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Marcinkiewicz strong laws for weighted sums under the sub-linear expectations

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Abstract. In this article, we establish complete convergence theorems for weighted sums in sub-linear expectations space. As corollaries, we obtain Marcinkiewicz strong laws for weighted sums under the sub-linear expectations. Our results extend and improve the corresponding results of Bai and Cheng (Statist. Probab. Lett. 46: 105-112, 2000) and Cuzick (J. Theoret. Probab. 8: 625-641, 1995) from classical probability space to sub-linear expectation space.

1. Introduction

In the classical probability theory, probability and expectation are both additive. But the uncertainty phenomenon can not be modeled using additive probabilities or additive expectations in many areas of applications. Non-additive probabilities and non-additive expectations are useful tools for studying uncertainties in statistics, measures of risk, superhedging in finance and non-linear stochastic calculus (see Denis and Martini[1] and Peng [2-5]). Peng [4-6] introduced the general framework of the sub-linear expectation in a general function space by relaxing the linear property of the classical expectation to the sub-additivity and positive homogeneity (cf. Definition 1.1 below). Under Peng's sub-linear expectation framework, many limit theorems have been established recently, including the central limit theorem and weak law of large numbers (see Peng [6-8]), strong law of large numbers (see Chen [9], Hu [10], Wu and Jiang [11], Tang et al. [12]), the law of the iterated logarithm (see Chen and Hu [13], Zhang [14], Donsker's invariance principle and Chung's law of the iterated logarithm (see Zhang [15]), Rosenthal's inequalities (see Zhang [16]) and Three series theorem (see Xu and Zhang [17]), complete convergence theorems (see Feng et al. [18], Wu and Jiang [19], Xi et al. [20]), self-normalized moderate deviation and law of the iterated logarithm (see Zhang [21]), and so on. Because sub-linear expectation and capacity are not additive, the study of the limit theorems under sub-linear expectation becomes much more complex and challenging. Extending the limit theorems in the traditional probability space to the case of sub-linear expectation space is of great significance in the theory and application. Complete convergence theorems are important limit theorems in probability theory. Many of related results have been obtained in the probability space. We

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refer the reader to Baum and Katz [22], Peligrad and Gut [23], Wang et al. [24], Huang et al. [25] and Wang et al. [26]. Complete convergence for weighted sums are also important in sub-linear expectation space, which can be applied to nonparametric regression models (see Xi et al. [20]). In fact, the limiting behavior of weighted sums is very important in many statistical problems such as least-squares estimators, nonparametric regression function estimators and jackknife estimators among others. We will establish stronger complete convergence theorems for weighted sums in sub-linear expectations space. As corollaries, we obtain Marcinkiewicz strong laws for weighted sums under the sub-linear expectations.

We use the framework and notations of Peng [6]. Let $(\Omega, \mathcal{F})$ be a given measurable space and let $\mathcal{H}$ be a linear space of real functions defined on $(\Omega, \mathcal{F})$ such that if $X_1, \cdots, X_n \in \mathcal{H}$ then $\varphi(X_1, \cdots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{l.Lip}(\mathbb{R}^n)$, where $C_{l.Lip}(\mathbb{R}^n)$ denotes the linear space of (local Lipschitz) functions φ satisfying

$$|\varphi(\boldsymbol{x}) - \varphi(\boldsymbol{y})| \leq C(1 + |\boldsymbol{x}|^m + |\boldsymbol{y}|^m)|\boldsymbol{x} - \boldsymbol{y}|, \quad \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n,$$

for some $C > 0, m \in \mathbb{N}$ depending on φ. $\mathcal{H}$ is considered as a space of "random variables". If X is an element of set $\mathcal{H}$, then we denote $X \in \mathcal{H}$.

Definition 1.1. *(Peng [6]) A sub-linear expectation $\widehat{\mathbb{E}}$ on $\mathcal{H}$ is a function $\widehat{\mathbb{E}} : \mathcal{H} \to \bar{\mathbb{R}}$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have*
(a) Monotonicity: If $X \geq Y$ then $\widehat{\mathbb{E}}[X] \geq \widehat{\mathbb{E}}[Y]$;
(b) Constant preserving: $\widehat{\mathbb{E}}[c] = c$;
(c) Sub-additivity: $\widehat{\mathbb{E}}[X + Y] \leq \widehat{\mathbb{E}}[X] + \widehat{\mathbb{E}}[Y]$ whenever $\widehat{\mathbb{E}}[X] + \widehat{\mathbb{E}}[Y]$ is not of the form $+\infty - \infty$ or $-\infty + \infty$;
(d) Positive homogeneity: $\widehat{\mathbb{E}}[\lambda X] = \lambda \widehat{\mathbb{E}}[X], \lambda > 0$.
Here $\bar{\mathbb{R}} = [-\infty, +\infty]$. The triple $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ is called a sub-linear expectation space. Given a sub-linear expectation $\widehat{\mathbb{E}}$, let us denote the conjugate expectation $\widehat{\mathcal{E}}$ of $\widehat{\mathbb{E}}$ by

$$\widehat{\mathcal{E}}[X] := -\widehat{\mathbb{E}}[-X], \quad \forall X \in \mathcal{H}.$$

