Filomat 37:23 (2023), 7905–7918
https://doi.org/10.2298/FIL2323905W

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Derivation extensions on Leibniz triple systems

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Abstract. In this paper, we first define a concrete representation on an abelian extension of a Leibniz triple system $\mathfrak{L}$ by a Leibniz triple system A. Using this new representation we construct the third-order cohomology classes by derivations of A and $\mathfrak{L}$, which characterize the splitting property of above abelian extensions. Then we study the obstruction for extensibility of derivation pairs. We prove that the set of compatible derivation pairs can define a Lie algebra, whose representation can also characterize the extensibility of the compatible derivation pairs.

1. Introduction

The Kolesnikov-Pozhideav algorithm ([9]) is used to convert identities for algebras into identities for dialgebras. For example, associative dialgebras can be obtained from associative algebras and Leibniz algebras can be obtained from Lie algebras by this algorithm. In [3] Bremner and Sánchez-Ortega introduced Leibniz triple systems by applying Kolesnikov-Pozhideav algorithm on Lie triple systems. Leibniz triple systems are the natural analogues of Lie triple systems in the context of dialgebras. Therefore, one may consider generalizing some properties of Lie triple systems to Leibniz triple systems. At present, the root system theories for Leibniz triple systems were introduced in [1, 4]. The representation theory and Levi's theorem for Leibniz triple systems were determined in [11]. In [14], we considered the cohomology theory of Leibniz triple systems.

Derivations are very important subjects in the research of algebras. For instance, the authors constructed a homotopy Lie algebra out of a graded Lie algebra with a special derivation, see [13]. One could construct deformation formula on associative algebras by noncommuting derivations [5]. Since derivations can be considered as infinitesimals of automorphisms and in [2], the authors studied extension of a pair of automorphisms of Lie algebras, they considered the extension of a pair of automorphisms. Naturally, one can consider the extension of a pair of derivations. The authors studied algebras with derivations from operadic point of view, see [8, 10]. In [12], the authors investigated Lie algebras with derivations from cohomologies point of view and extensions, deformation problems were considered. Extension of a pair of derivations on 3-Lie algebras, Leibniz algebras, associative algebras and Lie triple systems have been studied, refer to [6, 7, 15, 16]. We attempt to consider Leibniz triple systems with derivations. Inspired by

2020 *Mathematics Subject Classification*. Primary 17A32; Secondary 17A40, 17B40, 17B56
Keywords. Leibniz triple system; cohomology; derivation; abelian extension
Received: 01 November 2022; Accepted: 03 April 2023
Communicated by Dijana Mosić
This work is supported by NSF of Jilin Province (No. YDZJ202201ZYTS589), NNSF of China (Nos. 12271085, 12071405), CSC of China (No. 202106625001) and the Fundamental Research Funds for the Central Universities.
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[16], we use a pair of derivation (D_a, D_l) to construct 3-cocycles on Leibniz triple systems. This construction leads to a Lie algebra G_A, where A is an $\mathfrak{L}$-module for the Leibniz triple system $\mathfrak{L}$, and the space of first-order cohomology classes admits a certain representation of the Lie algebra G_A, then the certain representation can be used to characterize the extensibility of the compatible derivation pairs.

This paper is organized as follows. In Section 2, we recall some basic definitions and properties of Leibniz triple systems, and for an abelian extension we use the third-order cohomology group to characterize the splitting property. In Section 3, first, we characterize the extensibility of a pair $(D_a, D_l) \in \mathrm{Der}(A) \times \mathrm{Der}(\mathfrak{L})$ by a necessary condition. Then we define compatible derivation pairs, which are denoted by G_A. Later, we study the obstruction for extensibility of $(D_a, D_l) \in G_A$. Finally, we prove that G_A is a Lie algebra, whose representation can also describe the extensibility of $(D_a, D_l) \in G_A$.

In this paper, all Leibniz triple systems are defined over a fixed but arbitrary field $\mathbb{F}$.

2. Abelian extension of Leibniz triple systems

In this section, we first recall some basic definitions and properties of Leibniz triple systems, then we show that the trivial third-order cohomology group is a sufficient condition for an abelian extension to be split.

Definition 2.1. [3] *A Leibniz triple system is a vector space $\mathfrak{L}$ endowed with a trilinear operation $\{\cdot,\cdot,\cdot\} : \mathfrak{L}\times\mathfrak{L}\times\mathfrak{L} \longrightarrow \mathfrak{L}$ satisfying*

$$\{a,b,\{c,d,e\}\} = \{\{a,b,c\},d,e\} - \{\{a,b,d\},c,e\} - \{\{a,b,e\},c,d\} + \{\{a,b,e\},d,c\},$$
$$\{a,\{b,c,d\},e\} = \{\{a,b,c\},d,e\} - \{\{a,c,b\},d,e\} - \{\{a,d,b\},c,e\} + \{\{a,d,c\},b,e\},$$

for all $a,b,c,d,e \in \mathfrak{L}$.

A Leibniz triple system can be given by a Lie triple system with the same ternary product. Also, a Leibniz algebra L with product $[\cdot,\cdot]$ becomes a Leibniz triple system when $\{x,y,z\} := [[x,y],z]$, for all $x,y,z \in L$. More examples refer to [3]. Denote by $\mathrm{End}(\mathfrak{L})$ the set consisting of all linear maps on a Leibniz triple system $\mathfrak{L}$.

Definition 2.2. [3] *Let $\mathfrak{L}$ be a Leibniz triple system. A linear map $D : \mathfrak{L} \longrightarrow \mathfrak{L}$ is called a derivation of $\mathfrak{L}$, if for all $a,b,c \in \mathfrak{L}$,*

$$D(\{a,b,c\}) = \{D(a),b,c\} + \{a,D(b),c\} + \{a,b,D(c)\}.$$

Denote by $\mathrm{Der}(\mathfrak{L})$ *the space of derivations of $\mathfrak{L}$.*

Definition 2.3. [11] *Let $\mathfrak{L}$ be a Leibniz triple system and V a vector space. V is called an $\mathfrak{L}$-module, if $\mathfrak{L}\dotplus V$ is a Leibniz triple system such that* (1) *$\mathfrak{L}$ is a subsystem,* (2) *$\{a,b,c\} \in V$ if any one of $a,b,c \in V$;* (3) *$\{a,b,c\} = 0$ if any two of $a,b,c \in V$.*

Definition 2.4. [11] *Let $\mathfrak{L}$ be a Leibniz triple system and V a vector space. Suppose $l,m,r : \mathfrak{L}\times\mathfrak{L} \longrightarrow End(V)$ are bilinear maps such that*

$$\begin{aligned}
l(a,\{b,c,d\}) &= l(\{a,b,c\},d) - l(\{a,c,b\},d) - l(\{a,d,b\},c) + l(\{a,d,c\},b),\\
m(a,d)l(b,c) &= m(\{a,b,c\},d) - m(\{a,c,b\},d) - r(c,d)m(a,b) + r(b,d)m(a,c),\\
m(a,d)m(b,c) &= r(c,d)l(a,b) - r(c,d)m(a,b) - m(\{a,c,b\},d) + r(b,d)l(a,c),\\
m(a,d)r(b,c) &= r(c,d)m(a,b) - r(c,d)l(a,b) - r(b,d)l(a,c) + m(\{a,c,b\},d),\\
r(\{a,b,c\},d) &= r(c,d)r(a,b) - r(c,d)r(b,a) - r(b,d)r(c,a) + r(a,d)r(c,b),\\
l(a,b)l(c,d) &= l(\{a,b,c\},d) - l(\{a,b,d\},c) - r(c,d)l(a,b) + r(d,c)l(a,b),\\
l(a,b)m(c,d) &= m(\{a,b,c\},d) - r(c,d)l(a,b) - l(\{a,b,d\},c) + m(\{a,b,d\},c),\\
l(a,b)r(c,d) &= r(c,d)l(a,b) - m(\{a,b,c\},d) - m(\{a,b,d\},c) + l(\{a,b,d\},c),
\end{aligned}$$

$$m(a,\{b,c,d\}) = r(c,d)m(a,b) - r(b,d)m(a,c) - r(b,c)m(a,d) + r(c,b)m(a,d),$$
$$r(a,\{b,c,d\}) = r(c,d)r(a,b) - r(b,d)r(a,c) - r(b,c)r(a,d) + r(c,b)r(a,d),$$

for all $a,b,c,d \in \mathfrak{L}$. Then (r,m,l) is called a representation of $\mathfrak{L}$ on V.

Remark 2.5. [11] *Let $\mathfrak{L}$ be a Leibniz triple system and V an $\mathfrak{L}$-module. Then $\mathfrak{L}\dot{+}V$ is a Leibniz triple system, with*

$$\{x+u, y+v, z+w\}_{\mathfrak{L}\dot{+}V} = \{x,y,z\}_{\mathfrak{L}} + l(x,y)(w) + r(y,z)(u) + m(x,z)(v),$$

where $x,y,z \in \mathfrak{L},\ u,v,w \in V$, and (r,m,l) is a representation of $\mathfrak{L}$ on V.

