\theta) = 0.$$

As $\mathbf{d} \to 0$, we get

$$\begin{aligned}\mu(\mathcal{T}\mathbf{M}) &\leq 0 + \psi_\eta\left(\omega(\mathbf{M})\right) \\ &\quad + \frac{\mathcal{D}_\eta}{\left|\Gamma_{P,Q}(\alpha)\right|}\left\{\psi_{\mathcal{U}}(\mathbf{r}_0)\left(\mathcal{D}_{\mathcal{F}}\mu(\mathbf{M}) + 0\right) + \left(\mathbf{D}_{\mathcal{F}}\mathbf{r}_0 + \hat{\mathcal{F}}\right)\left(\psi_{\mathcal{U}}\left(\mu(\mathbf{M})\right) + 0\right)\right\} \\ &= \psi_\eta\left(\mu(\mathbf{M})\right) + \frac{\mathcal{D}_\eta}{\left|\Gamma_{P,Q}(\alpha)\right|}\left\{\psi_{\mathcal{U}}(\mathbf{r}_0)\mathcal{D}_{\mathcal{F}}\mu(\mathbf{M}) + \left(\mathcal{D}_{\mathcal{F}}\mathbf{r}_0 + \hat{\mathcal{F}}\right)\psi_{\mathcal{U}}\left(\mu(\mathbf{M})\right)\right\} \\ &< \hat{L}\mu(\mathbf{M}) + \mathcal{L}\mu(\mathbf{M}) + \mathcal{N}\mu(\mathbf{M})\end{aligned}$$

where $\mathcal{L} = \frac{\mathcal{D}_\eta\left(\mathcal{D}_{\mathcal{F}}\mathbf{r}_0 + \hat{\mathcal{F}}\right)}{\left|\Gamma_{P,Q}(\alpha)\right|}$ and $\mathcal{N} = \frac{\mathcal{D}_\eta\mathcal{D}_{\mathcal{F}}\psi_{\mathcal{U}}(\mathbf{r}_0)}{\left|\Gamma_{P,Q}(\alpha)\right|}$.
Therefore

$$\mu(\mathcal{T}\mathbf{M}) \leq \hat{\ell}\left(\mu(\mathbf{M})\right),$$

where $\hat{\ell}(\theta) = \hat{L}\theta + \mathcal{L}\theta + \mathcal{N}\theta$ for all$\theta \geq 0$ and it can be observed that $\hat{\ell}(\theta) < \theta$ for all $\theta > 0$ and $\hat{\ell}(0) = 0$ by assumption (9).

Therefore, by Corollary 3.4 there exists at least one fixed point of $\mathcal{T}$ in $\bar{\mathbf{B}}(0, \mathbf{r}_0)$, which is a solution for (1). □

5. Illustrative example

Example 5.1. Consider the following (P, Q)-integral equation:

$$\text{Ł}(\theta) = \frac{\theta^2}{36} + \frac{\text{Ł}(\theta)}{3} + \left(\frac{\theta}{3} + \frac{\text{Ł}(\theta)}{6}\right)\int_0^\theta (P\theta - Q\theta_1)_{P,Q}^{(\alpha-1)} \cdot \frac{\text{Ł}(\theta_1)}{3 + \theta_1^2} d_{P,Q}\theta_1 \tag{6}$$

where $\theta \in I = [0,1], \alpha > 1$ and $0 < Q < P \leq 1$.
Here,

$$\bar{a}(\theta) = \bar{b}(\theta) = \theta,$$

$$\eta(\theta, Ł, Ł') = \frac{\theta^2}{36} + \frac{Ł}{3} + Ł'\Gamma_{P,Q}(\alpha),$$

$$\mathcal{F}(\theta, Ł) = \frac{\theta}{3} + \frac{Ł}{6}$$

and

$$\mathcal{U}(\theta, Ł) = \frac{Ł}{3+\theta^2},$$

where $\theta \in I$ and $Ł \in C(I)$.
Assumption (1) is trivial. For all $\theta \in I$ and $Ł, Ł', Z, W \in \mathbb{R}$ we have

$$\begin{aligned} &\left|\eta(\theta, Ł, Ł') - \eta(\theta, Z, W)\right| \\ &= \left|\frac{Ł - Z}{3} + (Ł' - W)\Gamma_{P,Q}(\alpha)\right| \\ &\leq \frac{|Ł - Z|}{3} + |Ł' - W| \left|\Gamma_{P,Q}(\alpha)\right|. \end{aligned}$$