From the definition, we can easily get that $\widehat{\mathcal{E}}[X] \leq \widehat{\mathbb{E}}[X]$, $\widehat{\mathbb{E}}[X + c] = \widehat{\mathbb{E}}[X] + c$, $\widehat{\mathbb{E}}[X - Y] \geq \widehat{\mathbb{E}}[X] - \widehat{\mathbb{E}}[Y]$ and $|\widehat{\mathbb{E}}[X] - \widehat{\mathbb{E}}[Y]| \leq \widehat{\mathbb{E}}[|X - Y|]$. Further, if $\widehat{\mathbb{E}}[|X|]$ is finite, then $\widehat{\mathcal{E}}[X]$ and $\widehat{\mathbb{E}}[X]$ are both finite.

Definition 1.2. *(Peng [6])*

(i) (Identical distribution) Let $\boldsymbol{X_1}$ and $\boldsymbol{X_2}$ be two n-dimensional random vectors defined respectively in sub-linear expectation spaces $(\Omega_1, \mathcal{H}_1, \widehat{\mathbb{E}}_1)$ and $(\Omega_2, \mathcal{H}_2, \widehat{\mathbb{E}}_2)$. They are called identically distributed, denoted by $\boldsymbol{X_1} \stackrel{d}{=} \boldsymbol{X_2}$, if $\widehat{\mathbb{E}}_1[\varphi(\boldsymbol{X_1})] = \widehat{\mathbb{E}}_2[\varphi(\boldsymbol{X_2})], \quad \forall \varphi \in C_{l.Lip}(\mathbb{R}^n)$, whenever the sub-expectations are finite.

(ii) (Independence) In a sub-linear expectation space $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$, a random vector $\boldsymbol{Y} = (Y_1, Y_2, \cdots, Y_n), \ Y_i \in \mathcal{H}$ is said to be independent to another random vector $\boldsymbol{X} = (X_1, X_2, \cdots, X_m), \ X_i \in \mathcal{H}$ under $\widehat{\mathbb{E}}$ if for each test function $\varphi \in C_{l.Lip}(\mathbb{R}^m \times \mathbb{R}^n)$ we have $\widehat{\mathbb{E}}[\varphi(\boldsymbol{X}, \boldsymbol{Y})] = \widehat{\mathbb{E}}[\widehat{\mathbb{E}}[\varphi(\boldsymbol{x}, \boldsymbol{Y})]|_{\boldsymbol{x}=\boldsymbol{X}}]$, whenever $\bar{\varphi}(\boldsymbol{x}) := \widehat{\mathbb{E}}[|\varphi(\boldsymbol{x}, \boldsymbol{Y})|] < \infty$ for all $\boldsymbol{x}$ and $\widehat{\mathbb{E}}[|\bar{\varphi}(\boldsymbol{X})|] < \infty$.

(iii) (IID random variables) A sequence of random variables $\{X_n; n \geq 1\}$ is said to be independent if X_{i+1} is independent to $(X_1, X_2, \cdots, X_i)$ for each $i \geq 1$, and it is said to be identically distributed if $X_i \stackrel{d}{=} X_1$, for each $i \geq 1$.

Next, we introduce the capacities corresponding to the sub-linear expectations. Let $\mathcal{G} \subset \mathcal{F}$. A function $V : \mathcal{G} \to [0, 1]$ is called a capacity if

$$V(\phi) = 0, \ V(\Omega) = 1, \ \text{ and } V(A) \leq V(B) \ \ \forall A \subseteq B, A, B \in \mathcal{G}.$$

It is called to be sub-additive if $V(A \cup B) \leq V(A) + V(B)$ for all $A, B \in \mathcal{G}$ with $A \cup B \in \mathcal{G}$.