Definition 2.6. [14] *Let V be an $\mathfrak{L}$-module. A $(2n+1)$-linear map $f : \underbrace{\mathfrak{L}\otimes\cdots\otimes\mathfrak{L}}_{(2n+1)\text{ times}} \to V$ is called a $(2n+1)$-cochain of $\mathfrak{L}$ on V, for $n \geq 0$. Denote by $C^{2n+1}(\mathfrak{L},V)$ the set of all $(2n+1)$-cochains.*

Definition 2.7. [14] *Let $\mathfrak{L}$ be a Leibniz triple system and (r,m,l) a representation of $\mathfrak{L}$ on V.*
A 1-coboundary operator of $\mathfrak{L}$ on V is defined by

$$\delta^1 : C^1(\mathfrak{L},V) \to C^3(\mathfrak{L},V)$$
$$f \mapsto \delta^1 f$$

where

$$\delta^1 f(x_1,x_2,x_3) = r(x_2,x_3)f(x_1) + m(x_1,x_3)f(x_2) + l(x_1,x_2)f(x_3) - f(\{x_1,x_2,x_3\}). \tag{1}$$

A 3-coboundary operator of $\mathfrak{L}$ on V consists a pair of maps (δ^3_1, δ^3_2),

$$\delta^3_i : C^3(\mathfrak{L},V) \to C^5(\mathfrak{L},V)$$
$$f \mapsto \delta^3_i f$$

where

$$\begin{aligned}
&\delta^3_1 f(x_1,x_2,x_3,x_4,x_5)\\
&= f(x_1,x_2,\{x_3,x_4,x_5\}) - f(\{x_1,x_2,x_3\},x_4,x_5) + f(\{x_1,x_2,x_4\},x_3,x_5) + f(\{x_1,x_2,x_5\},x_3,x_4)\\
&\quad - f(\{x_1,x_2,x_5\},x_4,x_3) + l(x_1,x_2)f(x_3,x_4,x_5) - r(x_4,x_5)f(x_1,x_2,x_3) + r(x_3,x_5)f(x_1,x_2,x_4)\\
&\quad + r(x_3,x_4)f(x_1,x_2,x_5) - r(x_4,x_3)f(x_1,x_2,x_5),
\end{aligned} \tag{2}$$

$$\begin{aligned}
&\delta^3_2 f(x_1,x_2,x_3,x_4,x_5)\\
&= f(x_1,\{x_2,x_3,x_4\},x_5) - f(\{x_1,x_2,x_3\},x_4,x_5) + f(\{x_1,x_3,x_2\},x_4,x_5) + f(\{x_1,x_4,x_2\},x_3,x_5)\\
&\quad - f(\{x_1,x_4,x_3\},x_2,x_5) + m(x_1,x_5)f(x_2,x_3,x_4) - r(x_4,x_5)f(x_1,x_2,x_3) + r(x_4,x_5)f(x_1,x_3,x_2)\\
&\quad + r(x_3,x_5)f(x_1,x_4,x_2) - r(x_2,x_5)f(x_1,x_4,x_3).
\end{aligned} \tag{3}$$

Let $\mathfrak{L}$ be a Leibniz triple system and V an $\mathfrak{L}$-module. The set

$$Z^1(\mathfrak{L},V) = \{f \in C^1(\mathfrak{L},V) \mid \delta^1 f = 0\}$$

is called the space of 1-cocycles of $\mathfrak{L}$ on V. The set

$$Z^3(\mathfrak{L},V) = \{f \in C^3(\mathfrak{L},V) \mid \delta^3_1 f = \delta^3_2 f = 0\}$$

is called the space of 3-cocycles of $\mathfrak{L}$ on V. The set

$$B^3(\mathfrak{L},V) = \{\delta^1 f \mid f \in C^1(\mathfrak{L},V)\}$$

is called the space of 3-coboundaries of $\mathfrak{L}$ on V.

In [14], it is proved that $\delta_i^3\delta^1 = 0$ $(i = 1, 2)$, then the 1-cohomology space and 3-cohomology space of $\mathfrak{L}$ can be defined as

$$H^1(\mathfrak{L}, V) := Z^1(\mathfrak{L}, V).$$
$$H^3(\mathfrak{L}, V) := Z^3(\mathfrak{L}, V)/B^3(\mathfrak{L}, V).$$

Next, we will use $H^3(\mathfrak{L}, V)$ to characterize the splitting property of abelian extensions.

Definition 2.8. *Let $\mathfrak{L}$ and A be Leibniz triple systems. If*

$$0 \longrightarrow A \hookrightarrow \tilde{\mathfrak{L}} \xrightarrow{\pi} \mathfrak{L} \longrightarrow 0$$

is an exact sequence of Leibniz triple systems, and $\{A, A, \tilde{\mathfrak{L}}\} = \{A, \tilde{\mathfrak{L}}, A\} = \{\tilde{\mathfrak{L}}, A, A\} = 0$, then we call $\tilde{\mathfrak{L}}$ an abelian extension of $\mathfrak{L}$ by A. A linear map $s : \mathfrak{L} \to \tilde{\mathfrak{L}}$ is called a section of π if it satisfies $\pi \circ s = \mathrm{id}_{\mathfrak{L}}$. If there exists a section which is also a homomorphism between Leibniz triple systems, we say that the abelian extension is split.

Let $\tilde{\mathfrak{L}}$ be an abelian extension of $\mathfrak{L}$ by A. We construct a representation of $\mathfrak{L}$ on A. Fix any section $s : \mathfrak{L} \longrightarrow \tilde{\mathfrak{L}}$ of π and define $r_A, m_A, l_A : \mathfrak{L} \times \mathfrak{L} \longrightarrow \mathrm{End}(A)$ by

$$r_A(x, y)(v) = \{v, s(x), s(y)\}_{\tilde{\mathfrak{L}}}, m_A(x, y)(v) = \{s(x), v, s(y)\}_{\tilde{\mathfrak{L}}}, l_A(x, y)(v) = \{s(x), s(y), v\}_{\tilde{\mathfrak{L}}}, \tag{4}$$

for all $x, y \in \mathfrak{L}$, $v \in A$. It is easy to check that (r_A, m_A, l_A) is independent of the choice of s. Note that

$$\{s(x), s(y), s(z)\}_{\tilde{\mathfrak{L}}} - s(\{x, y, z\}_{\mathfrak{L}}) \in A, \ \forall\, x, y, z \in \mathfrak{L}.$$

Then one deduces that (r_A, m_A, l_A) is a representation of $\mathfrak{L}$ on A.

For a fixed section s, consider the map $\omega : \mathfrak{L} \times \mathfrak{L} \times \mathfrak{L} \longrightarrow A$,

$$\omega(x, y, z) = \{s(x), s(y), s(z)\}_{\tilde{\mathfrak{L}}} - s(\{x, y, z\}_{\mathfrak{L}}), \tag{5}$$

for all $x, y, z \in \mathfrak{L}$. It is routine to check that ω is a 3-cocycle associated to (r_A, m_A, l_A). The proof is similar to that of [14, Theorem 3.3]. One may notice that ω does depend on a certain section, however, we will show that the cohomology class of ω does not.

Lemma 2.9. *If s_1 and s_2 are sections of π, then $\omega_1 - \omega_2 = \delta^1\lambda$, where $\lambda = s_1 - s_2$ and ω_i is defined by Eq.* (5) *corresponding to s_i, for $i = 1, 2$.*

Proof. Note that $\lambda(x) = s_1(x) - s_2(x) \in ker\pi = A$, for all $x \in \mathfrak{L}$. Then $\lambda \in C^1(\mathfrak{L}, A)$ and

