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Some new parameterized inequalities based on Riemann–Liouville fractional integrals

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Abstract. In this article, we first obtain an identity that we will use throughout the article. With the help of this equality, new inequalities involving a real parameter are established for Riemann–Liouville fractional integrals. For this purpose, properties of the differentiable convex function, Hölder inequality, and power-mean inequality are used. In addition, new results are established with special choices of parameters in all proven inequalities. Our results are supported by examples and graphs. It is shown that some of these results generalize the trapezoid type and Newton-type inequalities.

1. Introduction

Integral inequalities have provided solutions to many problems in mathematics and related disciplines. Especially, a lot of research has been devoted to Hermite–Hadamard, Trapezoid, Midpoint, Simpson, and Newton-type inequalities. These inequalities are applied to pure mathematics and solving real-life problems. On the other hand, fractional calculus has become an important research area in integral inequalities by adding fractional derivatives and integrals to the literature.

In [10], Dragomir and Agarwal first introduced trapezoid type inequalities in 1998. The inequality obtained from the right side of the Hermite–Hadamard inequality, named trapezoid type inequalities, has directed many studies. Cerone et al. presented a generalization of trapezoid inequality for mappings of bounded variation in [7]. In [5], the generalization of trapezoid inequality via mappings of two independent variables with bounded variation and some applications were given by Budak and Sarıkaya. Budak and Noeiaghdam investigated some new perturbed trapezoid type inequalities via mappings whose first derivatives either are of bounded variation or Lipschitzian in [4]. Apart from these studies, there are many works on trapezoid type inequalities in the literature. For more information about these type results, one can refer to [1, 2, 8, 20].

Newton-type inequalities created from the well-known Simpson's second rule (Simpson's 3/8 rule) have been the focus of many researchers. For instance; Gao and Shi obtained some Newton-type inequality through convex mappings in [15]. In addition, the authors gave some applications for special cases of real mappings. Erden et al. established some error estimates of Newton-type cubature formula with the aid of

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bounded variation and Lipschitzian functions in [11]. Noor et al. presented some inequalities for Simpson 3/8 rule inequality for harmonic convexity and p-harmonic convexity in [21] and [22], respectively. For more Newton-type inequalities, we can also refer to [3, 12, 16, 24].

With the help of fractional integrals, the researchers obtained many new trapezoid and Newton-type integral inequalities. To illustrate it: Sarikaya et al. presented new Hermite–Hadamard type inequalities and trapezoid type based on Riemann–Liouville fractional integrals in [25]. Dragomir presented some trapezoid type inequalities with the help of the Riemann–Liouville fractional integrals of mappings of bounded variation and of Hölder continuous functions in [9]. Budak et al. obtained some new generalized inequalities via differentiable convex functions in the case of the some parameters and generalized fractional integrals in [6]. The authors demonstrated that these results reduce to several new Simpson, midpoint, and trapezoid type inequalities. Kunt et al. gave new fractional trapezoid and midpoint type inequalities for the differentiable convex functions in [19]. Sitthiwirattham et al. investigated Simpson's second rule inequalities for differentiable convex functions based on the Riemann–Liouville fractional integrals in [27]. Iftikhar et al. established new Newton-type inequalities for functions whose the local fractional derivatives in modulus and their some powers are generalized convex functions in [18]. You et al. investigated some new inequalities of Simpson's type based on differentiable convex functions in case of the some parameters and generalized fractional integrals in [28]. In [17], the authors dealt with Simpson's second-type inequalities with help of the coordinated convex functions. In addition, the researchers presented Simpson's second-type integral inequalities via two-variable functions whose second-order partial derivatives in modulus are co-ordinated convex.

Inspired by the above literature, we create parameterized inequalities based on Riemann–Liouville fractional integrals in this study. This article consists of 5 sections. In Section 2, fundamental information about fractional integrals and related inequalities is given. In Section 3, an identity depending on a real parameter that we use throughout the article is obtained. In Section 4, a new inequality is discussed by making use of the convexity of the differentiable convex function. Moreover, two more inequalities are established by utilizing the Hölder and power-mean inequalities. In special cases of these inequalities, Newton-type and trapezoid type inequalities are obtained. By giving some examples, we make the results better understood by the reader. We also show the accuracy of the examples with graphs. In the last part, suggestions for new studies are given to the reader.

2. Fractional Integrals and Related Inequalities

In this section, we recall some basic notations and notions of the fractional integrals. We also recall some inequalities via different fractional integrals.

Definition 2.1 (see, [13, 14]). *Let $f \in L_1[a,b]$. $\alpha > 0$, $a \geq 0$ and Γ is Gamma function. The Riemann-Liouville integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ order α are defined as*

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(a)} \int_a^x (x-t)^{\alpha-1} f(t)\,dt, \ x > a$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\,dt, x < b.$$

In 2013, Sarıkaya et al. investigated the following fractional Hermite–Hadamard type inequality for the first time:

Theorem 2.2 (see, [25]). *For a positive convex function $f : I \subset \mathbb{R} \to \mathbb{R}$ with $f \in L_1[a,b]$ and $0 \leq a < b$, the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right] \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

After that Sarıkaya and Yıldırım presented the following new version of fractional Hermite–Hadamard inequality:

Theorem 2.3 (see, [26]). *For a positive convex function $f : I \subset \mathbb{R} \to \mathbb{R}$ with $f \in L_1[a,b]$, $0 \le a < b$ and $a, b \in I$, the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)+}f(b) + J^{\alpha}_{\left(\frac{a+b}{2}\right)-}f(a)\right] \le \frac{f(a)+f(b)}{2}. \tag{2}$$

Remark 2.4. *If $\alpha = 1$ in inequalities* (1) *and* (2)*, then we obtain the classical Hermite–Hadamard inequality (see, [23]):*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}.$$

3. An Identity

In this section, we present an identity that is used in the next section.

