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A reliable approach of Ramadan group integral and projected differential transform methods to system of nonlinear partial differential equations

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Abstract. In this paper, a hybrid method is presented via combination of the Ramadan Group Integral method (RGITM) with method of Projected Differential Transform(PDTM) for the purpose of solving nonlinear partial differential equations systems. The method's goal is to produce analytical solutions in series form. In comparison to existing methods, the suggested method makes handling such partial differential equations simple. The outcome demonstrated the method's effectiveness, accuracy, and validity. The technique can be easily applied to a wide variety of nonlinear issues, and it has the potential to both reduce the amount of computation required and deal with the flaw brought about by the nonlinear components that cannot be resolved by employing recognized integral transforms. Examples will be looked at to help illustrate the proposed analysis.

1. Introduction

The nonlinear equations are used in a wide range of scientific and engineering applications, including fluid dynamics, plasma physics, hydrodynamics, solid state physics, optical fibers, and other areas. Since these systems are too complex to be exactly answered, it is still highly challenging to find closed-form answers to the majority of issues. Such problems have been addressed using a wide range of analytical and numerical techniques, such as Adomian decomposition approach [7], approach of homotopy perturbation [8, 11], first integral technique [12], Jacobi's elliptic function method [13], variational iteration method [21], modified B-spline differential quadrature approach [22], He-Laplace technique [23] and many other techniques.

However, there are many other approaches introduced a hybrid method to reduce computational restrictions. A hybrid approach is a combination of two or more procedures that is used to solve various types of differential equations, which include: partial differential equations [24, 25], fractional differential equations [26], Integro-differential equations [28]. Many hybrid methods are featured in the literature such as: Laplace homotopy perturbation method (LHPM) [29], Laplace Adominan decomposition method

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(LADM) [30, 31], homotopy perturbation and Sumudu transform Method [32], Laplace differential transform method (LDTM) [33].

In the above mentioned hybrid techniques many integral transform methods have been proposed, an integral transform is a specific kind of mathematical operator, and numerous integral transform approaches have been put forth. Any transform T that has the following form is an integral transform in mathematics.

$$T(f(u)) = \int_{t_1}^{t_2} K(t,u)f(t)dt.$$

There are many different integral transforms, and the choice of the function K specifies each one. Here we concern with RG- transform, we refer the reader to the papers [14–18] and references cited therein. For piece-wise continuous functions on the subintervals of $[0,\infty)$ of exponential order , RG-Transform is defined as follows:

Definition 1.1 (See [17]). *Think about the functions in the set A described by*

$$A = \{f(t) \mid \exists, t_1, t_2 > 0, | f(t) |\leq M \exp^{\frac{|t|}{t_i}}, if \;\; t \in (-1)^i \times [0,\infty), i = 1,2\}.$$

Noting that the constant M must be a finite number, but t_1 and t_2 may be finite or infinite numbers.
The definition of the RGT is

$$K(s,u) = RG(f(t)) = \begin{cases} \int_0^\infty e^{-st} f(ut)dt, & 0 \leq u < t_2 \\ \\ \int_0^\infty e^{-st} f(ut)dt, & -t_1 \leq u < 0, \end{cases}$$

Consequently, a novel hybrid technique which incorporates Ramadan integral transform along with PDTM , is thus described to offer a well-founded method for solving a system of nonlinear partial differential equations. Our technique is simple to use and requires minimal computational effort to solve systems of partial differential equations.

2. Preliminaries

In this section, We provide fundamental definitions and theorems to allow the reader to understand RGT and its basic properties.

Definition 2.1 (See [17]). *If $F(s)$ and $G(u)$ are the Laplace and Sumudu integrals transforms respectively of $f(t)$, thus, we have the following relationships*

$$F(s) = K(s,1), G(u) = K(1,u) \text{ and } K(s,u) = \frac{1}{u}F(\frac{s}{u}).$$

Theorem 2.2 (See [14, Theorem 3.1]). *Let $K_1(s,u)$ and $K_2(s,u)$ are RGT of the corresponding functions $f(t)$ and $g(t)$, then*

$$RG[(f * g)(t),(s,u)] = uK_1(s,u)K_2(s,u). \tag{1}$$

Where $*$ denotes the convolution of $f(t)$ and $g(t)$.