$$\begin{aligned}
&\omega_1(x, y, z) - \omega_2(x, y, z)\\
=& \{s_1(x), s_1(y), s_1(z)\}_{\tilde{\mathfrak{L}}} - s_1(\{x, y, z\}_{\mathfrak{L}}) - \{s_2(x), s_2(y), s_2(z)\}_{\tilde{\mathfrak{L}}} + s_2(\{x, y, z\}_{\mathfrak{L}})\\
=& \{s_2(x) + \lambda(x), s_2(y) + \lambda(y), s_2(z) + \lambda(z)\}_{\tilde{\mathfrak{L}}} - (s_2(\{x, y, z\}_{\mathfrak{L}} + \lambda(\{x, y, z\}_{\mathfrak{L}}))\\
&- \{s_2(x), s_2(y), s_2(z)\}_{\tilde{\mathfrak{L}}} + s_2(\{x, y, z\}_{\mathfrak{L}})\\
=& \{s_2(x), s_2(y), s_2(z)\}_{\tilde{\mathfrak{L}}} + \{s_2(x), s_2(y), \lambda(z)\}_{\tilde{\mathfrak{L}}} + \{s_2(x), \lambda(y), s_2(z)\}_{\tilde{\mathfrak{L}}} + \{s_2(x), \lambda(y), \lambda(z)\}_{\tilde{\mathfrak{L}}}\\
&+ \{\lambda(x), s_2(y), s_2(z)\}_{\tilde{\mathfrak{L}}} + \{\lambda(x), s_2(y), \lambda(z)\}_{\tilde{\mathfrak{L}}} + \{\lambda(x), \lambda(y), s_2(z)\}_{\tilde{\mathfrak{L}}} + \{\lambda(x), \lambda(y), \lambda(z)\}_{\tilde{\mathfrak{L}}}\\
&- (s_2(\{x, y, z\}_{\mathfrak{L}} + \lambda(\{x, y, z\}_{\mathfrak{L}})) - \{s_2(x), s_2(y), s_2(z)\}_{\tilde{\mathfrak{L}}} + s_2(\{x, y, z\}_{\mathfrak{L}})\\
=& \{s_2(x), s_2(y), \lambda(z)\}_{\tilde{\mathfrak{L}}} + \{s_2(x), \lambda(y), s_2(z)\}_{\tilde{\mathfrak{L}}} + \{\lambda(x), s_2(y), s_2(z)\}_{\tilde{\mathfrak{L}}} - \lambda(\{x, y, z\}_{\mathfrak{L}})\\
=& -\lambda(\{x, y, z\}_{\mathfrak{L}}) + l_A(x, y)(\lambda(z)) + m_A(x, z)(\lambda(y)) + r_A(y, z)(\lambda(x))\\
=& (\delta^1\lambda)(x, y, z),
\end{aligned}$$

which completes the proof. □

By Lemma 2.9, one has the following proposition.

Proposition 2.10. *The cohomology class* $[\omega]$ *does not depend on the choice of* s.

Proposition 2.11. *If* (r, m, l) *is a representation of* $\mathfrak{L}$ *on* V *and* f *is a* 3*-cocycle, then* $\mathfrak{L}\dotplus V$ *is a Leibniz triple system with the bracket given by*

$$\{x+u, y+v, z+w\}_{\mathfrak{L}\dotplus V} = \{x, y, z\}_{\mathfrak{L}} + f(x, y, z) + l(x, y)(w) + r(y, z)(u) + m(x, z)(v),$$

where $x, y, z \in \mathfrak{L}$, $u, v, w \in V$.

Proof. It follows by combining Remark 2.5 and $\delta_1^3 f = \delta_2^3 f = 0$ in Eqs. (2) and (3). □

By Proposition 2.11, one could check that the canonical projection $\pi : \mathfrak{L}\dotplus V \longrightarrow \mathfrak{L}$ is a homomorphism between Leibniz triple systems. Then we have the following corollary.

Corollary 2.12. *Retain all the notions and assumptions in Proposition* 2.11*. Then there is an abelian extension* $\tilde{\mathfrak{L}} = \mathfrak{L}\dotplus V$ *of the Leibniz triple system* $\mathfrak{L}$ *by* V.

Theorem 2.13. *Let* (r_A, m_A, l_A) *be a representation of a Leibniz triple system* $\mathfrak{L}$ *on a Leibniz triple system* A *that satisfies Eq.* (4)*. If* $H^3(\mathfrak{L}, A) = 0$ *then the abelian extension of* $\mathfrak{L}$ *by* A *is split.*

Proof. It suffices to show that there is a section of π which is also a homomorphism. Let s be any section of π. Recall that the map ω defined by Eq. (5) is a 3-cocycle. Since $H^3(\mathfrak{L}, A) = 0$, there exists $\alpha \in C^1(\mathfrak{L}, A)$, such that, for any $x, y, z \in \mathfrak{L}$,

$$\omega(x, y, z) = \delta^1\alpha(x, y, z) = -\alpha(\{x, y, z\}_{\mathfrak{L}}) + l_A(x, y)(\alpha(z)) + r_A(y, z)(\alpha(x)) + m_A(x, z)(\alpha(y)).$$

Define a linear map $s' : \mathfrak{L} \longrightarrow \tilde{\mathfrak{L}}$ by $s' = s - \alpha$. Then s' is also a section of π, and for any $x, y, z \in \mathfrak{L}$,

$$\begin{aligned}
&\{s'(x), s'(y), s'(z)\}_{\tilde{\mathfrak{L}}}\\
&= \{s(x) - \alpha(x), s(y) - \alpha(y), s(z) - \alpha(z)\}_{\tilde{\mathfrak{L}}}\\
&= \{s(x), s(y), s(z)\}_{\tilde{\mathfrak{L}}} - \{s(x), s(y), \alpha(z)\} - \{\alpha(x), s(y), s(z)\} - \{s(x), \alpha(y), s(z)\}\\
&= \{s(x), s(y), s(z)\}_{\tilde{\mathfrak{L}}} - l_A(x, y)(\alpha(z)) - m_A(x, z)(\alpha(y)) - r_A(y, z)(\alpha(x))\\
&= s(\{x, y, z\}_{\mathfrak{L}}) + \omega(x, y, z) - l_A(x, y)(\alpha(z)) - m_A(x, z)(\alpha(y)) - r_A(y, z)(\alpha(x))\\
&= s(\{x, y, z\}_{\mathfrak{L}}) - \alpha(\{x, y, z\}_{\mathfrak{L}})\\
&= s'(\{x, y, z\}_{\mathfrak{L}}).
\end{aligned}$$

Hence, s' is a homomorphism. □

3. Extensibility of derivations

Throughout this section, $\mathfrak{L}$ and A denote Leibniz triple systems, and $\tilde{\mathfrak{L}}$ is an abelian extension of $\mathfrak{L}$ by A. Let (r_A, m_A, l_A) denote the representation defined by Eq. (4). First, we give a necessary condition for the extensibility of a pair $(D_a, D_l) \in \mathrm{Der}(A) \times \mathrm{Der}(\mathfrak{L})$ and use it to define compatible derivation pairs, which are denoted by G_A. Then we study the obstruction for extensibility of derivation pairs belonging to G_A, see Theorem 3.6. Finally, we show that G_A is a Lie algebra, whose representation can also characterize the extensibility of (D_a, D_l).

Definition 3.1. *Keep notations as above. A pair* $(D_a, D_l) \in \mathrm{Der}(A)\times\mathrm{Der}(\mathfrak{L})$ *is called extensible if there is a derivation* $D_{\tilde{l}} \in \mathrm{Der}(\tilde{\mathfrak{L}})$ *such that the diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \hookrightarrow & \tilde{\mathfrak{L}} & \xrightarrow{\pi} & \mathfrak{L} & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle D_a} & & \downarrow{\scriptstyle D_{\tilde{l}}} & & \downarrow{\scriptstyle D_l} & & \\
0 & \longrightarrow & A & \hookrightarrow & \tilde{\mathfrak{L}} & \xrightarrow{\pi} & \mathfrak{L} & \longrightarrow & 0
\end{array} \tag{6}$$

is commutative.

Proposition 3.2. *Keep notations as above. Then (D_a, D_l) is extensible only if*

$$\begin{aligned} D_a l_A(x,y) - l_A(x,y)D_a &= l_A(D_l(x),y) + l_A(x,D_l(y)),\\ D_a m_A(x,y) - m_A(x,y)D_a &= m_A(D_l(x),y) + m_A(x,D_l(y)),\\ D_a r_A(x,y) - r_A(x,y)D_a &= r_A(D_l(x),y) + r_A(x,D_l(y)). \end{aligned} \tag{7}$$

Proof. Since (D_a, D_l) is extensible, there exists a derivation $D_{\tilde{l}} \in \mathrm{Der}(\tilde{\mathfrak{L}})$ such that the diagram (6) commutes. Then $D_{\tilde{l}}(s(x)) - s(D_l(x)) \in A$, for $x \in \mathfrak{L}$. Since $D_{\tilde{l}}|_A = D_a$ and $D_{\tilde{l}} \in \mathrm{Der}(\tilde{\mathfrak{L}})$, we obtain

$$\begin{aligned} D_a(l_A(x,y)(v)) - l_A(x,y)(D_a(v)) &= D_a(\{s(x),s(y),v\}_{\tilde{\mathfrak{L}}}) - \{s(x),s(y),D_a(v)\}_{\tilde{\mathfrak{L}}}\\ &= D_{\tilde{l}}(\{s(x),s(y),v\}_{\tilde{\mathfrak{L}}}) - \{s(x),s(y),D_a(v)\}_{\tilde{\mathfrak{L}}}\\ &= \{D_{\tilde{l}}(s(x)),s(y),v\}_{\tilde{\mathfrak{L}}} + \{s(x),D_{\tilde{l}}(s(y)),v\}_{\tilde{\mathfrak{L}}}\\ &\quad + \{s(x),s(y),D_{\tilde{l}}(v)\}_{\tilde{\mathfrak{L}}} - \{s(x),s(y),D_a(v)\}_{\tilde{\mathfrak{L}}}\\ &= \{D_{\tilde{l}}(s(x)) - s(D_l(x)),s(y),v\}_{\tilde{\mathfrak{L}}} + \{s(D_l(x)),s(y),v\}_{\tilde{\mathfrak{L}}}\\ &\quad + \{s(x),D_{\tilde{l}}(s(y)) - s(D_l(y)),v\}_{\tilde{\mathfrak{L}}} + \{s(x),s(D_l(y)),v\}_{\tilde{\mathfrak{L}}}\\ &\quad + \{s(x),s(y),D_a(v)\}_{\tilde{\mathfrak{L}}} - \{s(x),s(y),D_a(v)\}_{\tilde{\mathfrak{L}}}\\ &= \{s(D_l(x)),s(y),v\}_{\tilde{\mathfrak{L}}} + \{s(x),s(D_l(y)),v\}_{\tilde{\mathfrak{L}}}\\ &= l_A(D_l(x),y)(v) + l_A(x,D_l(y))(v). \end{aligned}$$