Let $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ be a sub-linear expectation space. We define $(\mathbb{V}, \mathcal{V})$ as a pair of capacities with the properties that

$$\widehat{\mathbb{E}}[f] \leq \mathbb{V}(A) \leq \widehat{\mathbb{E}}[g], \text{ if } f \leq I(A) \leq g, \ f, g \in \mathcal{H} \text{ and} A \in \mathcal{F}, \tag{1.1}$$

$$\mathbb{V} \text{ is sub-additive}$$

and $\mathcal{V}(A) := 1 - \mathbb{V}(A^c),\ \forall A \in \mathcal{F}$. It is obvious that

$$\mathcal{V}(A \cup B) \leq \mathcal{V}(A) + \mathbb{V}(B)$$

We call $\mathbb{V}$ and $\mathcal{V}$ the upper and lower capacity, respectively. In general, we choose $\mathbb{V}$ as

$$\mathbb{V}(A) := \inf\{\widehat{\mathbb{E}}[\xi] : I(A) \leq \xi, \xi \in \mathcal{H}\},\ \forall A \in \mathcal{F}. \tag{1.2}$$

To distinguish this capacity from others, we denote it by $\widehat{\mathbb{V}}$ and $\widehat{\mathcal{V}}(A) = 1 - \widehat{\mathbb{V}}(A^c)$. $\widehat{\mathbb{V}}$ is the largest capacity satisfying (1.1).

It should be known that (1.1) implies Markov inequality: $\forall X \in \mathcal{H}$,

$$\mathbb{V}(|X| \geq x) \leq \widehat{\mathbb{E}}[|X|^p]/x^p,\ \ \forall x > 0, p > 0$$

from $I(|X| \geq x) \leq |X|^p/x^p \in \mathcal{H}$. By Lemma 4.1 in Zhang [14], we have Hölder inequality: $\forall X, Y \in \mathcal{H},\ p, q > 1$, satisfying $p^{-1} + q^{-1} = 1$,

$$\widehat{\mathbb{E}}[|XY|] \leq (\widehat{\mathbb{E}}[|X|^p])^{\frac{1}{p}} (\widehat{\mathbb{E}}[|Y|^q])^{\frac{1}{q}},$$

particularly, Jensen inequality:

$$(\widehat{\mathbb{E}}[|X|^r])^{\frac{1}{r}} \leq (\widehat{\mathbb{E}}[|X|^s])^{\frac{1}{s}},\ \ \text{for } 0 < r \leq s.$$

Definition 1.3. *(Zhang [14]) A function $V : \mathcal{F} \to [0, 1]$ is called to be countably sub-additive if*

$$V\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} V(A_n),\ \ \forall A_n \in \mathcal{F}.$$

We define the Choquet integrals/expecations $(C_{\mathbb{V}}, C_{\mathcal{V}})$ by

$$C_V(X) := \int_0^{\infty} V(X \geq x)dx + \int_{-\infty}^{0} (V(X \geq x) - 1)dx$$

with V being replaced by $\mathbb{V}$ and $\mathcal{V}$, respectively. If $\lim_{c\to+\infty} \widehat{\mathbb{E}}[(|X| - c)^+] = 0$, then $\widehat{\mathbb{E}}[|X|] \leq C_{\mathbb{V}}(|X|)$ (Lemma 4.5(iii) of Zhang [14]).

Throughout this paper, C stands for a positive constant which may differ from one place to another and $I(.)$ denote an indicator function.

2. Main results

Theorem 2.1. *Let $\{X, X_n; n \geq 1\}$ be a sequence of independent and identically distributed random variables with $\widehat{\mathbb{E}}[X] = \widehat{\mathcal{E}}[X] = 0$ in a sub-linear expectation space $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$. Assume that $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ is an array of positive real numbers satisfying $\sum_{i=1}^{n} |a_{ni}|^{\alpha} = O(n)$ and $T_n = a_{ni}X_i$. Let $\frac{1}{p} = \frac{1}{\alpha} + \frac{1}{\beta}, 1 \leq p < 2, 1 < \alpha, \beta < \infty$. If $\widehat{\mathbb{E}}[|X|^{\beta}] \leq C_V[|X|^{\beta}] < \infty$, Then for any $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} n^{-1} \mathbb{V}\left(\max_{1\leq j\leq n} |T_j| > \varepsilon n^{\frac{1}{p}}\right) < \infty. \tag{2.1}$$

Corollary 2.2. *Under the conditions of Theorem 2.1, suppose that V is countably sub-additive and* $\{a_i, i \geq 1\}$ *is a sequence of positive real numbers satisfying* $\sum_{i=1}^{n} |a_i|^{\alpha} = O(n)$. *Then for any* $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{-1} \mathbb{V}\left(\max_{1 \leq j \leq n} | \sum_{i=1}^{j} |a_i X_i| > \varepsilon n^{\frac{1}{p}} \right) < \infty \tag{2.2}$$

and

$$\lim_{n \to \infty} \frac{\sum_{i=1}^{n} |a_i X_i|}{n^{\frac{1}{p}}} = 0 \ \ a.s. \ \mathbb{V}. \tag{2.3}$$

In Theorem 2.1, let $p = 1$, we can easily get the following theorem.