Lemma 3.1. *If $f : I \subset \mathbb{R} \to \mathbb{R}$ is a differentiable mapping on I° with $f \in L_1[a,b]$ and $\lambda \in \mathbb{R}$, then we have*

$$\begin{aligned} &\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{a+}f\left(\frac{2a+b}{3}\right) + J^{\alpha}_{\frac{2a+b}{3}+}f\left(\frac{a+2b}{3}\right) + J^{\alpha}_{\frac{a+2b}{3}+}f(b)\right] \\ &-\frac{1}{3}\left[(1-\lambda)(f(a)+f(b)) + \left(\lambda+\frac{1}{2}\right)\left(f\left(\frac{2a+b}{3}\right) + f\left(\frac{a+2b}{3}\right)\right)\right] = \frac{b-a}{9}\left[I_1+I_2+I_3\right], \end{aligned} \tag{3}$$

where

$$\begin{aligned} I_1 &= \int_0^1 (t^{\alpha}-\lambda) f'\left(ta+(1-t)\frac{2a+b}{3}\right)dt, \\ I_2 &= \int_0^1 \left(t^{\alpha}-\frac{1}{2}\right) f'\left(t\frac{2a+b}{3}+(1-t)\frac{a+2b}{3}\right)dt, \end{aligned}$$

and

$$I_3 = \int_0^1 (t^{\alpha}-(1-\lambda)) f'\left(t\frac{a+2b}{3}+(1-t)b\right)dt.$$

Proof. By utilizing integration by parts and change of variables, we derive

$$\begin{aligned} I_1 &= \int_0^1 (t^{\alpha}-\lambda) f'\left(ta+(1-t)\frac{2a+b}{3}\right)dt \\ &= -\frac{3}{(b-a)}(t^{\alpha}-\lambda) f\left(ta+(1-t)\frac{2a+b}{3}\right)\Bigg|_0^1 + \frac{3\alpha}{b-a}\int_0^1 t^{\alpha-1} f\left(ta+(1-t)\frac{2a+b}{3}\right)dt \\ &= -\frac{3(1-\lambda)}{(b-a)}f(a) - \frac{3\lambda}{b-a}f\left(\frac{2a+b}{3}\right) + \frac{3\alpha}{b-a}\int_a^{\frac{2a+b}{3}}\left(\frac{2a+b}{3}-x\right)^{\alpha-1}\left(\frac{3}{b-a}\right)^{\alpha-1} f(x)\,dx \end{aligned} \tag{4}$$

$$= \frac{3^{\alpha+1}\Gamma(\alpha+1)}{(b-a)^{\alpha+1}} J_{a+}^{\alpha} f\left(\frac{2a+b}{3}\right) - \left[\frac{3(1-\lambda)}{b-a} f(a) + \frac{3\lambda}{b-a} f\left(\frac{2a+b}{3}\right)\right].$$

By calculating similar to I_1, we get

$$\begin{aligned} I_2 &= \int_0^1 \left(t^{\alpha} - \frac{1}{2}\right) f'\left(t\frac{2a+b}{3} + (1-t)\frac{a+2b}{3}\right) dt \qquad (5) \\ &= \frac{3^{\alpha+1}\Gamma(\alpha+1)}{(b-a)^{\alpha+1}} J_{\frac{2a+b}{3}+}^{\alpha} f\left(\frac{a+2b}{3}\right) - \left[\frac{3}{2(b-a)} f\left(\frac{2a+b}{3}\right) + \frac{3}{2(b-a)} f\left(\frac{a+2b}{3}\right)\right] \end{aligned}$$

and

$$\begin{aligned} I_3 &= \int_0^1 \left(t^{\alpha} - (1-\lambda)\right) f'\left(t\frac{a+2b}{3} + (1-t)b\right) dt \qquad (6) \\ &= \frac{3^{\alpha+1}\Gamma(\alpha+1)}{(b-a)^{\alpha+1}} J_{\frac{a+2b}{3}+}^{\alpha} f(b) - \left[\frac{3\lambda}{(b-a)} f\left(\frac{a+2b}{3}\right) + \frac{3(1-\lambda)}{(b-a)} f(b)\right]. \end{aligned}$$

So, we derive the desired result by adding the equalities (4)–(6) and multiplying the resultant equality by $\frac{b-a}{9}$. □

4. Parametrized Fractional Inequalities for Differentiable Convex Functions

In this part, we derive some new parametrized inequalities for differentiable convex mappings involving Riemann–Liouville fractional integrals. Further we use the following notations:

$$\begin{aligned} A_1(\alpha,\lambda) &= \int_0^1 |t^{\alpha} - \lambda| \, dt = \begin{cases} \frac{1}{\alpha+1} - \lambda, & -1 \le \lambda \le 0 \\ \frac{2\alpha}{\alpha+1}\lambda^{\frac{1}{\alpha}+1} + \frac{1}{\alpha+1} - \lambda, & 0 < \lambda \le 1, \end{cases} \qquad (7) \\ A_2(\alpha,\lambda) &= \int_0^1 |t^{\alpha} - \lambda| \, t dt = \begin{cases} \frac{1}{\alpha+2} - \frac{\lambda}{2}, & -1 \le \lambda \le 0 \\ \frac{\alpha}{\alpha+2}\lambda^{\frac{2}{\alpha}+1} + \frac{1}{\alpha+2} - \frac{\lambda}{2}, & 0 < \lambda \le 1, \end{cases} \\ A_3(\alpha) &= \int_0^1 \left|t^{\alpha} - \frac{1}{2}\right| dt = \frac{2\alpha}{\alpha+1}\left(\frac{1}{2}\right)^{\frac{\alpha+1}{\alpha}} + \frac{1}{\alpha+1} - \frac{1}{2}, \\ A_4(\alpha) &= \int_0^1 \left|t^{\alpha} - \frac{1}{2}\right| t dt = \frac{\alpha}{\alpha+2}\left(\frac{1}{2}\right)^{\frac{\alpha+2}{\alpha}} + \frac{1}{\alpha+2} - \frac{1}{4}, \\ A_5(\alpha,\lambda) &= \int_0^1 |t^{\alpha} - (1-\lambda)| \, dt = \begin{cases} 1 - \lambda - \frac{1}{\alpha+1}, & -1 \le \lambda \le 0 \\ \frac{2\alpha}{\alpha+1}(1-\lambda)^{\frac{1}{\alpha}+1} + \frac{1}{\alpha+1} - 1 + \lambda, & 0 < \lambda \le 1, \end{cases} \\ A_6(\alpha,\lambda) &= \int_0^1 |t^{\alpha} - (1-\lambda)| \, t dt = \begin{cases} \frac{1-\lambda}{2} - \frac{1}{\alpha+2}, & -1 \le \lambda \le 0 \\ \frac{\alpha}{\alpha+2}(1-\lambda)^{\frac{2}{\alpha}+1} + \frac{1}{\alpha+2} - \frac{1-\lambda}{2}, & 0 < \lambda \le 1. \end{cases} \end{aligned}$$

Theorem 4.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° with $f \in L[a,b]$ and $-1 \leq \lambda \leq 1$. If $\left|f'\right|$ is a convex mapping, then we get the following parametrized inequality:*

$$\begin{aligned} & \left| \frac{3^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} \left[J^\alpha_{a+} f\left(\frac{2a+b}{3}\right) + J^\alpha_{\frac{2a+b}{3}+} f\left(\frac{a+2b}{3}\right) + J^\alpha_{\frac{a+2b}{3}+} f(b) \right] \right. \\ & \left. - \frac{1}{3}\left[(1-\lambda)(f(a)+f(b)) + \left(\lambda+\frac{1}{2}\right)\left(f\left(\frac{2a+b}{3}\right) + f\left(\frac{a+2b}{3}\right)\right)\right] \right| \\ \leq \; & \frac{b-a}{27}\left[\left|f'(a)\right| \left(2A_1(\alpha,\lambda) + A_2(\alpha,\lambda) + A_3(\alpha) + A_4(\alpha) + A_6(\alpha,\lambda)\right) \right. \\ & \left. + \left|f'(b)\right| \left(A_1(\alpha,\lambda) - A_2(\alpha,\lambda) + 2A_3(\alpha) - A_4(\alpha) + 3A_5(\alpha,\lambda) - A_6(\alpha,\lambda)\right)\right], \end{aligned} \tag{8}$$

where A_i for $i = 1, 2, ..., 6$ are expressed as in (7).