Table 1: Ramadan Group Transform of several functions

Original function	transformed function
1	$\frac{1}{s}$
t	$\frac{u}{s^2}$
$\frac{t^{n-1}}{(n-1)!}, \ n = 1, 2, ...$	$\frac{u^{n-1}}{s^n}$
e^{at}	$\frac{1}{s-au}$
$sin(wt)$	$\frac{wu}{s^2+u^2w^2}$
te^{at}	$\frac{u}{(s-au)^2}$
$e^{at}cos(wt)$	$\frac{s-au}{(s-au)^2+u^2w^2}$

Theorem 2.3 (See [17, Theorem 2]). *Let $n \geq 1$ and $K(s, u)$ is the RGT of the function $f(t)$, then the following is how to get the RGT of the n^{th} derivative of f(t):*

$$RG[(f^n(t), (s, u)] = \frac{s^n K(s, u)}{u^n} - \sum_{k=0}^{n-1} \frac{s^{n-k-1} f^k(0)}{u^{n-k}}.$$

We display the Ramadan Group Transform of several functions in the following table.

The differential transform method (DTM), initially suggested by J.K. Zhou [5] in 1986, is an iterative process for producing an analytical Taylor's series solution to a differential equation. Its primary applications include linear and non-linear starting value issues in electric circuit analysis.

The projected differential transform (PDTM) is a modification of DTM , where the projected differential transform of a function $y(x, t)$ with regard to the variable $'x'$ at x_0, is defined as:[1, 19, 20]

$$Y_k(t) = \frac{1}{k!}\Big[\frac{\partial^k y(x, t)}{\partial x^k}\Big]_{x=x_0} \quad k \geq 0, \tag{2}$$

where $y(x, t)$ is the original function and $Y_k(t)$ is the transformed function. The following definition describes the $Yk(t)$ inverse projected differential transform:

$$y(x, t) = \sum_{k=0}^{k=\infty} Y_k(t)(x - x_0)^k. \tag{3}$$

Substituting equation (2) in (3), we get

$$y(x, t) = \sum_{k=0}^{k=\infty} \frac{1}{k!}\Big[\frac{\partial^k y(x, t)}{\partial x^k}\Big]_{x=x_0}(x - x_0)^k. \tag{4}$$

When x_0 are taken as $x_0 = 0$ then equation (4) can be expressed as

$$y(x, t) = \sum_{k=0}^{k=\infty} Y_k(t)(x)^k. \tag{5}$$

In real application, the function $y(x, t)$ by finite series of equation (5) can be expressed as

$$y(x, t) = \sum_{k=0}^{n} Y_k(t)(x)^k.$$

Usually, the values of n is decided by convergence of the series coefficients. Some basic formulas of PDTM are listed in Table 2

Table 2: Some basic formulas of projected Differential transform method (PDTM)

Original function	transformed function
$y(x,t) = f(x,t) \pm g(x,t)$	$Y_k(t) = F_k(t) \pm G_k(t)$
$y(x,t) = \alpha f(x,t)$	$Y_k(t) = \alpha F_k(t)$
$y(x,t) = \frac{\partial^r f(x,t)}{\partial x^r}$	$Y_k(t) = (k+1)(k+2)....(k+r)F_{k+r}(t)$
$y(x,t) = (x-x_0)^r (t-t_0)^v$	$Y_k(t) = (t-t_0)^v \delta(k-r,t)$ where $\delta(k-r,t) = \begin{cases} 1, k = r \\ 0, r \neq k \end{cases}$
$y(x,t) = f(x,t)g(x,t)$	$Y_k(t) = \sum_{r=0}^{k} F_r(t) G_{k-r}(t)$
$y(x,t) = sin(ax+\alpha)$	$Y_k(t) = \frac{a^k}{k!}\left[sin(\frac{k\Pi}{2}+\alpha)\right]$
$y(x,t) = cos(ax+\alpha)$	$Y_k(t) = \frac{a^k}{k!}\left[cos(\frac{k\Pi}{2}+\alpha)\right]$

3. Coupling of Ramadan group transform with projected differential transform method

Think about the system of m nonlinear nonhomogeneous partial differential equations.