Theorem 2.3. *Let* $\{X, X_n; n \geq 1\}$ *be a sequence of independent and identically distributed random variables with* $\widehat{\mathbb{E}}[X] = \widehat{\mathcal{E}}[X] = 0$ *in a sub-linear expectation space* $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$. *Assume that* $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ *is an array of positive real numbers satisfying* $\sum_{i=1}^{n} |a_{ni}|^{\alpha} = O(n), 1 < \alpha < \infty$ *and* $T_n = a_{ni} X_i$. *Let* $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. *If* $\widehat{\mathbb{E}}[|X|^{\beta}] \leq C_V[|X|^{\beta}] < \infty$, *Then for any* $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{-1} \mathbb{V}\left(\max_{1 \leq j \leq n} |T_j| > \varepsilon n^{\frac{1}{p}} \right) < \infty. \tag{2.4}$$

Corollary 2.4. *Under the conditions of Theorem 2.3, suppose that V is countably sub-additive and* $\{a_i, i \geq 1\}$ *is a sequence of positive real numbers satisfying* $\sum_{i=1}^{n} |a_i|^{\alpha} = O(n)$. *Then for any* $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{-1} \mathbb{V}\left(\max_{1 \leq j \leq n} | \sum_{i=1}^{j} |a_i X_i| > \varepsilon n \right) < \infty \tag{2.5}$$

and

$$\lim_{n \to \infty} \frac{\sum_{i=1}^{n} |a_i X_i|}{n} = 0 \ \ a.s. \ \mathbb{V}. \tag{2.6}$$

Remark 2.5 From our Theorem 2.1 and Corollary 2.2, we can see that the result of our Theorem 2.1 is stronger than the corresponding result in Bai and Cheng [27]. Our results not only extend the corresponding result in Bai and Cheng [27] from classical probability space to sub-linear expectation space, but also improve the corresponding result.

Remark 2.6 Our Corollary 2.4 extends the corresponding result of Cuzick [28] from classical probability space to sub-linear expectation space.

3. Proof of main result

In order to prove our results, we need the following lemmas.

Lemma 3.1. *(Rosenthal-type inequality (see Zhang[16])) Let* $\{X_n; n \geq 1\}$ *be a sequence of independent random variables in* $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ *and* $S_n = \sum_{i=1}^{n} X_i$. *Suppose* $p \geq 2$. *Then*

$$\widehat{\mathbb{E}}\left[\max_{1 \leq k \leq n} |S_k|^p\right] \leq C_p \left\{ \sum_{k=1}^{n} \widehat{\mathbb{E}}[|X_k|^p] + \left(\sum_{k=1}^{n} \widehat{\mathbb{E}}[X_k^2] \right)^{p/2} \right\} + C_p \left(\sum_{k=1}^{n} [(\widehat{\mathcal{E}}[X_k])^- + (\widehat{\mathbb{E}}[X_k])^+] \right)^p .$$

Lemma 3.2. *(Borel-Cantellis Lemma (see Zhang[14])) Let $\{A_n; n \geq 1\}$ be a sequence of events in $\mathcal{F}$. Suppose that V is a countably sub-additive capacity. If $\sum_{n=1}^{\infty} V(A_n) < \infty$, then $V(A_n; i.o.) = 0$, where $V(A_n; i.o.) = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m$.*

Proof of Theorem 2.1 For $0 < \mu < 1$, let $g(x) \in C_{l.Lip}(\mathbb{R}), 0 \leq g(x) \leq 1$ for all x, $g(x) = 1$ if $x \leq \mu$, $g(x) = 0$ if $x > 1$. Then

$$I(|x| \leq \mu) \leq g(|x|) \leq I(|x| \leq 1),\ I(|x| > 1) \leq 1 - g(|x|) \leq I(|x| > \mu). \tag{3.1}$$