Proof. Taking absolute value of (3), then we get

$$\begin{aligned} & \left| \frac{3^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} \left[J^\alpha_{a+} f\left(\frac{2a+b}{3}\right) + J^\alpha_{\frac{2a+b}{3}+} f\left(\frac{a+2b}{3}\right) + J^\alpha_{\frac{a+2b}{3}+} f(b) \right] \right. \\ & \left. - \frac{1}{3}\left[(1-\lambda)(f(a)+f(b)) + \left(\lambda+\frac{1}{2}\right)\left(f\left(\frac{2a+b}{3}\right) + f\left(\frac{a+2b}{3}\right)\right)\right] \right| \\ \leq \; & \frac{b-a}{9}\left[\int_0^1 |t^\alpha - \lambda| \left| f'\left(t\frac{a+2b}{3} + (1-t)b\right)\right| dt + \int_0^1 \left|t^\alpha - \frac{1}{2}\right| \left| f'\left(t\frac{2a+b}{3} + (1-t)\frac{a+2b}{3}\right)\right| dt \right. \\ & \left. + \int_0^1 |(t^\alpha - (1-\lambda))| \left| f'\left(t\frac{a+2b}{3} + (1-t)b\right)\right| dt \right] \\ = \; & \frac{b-a}{9}\left[\int_0^1 |t^\alpha - \lambda| \left| f'\left(\frac{2+t}{3}a + \frac{1-t}{3}b\right)\right| dt + \int_0^1 \left|t^\alpha - \frac{1}{2}\right| \left| f'\left(\frac{1+t}{3}a + \frac{2-t}{3}b\right)\right| dt \right. \\ & \left. + \int_0^1 |t^\alpha - (1-\lambda)| \left| f'\left(\frac{t}{3}a + \frac{3-t}{3}b\right)\right| dt \right]. \end{aligned} \tag{9}$$

Using $\left|f'\right|$, we posses

$$\begin{aligned} & \left| \frac{3^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} \left[J^\alpha_{a+} f\left(\frac{2a+b}{3}\right) + J^\alpha_{\frac{2a+b}{3}+} f\left(\frac{a+2b}{3}\right) + J^\alpha_{\frac{a+2b}{3}+} f(b) \right] \right. \\ & \left. - \frac{1}{3}\left[(1-\lambda)(f(a)+f(b)) + \left(\lambda+\frac{1}{2}\right)\left(f\left(\frac{2a+b}{3}\right) + f\left(\frac{a+2b}{3}\right)\right)\right] \right| \\ \leq \; & \frac{b-a}{9}\left[\int_0^1 |t^\alpha - \lambda| \left(\frac{2+t}{3}\left|f'(a)\right| + \frac{1-t}{3}\left|f'(b)\right| \right) dt \right. \\ & \left. + \int_0^1 \left|t^\alpha - \frac{1}{2}\right| \left(\frac{1+t}{3}\left|f'(a)\right| + \frac{2-t}{3}\left|f'(b)\right| \right) dt + \int_0^1 |t^\alpha - (1-\lambda)| \left(\frac{t}{3}\left|f'(a)\right| + \frac{3-t}{3}\left|f'(b)\right| \right) dt \right] \\ = \; & \frac{b-a}{27}\left[(2A_1(\alpha,\lambda) + A_2(\alpha,\lambda)) \left|f'(a)\right| + (A_1(\alpha,\lambda) - A_2(\alpha,\lambda)) \left|f'(b)\right| \right. \end{aligned}$$

$$
\begin{aligned}
& + (A_3(\alpha) + A_4(\alpha)) \left|f'(a)\right| + (2A_3(\alpha) - A_4(\alpha)) \left|f'(b)\right| \\
& + A_6(\alpha,\lambda) \left|f'(a)\right| + (3A_5(\alpha,\lambda) - A_6(\alpha,\lambda)) \left|f'(b)\right| \\
= & \frac{b-a}{27} \Big[\left|f'(a)\right| (2A_1(\alpha,\lambda) + A_2(\alpha,\lambda) + A_3(\alpha) + A_4(\alpha) + A_6(\alpha,\lambda)) \\
& + \left|f'(b)\right| (A_1(\alpha,\lambda) - A_2(\alpha,\lambda) + 2A_3(\alpha) - A_4(\alpha) + 3A_5(\alpha,\lambda) - A_6(\alpha,\lambda)) \Big].
\end{aligned}
$$

Thus the proof ends. □

Remark 4.2. *If $\lambda = \frac{5}{8}$, then inequality* (8) *turns into the inequality given by Sitthiwirattham et al. in [27, Theorem 4].*

From Theorem 4.1 we immediately get the following result.

Corollary 4.3. *If $\lambda = -\frac{1}{2}$, then* (8) *reduces to the inequality*

$$
\begin{aligned}
& \left| \frac{3^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}} \left[J_{a+}^{\alpha} f\left(\frac{2a+b}{3}\right) + J_{\frac{2a+b}{3}+}^{\alpha} f\left(\frac{a+2b}{3}\right) + J_{\frac{a+2b}{3}+}^{\alpha} f(b) \right] - \frac{f(a)+f(b)}{2} \right| \\
\leq & \frac{b-a}{27} \left[\left|f'(a)\right| \left(\frac{5\alpha^2+31\alpha+38}{4(\alpha+1)(\alpha+2)} + \frac{2\alpha}{\alpha+1}\left(\frac{1}{2}\right)^{\frac{1}{\alpha}(\alpha+1)} + \frac{\alpha}{\alpha+2}\left(\frac{1}{2}\right)^{\frac{1}{\alpha}(\alpha+2)} \right) \right. \\
& \left. + \left|f'(b)\right| \left(\frac{13\alpha+22}{4(\alpha+2)} + \frac{4\alpha}{\alpha+1}\left(\frac{1}{2}\right)^{\frac{1}{\alpha}(\alpha+1)} - \frac{\alpha}{\alpha+2}\left(\frac{1}{2}\right)^{\frac{1}{\alpha}(\alpha+2)} \right) \right].
\end{aligned}
\tag{10}
$$