$$Ły_i(x,t) + \Re y_i(x,t) + \aleph y_i(x,t) = \phi_i, i = 1,2,....m \tag{6}$$

with initiating conditions

$$y_i(x,0) = g_{i_0}(x), y_{i_t}(x,0) = g_{i_1}(x), ..., y_{i_t}^{(n-1)}(x,0) = g_{i_{(n-1)}}(x), \tag{7}$$

and spatial conditions

$$y_i(0,t) = h_{i_0}(t), y_{i_x}(0,t) = h_{i_1}(t), \tag{8}$$

where Ł is the n^th order derivative w.r.t $'t'$, $\Re$ is linear operator and $\aleph$ is non-linear operator. And $\phi_i = \phi_i(x,t)$ and $y_i = y_i(x,t)$ are known and unknown functions respectively. First we apply Ramadan group transformation on equation (6), with regard to $'t'$, therefore we get

$$RG[Ły_i(x,t)] = -RG[\Re y_i(x,t) + \aleph y_i(x,t)] + RG[\phi_i], i = 1,2,....m$$

By using I.C. (7), we get

$$\frac{s^n}{u^n} RG[y_i(x,t)] = \sum_{j=0}^{n-1} \frac{s^{n-j-1} g_{i_{(j)}}(x)}{u^{n-j}} + RG[\phi_i] - RG[\Re y_i(x,t) + \aleph y_i(x,t)],$$

Now, multiplying by $\frac{u^n}{s^n}$ on both sides, we get

$$RG[y_i(x,t)] = \hat{f}(x,s,u) - \frac{u^n}{s^n} RG[\Re y_i(x,t) + \aleph y_i(x,t)], \tag{9}$$

where

$$\hat{f}_i(x,s,u) = \frac{u^n}{s^n}\sum_{j=0}^{n-1}\frac{s^{n-j-1}g_{i_{(j)}}(x)}{u^{n-j}} + \frac{u^n}{s^n}RG[\phi_i].$$

Second, the equation (10) is subjected to inverse RG transform in the second step, getting

$$y_i(x,t) = \hat{f}_i(x,t) - RG^{-1}\Big[\frac{u^n}{s^n}RG[\Re y_i(x,t) + \aleph y_i(x,t)]\Big], \tag{10}$$

Now, we use PDTM to equations (10) and (8) with regard to $'x'$, we obtain

$$Y_{i_k}(t) = \hat{F}_{i_k}(t) - RG^{-1}\Big[\frac{u^n}{s^n}RG[V_{i_k}(t) + \Re Y_{i_k}(t)]\Big], \tag{11}$$

and

$$Y_i(t) = h_{i_0}(t), Y_{i_x}(t) = h_{i_1}(t), \tag{12}$$

where $\hat{F}_{i_k}(t)$, and $V_{i_k}(t)$ are the transformed functions of $\hat{f}_i(x,t)$, and $\aleph y_i(x,t)$ respectively. By the above recurrence equations (11) and the initial conditions (12), the solution can be formulated as

$$y_i(x,t) = \sum_{k=0}^{\infty} Y_{i_k}(t)x^k, i = 1, 2, ..., m$$

4. Numerical results

In this section, we solve some instances using our novel methodology. Examples are provided to demonstrate the effectiveness of our suggested technique. We take into account RGTM with respect to the variable $'t'$ and PDTM with regard to the variable $'x'$ for all illustrative examples. .

Example 4.1. *Take into account the system below.[1]*

$$\begin{aligned} y_t + vy_x + y = 1,\\ v_t + yv_x - v = -1. \end{aligned} \tag{13}$$

Given initial conditions;

$$\begin{aligned} y(x,0) = e^x, \;\; v(x,0) = e^{-x},\\ y(0,t) = e^{-t}, \;\; v(0,t) = e^t. \end{aligned}$$

First, we apply RG-transform to (13) *with regard to* $'t'$*, we have*

$$\begin{aligned} \frac{s}{u}RG[y(x,t)] - \frac{1}{u}y(x,0) = \frac{1}{s} - RG[vy_x + y],\\ \frac{s}{u}RG[v(x,t)] - \frac{1}{u}v(x,0) = \frac{-1}{s} - RG[yv_x - v], \end{aligned}$$

using initial conditions, we get

$$\begin{aligned} RG[y(x,t)] = \frac{e^x}{s} + \frac{u}{s^2} - \frac{u}{s}RG[vy_x + y],\\ RG[v(x,t)] = \frac{e^{-x}}{s} - \frac{u}{s^2} - \frac{s}{u}RG[yv_x - v]. \end{aligned}$$

Now, by taking inverse RG-transform to the previous system, we reach to

$$y(x,t) = e^x + t - RG^{-1}\Big[\frac{u}{s}RG[vy_x + y]\Big],$$
$$v(x,t) = e^{-x} - t - RG^{-1}\Big[\frac{s}{u}RG[yv_x - v]\Big].$$