For $i \geq 1$, let $Y_i = X_i g\left(\frac{|X_i|}{n^{\frac{1}{p}}}\right)$ and $T_j^{(n)} = \sum_{i=1}^{j}(a_{ni}Y_i - \widehat{\mathbb{E}}[a_{ni}Y_i])$. Then for any $\varepsilon > 0$, we can easily get

$$\begin{aligned}
&\mathbb{V}\left(\max_{1\leq j\leq n}|T_j| > \varepsilon n^{\frac{1}{p}}\right)\\
&\leq \mathbb{V}\left(\max_{1\leq i\leq n}|X_i| > n^{\frac{1}{p}}\right) + \mathbb{V}\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_{ni}Y_i\right| > \varepsilon n^{\frac{1}{p}}\right)\\
&\leq \sum_{i=1}^{n}\mathbb{V}\left(|X_i| > n^{\frac{1}{p}}\right) + \mathbb{V}\left(\max_{1\leq j\leq n}\left|T_j^{(n)}\right| > \varepsilon n^{\frac{1}{p}} - \max_{1\leq j\leq n}\left|\sum_{i=1}^{j}\widehat{\mathbb{E}}[a_{ni}Y_i]\right|\right).
\end{aligned} \tag{3.2}$$

We first show that, when $n \to \infty$,

$$n^{-\frac{1}{p}}\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}\widehat{\mathbb{E}}[a_{ni}Y_i]\right| \to 0. \tag{3.3}$$

$\forall 1 \leq k \leq \alpha$, by $\sum_{i=1}^{n}|a_{ni}|^{\alpha} = O(n)$ and Hölder inequality, we have

$$\sum_{i=1}^{n}|a_{ni}|^{k} \leq (\sum_{i=1}^{n}|a_{ni}|^{\alpha})^{\frac{k}{\alpha}}(\sum_{i=1}^{n}1)^{1-\frac{k}{\alpha}} \leq Cn. \tag{3.4}$$

By $\widehat{\mathbb{E}}[X_i] = 0$, $\frac{1}{p} = \frac{1}{\alpha} + \frac{1}{\beta}, \beta > 1$ and (3.4), we have

$$\begin{aligned}
n^{-\frac{1}{p}}\max_{1\leq j\leq n}|\sum_{i=1}^{j}\widehat{\mathbb{E}}[a_{ni}Y_i]| &\leq n^{-\frac{1}{p}}\sum_{i=1}^{n}|\widehat{\mathbb{E}}[a_{ni}Y_i]|\\
&= n^{-\frac{1}{p}}\sum_{i=1}^{n}|\widehat{\mathbb{E}}[a_{ni}X_i] - \widehat{\mathbb{E}}[a_{ni}Y_i]|\\
&\leq n^{-\frac{1}{p}}\sum_{i=1}^{n}\widehat{\mathbb{E}}|a_{ni}|[|X_i - Y_i|]\\
&\leq n^{-\frac{1}{p}}\sum_{i=1}^{n}|a_{ni}|\widehat{\mathbb{E}}\left[|X_i|\left(1 - g\left(\frac{|X_i|}{n^{\frac{1}{p}}}\right)\right)\right]\\
&= Cn^{1-\frac{1}{p}}\widehat{\mathbb{E}}\left[|X|\left(1 - g\left(\frac{|X|}{n^{\frac{1}{p}}}\right)\right)\right]\\
&\leq Cn^{1-\frac{1}{p}}n^{-\frac{1}{p}(\beta-1)}\widehat{\mathbb{E}}\left[|X|^{\beta}\left(1 - g\left(\frac{|X|}{n^{\frac{1}{p}}}\right)\right)\right]\\
&= Cn^{1-\frac{\beta}{p}}\widehat{\mathbb{E}}\left[|X|^{\beta}\left(1 - g\left(\frac{|X|}{n^{\frac{1}{p}}}\right)\right)\right] \to 0 \ \text{ as } n \to \infty.
\end{aligned}$$

Hence (3.3) holds. For n large enough, by (3.2) and (3.3), we have

$$\mathbb{V}\left(\max_{1\leq j\leq n}|T_j| > \varepsilon n^{\frac{1}{p}}\right) \leq \sum_{i=1}^{n}\mathbb{V}\left(|X_i| > n^{\frac{1}{p}}\right) + \mathbb{V}\left(\max_{1\leq j\leq n}\left|T_j^{(n)}\right| > \frac{\varepsilon}{2}n^{\frac{1}{p}}\right).$$

Hence we only need to prove

$$I =: \sum_{n=1}^{\infty} n^{-1} \sum_{i=1}^{n} \mathbb{V}\left(|X_i| > n^{\frac{1}{p}}\right) < \infty \tag{3.5}$$

and

$$II =: \sum_{n=1}^{\infty} n^{-1} \mathbb{V}\left(\max_{1\le j\le n}\left|T_j^{(n)}\right| > \frac{\varepsilon}{2} n^{\frac{1}{p}}\right) < \infty. \tag{3.6}$$