Example 4.4. *If the function $f : [a,b] = [0,1] \to \mathbb{R}$ is defined as $f(x) = \frac{x^3}{3}$, then the left-hand side of the inequality* (10) *becomes to*

$$
\begin{aligned}
& \left| \frac{3^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}} \left[J_{a+}^{\alpha} f\left(\frac{2a+b}{3}\right) + J_{\frac{2a+b}{3}+}^{\alpha} f\left(\frac{a+2b}{3}\right) + J_{\frac{a+2b}{3}+}^{\alpha} f(b) \right] - \frac{f(a)+f(b)}{2} \right| \\
= & \left| 3^{\alpha-1}\Gamma(\alpha+1) \left[J_{0+}^{\alpha} f\left(\frac{1}{3}\right) + J_{\frac{1}{3}+}^{\alpha} f\left(\frac{2}{3}\right) + J_{\frac{2}{3}+}^{\alpha} f(1) \right] - \frac{f(0)+f(1)}{2} \right| \\
= & \left| 3^{\alpha-1}\alpha \left[\int_0^{\frac{1}{3}} \left(\frac{1}{3}-t\right)^{\alpha-1} \frac{t^3}{3}dt + \int_{\frac{1}{3}}^{\frac{2}{3}} \left(\frac{2}{3}-t\right)^{\alpha-1} \frac{t^3}{3}dt + \int_{\frac{2}{3}}^{1} (1-t)^{\alpha-1} \frac{t^3}{3}dt \right] - \frac{1}{6} \right| \\
= & \left| 3^{\alpha-1}\alpha \left[\frac{2\cdot 3^{-\alpha-3}}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)} + \frac{3^{-\alpha-4}(\alpha+4)\left(\alpha^2+5\alpha+12\right)}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)} + \left| \frac{2\cdot 3^{-\alpha-4}\left(4\alpha^3+30\alpha^2+80\alpha+81\right)}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)} - \frac{1}{6} \right| \right] \right| \\
= & \left| \frac{3\alpha^3+23\alpha^2+64\alpha+72}{81(\alpha+1)(\alpha+2)(\alpha+3)} - \frac{1}{6} \right|.
\end{aligned}
$$

Using the facts that

$$
\left|f'(0)\right| \left(\frac{5\alpha^2+31\alpha+38}{4(\alpha+1)(\alpha+2)} + \frac{2\alpha}{\alpha+1}\left(\frac{1}{2}\right)^{\frac{1}{\alpha}(\alpha+1)} + \frac{\alpha}{\alpha+2}\left(\frac{1}{2}\right)^{\frac{1}{\alpha}(\alpha+2)} \right) = 0
\tag{11}
$$

and

$$
\left|f'(1)\right| \left(\frac{13\alpha+22}{4(\alpha+2)} + \frac{4\alpha}{\alpha+1}\left(\frac{1}{2}\right)^{\frac{1}{\alpha}(\alpha+1)} - \frac{\alpha}{\alpha+2}\left(\frac{1}{2}\right)^{\frac{1}{\alpha}(\alpha+2)} \right)
\tag{12}
$$

$$= \left(\frac{13\alpha+22}{4(\alpha+2)}+\frac{4\alpha}{\alpha+1}\left(\frac{1}{2}\right)^{\frac{1}{\alpha}(\alpha+1)}-\frac{\alpha}{\alpha+2}\left(\frac{1}{2}\right)^{\frac{1}{\alpha}(\alpha+2)}\right),$$

we obtain that the right-hand side of (10) *can be calculated as*

$$\frac{b-a}{27}\left[\left|f'(a)\right|\left(\frac{5\alpha^2+31\alpha+38}{4(\alpha+1)(\alpha+2)}+\frac{2\alpha}{\alpha+1}\left(\frac{1}{2}\right)^{\frac{1}{\alpha}(\alpha+1)}+\frac{\alpha}{\alpha+2}\left(\frac{1}{2}\right)^{\frac{1}{\alpha}(\alpha+2)}\right)\right.$$
$$\left.+\left|f'(b)\right|\left(\frac{13\alpha+22}{4(\alpha+2)}+\frac{4\alpha}{\alpha+1}\left(\frac{1}{2}\right)^{\frac{1}{\alpha}(\alpha+1)}-\frac{\alpha}{\alpha+2}\left(\frac{1}{2}\right)^{\frac{1}{\alpha}(\alpha+2)}\right)\right]$$
$$= \frac{1}{27}\left(\frac{13\alpha+22}{4(\alpha+2)}+\frac{4\alpha}{\alpha+1}\left(\frac{1}{2}\right)^{\frac{1}{\alpha}(\alpha+1)}-\frac{\alpha}{\alpha+2}\left(\frac{1}{2}\right)^{\frac{1}{\alpha}(\alpha+2)}\right).$$

As a result, we get the following inequality

$$\left|\frac{3\alpha^3+23\alpha^2+64\alpha+72}{81(\alpha+1)(\alpha+2)(\alpha+3)}-\frac{1}{6}\right| \le \frac{1}{27}\left(\frac{13\alpha+22}{4(\alpha+2)}+\frac{4\alpha}{\alpha+1}\left(\frac{1}{2}\right)^{\frac{1}{\alpha}(\alpha+1)}-\frac{\alpha}{\alpha+2}\left(\frac{1}{2}\right)^{\frac{1}{\alpha}(\alpha+2)}\right).$$

To illustrate the accuracy of this result, one can refer to Figure 1.

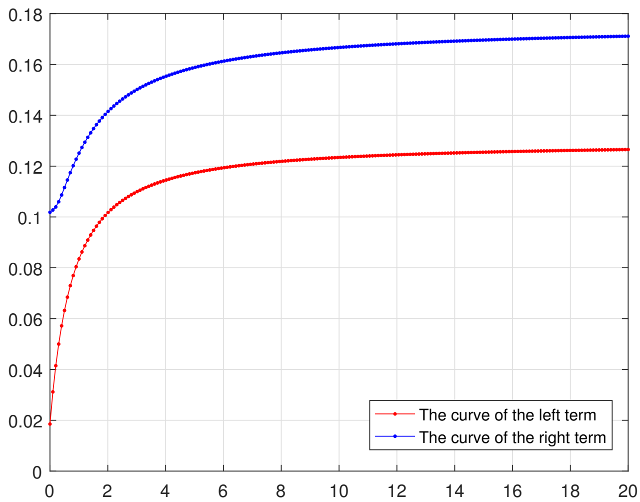

Figure 1: Graph for the result of Example 4.4 computed and plotted in MATLAB program.