Next, we apply PDTM with regard to 'x', we get

$$Y_k(t) = \frac{1}{k!}[\frac{d^k}{dx^k}e^x]_{x=0} + t\delta(k-0,t) - RG^{-1}\Big[[\frac{u}{s}RG[\sum_{r=0}^{r=k}(r+1)Y_{r+1}(t)V_{k-r}(t) + Y_k(t)]\Big],$$
$$V_k(t) = \frac{1}{k!}[\frac{d^k}{dx^k}e^{-x}]_{x=0} - t\delta(k-0,t) - RG^{-1}\Big[\frac{u}{s}RG[\sum_{r=0}^{r=k}(r+1)V_{r+1}(t)Y_{k-r}(t) - V_k(t)]\Big], \quad (14)$$

also, applying PDTM to $y(0,t) = e^{-t}$,and $v(0,t) = e^t$, we have

$$Y_0(t) = e^{-t} \ \ and \ \ V_0(t) = e^t.$$

FIRST, substituting for $k = 0$ in system (14), we obtain

$$Y_0(t) = 1 + t - RG^{-1}\Big[\frac{u}{s}RG[Y_1(t)V_0(t) + Y_0(t)]\Big],$$
$$V_0(t) = 1 - t - RG^{-1}[\frac{u}{s}RG[V_1(t)Y_0(t) - V_0(t)]\Big],$$

and so,

$$RG^{-1}\Big[\frac{s}{u}RG[-e^{-t} + t + 1]\Big] = Y_1(t)e^t + e^{-t},$$
$$RG^{-1}\Big[\frac{s}{u}RG[-e^t + -t + 1]\Big] = V_1(t)e^{-t} - e^t,$$

thus, we get $Y_1(t) = e^{-t}$ and $V_1(t) = -e^t$. Second, substituting for $k = 1$ in system (14), we obtain

$$Y_1(t) = 1 - RG^{-1}\Big[\frac{u}{s}RG[Y_1(t)V_1(t) + 2Y_2(t)V_0(t) + Y_1(t)]\Big],$$
$$V_1(t) = 1 - RG^{-1}\Big[\frac{u}{s}RG[V_1(t)Y_1(t) + 2V_2(t)Y_0(t) - V_1(t)]\Big],$$

then, we have

$$RG^{-1}\Big[\frac{s}{u}RG[-e^{-t} + 1]\Big] = -1 + 2e^tY_2(t) + e^{-t},$$
$$RG^{-1}\Big[\frac{s}{u}RG[e^t - 1]\Big] = -1 + 2e^{-t}V_2(t) + e^t,$$

so, we obtain $Y_2(t) = \frac{e^{-t}}{2}$ and $V_2(t) = \frac{e^t}{2}$. Now, substituting for $k = 2,3,4,...$ in system (14), the following approximations are obtained successively

$$Y_3(t) = \frac{e^{-t}}{3!}, Y_4(t) = \frac{e^{-t}}{4!}, Y_5(t) = \frac{e^{-t}}{5!}, ...$$
$$V_3(t) = \frac{-e^t}{3!}, V_4(t) = \frac{e^t}{4!}, V_5(t) = \frac{-e^t}{5!}, ...$$

The solution is finally provided by

$$y(x,t) = \sum_{k=0}^{\infty} Y_k(t)x^k = e^{-t}[1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ...] = e^{x-t},$$
$$v(x,t) = \sum_{k=0}^{\infty} V_k(t)x^k = e^t[1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + ...] = e^{t-x},$$

which is the exact solution.

Example 4.2. *Take coupled Burger's equation into consideration.[2, 3]*

$$y_t - y_{xx} - 2yy_x + (yv)_x = 0,$$
$$v_t - v_{xx} - 2vv_x + (yv)_x = 0. \tag{15}$$

With initial conditions;

$$y(x,0) = sinx = v(x,0),$$
$$y(0,t) = 0 = v(0,t), \quad y_x(0,t) = e^{-t} = v_x(0,t).$$

First, we apply RG-transform to (15) *with regard to 't', hence*

$$\frac{s}{u}RG[y(x,t)] - \frac{1}{u}y(x,0) = RG[y_{xx} + 2yy_x - (yv)_x],$$
$$\frac{s}{u}RG[v(x,t)] - \frac{1}{u}v(x,0) = RG[v_{xx} + 2vv_x - (yv)_x],$$

using initial conditions, we get

$$RG[y(x,t)] = \frac{sinx}{s} + \frac{u}{s}RG[y_{xx} + 2yy_x - (yv)_x],$$
$$RG[v(x,t)] = \frac{sinx}{s} + \frac{s}{u}RG[v_{xx} + 2vv_x - (yv)_x].$$