If we choose $\psi_\eta(\theta) = \frac{\theta}{3}$ where $\theta \geq 0$ and $\mathcal{D}_\eta = \left|\Gamma_{P,Q}(\alpha)\right|$, then

$$\left|\eta(\theta, Ł, Ł') - \eta(\theta, Z, W)\right| \leq \psi_\eta(|Ł - Z|) + \mathcal{D}_\eta |Ł' - W|.$$

Thus, assumption (2) is satisfied. Again, for all $Ł, Ł' \in C(I)$ we have

$$|\mathcal{U}(\theta, Ł) - \mathcal{U}(\theta, Ł')| = \left|\frac{Ł}{3+\theta^2} - \frac{Ł'}{3+\theta^2}\right| \leq \frac{1}{3+\theta^2}|Ł - Ł'| \leq \frac{1}{3}|Ł - Ł'| = \psi_{\mathcal{U}}(|Ł - Ł'|),$$

where $\psi_{\mathcal{U}}(\theta) = \frac{\theta}{3}$ for all $\theta \geq 0$. Also, $\mathcal{U}(\theta, 0) = 0$ for all $\theta \in I$ and $\psi_{\mathcal{U}}(\theta) < \theta$ for all $\theta > 0$. Thus, assumption (4) is satisfied. For this example we have $\hat{\eta} = \frac{1}{36}$ and $\hat{\mathcal{F}} = \frac{1}{3}$. Now

$$\psi_\eta(\mathbf{r}_0) + \frac{\mathcal{D}_\eta \psi_{\mathcal{U}}(\mathbf{r}_0)}{\left|\Gamma_{P,Q}(\alpha)\right|}\left(\mathcal{D}_{\mathcal{F}}\mathbf{r}_0 + \hat{\mathcal{F}}\right) + \hat{\eta} \leq \mathbf{r}_0$$

is equivalent to

$$\frac{\mathbf{r}_0}{3} + \frac{\mathbf{r}_0}{3}\left(\frac{\mathbf{r}_0}{6} + \frac{1}{3}\right) + \frac{1}{36} \leq \mathbf{r}_0,$$

i.e.,

$$2\mathbf{r}_0^2 - 20\mathbf{r}_0 + 1 \leq 0.$$

The inequality holds for $\mathbf{r}_0 \in \left[\frac{5-7\sqrt{2}}{2}, \frac{5+7\sqrt{2}}{2}\right]$. Since $\psi_\eta(\theta) = \frac{\theta}{3} < \theta$ for all $\theta > 0$ and ψ_η is continuous, therefore assumption (6) is satisfied.
Again since

$$|\bar{a}(\theta) - \bar{a}(\theta_1)| = |\theta - \theta_1| = \psi_{\bar{a}}(|\theta - \theta_1|),$$

where $\psi_{\bar{a}}(\theta) = \theta$ which is non-decreasing and $\lim_{\theta \to 0^+} \psi_{\bar{a}}(\theta) = 0$, therefore, assumption (7) is satisfied.