Remark 4.5. *If $\alpha = 1$, then Corollary 4.3 turns into the inequality given by Dragomir and Agarwal in [10].*

Corollary 4.6. *If $\lambda = 1$, then* (8) *reduces to the inequality*

$$\left|\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f\left(\frac{2a+b}{3}\right)+J_{\frac{2a+b}{3}+}^{\alpha}f\left(\frac{a+2b}{3}\right)+J_{\frac{a+2b}{3}+}^{\alpha}f(b)\right]-\frac{1}{2}\left(f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)\right)\right| \quad (13)$$
$$\le \frac{b-a}{27}\left[\left|f'(a)\right|\left(\frac{7\alpha^2+21\alpha+10}{4(\alpha+1)(\alpha+2)}+\frac{2\alpha}{\alpha+1}\left(\frac{1}{2}\right)^{\frac{1}{\alpha}(\alpha+1)}+\frac{\alpha}{\alpha+2}\left(\frac{1}{2}\right)^{\frac{1}{\alpha}(\alpha+2)}\right)\right.$$
$$\left.+\left|f'(b)\right|\left(\frac{-\alpha^2+9\alpha+26}{4(\alpha+1)(\alpha+2)}+\frac{4\alpha}{\alpha+1}\left(\frac{1}{2}\right)^{\frac{1}{\alpha}(\alpha+1)}-\frac{\alpha}{\alpha+2}\left(\frac{1}{2}\right)^{\frac{1}{\alpha}(\alpha+2)}\right)\right].$$

Example 4.7. *Considering the conditions of Example 4.4, the first term of* (13) *can be presented as*

$$\left|\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f\left(\frac{2a+b}{3}\right)+J_{\frac{2a+b}{3}+}^{\alpha}f\left(\frac{a+2b}{3}\right)+J_{\frac{a+2b}{3}+}^{\alpha}f(b)\right]-\frac{1}{2}\left(f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)\right)\right| \tag{14}$$
$$=\left|3^{\alpha-1}\Gamma(\alpha+1)\left[J_{0+}^{\alpha}f\left(\frac{1}{3}\right)+J_{\frac{1}{3}+}^{\alpha}f\left(\frac{2}{3}\right)+J_{\frac{2}{3}+}^{\alpha}f(1)\right]-\frac{1}{2}\left(f\left(\frac{1}{3}\right)+f\left(\frac{2}{3}\right)\right)\right|$$
$$=\left|\frac{3\alpha^3+23\alpha^2+64\alpha+72}{81(\alpha+1)(\alpha+2)(\alpha+3)}-\frac{1}{18}\right|.$$

On the other hand, the second term of (13) *we can express as*

$$\frac{b-a}{27}\left[\left|f'(a)\right|\left(\frac{7\alpha^2+21\alpha+10}{4(\alpha+1)(\alpha+2)}+\frac{2\alpha}{\alpha+1}\left(\frac{1}{2}\right)^{\frac{1}{\alpha}(\alpha+1)}+\frac{\alpha}{\alpha+2}\left(\frac{1}{2}\right)^{\frac{1}{\alpha}(\alpha+2)}\right)\right. \tag{15}$$
$$\left.+\left|f'(b)\right|\left(\frac{-\alpha^2+9\alpha+26}{4(\alpha+1)(\alpha+2)}+\frac{4\alpha}{\alpha+1}\left(\frac{1}{2}\right)^{\frac{1}{\alpha}(\alpha+1)}-\frac{\alpha}{\alpha+2}\left(\frac{1}{2}\right)^{\frac{1}{\alpha}(\alpha+2)}\right)\right]$$
$$=\frac{1}{27}\left(\frac{-\alpha^2+9\alpha+26}{4(\alpha+1)(\alpha+2)}+\frac{4\alpha}{\alpha+1}\left(\frac{1}{2}\right)^{\frac{1}{\alpha}(\alpha+1)}-\frac{\alpha}{\alpha+2}\left(\frac{1}{2}\right)^{\frac{1}{\alpha}(\alpha+2)}\right).$$

Finally, substituting (14) *and* (15) *into* (13), *we obtain*

$$\left|\frac{3\alpha^3+23\alpha^2+64\alpha+72}{81(\alpha+1)(\alpha+2)(\alpha+3)}-\frac{1}{18}\right|\le\frac{1}{27}\left(\frac{-\alpha^2+9\alpha+26}{4(\alpha+1)(\alpha+2)}+\frac{4\alpha}{\alpha+1}\left(\frac{1}{2}\right)^{\frac{1}{\alpha}(\alpha+1)}-\frac{\alpha}{\alpha+2}\left(\frac{1}{2}\right)^{\frac{1}{\alpha}(\alpha+2)}\right).$$

The left and right sides of this inequality can be seen in Figure 2.

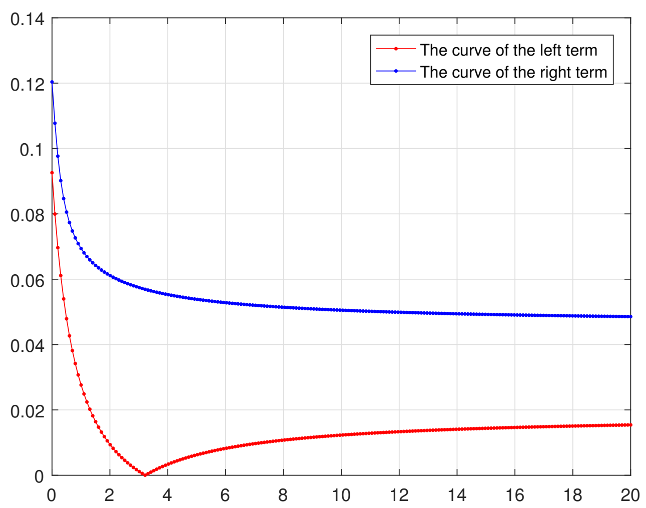

Figure 2: Graph for the result of Example 4.4 computed and plotted in MATLAB program.

Corollary 4.8. *If* $\alpha = 1$, *then Corollary 4.6 reduces to the inequality*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{1}{2}\left(f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)\right)\right|\le\frac{5(b-a)}{72}\left(\left|f'(a)\right|+f'(b)\right).$$

Theorem 4.9. *Let all the conditions of Lemma 3.1 are satisfied. If $\left|f'\right|^q$, $q \geq 1$ is convex mapping, then the parametrized inequality*

$$\begin{aligned}
&\left|\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f\left(\frac{2a+b}{3}\right)+J_{\frac{2a+b}{3}+}^{\alpha}f\left(\frac{a+2b}{3}\right)+J_{\frac{a+2b}{3}+}^{\alpha}f(b)\right]\right. \\
&\left.-\frac{1}{3}\left[(1-\lambda)(f(a)+f(b))+\left(\lambda+\frac{1}{2}\right)\left(f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)\right)\right]\right| \\
\leq\ & \frac{b-a}{9}\left[A_1^{1-\frac{1}{q}}(\alpha,\lambda)\left(\frac{\left|f'(a)\right|^q(2A_1(\alpha,\lambda)+A_2(\alpha,\lambda))+\left|f'(b)\right|^q(A_1(\alpha,\lambda)-A_2(\alpha,\lambda))}{3}\right)^{\frac{1}{q}}\right. \\
&+A_3^{1-\frac{1}{q}}(\alpha)\left(\frac{\left|f'(a)\right|^q(A_3(\alpha)+A_4(\alpha))+\left|f'(b)\right|^q(2A_3(\alpha,\lambda)-A_4(\alpha,\lambda))}{3}\right)^{\frac{1}{q}} \\
&\left.+A_5^{1-\frac{1}{q}}(\alpha,\lambda)\left(\frac{\left|f'(a)\right|^q(A_6(\alpha,\lambda))+\left|f'(b)\right|^q(3A_5(\alpha,\lambda)-A_6(\alpha,\lambda))}{3}\right)^{\frac{1}{q}}\right]
\end{aligned} \tag{16}$$

holds.