By (3.1) and $C_V[|X|^\beta] < \infty$, we get

$$\begin{aligned} I &= \sum_{n=1}^{\infty} n^{-1} \sum_{i=1}^{n} \mathbb{V}\left(|X_i| > n^{\frac{1}{p}}\right) \\ &\le \sum_{n=1}^{\infty} n^{-1} \sum_{i=1}^{n} \widehat{\mathbb{E}}\left[1 - g\left(\frac{|X_i|}{n^{\frac{1}{p}}}\right)\right] \\ &= \sum_{n=1}^{\infty} \widehat{\mathbb{E}}\left[1 - g\left(\frac{|X|}{n^{\frac{1}{p}}}\right)\right] \\ &\le \sum_{n=1}^{\infty} \mathbb{V}\left(|X| > \mu n^{\frac{1}{p}}\right) \\ &\le C_V[|X|^p] \le C_V[|X|^\beta] < \infty. \end{aligned} \tag{3.7}$$

For $q > 2$, by Lemma 3.1, we have

$$\begin{aligned} II &\le \sum_{n=1}^{\infty} C n^{-1-\frac{q}{p}} \widehat{\mathbb{E}}\left[\max_{1\le j\le n}\left|T_j^{(n)}\right|^q\right] \\ &\le C \sum_{n=1}^{\infty} n^{-1-\frac{q}{p}} \sum_{i=1}^{n} |a_{ni}|^q \widehat{\mathbb{E}}[|Y_i|^q] + C \sum_{n=1}^{\infty} n^{-1-\frac{q}{p}} \left(\sum_{i=1}^{n} |a_{ni}|^2 \widehat{\mathbb{E}}[|Y_i|^2]\right)^{q/2} \\ &\quad + C \sum_{n=1}^{\infty} n^{-1-\frac{q}{p}} \left(\sum_{i=1}^{n}\left[(\widehat{\mathbb{E}}[a_{ni}Y_i - \widehat{\mathbb{E}}[a_{ni}Y_i]])^+ + (\widehat{\mathcal{E}}[a_{ni}Y_i - \widehat{\mathbb{E}}[a_{ni}Y_i]])^-\right]\right)^q \\ &=: II_1 + II_2 + II_3. \end{aligned} \tag{3.8}$$

$\forall k \ge \alpha$, by $\sum_{i=1}^n |a_{ni}|^\alpha = O(n)$ and Hölder inequality, we have

$$\sum_{i=1}^{n} |a_{ni}|^k \le (\sum_{i=1}^{n} |a_{ni}|^\alpha)^{\frac{k}{\alpha}} \le C n^{\frac{k}{\alpha}}. \tag{3.9}$$

Take $\max(2, \alpha, \beta) \le q < \frac{\alpha\beta}{p}$, by (3.9), we have

$$\begin{aligned} II_1 &= C \sum_{n=1}^{\infty} n^{-1-\frac{q}{p}} \sum_{i=1}^{n} |a_{ni}|^q \widehat{\mathbb{E}}\left[|X|^q g\left(\frac{|X|}{n^{\frac{1}{p}}}\right)\right] \\ &\le C \sum_{n=1}^{\infty} n^{-1-\frac{q}{p}+\frac{q}{\alpha}} n^{\frac{1}{p}(q-\beta)} \widehat{\mathbb{E}}[|X|^\beta] \\ &= C \sum_{n=1}^{\infty} n^{-1+\frac{q}{\alpha}-\frac{\beta}{p}} < \infty. \end{aligned} \tag{3.10}$$

When $\beta \geq 2$ and $1 < \alpha < 2$, by (3.9), then we get

$$\begin{aligned} II_2 &= C\sum_{n=1}^{\infty} n^{-1-\frac{q}{p}} \left(\sum_{i=1}^{n} |a_{ni}|^2\right)^{q/2} \left(\widehat{\mathbb{E}}\left[|X|^2 g\left(\frac{|X|}{n^{\frac{1}{p}}}\right)\right]\right)^{q/2} \\ &\leq C\sum_{n=1}^{\infty} n^{-1-\frac{q}{p}} (n^{\frac{2}{\alpha}})^{q/2} \\ &= C\sum_{n=1}^{\infty} n^{-1-\frac{q}{p}+\frac{q}{\alpha}} < \infty. \end{aligned} \tag{3.11}$$