From

$$0 < \psi_{\mathcal{U}}(\mathbf{r}_0) < \frac{\left|\Gamma_{P,Q}(\alpha)\right|}{\mathcal{D}_{\mathcal{F}}\mathcal{D}_{\eta}},$$

we have $0 < \frac{\mathbf{r}_0}{3} < 6$, i.e., $0 < \mathbf{r}_0 < 18$, where $\mathcal{D}_{\mathcal{F}} = \frac{1}{6}$.
Again, from

$$\frac{\mathcal{D}_{\eta}}{\left|\Gamma_{P,Q}(\alpha)\right|}\left(\mathcal{D}_{\mathcal{F}}\mathbf{r}_0 + \hat{\mathcal{F}}\right) < 1$$

we get $\mathbf{r}_0 < 4$.
Since

$$\left[\frac{5 - 7\sqrt{2}}{2}, \frac{5 + 7\sqrt{2}}{2}\right] \cap (0, 4) \neq \emptyset.$$

Also, $\hat{L} = \frac{1}{3}$, $\mathcal{L} = \frac{\mathbf{r}_0}{6} + \frac{1}{3}$, $\mathcal{N} = \frac{\mathbf{r}_0}{18}$ and

$$\hat{L} + \mathcal{L} + \mathcal{N} = \frac{2}{3} + \frac{2\mathbf{r}_0}{9} < 1$$

for $\mathbf{r}_0 = \frac{1}{2}$. Therefore, by Theorem 4.1 the equation (6) has at least one solution in $C(I)$.

References

[1] R.P. Agarwal, D. O'Regan, Fixed point theory and applications, Cambridge University Press, (2004).
[2] R.P. Ararwal, Certain fractional q-integrals and q-derivatives, Proc. Cambridge Philos. Soc., 66 (1969) 365-370.
[3] A. Aghajani, R. Allahyari, M. Mursaleen, A generalization of Darbo's theorem with application to the solvability of systems of integral equations, J. Comput. Appl. Math., 260 (2014) 68-77.
[4] W. Al-Salam, Some fractional q-integrals and q-derivatives, Proc. Edinb. Math. Soc., 15(2) (1966/1967) 135-140.
[5] M.H. Annaby, Z.S. Mansour, q-fractional calculus and equations, Lecture notes in Mathematics, 2056, Springer-Verlag, Berlin, 2012.
[6] R. Arab, R. Allahyari, A.S. Haghighi, Existence of solutions of infinite systems of integral equations in two variables via measure of noncompactness, App. Math. Comput., 246 (2014) 283-291.
[7] J. Banaś, M. Mursaleen, *Sequence spaces and measures of noncompactness with applications to differential and integral equations*, Springer, New Delhi, 2014.
[8] A. Bensoussa, G.Da Prato, M.C. Delfour, S.K. Mitter, Representation and Control of Infinite Dimensional Systems, 2nd edn. Birkhäuser, Boston, (2007).
[9] G. Darbo, Punti uniti in trasformazioni a codominio non compatto (Italian), Rend. Sem. Mat. Univ. Padova, 24 (1955) 84-92.
[10] M.A. Darwish, On quadratic integral equation of fractional orders, J. Math. Anal. Appl., 311 (2005) 112-119.
[11] F.H. Jackson, On q-functions and a certain difference operator, Trans. Roy. Soc. Edin., 46 (1908) 253-281.
[12] F.H. Jackson, On q-definite integrals, Quart. J. Pure and Appl. Math., 41 (1910) 193-203.
[13] M. Jleli, M. Mursaleen, B. Samet, Q-integral equations of fractional orders, Electronic J. Diff. Equa., 17 (2016) 1-14.
[14] K. Kuratowski, Sur les espaces complets, Fund. Math., 15 (1930) 301-309.
[15] H.K. Nashine, R. Arab, R.P. Agarwal, M. De la Sen, Positive solutions of fractional integral equations by the technique of measure of noncompactness, J. Inequalities Appl., (2017) 2017:225, DOI 10.1186/s13660-017-1497-6.
[16] P. Njionou Sadjang, On the fundamental theorem of (p, q)-calculus and some (p, q)-Taylor formulas, Results Math., (2018) 73:39, doi.10.1007/s00025-018-0783-z.
[17] P.M. Rajković, S.D. Marinković, On q-analogues of Caputo derivative and Mittag-Leffler function, Fract. Calc. Appl. Anal., 10 (2007) 359-373.