Proof. With help of the power-mean inequality in (9), we get

$$\begin{aligned}
&\left|\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f\left(\frac{2a+b}{3}\right)+J_{\frac{2a+b}{3}+}^{\alpha}f\left(\frac{a+2b}{3}\right)+J_{\frac{a+2b}{3}+}^{\alpha}f(b)\right]\right. \\
&\left.-\frac{1}{3}\left[(1-\lambda)(f(a)+f(b))+\left(\lambda+\frac{1}{2}\right)\left(f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)\right)\right]\right| \\
\leq\ & \frac{b-a}{9}\left[\left(\int_0^1\left|t^{\alpha}-\lambda\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left|t^{\alpha}-\lambda\right|\left|f'\left(\frac{2+t}{3}a+\frac{1-t}{3}b\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&+\left(\int_0^1\left|t^{\alpha}-\frac{1}{2}\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left|t^{\alpha}-\frac{1}{2}\right|\left|f'\left(\frac{1+t}{3}a+\frac{2-t}{3}b\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\left.+\left(\int_0^1\left|t^{\alpha}-(1-\lambda)\right|\right)^{1-\frac{1}{q}}\left(\int_0^1\left|t^{\alpha}-(1-\lambda)\right|\left|f'\left(\frac{t}{3}a+\frac{3-t}{3}b\right)\right|^q dt\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Utilizing the convexity of $\left|f'\right|^q$, we derive

$$\begin{aligned}
&\left|\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f\left(\frac{2a+b}{3}\right)+J_{\frac{2a+b}{3}+}^{\alpha}f\left(\frac{a+2b}{3}\right)+J_{\frac{a+2b}{3}+}^{\alpha}f(b)\right]\right. \\
&\left.-\frac{1}{3}\left[(1-\lambda)(f(a)+f(b))+\left(\lambda+\frac{1}{2}\right)\left(f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)\right)\right]\right| \\
\leq\ & \frac{b-a}{9}\left[\left(\int_0^1\left|t^{\alpha}-\lambda\right|dt\right)^{1-\frac{1}{q}}\left(\left|f'(a)\right|^q\int_0^1\frac{2+t}{3}\left|t^{\alpha}-\lambda\right|dt+\left|f'(b)\right|^q\int_0^1\frac{1-t}{3}\left|t^{\alpha}-\lambda\right|dt\right)^{\frac{1}{q}}\right.
\end{aligned}$$

$$+\left(\int_0^1 \left|t^\alpha-\frac{1}{2}\right|dt\right)^{1-\frac{1}{q}}\left(\left|f'(a)\right|^q\int_0^1\frac{1+t}{3}\left|t^\alpha-\frac{1}{2}\right|dt+\left|f'(b)\right|^q\int_0^1\frac{2-t}{3}\left|t^\alpha-\frac{1}{2}\right|dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 \left|t^\alpha-(1-\lambda)\right|\right)^{1-\frac{1}{q}}\left(\left|f'(a)\right|^q\int_0^1\frac{t}{3}\left|t^\alpha-(1-\lambda)\right|dt+\left|f'(b)\right|^q\int_0^1\frac{3-t}{3}\left|t^\alpha-(1-\lambda)\right|dt\right)^{\frac{1}{q}}\Bigg]$$

$$= \frac{b-a}{9}\left[A_1^{1-\frac{1}{q}}(\alpha,\lambda)\left(\frac{\left|f'(a)\right|^q(2A_1(\alpha,\lambda)+A_2(\alpha,\lambda))+\left|f'(b)\right|^q(A_1(\alpha,\lambda)-A_2(\alpha,\lambda))}{3}\right)^{\frac{1}{q}}\right.$$

$$+A_3^{1-\frac{1}{q}}(\alpha)\left(\frac{\left|f'(a)\right|^q(A_3(\alpha)+A_4(\alpha))+\left|f'(b)\right|^q(2A_3(\alpha,\lambda)-A_4(\alpha,\lambda))}{3}\right)^{\frac{1}{q}}$$

$$\left.+A_5^{1-\frac{1}{q}}(\alpha,\lambda)\left(\frac{\left|f'(a)\right|^q(A_6(\alpha,\lambda))+\left|f'(b)\right|^q(3A_5(\alpha,\lambda)-A_6(\alpha,\lambda))}{3}\right)^{\frac{1}{q}}\right].$$

So, the proof ends. □

Remark 4.10. *If $\lambda=\frac{5}{8}$, then inequality* (16) *turns into the inequality given by Sitthiwirattham et al. in [27, Theorem 5].*

Corollary 4.11. *If $\lambda=-\frac{1}{2}$, then inequality* (16) *reduces to the inequality*

$$\left|\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J_{a+}^\alpha f\left(\frac{2a+b}{3}\right)+J_{\frac{2a+b}{3}+}^\alpha f\left(\frac{a+2b}{3}\right)+J_{\frac{a+2b}{3}+}^\alpha f(b)\right]-\frac{f(a)+f(b)}{2}\right|$$

$$\leq \frac{b-a}{9}\left\{\left(\frac{3+\alpha}{2(\alpha+1)}\right)^{1-\frac{1}{q}}\left[\frac{1}{3}\left(\left|f'(a)\right|^q\left(\frac{5\alpha^2+27\alpha+30}{4(\alpha+1)(\alpha+2)}\right)+\left|f'(b)\right|^q\left(\frac{\alpha^2+3\alpha+6}{4(\alpha+1)(\alpha+2)}\right)\right)\right]^{\frac{1}{q}}\right.$$

$$+\left(\frac{2\alpha}{\alpha+1}\left(\frac{1}{2}\right)^{\frac{\alpha+1}{\alpha}}+\frac{1-\alpha}{2(\alpha+1)}\right)^{1-\frac{1}{q}}\left[\frac{1}{3}\left(\left|f'(a)\right|^q\left(\frac{2\alpha}{\alpha+1}\left(\frac{1}{2}\right)^{\frac{\alpha+1}{\alpha}}+\frac{\alpha}{\alpha+2}\left(\frac{1}{2}\right)^{\frac{\alpha+2}{\alpha}}+\frac{-3\alpha^2-\alpha+6}{4(\alpha+1)(\alpha+2)}\right)\right.\right.$$