Now, by taking inverse RG-transform to the previous system, we obtain

$$y(x,t) = sinx + RG^{-1}\Big[\frac{u}{s}RG[y_{xx} + 2yy_x - (yv)_x]\Big],$$
$$v(x,t) = sinx + RG^{-1}\Big[\frac{s}{u}RG[v_{xx} + 2vv_x - (yv)_x]\Big].$$

Next, we apply PDTM with respect to 'x', we get

$$Y_k(t) = \frac{1}{k!}sin(\frac{k\Pi}{2}) + RG^{-1}\Big[[\frac{u}{s}RG[(k+1)(k+2)Y_{k+2}(t) + \sum_{r=0}^{r=k} 2(r+1)Y_{r+1}(t)Y_{k-r}(t) -$$
$$(r+1)Y_{r+1}(t)V_{k-r}(t) - (r+1)V_{r+1}(t)Y_{k-r}(t)]\Big],$$
$$V_k(t) = \frac{1}{k!}sin(\frac{k\Pi}{2}) + RG^{-1}\Big[[\frac{u}{s}RG[(k+1)(k+2)V_{k+2}(t) + \sum_{r=0}^{r=k} 2(r+1)V_{r+1}(t)V_{k-r}(t) -$$
$$(r+1)V_{r+1}(t)Y_{k-r}(t) - (r+1)Y_{r+1}(t)V_{k-r}(t)]\Big], \tag{16}$$

also, applying PDTM to initial conditions $y(0,t) = 0 = v(0,t)$,*and* $v_x(0,t) = e^{-t} = y_x(0,t)$, *we get*

$$Y_0(t) = 0 = V_0(t) \;\; and \;\; V_1(t) = e^{-t} = Y_1(t). \tag{17}$$

FIRST, substituting for $k = 0$ *in system* (16), *we obtain*

$$Y_0(t) = 0 + RG^{-1}\Big[\frac{u}{s}RG[2Y_2(t) + 2Y_1(t)Y_0(t) - Y_1(t)V_0(t) - V_1(t)Y_0(t)]\Big],$$
$$V_0(t) = 0 + RG^{-1}[\frac{u}{s}RG[2V_2(t) + 2V_1(t)V_0(t) - V_1(t)Y_0(t) - Y_1(t)V_0(t)]\Big],$$

using (17), *we obtain*

$$Y_2(t) = 0 = V_2(t). \tag{18}$$

Second, substituting for $k = 1$ in system (16), *we obtain*

$$Y_1(t) = 1 + RG^{-1}\Big[\frac{u}{s}RG[6Y_3(t) + 2Y_1(t)Y_1(t) - Y_1(t)V_1(t) - V_1(t)Y_1(t) + 4Y_2(t)Y_0(t) - 2Y_2(t)V_0(t) - 2V_2(t)Y_0(t)]\Big]$$

$$V_1(t) = 1 + RG^{-1}\Big[\frac{u}{s}RG[6V_3(t) + 2V_1(t)V_1(t) - V_1(t)Y_1(t) - Y_1(t)V_1(t) + 4V_2(t)V_0(t) - 2V_2(t)Y_0(t) - 2Y_2(t)V_0(t)]\Big],$$

then, we have

$$RG^{-1}\Big[\frac{s}{u}RG[e^{-t} - 1]\Big] = 6Y_3(t) + 2e^{-2t} - e^{-2t} - e^{-2t},$$
$$RG^{-1}\Big[\frac{s}{u}RG[e^{-t} - 1]\Big] = 6YV_3(t) + 2e^{-2t} - e^{-2t} - e^{-2t},$$

so, we obtain $Y_3(t) = \frac{-e^{-t}}{6}$ *and* $V_3(t) = \frac{-e^{-t}}{6}$. *Now, substituting for* $k = 2, 3, 4, ...$ *in system* (16), *the following approximations are obtained successively*

$$Y_4(t) = 0, Y_5(t) = \frac{e^{-t}}{5!}, Y_6(t) = 0, Y_7(t) = \frac{-e^{-t}}{7!}....$$
$$V_4(t) = 0, V_5(t) = \frac{e^{-t}}{5!}, V_6(t) = 0, V_7(t) = \frac{-e^{-t}}{7!}....$$

The solution is finally provided by

$$y(x,t) = \sum_{k=0}^{\infty} Y_k(t)x^k = e^{-t}[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...] = e^{-t}sinx,$$
$$v(x,t) = \sum_{k=0}^{\infty} V_k(t)x^k = e^{t}[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...] = e^{-t}sinx.$$

which is the exact solution.