Similarly, we have

$$\begin{aligned} D_a(m_A(x,y)(v)) - m_A(x,y)(D_a(v)) &= m_A(D_l(x),y)(v) + m_A(x,D_l(y))(v),\\ D_a(r_A(x,y)(v)) - r_A(x,y)(D_a(v)) &= r_A(D_l(x),y)(v) + r_A(x,D_l(y))(v), \end{aligned}$$

which completes the proof. □

Definition 3.3. *Keep notations as above. A pair $(D_a, D_l) \in \mathrm{Der}(A) \times \mathrm{Der}(\mathfrak{L})$ is called compatible if Eq. (7) holds. All such pairs are denoted by G_A.*

Proposition 3.2 says that an extensible derivation pair (D_a, D_l) is compatible, i.e., $(D_a, D_l) \in G_A$. Then a natural question is: when is $(D_a, D_l) \in G_A$ extensible? We need the following preparations.

For any pair $(D_a, D_l) \in \mathrm{Der}(A) \times \mathrm{Der}(\mathfrak{L})$ and ω defined by Eq. (5), define a 3-cochain $\mathrm{Ob}^{\omega}_{(D_a,D_l)} \in C^3(\mathfrak{L}, A)$ as

$$\mathrm{Ob}^{\omega}_{(D_a,D_l)} = D_a\omega - \omega(D_l \otimes \mathrm{id} \otimes \mathrm{id}) - \omega(\mathrm{id} \otimes D_l \otimes \mathrm{id}) - \omega(\mathrm{id} \otimes \mathrm{id} \otimes D_l), \tag{8}$$

or equivalently,

$$\mathrm{Ob}^{\omega}_{(D_a,D_l)}(x,y,z) = D_a\omega(x,y,z) - \omega(D_l(x),y,z) - \omega(x,D_l(y),z) - \omega(x,y,D_l(z)), \tag{9}$$

for all $x, y, z \in \mathfrak{L}$.

Lemma 3.4. *Keep notations as above. Then $\mathrm{Ob}^{\omega}_{(D_a,D_l)}$ does not depend on the choice of sections of π.*

Proof. For $i = 1, 2$, let s_i be sections of π, ω_i be defined by Eq. (5), and $\mathrm{Ob}^{\omega_i}_{(D_a,D_l)}$ be defined by Eq. (9). Then

$$\begin{aligned} &\mathrm{Ob}^{\omega_1}_{(D_a,D_l)}(x,y,z) - \mathrm{Ob}^{\omega_2}_{(D_a,D_l)}(x,y,z)\\ =&D_a(\omega_1(x,y,z)) - \omega_1(D_l(x),y,z) - \omega_1(x,D_l(y),z) - \omega_1(x,y,D_l(z))\\ &- D_a(\omega_2(x,y,z)) + \omega_2(D_l(x),y,z) + \omega_2(x,D_l(y),z) + \omega_2(x,y,D_l(z))\\ =&\underbrace{D_a(\omega_1(x,y,z) - \omega_2(x,y,z))}_{I_1} - \underbrace{(\omega_1(D_l(x),y,z) - \omega_2(D_l(x),y,z))}_{I_2}\\ &- \underbrace{(\omega_1(x,D_l(y),z) - \omega_2(x,D_l(y),z))}_{I_3} - \underbrace{(\omega_1(x,y,D_l(z)) - \omega_2(x,y,D_l(z)))}_{I_4}. \end{aligned} \tag{10}$$

Define a map $\lambda : \mathfrak{L} \longrightarrow A$ by $\lambda(x) = s_1(x) - s_2(x)$, for all $x \in \mathfrak{L}$. By Lemma 2.9, we have

$$\omega_1(x, y, z) - \omega_2(x, y, z) = -\lambda(\{x, y, z\}) + l_A(x, y)(\lambda(z)) + m_A(x, z)(\lambda(y)) + r_A(y, z)(\lambda(x)),$$

for any $x, y, z \in \mathfrak{L}$. Therefore, one obtains

$$I_1 = D_a(-\lambda(\{x, y, z\}) + l_A(x, y)(\lambda(z)) + m_A(x, z)(\lambda(y)) + r_A(y, z)(\lambda(x))).$$

Similarly, we have

$$I_2 = -\lambda(\{D_l(x), y, z\}) + l_A(D_l(x), y)(\lambda(z)) + m_A(D_l(x), z)(\lambda(y)) + r_A(y, z)(\lambda(D_l(x))),$$

$$I_3 = -\lambda(\{x, D_l(y), z\}) + l_A(x, D_l(y))(\lambda(z)) + m_A(x, z)(\lambda(D_l(y))) + r_A(D_l(y), z)(\lambda(x)),$$

and

$$I_4 = -\lambda(\{x, y, D_l(z)\}) + l_A(x, y)(\lambda(D_l(z))) + m_A(x, D_l(z))(\lambda(y)) + r_A(y, D_l(z))(\lambda(x)).$$

Substituting I_1, I_2, I_3, I_4 into Eq. (10), one has

$$\begin{aligned}
&\mathrm{Ob}^{\omega_1}_{(D_a,D_l)}(x, y, z) - \mathrm{Ob}^{\omega_2}_{(D_a,D_l)}(x, y, z)\\
&= \Big(D_a l_A(x, y) - l_A(D_l(x), y) - l_A(x, D_l(y))\Big)(\lambda(z)) + \Big(D_a r_A(y, z) - r_A(D_l(y), z) - r_A(y, D_l(z))\Big)(\lambda(x))\\
&\quad + \Big(D_a m_A(x, z) - m_A(D_l(x), z) - m_A(x, D_l(z))\Big)(\lambda(y)) - l_A(x, y)(\lambda(D_l(z))) - m_A(x, z)(\lambda(D_l(y)))\\
&\quad - r_A(y, z)(\lambda(D_l(x))) - D_a(\lambda(\{x, y, z\})) + \lambda(D_l(\{x, y, z\}))\\
&\overset{(7)}{=} l_A(x, y)D_a(\lambda(z)) + m_A(x, z)D_a(\lambda(y)) + r_A(y, z)D_a(\lambda(x)) - r_A(y, z)(\lambda(D_l(x))) - l_A(x, y)(\lambda(D_l(z)))\\
&\quad - m_A(x, z)(\lambda(D_l(y))) - D_a(\lambda(\{x, y, z\})) + \lambda(D_l(\{x, y, z\}))\\
&= \delta^1(D_a \circ \lambda - \lambda \circ D_l)(x, y, z) = 0.
\end{aligned}$$

The proof is finished. □

By the independency of $\mathrm{Ob}^{\omega}_{(D_a,D_l)}$ on sections, we use the notation $\mathrm{Ob}^{\tilde{\mathfrak{L}}}_{(D_a,D_l)}$ instead of $\mathrm{Ob}^{\omega}_{(D_a,D_l)}$. In what follows, we will use $\mathrm{Ob}^{\tilde{\mathfrak{L}}}_{(D_a,D_l)}$ to obtain a necessary and sufficient condition for (D_a, D_l) to be extensible. First, we have

Lemma 3.5. *For $(D_a, D_l) \in G_A$, $\mathrm{Ob}^{\tilde{\mathfrak{L}}}_{(D_a,D_l)}$ is a 3-cocycle.*

Proof. It suffices to show that $\delta^3_i \mathrm{Ob}^{\omega}_{(D_a,D_l)} = 0$ $(i = 1, 2)$. Since ω is a 3-cocycle, $\delta^3_i\omega = 0$ $(i = 1, 2)$, here we use $\delta^3_1\omega = 0$ to prove $\delta^3_1 \mathrm{Ob}^{\omega}_{(D_a,D_l)} = 0$ as an example. By Eq. (2) it follows that, for any $x_1, x_2, x_3, x_4, x_5 \in \mathfrak{L}$,

$$\begin{aligned}
0 &= \delta^3_1\omega(x_1, x_2, x_3, x_4, x_5)\\
&= \omega(x_1, x_2, \{x_3, x_4, x_5\}) - \omega(\{x_1, x_2, x_3\}, x_4, x_5) + \omega(\{x_1, x_2, x_4\}, x_3, x_5) + \omega(\{x_1, x_2, x_5\}, x_3, x_4)\\
&\quad - \omega(\{x_1, x_2, x_5\}, x_4, x_3) + l_A(x_1, x_2)\omega(x_3, x_4, x_5) - r_A(x_4, x_5)\omega(x_1, x_2, x_3) + r_A(x_3, x_5)\omega(x_1, x_2, x_4)\\
&\quad + r_A(x_3, x_4)\omega(x_1, x_2, x_5) - r_A(x_4, x_3)\omega(x_1, x_2, x_5).
\end{aligned} \tag{11}$$