When $\beta \geq 2$ and $\alpha \geq 2$, by (3.4), then we have

$$\begin{aligned} II_2 &= C\sum_{n=1}^{\infty} n^{-1-\frac{q}{p}} \left(\sum_{i=1}^{n} |a_{ni}|^2\right)^{q/2} \left(\widehat{\mathbb{E}}\left[|X|^2 g\left(\frac{|X|}{n^{\frac{1}{p}}}\right)\right]\right)^{q/2} \\ &\leq C\sum_{n=1}^{\infty} n^{-1-\frac{q}{p}} n^{q/2} \\ &= C\sum_{n=1}^{\infty} n^{-1-\frac{q}{p}+\frac{q}{2}} < \infty. \end{aligned} \tag{3.12}$$

When $1 < \beta < 2$ and $1 < \alpha < 2$, by (3.9), then we get

$$\begin{aligned} II_2 &= C\sum_{n=1}^{\infty} n^{-1-\frac{q}{p}} \left(\sum_{i=1}^{n} |a_{ni}|^2\right)^{q/2} \left(\widehat{\mathbb{E}}\left[|X|^2 g\left(\frac{|X|}{n^{\frac{1}{p}}}\right)\right]\right)^{q/2} \\ &\leq C\sum_{n=1}^{\infty} n^{-1-\frac{q}{p}} (n^{\frac{2}{\alpha}})^{q/2} (n^{\frac{2-\beta}{p}})^{q/2} \left(\widehat{\mathbb{E}}\left[|X|^\beta g\left(\frac{|X|}{n^{\frac{1}{p}}}\right)\right]\right)^{q/2} \\ &= C\sum_{n=1}^{\infty} n^{-1-q(\frac{\beta}{2p}-\frac{1}{\alpha})} < \infty. \end{aligned} \tag{3.13}$$

When $1 < \beta < 2$ and $\alpha \geq 2$, by (3.4), then we have

$$\begin{aligned} II_2 &= C\sum_{n=1}^{\infty} n^{-1-\frac{q}{p}} \left(\sum_{i=1}^{n} |a_{ni}|^2\right)^{q/2} \left(\widehat{\mathbb{E}}\left[|X|^2 g\left(\frac{|X|}{n^{\frac{1}{p}}}\right)\right]\right)^{q/2} \\ &\leq C\sum_{n=1}^{\infty} n^{-1-\frac{q}{p}} n^{q/2} (n^{\frac{2-\beta}{p}})^{q/2} \left(\widehat{\mathbb{E}}\left[|X|^\beta g\left(\frac{|X|}{n^{\frac{1}{p}}}\right)\right]\right)^{q/2} \\ &\leq C\sum_{n=1}^{\infty} n^{-1+\frac{q}{2}-\frac{\beta q}{2p}} < \infty. \end{aligned} \tag{3.14}$$

Now we consider II_3. By the fact $\widehat{\mathbb{E}}[X + C] = \widehat{\mathbb{E}}[X] + C$, we have $\widehat{\mathbb{E}}[a_{ni}Y_i - \widehat{\mathbb{E}}[a_{ni}Y_i]] = 0$. Hence, by