$$\left.\left.+\left|f'(b)\right|^q\left(\frac{4\alpha}{\alpha+1}\left(\frac{1}{2}\right)^{\frac{\alpha+1}{\alpha}}-\frac{\alpha}{\alpha+2}\left(\frac{1}{2}\right)^{\frac{\alpha+2}{\alpha}}+\frac{-3\alpha^2-5\alpha+6}{4(\alpha+1)(\alpha+2)}\right)\right]^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{3\alpha+1}{2(\alpha+1)}\right)^{1-\frac{1}{q}}\left[\frac{1}{3}\left(\left|f'(a)\right|^q\left(\frac{3\alpha+2}{4(\alpha+2)}\right)+\left|f'(b)\right|^q\left(\frac{15\alpha^2+37\alpha+10}{4(\alpha+1)(\alpha+2)}\right)\right)\right]^{\frac{1}{q}}\right\}.$$

Corollary 4.12. *If $\alpha=1$, then Corollary 4.11 reduces to the inequality*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{f(a)+f(b)}{2}\right|$$

$$\leq \frac{b-a}{9}\left[\left(\frac{31\left|f'(a)\right|^q+5\left|f'(b)\right|^q}{36}\right)^{\frac{1}{q}}+\frac{1}{4}\left(\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{5\left|f'(a)\right|^q+31\left|f'(b)\right|^q}{36}\right)^{\frac{1}{q}}\right].$$

Corollary 4.13. *If $\lambda = 1$, then inequality (16) reduces to the inequality*

$$\begin{aligned}
&\left|\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f\left(\frac{2a+b}{3}\right)+J_{\frac{2a+b}{3}+}^{\alpha}f\left(\frac{a+2b}{3}\right)+J_{\frac{a+2b}{3}+}^{\alpha}f(b)\right]-\frac{1}{2}\left(f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)\right)\right|\\
\leq\;& \frac{b-a}{9}\left\{\left(\frac{\alpha}{\alpha+1}\right)^{1-\frac{1}{q}}\left[\frac{1}{3}\left(\left|f'(a)\right|^{q}\left(\frac{5\alpha^{2}+9\alpha}{2(\alpha+1)(\alpha+2)}\right)+\left|f'(b)\right|^{q}\left(\frac{\alpha^{2}+3\alpha}{2(\alpha+1)(\alpha+2)}\right)\right)\right]^{\frac{1}{q}}\right.\\
&+\left(\frac{1-\alpha}{2\alpha+2}+\frac{2\alpha}{\alpha+1}\left(\frac{1}{2}\right)^{\frac{1}{\alpha}(\alpha+1)}\right)^{1-\frac{1}{q}}\left[\frac{1}{3}\left(\left|f'(a)\right|^{q}\left(\frac{-3\alpha^{2}-\alpha+6}{4(\alpha+1)(\alpha+2)}+\frac{2\alpha}{\alpha+1}\left(\frac{1}{2}\right)^{\frac{1}{\alpha}(\alpha+1)}+\frac{\alpha}{\alpha+2}\left(\frac{1}{2}\right)^{\frac{1}{\alpha}(\alpha+2)}\right)\right.\right.\\
&+\left.\left.\left|f'(b)\right|^{q}\left(\frac{-3\alpha^{2}-5\alpha+6}{4(\alpha+1)(\alpha+2)}+\frac{4\alpha}{\alpha+1}\left(\frac{1}{2}\right)^{\frac{1}{\alpha}(\alpha+1)}-\frac{\alpha}{\alpha+2}\left(\frac{1}{2}\right)^{\frac{1}{\alpha}(\alpha+2)}\right)\right)\right]^{\frac{1}{q}}\\
&+\left.\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left[\frac{1}{3}\left(\left|f'(a)\right|^{q}\left(\frac{1}{\alpha+2}\right)+\left|f'(b)\right|^{q}\left(\frac{2\alpha+5}{(\alpha+1)(\alpha+2)}\right)\right)\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

Corollary 4.14. *If $\alpha = 1$, then Corollary 4.13 reduces to the inequality*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_{a}^{b}f(x)\,dx-\frac{1}{2}\left(f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)\right)\right|\\
\leq\;& \frac{b-a}{18}\left\{\left(\frac{7\left|f'(a)\right|^{q}+2\left|f'(b)\right|^{q}}{9}\right)^{\frac{1}{q}}+\frac{1}{2}\left(\frac{\left|f'(a)\right|^{q}+\left|f'(b)\right|^{q}}{2}\right)^{\frac{1}{q}}\right.\\
&+\left.\left(\frac{2\left|f'(a)\right|^{q}+7\left|f'(b)\right|^{q}}{9}\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Theorem 4.15. *Assume that the conditions of Lemma 3.1 are valid. If $\left|f'\right|^{q}$, $q > 1$ is a convex function, then we have*

$$\begin{aligned}
&\left|\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f\left(\frac{2a+b}{3}\right)+J_{\frac{2a+b}{3}+}^{\alpha}f\left(\frac{a+2b}{3}\right)+J_{\frac{a+2b}{3}+}^{\alpha}f(b)\right]\right.\\
&-\left.\frac{1}{3}\left[(1-\lambda)(f(a)+f(b))+\left(\lambda+\frac{1}{2}\right)\left(f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)\right)\right]\right|\\
\leq\;& \frac{b-a}{9}\left[A_{7}(\alpha,\lambda,p)\left(\frac{5\left|f'(a)\right|^{q}+\left|f'(b)\right|^{q}}{6}\right)^{\frac{1}{q}}\right.\\
&+\left.A_{8}(\alpha,p)\left(\frac{\left|f'(a)\right|^{q}+\left|f'(b)\right|^{q}}{2}\right)^{\frac{1}{q}}+A_{9}(\alpha,\lambda,p)\left(\frac{\left|f'(a)\right|^{q}+5\left|f'(b)\right|^{q}}{6}\right)^{\frac{1}{q}}\right],
\end{aligned}\tag{17}$$

where $q^{-1} + p^{-1} = 1$ and

$$\begin{aligned}
A_{7}(\alpha,\lambda,p) &= \left(\int_{0}^{1}\left|t^{\alpha}-\lambda\right|^{p}dt\right)^{\frac{1}{p}},\\
A_{8}(\alpha,p) &= \left(\int_{0}^{1}\left|t^{\alpha}-\frac{1}{2}\right|^{p}dt\right)^{\frac{1}{p}},
\end{aligned}$$

and

$$A_9(\alpha,\lambda,p)=\left(\int_0^1 |t^\alpha-(1-\lambda)|^p\,dt\right)^{\frac{1}{p}}.$$