Then for any $x_1, x_2, x_3, x_4, x_5 \in \mathfrak{L}$,

$$\begin{aligned}
&(\delta^3_1 \mathrm{Ob}^{\omega}_{(D_a,D_l)})(x_1, x_2, x_3, x_4, x_5)\\
=\ & \mathrm{Ob}^{\omega}_{(D_a,D_l)}(x_1, x_2, \{x_3, x_4, x_5\}) - \mathrm{Ob}^{\omega}_{(D_a,D_l)}(\{x_1, x_2, x_3\}, x_4, x_5) + \mathrm{Ob}^{\omega}_{(D_a,D_l)}(\{x_1, x_2, x_4\}, x_3, x_5)\\
&+ \mathrm{Ob}^{\omega}_{(D_a,D_l)}(\{x_1, x_2, x_5\}, x_3, x_4) - \mathrm{Ob}^{\omega}_{(D_a,D_l)}(\{x_1, x_2, x_5\}, x_4, x_3) + l_A(x_1, x_2)\mathrm{Ob}^{\omega}_{(D_a,D_l)}(x_3, x_4, x_5)
\end{aligned}$$

$$
\begin{aligned}
&- r_A(x_4,x_5)\mathrm{Ob}^{\omega}_{(D_a,D_l)}(x_1,x_2,x_3) + r_A(x_3,x_5)\mathrm{Ob}^{\omega}_{(D_a,D_l)}(x_1,x_2,x_4)\\
&+ r_A(x_3,x_4)\mathrm{Ob}^{\omega}_{(D_a,D_l)}(x_1,x_2,x_5) - r_A(x_4,x_3)\mathrm{Ob}^{\omega}_{(D_a,D_l)}(x_1,x_2,x_5)\\
=\ & \underbrace{D_a(\omega(x_1,x_2,\{x_3,x_4,x_5\}))}_{(a_1)} \underbrace{-\omega(D_l(x_1),x_2,\{x_3,x_4,x_5\})}_{(b_1)} \underbrace{-\omega(x_1,D_l(x_2),\{x_3,x_4,x_5\})}_{(c_1)} \underbrace{-\omega(x_1,x_2,\{D_l(x_3),x_4,x_5\})}_{(d_1)}\\
& \underbrace{-\omega(x_1,x_2,\{x_3,D_l(x_4),x_5\})}_{(e_1)} \underbrace{-\omega(x_1,x_2,\{x_3,x_4,D_l(x_5)\})}_{(f_1)} \underbrace{-D_a(\omega(\{x_1,x_2,x_3\},x_4,x_5))}_{(a_2)} \underbrace{+\omega(\{D_l(x_1),x_2,x_3\},x_4,x_5)}_{(b_2)}\\
& \underbrace{+\omega(\{x_1,D_l(x_2),x_3\},x_4,x_5)}_{(c_2)} \underbrace{+\omega(\{x_1,x_2,D_l(x_3)\},x_4,x_5)}_{(d_2)} \underbrace{+\omega(\{x_1,x_2,x_3\},D_l(x_4),x_5)}_{(e_2)} \underbrace{+\omega(\{x_1,x_2,x_3\},x_4,D_l(x_5))}_{(f_2)}\\
& \underbrace{+D_a(\omega(\{x_1,x_2,x_4\},x_3,x_5))}_{(a_3)} \underbrace{-\omega(\{D_l(x_1),x_2,x_4\},x_3,x_5)}_{(b_3)} \underbrace{-\omega(\{x_1,D_l(x_2),x_4\},x_3,x_5)}_{(c_3)} \underbrace{-\omega(\{x_1,x_2,D_l(x_4)\},x_3,x_5)}_{(e_3)}\\
& \underbrace{-\omega(\{x_1,x_2,x_4\},D_l(x_3),x_5)}_{(d_3)} \underbrace{-\omega(\{x_1,x_2,x_4\},x_3,D_l(x_5))}_{(f_3)} \underbrace{+D_a(\omega(\{x_1,x_2,x_5\},x_3,x_4))}_{(a_4)} \underbrace{-\omega(\{D_l(x_1),x_2,x_5\},x_3,x_4)}_{(b_4)}\\
& \underbrace{-\omega(\{x_1,D_l(x_2),x_5\},x_3,x_4)}_{(c_4)} \underbrace{-\omega(\{x_1,x_2,D_l(x_5)\},x_3,x_4)}_{(f_4)} \underbrace{-\omega(\{x_1,x_2,x_5\},D_l(x_3),x_4)}_{(d_4)} \underbrace{-\omega(\{x_1,x_2,x_5\},x_3,D_l(x_4))}_{(e_4)}\\
& \underbrace{-D_a(\omega(\{x_1,x_2,x_5\},x_4,x_3))}_{(a_5)} \underbrace{+\omega(\{D_l(x_1),x_2,x_5\},x_4,x_3)}_{(b_5)} \underbrace{+\omega(\{x_1,D_l(x_2),x_5\},x_4,x_3)}_{(c_5)} \underbrace{+\omega(\{x_1,x_2,D_l(x_5)\},x_4,x_3)}_{(f_5)}\\
& \underbrace{+\omega(\{x_1,x_2,x_5\},D_l(x_4),x_3)}_{(e_5)} \underbrace{+\omega(\{x_1,x_2,x_5\},x_4,D_l(x_3))}_{(d_5)} +l_A(x_1,x_2)D_a\omega(x_3,x_4,x_5) \underbrace{-l_A(x_1,x_2)\omega(D_l(x_3),x_4,x_5)}_{(d_6)}\\
& - \underbrace{l_A(x_1,x_2)\omega(x_3,D_l(x_4),x_5)}_{(e_6)} \underbrace{-l_A(x_1,x_2)\omega(x_3,x_4,D_l(x_5))}_{(f_6)} -r_A(x_4,x_5)D_a\omega(x_1,x_2,x_3)\\
& \underbrace{+r_A(x_4,x_5)\omega(D_l(x_1),x_2,x_3)}_{(b_7)} \underbrace{+r_A(x_4,x_5)\omega(x_1,D_l(x_2),x_3)}_{(c_7)} \underbrace{+r_A(x_4,x_5)\omega(x_1,x_2,D_l(x_3))}_{(d_7)}\\
& + r_A(x_3,x_5)D_a\omega(x_1,x_2,x_4) \underbrace{-r_A(x_3,x_5)\omega(D_l(x_1),x_2,x_4)}_{(b_8)} \underbrace{-r_A(x_3,x_5)\omega(x_1,D_l(x_2),x_4)}_{(c_8)}\\
& \underbrace{-r_A(x_3,x_5)\omega(x_1,x_2,D_l(x_4))}_{(e_8)} +r_A(x_3,x_4)D_a\omega(x_1,x_2,x_5) \underbrace{-r_A(x_3,x_4)\omega(D_l(x_1),x_2,x_5)}_{(b_9)}\\
& \underbrace{-r_A(x_3,x_4)\omega(x_1,D_l(x_2),x_5)}_{(c_9)} \underbrace{-r_A(x_3,x_4)\omega(x_1,x_2,D_l(x_5))}_{(f_9)} -r_A(x_4,x_3)D_a\omega(x_1,x_2,x_5)\\
& \underbrace{+r_A(x_4,x_3)\omega(D_l(x_1),x_2,x_5)}_{(b_{10})} \underbrace{+r_A(x_4,x_3)\omega(x_1,D_l(x_2),x_5)}_{(c_{10})} \underbrace{+r_A(x_4,x_3)\omega(x_1,x_2,D_l(x_5))}_{(f_{10})}\\
\overset{(11)}{=}\ & \underbrace{l_A(D_l(x_1),x_2)\omega(x_3,x_4,x_5)}_{(b_6)} \underbrace{+l_A(x_1,D_l(x_2))\omega(x_3,x_4,x_5)}_{(c_6)} \underbrace{+r_A(D_l(x_3),x_5)\omega(x_1,x_2,x_4)}_{(d_8)}\\
& \underbrace{+r_A(D_l(x_3),x_4)\omega(x_1,x_2,x_5)}_{(d_9)} \underbrace{-r_A(x_4,D_l(x_3))\omega(x_1,x_2,x_5)}_{(d_{10})} \underbrace{-r_A(D_l(x_4),x_5)\omega(x_1,x_2,x_3)}_{(e_7)}
\end{aligned}
$$