$\widehat{\mathbb{E}}[X_i] = \widehat{\mathcal{E}}[X_i] = 0$, we have

$$\begin{aligned}
II_3 &= C\sum_{n=1}^{\infty} n^{-1-\frac{q}{p}}\left(\sum_{i=1}^{n}\left[(\widehat{\mathcal{E}}[a_{ni}Y_i - \widehat{\mathbb{E}}[a_{ni}Y_i]])^-\right]\right)^q \\
&\le C\sum_{n=1}^{\infty} n^{-1-\frac{q}{p}}\left(\sum_{i=1}^{n}|-\widehat{\mathbb{E}}[-a_{ni}Y_i + \widehat{\mathbb{E}}[a_{ni}Y_i]]|\right)^q \\
&= C\sum_{n=1}^{\infty} n^{-1-\frac{q}{p}}\left(\sum_{i=1}^{n}|\widehat{\mathbb{E}}[-a_{ni}Y_i] + \widehat{\mathbb{E}}[a_{ni}Y_i]|\right)^q \\
&\le C\sum_{n=1}^{\infty} n^{-1-\frac{q}{p}}\left(\sum_{i=1}^{n}(|\widehat{\mathbb{E}}[-a_{ni}Y_i]| + |\widehat{\mathbb{E}}[a_{ni}Y_i]|)\right)^q \\
&\le C\sum_{n=1}^{\infty} n^{-1-\frac{q}{p}}\left(\sum_{i=1}^{n}|a_{ni}||\widehat{\mathbb{E}}[-Y_i]|\right)^q + C\sum_{n=1}^{\infty} n^{-1-\frac{q}{p}}\left(\sum_{i=1}^{n}|a_{ni}||\widehat{\mathbb{E}}[Y_i]|\right)^q \\
&= C\sum_{n=1}^{\infty} n^{-1-\frac{q}{p}}\left(\sum_{i=1}^{n}|a_{ni}||\widehat{\mathbb{E}}[-X_i] - \widehat{\mathbb{E}}[-Y_i]|\right)^q + C\sum_{n=1}^{\infty} n^{-1-\frac{q}{p}}\left(\sum_{i=1}^{n}|a_{ni}||\widehat{\mathbb{E}}[X_i] - \widehat{\mathbb{E}}[Y_i]|\right)^q \\
&\le C\sum_{n=1}^{\infty} n^{-1-\frac{q}{p}}\left(\sum_{i=1}^{n}|a_{ni}|\widehat{\mathbb{E}}|-X_i-(-Y_i)|\right)^q + C\sum_{n=1}^{\infty} n^{-1-\frac{q}{p}}\left(\sum_{i=1}^{n}|a_{ni}|\widehat{\mathbb{E}}|X_i - Y_i|\right)^q \\
&\le C\sum_{n=1}^{\infty} n^{-1-\frac{q}{p}}\left(\sum_{i=1}^{n}|a_{ni}|\widehat{\mathbb{E}}\left[|X|\left(1-g\left(\frac{|X|}{n^{\frac{1}{p}}}\right)\right)\right]\right)^q \\
&\le C\sum_{n=1}^{\infty} n^{-1-\frac{q}{p}}\left(nn^{\frac{1-\beta}{p}}\widehat{\mathbb{E}}\left[|X|^\beta g\left(\frac{|X|}{n^{\frac{1}{p}}}\right)\right]\right)^q \\
&= C\sum_{n=1}^{\infty} n^{-1+q-\frac{q\beta}{p}} < \infty.
\end{aligned} \tag{3.15}$$

By (3.7), (3.9)-(3.14), we have $II < \infty$. We complete the proof of Theorem 2.1.

Proof of Corollary 2.2 The proof of (2.2) is the same as that of (2.1), so we omit it. By (2.2), we get

$$\begin{aligned}
\infty &> \sum_{n=1}^{\infty} n^{-1}\mathbb{V}\left(\max_{1\le j\le n}|\sum_{i=1}^{j}|a_iX_i|| > \varepsilon n^{\frac{1}{p}}\right) \\
&= \sum_{i=1}^{\infty}\sum_{n=2^i}^{2^{i+1}-1} n^{-1}\mathbb{V}\left(\max_{1\le j\le n}|\sum_{i=1}^{j}|a_iX_i|| > \varepsilon n^{\frac{1}{p}}\right) \\
&\ge \frac{1}{2}\sum_{i=1}^{\infty}\mathbb{V}\left(\max_{1\le j\le 2^i}|\sum_{i=1}^{j}|a_iX_i|| > \varepsilon 2^{\frac{i+1}{p}}\right).
\end{aligned}$$

By Borel-Cantelli Lemma, we have

$$\mathbb{V}\left(\max_{1\le j\le 2^i}|\sum_{i=1}^{j}|a_iX_i| > \varepsilon 2^{\frac{i+1}{p}}\ \ i.o.\right) = 0$$

Hence,

$$\lim_n \to \infty\frac{\max_{1\le j\le 2^i}|\sum_{i=1}^{j}|a_iX_i|}{2^{\frac{i+1}{p}}} = 0 \ \text{ a.s. } \mathbb{V}.$$

Combining

$$\max_{2^{i-1}\le n\le 2^i} \frac{|\sum_{i=1}^{n}|a_iX_i||}{n^{\frac{1}{p}}} \le 2^{\frac{2}{p}} \frac{\max_{1\le j\le 2^i}|\sum_{i=1}^{j}|a_iX_i||}{2^{\frac{i+1}{p}}},$$

we have

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n}|a_iX_i|}{n^{\frac{1}{p}}} = 0 \ \text{ a.s. } \mathbb{V}.$$

We complete the proof of Corollary 2.2.

From Theorem 2.1, we can obtain Theorem 2.3. The proof of Corollary 2.4 is similar to that of Corollary 2.2, so we omit it.

4. Conclusions

Optimal complete convergence theorems are established for weighted sums in sub-linear expectations space. As corollaries, Marcinkiewicz strong laws are obtained for weighted sums under the sub-linear expectations. These results extend and improve the corresponding results from classical probability space to sub-linear expectation space.

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