Proof. Using the Hölder's inequality, it follows from (9) that

$$\begin{aligned}
&\left|\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J_{a+}^\alpha f\left(\frac{2a+b}{3}\right)+J_{\frac{2a+b}{3}+}^\alpha f\left(\frac{a+2b}{3}\right)+J_{\frac{a+2b}{3}+}^\alpha f(b)\right]\right.\\
&\left.-\frac{1}{3}\left[(1-\lambda)(f(a)+f(b))+\left(\lambda+\frac{1}{2}\right)\left(f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)\right)\right]\right|\\
\leq\ &\frac{b-a}{9}\left[\left(\int_0^1 |t^\alpha-\lambda|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{2+t}{3}a+\frac{1-t}{3}b\right)\right|^q dt\right)^{\frac{1}{q}}\right.\\
&+\left(\int_0^1\left|t^\alpha-\frac{1}{2}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{1+t}{3}a+\frac{2-t}{3}b\right)\right|^q dt\right)^{\frac{1}{q}}\\
&\left.+\left(\int_0^1 |t^\alpha-(1-\lambda)|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{t}{3}a+\frac{3-t}{3}b\right)\right|^q dt\right)^{\frac{1}{q}}\right].
\end{aligned}$$

With the aid of the convexity of $|f'|^q$, $q>1$,we obtain

$$\begin{aligned}
&\left|\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J_{a+}^\alpha f\left(\frac{2a+b}{3}\right)+J_{\frac{2a+b}{3}+}^\alpha f\left(\frac{a+2b}{3}\right)+J_{\frac{a+2b}{3}+}^\alpha f(b)\right]\right.\\
&\left.-\frac{1}{3}\left[(1-\lambda)(f(a)+f(b))+\left(\lambda+\frac{1}{2}\right)\left(f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)\right)\right]\right|\\
\leq\ &\frac{b-a}{9}\left[\left(\int_0^1 |t^\alpha-\lambda|^p\,dt\right)^{\frac{1}{p}}\left(|f'(a)|^q\int_0^1\frac{2+t}{3}dt+|f'(b)|^q\int_0^1\frac{1-t}{3}dt\right)^{\frac{1}{q}}\right.\\
&+\left(\int_0^1\left|t^\alpha-\frac{1}{2}\right|^p dt\right)^{\frac{1}{p}}\left(|f'(a)|^q\int_0^1\frac{1+t}{3}dt+|f'(b)|^q\int_0^1\frac{2-t}{3}dt\right)^{\frac{1}{q}}\\
&\left.+\left(\int_0^1 |t^\alpha-(1-\lambda)|^p\,dt\right)^{\frac{1}{p}}\left(|f'(a)|^q\int_0^1\frac{t}{3}dt+|f'(b)|^q\int_0^1\frac{3-t}{3}dt\right)^{\frac{1}{q}}\right]\\
=\ &\frac{b-a}{9}A_7(\alpha,\lambda,p)\left(\frac{5|f'(a)|^q+|f'(b)|^q}{6}\right)^{\frac{1}{q}}\\
&+A_8(\alpha,p)\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}+A_9(\alpha,\lambda,p)\left(\frac{|f'(a)|^q+5|f'(b)|^q}{6}\right)^{\frac{1}{q}}.
\end{aligned}$$

This calculation ends the proof. □

Remark 4.16. *If $\lambda = \frac{5}{8}$ in the inequality* (17)*, it turns into the inequality obtained by Sitthiwirattham et al. in [27, Theorem 6].*

Corollary 4.17. *If $\lambda = -\frac{1}{2}$, then inequality* (17) *reduces to the inequality*

$$\left|\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f\left(\frac{2a+b}{3}\right)+J_{\frac{2a+b}{3}+}^{\alpha}f\left(\frac{a+2b}{3}\right)+J_{\frac{a+2b}{3}+}^{\alpha}f(b)\right]-\frac{f(a)+f(b)}{2}\right|$$

$$\leq \frac{b-a}{9}\left[\left(\int_0^1\left(t^{\alpha}+\frac{1}{2}\right)^p dt\right)^{\frac{1}{p}}\left(\frac{5|f'(a)|^q+|f'(b)|^q}{6}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1\left|t^{\alpha}-\frac{1}{2}\right|^p dt\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}+\left(\int_0^1\left(\frac{3}{2}-t^{\alpha}\right)^p dt\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q+5|f'(b)|^q}{6}\right)^{\frac{1}{q}}\right].$$

Corollary 4.18. *If $\alpha = 1$, then Corollary 4.17 reduces to the inequality*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{f(a)+f(b)}{2}\right|$$

$$\leq \frac{b-a}{18}\left[\left(\frac{3^{p+1}-1}{2(p+1)}\right)^{\frac{1}{p}}\left[\left(\frac{5|f'(a)|^q+|f'(b)|^q}{6}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^q+5|f'(b)|^q}{6}\right)^{\frac{1}{q}}\right]\right.$$

$$\left.+\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}\right].$$

Corollary 4.19. *If $\lambda = 1$, then inequality* (17) *reduces to the inequality*

$$\left|\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f\left(\frac{2a+b}{3}\right)+J_{\frac{2a+b}{3}+}^{\alpha}f\left(\frac{a+2b}{3}\right)+J_{\frac{a+2b}{3}+}^{\alpha}f(b)\right]-\frac{1}{2}\left(f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)\right)\right|$$

$$\leq \frac{b-a}{9}\left[\left(\int_0^1(1-t^{\alpha})^p dt\right)^{\frac{1}{p}}\left(\frac{5|f'(a)|^q+|f'(b)|^q}{6}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1\left|t^{\alpha}-\frac{1}{2}\right|^p dt\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}+\frac{1}{\alpha p+1}\left(\frac{|f'(a)|^q+5|f'(b)|^q}{6}\right)^{\frac{1}{q}}\right].$$

Corollary 4.20. *If $\alpha = 1$, then Corollary 4.19 reduces to the inequality*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{1}{2}\left(f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)\right)\right|$$

$$\leq \frac{b-a}{9(p+1)^{\frac{1}{p}}}\left[\left(\frac{5|f'(a)|^q+|f'(b)|^q}{6}\right)^{\frac{1}{q}}+\frac{1}{2}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{|f'(a)|^q+5|f'(b)|^q}{6}\right)^{\frac{1}{q}}\right].$$

5. Conclusion

In this article, we obtained parameterized inequalities due to Riemann–Liouville integrals. It has been shown to generalize Newton's and Trapezoid type inequalities. New results can be obtained for different choices of real parameters. Curious readers can consider new inequalities using other fractional integrals. In addition, new inequalities can be established with the help of different types of convexities in the literature.

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