$$
\begin{aligned}
&\underbrace{+r_A(x_3,D_l(x_4))\omega(x_1,x_2,x_5)}_{(e_9)}\underbrace{-r_A(D_l(x_4),x_3)\omega(x_1,x_2,x_5)}_{(e_{10})}\underbrace{-r_A(x_4,D_l(x_5))\omega(x_1,x_2,x_3)}_{(f_7)}\\
&\underbrace{+r_A(x_3,D_l(x_5))\omega(x_1,x_2,x_4)}_{(f_8)}\underbrace{-D_a(l_A(x_1,x_2)\omega(x_3,x_4,x_5))}_{(a_6)}\underbrace{+D_a(r_A(x_4,x_5)\omega(x_1,x_2,x_3))}_{(a_7)}\\
&\underbrace{-D_a(r_A(x_3,x_5)\omega(x_1,x_2,x_4))}_{(a_8)}\underbrace{-D_a(r_A(x_3,x_4)\omega(x_1,x_2,x_5))}_{(a_9)}\underbrace{+D_a(r_A(x_4,x_3)\omega(x_1,x_2,x_5))}_{(a_{10})}\\
&+l_A(x_1,x_2)D_a\omega(x_3,x_4,x_5)-r_A(x_4,x_5)D_a\omega(x_1,x_2,x_3)+r_A(x_3,x_5)D_a\omega(x_1,x_2,x_4)\\
&+r_A(x_3,x_4)D_a\omega(x_1,x_2,x_5)-r_A(x_4,x_3)D_a\omega(x_1,x_2,x_5)\\
=\ &(-D_a(l_A(x_1,x_2)+l_A(x_1,x_2)D_a+l_A(D_l(x_1),x_2)+l_A(x_1,D_l(x_2)))\omega(x_3,x_4,x_5)\\
&-(-D_a(r_A(x_4,x_5)+r_A(x_4,x_5)D_a+r_A(D_l(x_4),x_5)+r_A(x_4,D_l(x_5)))\omega(x_1,x_2,x_3)\\
&+(-D_a(r_A(x_3,x_5)+r_A(x_3,x_5)D_a+r_A(D_l(x_3),x_5)+r_A(x_3,D_l(x_5)))\omega(x_1,x_2,x_4)\\
&+(-D_a(r_A(x_3,x_4)+r_A(x_3,x_4)D_a+r_A(D_l(x_3),x_4)+r_A(x_3,D_l(x_4)))\omega(x_1,x_2,x_5)\\
&-(-D_a(r_A(x_4,x_3)+r_A(x_4,x_3)D_a+r_A(D_l(x_4),x_3)+r_A(x_4,D_l(x_3)))\omega(x_1,x_2,x_5)\\
\overset{(7)}{=}\ &0.
\end{aligned}
$$

Similarly, it is straightforward to check that $\delta_2^3\mathrm{Ob}^{\omega}_{(D_a,D_l)}=0$, as required. □

Theorem 3.6. *Suppose $(D_a,D_l)\in G_A$. Then (D_a,D_l) is extensible if and only if $[\mathrm{Ob}^{\tilde{\mathfrak{L}}}_{(D_a,D_l)}]\in H^3(\mathfrak{L},A)$ is trivial.*

Proof. (⇒) Fix any section s of π. Suppose that (D_a,D_l) is extensible, then there exists a derivation $D_{\tilde{l}}\in\mathrm{Der}(\tilde{\mathfrak{L}})$ such that the associated diagram (6) is commutative. Since $\pi\circ D_{\tilde{l}}=D_l\circ\pi$, we have $D_{\tilde{l}}(s(x))-s(D_l(x))\in A$, for $x\in\mathfrak{L}$. Thus there is a map $\mu:\mathfrak{L}\longrightarrow A$ given by

$$\mu(x)=D_{\tilde{l}}(s(x))-s(D_l(x)).$$

It is sufficient to show that

$$\mathrm{Ob}^{\tilde{\mathfrak{L}}}_{(D_a,D_l)}(x,y,z)=(\delta^1\mu)(x,y,z),\tag{12}$$

for all $x,y,z\in\mathfrak{L}$, which will be proved by computing both hand sides of the following identity

$$
\begin{aligned}
&D_{\tilde{l}}\Big(\{s(x_1)+v_1,s(x_2)+v_2,s(x_3)+v_3\}_{\tilde{\mathfrak{L}}}\Big)\\
&=\{D_{\tilde{l}}(s(x_1)+v_1),s(x_2)+v_2,s(x_3)+v_3\}_{\tilde{\mathfrak{L}}}+\{s(x_1)+v_1,D_{\tilde{l}}(s(x_2)+v_2),s(x_3)+v_3\}_{\tilde{\mathfrak{L}}}\\
&\quad+\{s(x_1)+v_1,s(x_2)+v_2,D_{\tilde{l}}(s(x_3)+v_3)\}_{\tilde{\mathfrak{L}}},
\end{aligned}\tag{13}
$$

for any $x_1,x_2,x_3\in\mathfrak{L}$, $v_1,v_2,v_3\in A$.

At first, since $\tilde{\mathfrak{L}}$ is an abelian extension of $\mathfrak{L}$ by A, we have $\{A,A,\tilde{\mathfrak{L}}\}=\{A,\tilde{\mathfrak{L}},A\}=\{\tilde{\mathfrak{L}},A,A\}=0$ and

$$
\begin{aligned}
&\{s(x_1)+v_1,s(x_2)+v_2,s(x_3)+v_3\}_{\tilde{\mathfrak{L}}}\\
&=\{s(x_1),s(x_2),s(x_3)\}_{\tilde{\mathfrak{L}}}+\{s(x_1),s(x_2),v_3\}_{\tilde{\mathfrak{L}}}+\{s(x_1),v_2,s(x_3)\}_{\tilde{\mathfrak{L}}}+\{v_1,s(x_2),s(x_3)\}_{\tilde{\mathfrak{L}}}\\
&=\{s(x_1),s(x_2),s(x_3)\}_{\tilde{\mathfrak{L}}}+l_A(x_1,x_2)(v_3)+m_A(x_1,x_3)(v_2)+r_A(x_2,x_3)(v_1),
\end{aligned}
$$

and by Eq. (5),

$$
\begin{aligned}
\text{LHS of Eq. (13)}&=D_{\tilde{l}}\Big(\{s(x_1),s(x_2),s(x_3)\}_{\tilde{\mathfrak{L}}}+l_A(x_1,x_2)(v_3)+m_A(x_1,x_3)(v_2)+r_A(x_2,x_3)(v_1)\Big),\\
&=D_{\tilde{l}}\Big(s(\{x_1,x_2,x_3\}_{\mathfrak{L}})+\omega(x_1,x_2,x_3)+l_A(x_1,x_2)(v_3)+m_A(x_1,x_3)(v_2)+r_A(x_2,x_3)(v_1)\Big).
\end{aligned}
$$

Since the diagram (6) is commutative, it follows that

$$\begin{aligned} \text{LHS of Eq. (13)} &= s(D_l(\{x_1,x_2,x_3\}_{\mathfrak{L}})) + \mu(\{x_1,x_2,x_3\}_{\mathfrak{L}}) + D_a(\omega(x_1,x_2,x_3)) \\ &\quad + D_a(l_A(x_1,x_2)(v_3)) + D_a(r_A(x_2,x_3)(v_1)) + D_a(m_A(x_1,x_3)(v_2)) \\ &= s(\{D_l(x_1),x_2,x_3\}_{\mathfrak{L}}) + s(\{x_1,D_l(x_2),x_3\}_{\mathfrak{L}}) + s(\{x_1,x_2,D_l(x_3)\}_{\mathfrak{L}}) + \mu(\{x_1,x_2,x_3\}_{\mathfrak{L}}) \\ &\quad + D_a(\omega(x_1,x_2,x_3)) + D_a(l_A(x_1,x_2)(v_3)) + D_a(r_A(x_2,x_3)(v_1)) + D_a(m_A(x_1,x_3)(v_2)). \end{aligned} \tag{14}$$