Example 4.3. *Take into considerations the following system [4]*

$$\begin{aligned} y_t - w_x v_t - \frac{1}{2}w_t y_{xx} &= -4xt, \\ v_t - w_t y_{xx} &= 6t, \\ w_t - y_{xx} - v_x w_t &= 4xt - 2t - 2. \end{aligned} \tag{19}$$

Under initial conditions;

$$y(x,0) = x^2 + 1, \;\; v(x,0) = x^2 - 1, \;\; w(x,0) = x^2 - 1$$
$$y(0,t) = 1 - t^2, \;\; v(0,t) = t^2 - 1, \;\; w(0,t) = -t^2 - 1 \;\; y_x(0,t) = 0.$$

First, we apply RG-transform to (19) *with regard to 't', hence*

$$\frac{s}{u}RG[y(x,t)] - \frac{1}{u}y(x,0) = \frac{-4xu}{s^2} + RG[w_x v_t + \frac{1}{2}w_t y_{xx}],$$
$$\frac{s}{u}RG[v(x,t)] - \frac{1}{u}v(x,0) = \frac{6u}{s^2} + RG[w_t y_{xx}],$$
$$\frac{s}{u}RG[w(x,t)] - \frac{1}{u}w(x,0) = \frac{2(2x-1)u}{s^2} - \frac{2}{s} + RG[y_{xx} + v_x w_t].$$

using initial conditions, we get

$$RG[y(x,t)] = \frac{x^2+1}{s} - \frac{4xu^2}{s^3} + \frac{u}{s}RG[w_x v_t + \frac{1}{2}w_t y_{xx}],$$
$$RG[v(x,t)] = \frac{x^2-1}{s} + \frac{6u^2}{s^3} + \frac{s}{u}RG[w_t y_{xx}],$$
$$RG[w(x,t)] = \frac{x^2-1}{s} + \frac{(4x-2)u^2}{s^3} - \frac{2u}{s^2} + \frac{s}{u}RG[y_{xx} + v_x w_t].$$

Now, by taking inverse RG-transform to the previous system, we obtain

$$y(x,t) = x^2 + 1 - \frac{4xt^2}{2!} + RG^{-1}\Big[\frac{u}{s}RG[w_x v_t + \frac{1}{2}w_t y_{xx}]\Big],$$
$$v(x,t) = x^2 - 1 + \frac{6ut^2}{2!} + RG^{-1}\Big[\frac{s}{u}RG[w_t y_{xx}]\Big],$$
$$w(x,t) = x^2 - 1 + (2x-1)t^2 - 2t + RG^{-1}\Big[\frac{s}{u}RG[y_{xx} + v_x w_t]\Big].$$

Next, we apply Projected differential transform method (PDTM) with respect to 'x', we get

$$\begin{aligned}
Y_k(t) &= \delta(k-2,t) + \delta(k-0,t) - 2t^2\delta(k-1,t) + RG^{-1}\Big[\frac{u}{s}RG[\sum_{r=0}^{k}(r+1)W_{r+1}(t)\frac{\partial}{\partial t}V_{k-r}(t) \\
&\quad + \frac{1}{2}(r+1)(r+2)Y_{r+2}(t)\frac{\partial}{\partial t}W_{k-r}(t)]\Big], \\
V_k(t) &= \delta(k-2,t) - \delta(k-0,t) + 3t^2\delta(k-0,t) + RG^{-1}\Big[\frac{u}{s}RG[\sum_{r=0}^{k}(r+1)(r+2)Y_{r+2}(t)\frac{\partial}{\partial t}W_{k-r}(t)]\Big], \\
W_k(t) &= \delta(k-0,t)[-1-2t-t^2] + 2t^2\delta(k-1,t) + \delta(k-2,t) + RG^{-1}\Big[\frac{u}{s}RG[(k+1)(k+2)Y_{k+2}(t) \\
&\quad + \sum_{r=0}^{k}(r+1)V_{r+1}(t)\frac{\partial}{\partial t}W_{k-r}(t)]\Big]. \qquad (20)
\end{aligned}$$

also, applying PDTM to initial conditions, we get

$$Y_0(t) = 1 - t^2 \;\; \text{and} \;\; V_0(t) = t^2 - 1 \;\; W_0(t) = -t^2 - 1 \;\; Y_1(t) = 0.$$