Now we compute the right-hand side of Eq. (13). Note that, since $D_{\tilde{l}}|_A = D_a$, it holds that

$$\begin{aligned} D_{\tilde{l}}(s(x_i)+v_i) &= D_{\tilde{l}}(s(x_i)) + D_a(v_i) \\ &= D_{\tilde{l}}(s(x_i)) - s(D_l(x_i)) + s(D_l(x_i)) + D_a(v_i) \\ &= s(D_l(x_i)) + \mu(x_i) + D_a(v_i) \in s(\mathfrak{L}) \dot{+} A, \end{aligned}$$

where $i = 1,2,3$, which combining with $\{A,A,\tilde{\mathfrak{L}}\} = \{A,\tilde{\mathfrak{L}},A\} = \{\tilde{\mathfrak{L}},A,A\} = 0$ show

$$\begin{aligned} \text{RHS of Eq. (13)} &= \{s(D_l(x_1)) + \mu(x_1) + D_a(v_1), s(x_2)+v_2, s(x_3)+v_3\}_{\tilde{\mathfrak{L}}} \\ &\quad + \{s(x_1)+v_1, s(D_l(x_2)) + \mu(x_2) + D_a(v_2), s(x_3)+v_3\}_{\tilde{\mathfrak{L}}} \\ &\quad + \{s(x_1)+v_1, s(x_2)+v_2, s(D_l(x_3)) + \mu(x_3) + D_a(v_3)\}_{\tilde{\mathfrak{L}}} \\ &= \{s(D_l(x_1)), s(x_2), s(x_3)\}_{\tilde{\mathfrak{L}}} + \{s(D_l(x_1)), s(x_2), v_3\}_{\tilde{\mathfrak{L}}} + \{s(D_l(x_1)), v_2, s(x_3)\}_{\tilde{\mathfrak{L}}} \\ &\quad + \{\mu(x_1), s(x_2), s(x_3)\}_{\tilde{\mathfrak{L}}} + \{s(x_1), \mu(x_2), s(x_3)\}_{\tilde{\mathfrak{L}}} + \{D_a(v_1), s(x_2), s(x_3)\}_{\tilde{\mathfrak{L}}} \\ &\quad + \{s(x_1), s(D_l(x_2)), v_3\}_{\tilde{\mathfrak{L}}} + \{v_1, s(D_l(x_2)), s(x_3)\}_{\tilde{\mathfrak{L}}} + \{s(x_1), D_a(v_2), s(x_3)\}_{\tilde{\mathfrak{L}}} \\ &\quad + \{s(x_1), s(x_2), s(D_l(x_3))\}_{\tilde{\mathfrak{L}}} + \{s(x_1), s(x_2), \mu(x_3)\}_{\tilde{\mathfrak{L}}} + \{s(x_1), s(x_2), D_a(v_3)\}_{\tilde{\mathfrak{L}}} \\ &\quad + \{s(x_1), v_2, s(D_l(x_3))\}_{\tilde{\mathfrak{L}}} + \{v_1, s(x_2), s(D_l(x_3))\}_{\tilde{\mathfrak{L}}} + \{s(x_1), s(D_l(x_2)), s(x_3)\}_{\tilde{\mathfrak{L}}}. \end{aligned} \tag{15}$$

By Eq. (14) and Eq. (15), one has

$$\begin{aligned} & s(\{D_l(x_1),x_2,x_3\}_{\mathfrak{L}}) + s(\{x_1,D_l(x_2),x_3\}_{\mathfrak{L}}) + s(\{x_1,x_2,D_l(x_3)\}_{\mathfrak{L}}) + \mu(\{x_1,x_2,x_3\}_{\mathfrak{L}}) \\ &\quad + D_a(\omega(x_1,x_2,x_3)) + D_a(l_A(x_1,x_2)(v_3)) + D_a(r_A(x_2,x_3)(v_1)) + D_a(m_A(x_1,x_3)(v_2)) \\ &= \{s(D_l(x_1)), s(x_2), s(x_3)\}_{\tilde{\mathfrak{L}}} + l_A(D_l(x_1),x_2)(v_3) + m_A(D_l(x_1),x_3)(v_2) + r_A(x_2,x_3)(\mu(x_1)) \\ &\quad + r_A(x_2,x_3)(D_a(v_1)) + \{s(x_1), s(D_l(x_2)), s(x_3)\}_{\tilde{\mathfrak{L}}} + l_A(x_1,D_l(x_2))(v_3) + m_A(x_1,x_3)(\mu(x_2)) \\ &\quad + m_A(x_1,x_3)(D_a(v_2)) + r_A(D_l(x_2),x_3)(v_1) + \{s(x_1), s(x_2), s(D_l(x_3))\}_{\tilde{\mathfrak{L}}} + l_A(x_1,x_2)(\mu(x_3)) \\ &\quad + l_A(x_1,x_2)(D_a(v_3)) + m_A(x_1,D_l(x_3))(v_2) + r_A(x_2,D_l(x_3))(v_1). \end{aligned}$$

Then

$$\begin{aligned} 0 = &-\omega(D_l(x_1),x_2,x_3) - \omega(x_1,D_l(x_2),x_3) - \omega(x_1,x_2,D_l(x_3)) + D_a(\omega(x_1,x_2,x_3)) \\ &- l_A(x_1,x_2)(\mu(x_3)) - r_A(x_2,x_3)(\mu(x_1)) - m_A(x_1,x_3)(\mu(x_2)) + \mu(\{x_1,x_2,x_3\}_{\mathfrak{L}}) \\ &+ \Big(D_a l_A(x_1,x_2) - l_A(x_1,x_2)D_a - l_A(D_l(x_1),x_2) - l_A(x_1,D_l(x_2))\Big)(v_3) \\ &+ \Big(D_a r_A(x_2,x_3) - r_A(x_2,x_3)D_a - r_A(D_l(x_2),x_3) - r_A(x_2,D_l(x_3))\Big)(v_1) \\ &+ \Big(D_a m_A(x_1,x_3) - m_A(x_1,x_3)D_a - m_A(D_l(x_1),x_3) - m_A(x_1,D_l(x_3))\Big)(v_2). \end{aligned}$$

Since (D_a, D_l) is compatible and by the proof of Lemma 8,

$$\begin{aligned} &\Big(D_a l_A(x_1,x_2) - l_A(x_1,x_2)D_a - l_A(D_l(x_1),x_2) - l_A(x_1,D_l(x_2))\Big)(v_3) = 0, \\ &\Big(D_a r_A(x_2,x_3) - r_A(x_2,x_3)D_a - r_A(D_l(x_2),x_3) - r_A(x_2,D_l(x_3))\Big)(v_1) = 0, \\ &\Big(D_a m_A(x_1,x_3) - m_A(x_1,x_3)D_a - m_A(D_l(x_1),x_3) - m_A(x_1,D_l(x_3))\Big)(v_2) = 0. \end{aligned} \tag{16}$$

Thus we have

$$\begin{aligned} & D_a(\omega(x_1,x_2,x_3)) - \omega(D_l(x_1),x_2,x_3) - \omega(x_1,D_l(x_2),x_3) - \omega(x_1,x_2,D_l(x_3)) \\ & - l_A(x_1,x_2)(\mu(x_3)) - r_A(x_2,x_3)(\mu(x_1)) - m_A(x_1,x_3)(\mu(x_2)) + \mu(\{x_1,x_2,x_3\}_{\mathfrak{L}}) = 0, \end{aligned} \tag{17}$$

which is exactly Eq. (12) due to Eqs. (1) and (9). So $[\mathrm{Ob}^{\tilde{\mathfrak{L}}}_{(D_a,D_l)}] = 0$, as required.

(⇐) Suppose that $[\mathrm{Ob}^{\tilde{\mathfrak{L}}}_{(D_a,D_l)}]$ is trivial. Then there is a map $\mu : \mathfrak{L} \longrightarrow A$ such that $\mathrm{Ob}^{\tilde{\mathfrak{L}}}_{(D_a,D_l)} = \delta^1\mu$. For any element $s(x) + v \in \tilde{\mathfrak{L}}$, define $D_{\tilde{l}} : \tilde{\mathfrak{L}} \longrightarrow \tilde{\mathfrak{L}}$ by

$$D_{\tilde{l}}(s(x) + v) = s(D_l(x)) + \mu(x) + D_a(v),$$

then the associated diagram in (6) is commutative: for any $x \in \mathfrak{L}, v \in A$,

$$\begin{aligned} (\pi \circ D_{\tilde{l}})(s(x) + v) &= \pi(s(D_l(x)) + \mu(x) + D_a(v)) = D_l(x) = (D_l \circ \pi)(s(x) + v); \\ D_{\tilde{l}} \circ \iota(v) &= D_{\tilde{l}}(v) = D_a(v) = \iota \circ D_a(v). \end{aligned}$$

Moreover, since (D_a, D_l) is compatible satisfying Eq. (7), by Eq. (16) and Eq. (17), it follows that Eq. (13) holds by Eq. (14) and Eq. (15), that is, $D_{\tilde{l}} \in \mathrm{Der}(\tilde{\mathfrak{L}})$, as required. □

Hence, for a pair $(D_a, D_l) \in G_A$, the cohomology class $[\mathrm{Ob}^{\tilde{\mathfrak{L}}}_{(D_a,D_l)}]$ can be regarded as an obstruction to the extensibility of (D_a, D_l). We also have the following straightforward corollary.

Corollary 3.7. *If $H^3(\mathfrak{L}, A) = 0$, then $(D_a, D_l) \in G_A$ if and only if (D_a, D_l) is extensible.*

Recall that the condition $H^3(\mathfrak{L}, A) = 0$ is in general not equivalent to split property of extensions. However, we still have the following result.

Corollary 3.8. *Let $\tilde{\mathfrak{L}}$ be a split abelian extension of a Leibniz triple system $\mathfrak{L}$ by A. Then any pair $(D_a, D_l) \in \mathrm{Der}(A) \times \mathrm{Der}(\mathfrak{L})$ is compatible if and only if it is extensible.*

Proof. (⇐) It holds due to Proposition 3.2.