FIRST, substituting for $k = 0$ in system (20)*, we obtain*

$$Y_0(t) = 1 + RG^{-1}\Big[\frac{u}{s}RG[W_1(t)\frac{\partial}{\partial t}V_0(t) + Y_2(t)\frac{\partial}{\partial t}W_0(t)]\Big],$$
$$V_0(t) = -1 + 3t^2 + RG^{-1}[\frac{u}{s}RG[2Y_2(t)\frac{\partial}{\partial t}W_0(t)]\Big],$$
$$W_0(t) = -1 - 2t - t^2 + RG^{-1}[\frac{u}{s}RG[2Y_2(t) + V_1(t)\frac{\partial}{\partial t}W_0(t)]\Big].$$

and so, thus, we get $Y_2(t) = 1$, $W_1(t) = 0$ and $V_1(t) = 0$. Second, substituting for $k = 1$ in system (20)*, we obtain*

$$Y_1(t) = -2t^2 + RG^{-1}\Big[\frac{u}{s}RG[W_1(t)\frac{\partial}{\partial t}V_1(t) + 2W_2(t)\frac{\partial}{\partial t}V_0(t) + Y_2(t)\frac{\partial}{\partial t}W_1(t) + 3Y_3(t)\frac{\partial}{\partial t}W_0(t)]\Big],$$
$$V_1(t) = RG^{-1}[\frac{u}{s}RG[2Y_2(t)\frac{\partial}{\partial t}W_1(t) + 6Y_3(t)\frac{\partial}{\partial t}W_0(t)]\Big],$$
$$W_1(t) = t^2 + RG^{-1}[\frac{u}{s}RG[2(3)Y_3(t) + V_1(t)\frac{\partial}{\partial t}W_1(t) + 2V_2(t)\frac{\partial}{\partial t}W_0(t)]\Big].$$

then, we obtain $W_2(t) = 1$, $Y_3(t) = 0$, and $V_2(t) = 1$. Now, substituting for $k = 2, 3, 4, ...$ in system (14), the following approximations are obtained successively

$$\begin{aligned} Y_4(t) = 0, Y_5(t) = 0, ... \\ V_3(t) = 0, V_4(t) = 0, V_5(t) = 0, ... \\ W_3(t) = 0, W_4(t) = 0, W_5(t) = 0, ... \end{aligned}$$

The solution is finally provided by

$$\begin{aligned} y(x,t) &= \sum_{k=0}^{\infty} Y_k(t)x^k = 1 - t^2 + x^2, \\ v(x,t) &= \sum_{k=0}^{\infty} V_k(t)x^k = t^2 - 1 + x^2, \\ w(x,t) &= \sum_{k=0}^{\infty} W_k(t)x^k = x^2 - t^2 - 1, \end{aligned}$$

which is the exact solution.

Example 4.4. *Take into consideration the system of nonlinear partial differential equations below.[6]*

$$\begin{aligned} & y_t + 2vy_x - y = 2 \\ & v_t - 3yv_x + v = 3 \end{aligned} \tag{21}$$

With initial conditions;

$$\begin{aligned} y(x,0) = e^x, \quad v(x,0) = e^{-x}, \\ y(0,t) = e^t, \quad v(0,t) = e^{-t}. \end{aligned}$$

First, we apply RG-transform to (21) with regard to 't', hence

$$\begin{aligned} \frac{s}{u}RG[y(x,t)] - \frac{1}{u}y(x,0) = \frac{2}{s} + RG[-2vy_x + y], \\ \frac{s}{u}RG[v(x,t)] - \frac{1}{u}v(x,0) = \frac{3}{s} + RG[3yv_x - v], \end{aligned}$$

using initial conditions, we get

$$\begin{aligned} RG[y(x,t)] = \frac{e^x}{s} + \frac{2u}{s^2} + \frac{u}{s}RG[-2vy_x + y], \\ RG[v(x,t)] = \frac{e^{-x}}{s} + \frac{3u}{s^2} + \frac{s}{u}RG[3yv_x - v]. \end{aligned}$$

Now, by taking inverse RG-transform to the previous system, we obtain

$$\begin{aligned} y(x,t) = e^x + 2t + RG^{-1}\Big[\frac{u}{s}RG[-2vy_x + y]\Big], \\ v(x,t) = e^{-x} + 2t + RG^{-1}\Big[\frac{s}{u}RG[3yv_x - v]\Big]. \end{aligned}$$

Next, we apply (PDTM) with regard to 'x', we get

$$\begin{aligned} Y_k(t) &= \frac{1}{k!}[\frac{d^k}{dx^k}e^x]_{x=0} + 2t\delta(k-0,t) + RG^{-1}\Big[[\frac{u}{s}RG[Y_k(t) - 2\sum_{r=0}^{r=k}(r+1)Y_{r+1}(t)V_{k-r}(t)]\Big] \\ V_k(t) &= \frac{1}{k!}[\frac{d^k}{dx^k}e^{-x}]_{x=0} + 3t\delta(k-0,t) + RG^{-1}\Big[\frac{u}{s}RG[3\sum_{r=0}^{r=k}(r+1)V_{r+1}(t)Y_{k-r}(t) - V_k(t)]\Big], \end{aligned} \tag{22}$$

also, applying PDTM to $y(0,t) = e^t$,*and* $v(0,t) = e^{-t}$, *we get*

$$Y_0(t) = e^t \text{ and } V_0(t) = e^{-t}.$$

FIRST, substituting for $k = 0$ *in system* (22), *we obtain*

$$Y_0(t) = 1 + 2t + RG^{-1}\Big[\frac{u}{s}RG[-2Y_1(t)V_0(t) + Y_0(t)]\Big]$$

$$V_0(t) = 1 + 3t + RG^{-1}[\frac{u}{s}RG[3V_1(t)Y_0(t) - V_0(t)]\Big],$$

and so,

$$RG^{-1}\Big[\frac{s}{u}RG[e^t - 2t - 1]\Big] = -2Y_1(t)e^{-t} + e^t$$

$$RG^{-1}\Big[\frac{s}{u}RG[e^{-t} - 3t - 1]\Big] = 3V_1(t)e^t - e^{-t},$$

thus, we get $Y_1(t) = e^t$ *and* $V_1(t) = -e^{-t}$. *Second, substituting for* $k = 1$ *in system* (22), *we obtain*

$$Y_1(t) = 1 + RG^{-1}\Big[\frac{u}{s}RG[-2Y_1(t)V_1(t) - 4Y_2(t)V_0(t) + Y_1(t)]\Big]$$

$$V_1(t) = -1 + RG^{-1}\Big[\frac{u}{s}RG[3V_1(t)Y_1(t) + 6V_2(t)Y_0(t) - V_1(t)]\Big],$$

then, we have

$$RG^{-1}\Big[\frac{s}{u}RG[e^t - 1]\Big] = -4e^{-t}Y_2(t) + 2 + e^t$$

$$RG^{-1}\Big[\frac{s}{u}RG[-e^{-t} + 1]\Big] = -3 + 6e^tV_2(t) + e^{-t},$$

so, we obtain $Y_2(t) = \frac{e^t}{2}$ *and* $V_2(t) = \frac{e^{-t}}{2}$. *Now, substituting for* $k = 2, 3, 4, ...$ *in system* (14), *the following approximations are obtained successively*

$$Y_3(t) = \frac{e^t}{3!}, Y_4(t) = \frac{e^t}{4!}, Y_5(t) = \frac{e^t}{5!}, ...$$

$$V_3(t) = \frac{-e^{-t}}{3!}, V_4(t) = \frac{e^{-t}}{4!}, V_5(t) = \frac{-e^{-t}}{5!}, ...$$

The solution is finally provided by

$$y(x,t) = \sum_{k=0}^{\infty} Y_k(t)x^k = e^t[1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ...] = e^{x+t},$$

$$v(x,t) = \sum_{k=0}^{\infty} V_k(t)x^k = e^{-t}[1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + ...] = e^{-t-x},$$

which is the exact solution.

5. Conclusion

In order to solve systems of non-linear partial differential equations, the Ramadan Group Integral method (RGITM) and Projected differential transform method (PDTM) have been combined in this article. Four examples of the suggested method have been successfully applied, and accurate solutions to the equations are obtained, as opposed to approximate answers obtained by other conventional methods. As a result, the findings show that the strategy described is an effective way to solve nonlinear PDE systems with initial conditions. The fundamental concept presented in this study is sound enough to be applied to the solution of various kinds of equations. The current work can be extended to handle more complicated models, for instance [34].

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