(⇒) Since the extension is split there exists a section s' which is a homomorphism. Suppose that $(r_{s'}, m_{s'}, l_{s'})$ (resp., $\omega_{s'}$) is defined by Eq. (4) (resp., Eq. (5)) with respect to s'. Then we get $\omega_{s'} = 0$. By the definition of $\mathrm{Ob}^{\tilde{\mathfrak{L}}}_{(D_a,D_l)}$ (see Eq. (8)), we have $\mathrm{Ob}^{\omega_{s'}}_{(D_a,D_l)} = 0$. In view of Lemmas 3.4 and 3.5, one has $[\mathrm{Ob}^{\tilde{\mathfrak{L}}}_{(D_a,D_l)}] = [\mathrm{Ob}^{\omega_{s'}}_{(D_a,D_l)}] = 0$. Then by Theorem 3.6, we deduce that (D_a, D_l) is extensible as required. □

At the end of this section, we will show that G_A is a Lie algebra and construct a representation of G_A to characterize the extensibility of (D_a, D_l).

Proposition 3.9. *G_A is a Lie algebra.*

Proof. It suffices to prove that G_A is a subalgebra of $\mathrm{Der}(A) \times \mathrm{Der}(\mathfrak{L})$. Suppose $(D_{a_1}, D_{l_1}), (D_{a_2}, D_{l_2}) \in G_A$. Note that, for all $x, y \in \mathfrak{L}$, we have

$$\begin{aligned} & (D_{a_1}D_{a_2} - D_{a_2}D_{a_1})l_A(x,y) - l_A(x,y)(D_{a_1}D_{a_2} - D_{a_2}D_{a_1}) \\ &= D_{a_1}(D_{a_2}l_A(x,y)) - D_{a_2}(D_{a_1}l_A(x,y)) - l_A(x,y)D_{a_1}D_{a_2} + l_A(x,y)D_{a_2}D_{a_1} \\ &= \underbrace{D_{a_1}\Big(l_A(x,y)D_{a_2} + l_A(D_{l_2}(x),y) + l_A(x,D_{l_2}(y))\Big)}_{I_1} \\ & \quad - \underbrace{D_{a_2}\Big(l_A(x,y)D_{a_1} + l_A(D_{l_1}(x),y) + l_A(x,D_{l_1}(y))\Big)}_{I_2} \\ & \quad - l_A(x,y)D_{a_1}D_{a_2} + l_A(x,y)D_{a_2}D_{a_1}. \end{aligned} \tag{18}$$

By Eq. (7) it follows that

$$I_1 = l_A(x,y)D_{a_1}D_{a_2} + l_A(D_{l_1}(x),y)D_{a_2} + l_A(x,D_{l_1}(y))D_{a_2} + l_A(D_{l_2}(x),y)D_{a_1} + l_A(D_{l_1}D_{l_2}(x),y) + l_A(D_{l_2}(x),D_{l_1}(y)) + l_A(x,D_{l_2}(y))D_{a_1} + l_A(D_{l_1}(x),D_{l_2}(y)) + l_A(x,D_{l_1}D_{l_2}(y)), \quad (19)$$

$$I_2 = l_A(x,y)D_{a_2}D_{a_1} + l_A(D_{l_2}(x),y)D_{a_1} + l_A(x,D_{l_2}(y))D_{a_1} + l_A(D_{l_1}(x),y)D_{a_2} + l_A(D_{l_2}D_{l_1}(x),y) + l_A(D_{l_1}(x),D_{l_2}(y)) + l_A(x,D_{l_1}(y))D_{a_2} + l_A(D_{l_2}(x),D_{l_1}(y)) + l_A(x,D_{l_2}D_{l_1}(y)). \quad (20)$$

Then substituting Eq. (19) and Eq. (20) into Eq. (18) gives that

$$[D_{a_1},D_{a_2}]l_A(x,y) - l_A(x,y)[D_{a_1},D_{a_2}] = l_A\Big([D_{l_1},D_{l_2}](x),y\Big) + l_A\Big(x,[D_{l_1},D_{l_2}](y)\Big).$$

Similarly, we have

$$[D_{a_1},D_{a_2}]r_A(x,y) - r_A(x,y)[D_{a_1},D_{a_2}] = r_A\Big([D_{l_1},D_{l_2}](x),y\Big) + r_A\Big(x,[D_{l_1},D_{l_2}](y)\Big),$$

and

$$[D_{a_1},D_{a_2}]m_A(x,y) - m_A(x,y)[D_{a_1},D_{a_2}] = m_A\Big([D_{l_1},D_{l_2}](x),y\Big) + m_A\Big(x,[D_{l_1},D_{l_2}](y)\Big),$$

which implies that $[(D_{a_1},D_{l_1}),(D_{a_2},D_{l_2})] = ([D_{a_1},D_{a_2}],[D_{l_1},D_{l_2}])$ is compatible. □

Lemma 3.10. *Define a linear map* $\Phi : G_A \longrightarrow \mathrm{gl}(H^3(\mathfrak{L},A))$ *by*

$$\Phi(D_a,D_l)([\omega]) = [\mathrm{Ob}^{\omega}_{(D_a,D_l)}],\ \forall\ \omega \in Z^3(\mathfrak{L},A), \quad (21)$$

where $\mathrm{Ob}^{\omega}_{(D_a,D_l)}$ *is given by Eq.* (8). *Then* Φ *is a representation of* G_A *on* $H^3(\mathfrak{L},A)$.

Proof. Since (D_a,D_l) is compatible with respect to (r_A,m_A,l_A) by Lemma 3.5 it follows that $\mathrm{Ob}^{\omega}_{(D_a,D_l)}$ is a 3-cocycle whenever ω is a 3-cocycle. Therefore, it suffices to show that if $\delta^1\lambda$ is a 3-coboundary, then $\Phi(D_a,D_l)(\delta^1\lambda) = 0$, which implies that Φ is well-defined. In fact,

$$\begin{aligned}
&(\Phi(D_a,D_l)(\delta^1\lambda))(x,y,z)\\
&= \Big(D_a(\delta^1\lambda) - (\delta^1\lambda)(D_l\otimes\mathrm{id}\otimes\mathrm{id}) - (\delta^1\lambda)(\mathrm{id}\otimes D_l\otimes\mathrm{id}) - (\delta^1\lambda)(\mathrm{id}\otimes\mathrm{id}\otimes D_l)\Big)(x,y,z)\\
&= D_a\Big(-\lambda(\{x,y,z\}) + l_A(x,y)(\lambda(z)) + m_A(x,z)(\lambda(y)) + r_A(y,z)(\lambda(x))\Big)\\
&\quad - \Big(-\lambda(\{D_l(x),y,z\}) + l_A(D_l(x),y)(\lambda(z)) + m_A(D_l(x),z)(\lambda(y)) + r_A(y,z)(\lambda(D_l(x)))\Big)\\
&\quad - \Big(-\lambda(\{x,D_l(y),z\}) + l_A(x,D_l(y))(\lambda(z)) + m_A(x,z)(\lambda(D_l(y))) + r_A(D_l(y),z)(\lambda(x))\Big)\\
&\quad - \Big(-\lambda(\{x,y,D_l(z)\}) + l_A(x,y)(\lambda(D_l(z))) + m_A(x,D_l(z))(\lambda(y)) + r_A(y,D_l(z))(\lambda(x))\Big).
\end{aligned}$$

Since D_l is a derivation on $\mathfrak{L}$, we have

$$\lambda(\{D_l(x),y,z\}) + \lambda(\{x,D_l(y),z\}) + \lambda(\{x,y,D_l(z)\}) = \lambda(D_l(\{x,y,z\})).$$

Then

$$\begin{aligned}
&(\Phi(D_a,D_l)(\delta^1\lambda))(x,y,z)\\
&= D_a l_A(x,y)(\lambda(z)) + D_a m_A(x,z)(\lambda(y)) + D_a r_A(y,z)(\lambda(x)) - l_A(D_l(x),y)(\lambda(z)) - m_A(D_l(x),z)(\lambda(y))\\
&\quad - r_A(y,z)(\lambda(D_l(x)) - l_A(x,D_l(y))(\lambda(z)) - m_A(x,z)(\lambda(D_l(y))) - r_A(D_l(y),z)(\lambda(x)) - l_A(x,y)(\lambda(D_l(z)))\\
&\quad - m_A(x,D_l(z))(\lambda(y)) - r_A(y,D_l(z))(\lambda(x)) - D_a(\lambda(\{x,y,z\})) + \lambda(D_l(\{x,y,z\}))
\end{aligned}$$

$$= \Big(D_a l_A(x,y) - l_A(D_l(x),y) - l_A(x,D_l(y))\Big)(\lambda(z)) + \Big(D_a m_A(x,z) - m_A(D_l(x),z) - m_A(x,D_l(z))\Big)(\lambda(y))$$
$$+ \Big(D_a r_A(y,z) - r_A(D_l(y),z) - r_A(y,D_l(z))\Big)(\lambda(x)) - r_A(y,z)(\lambda(D_l(x))) - l_A(x,y)(\lambda(D_l(z)))$$
$$- m_A(x,z)(\lambda(D_l(y))) - D_a(\lambda(\{x,y,z\})) + \lambda(D_l(\{x,y,z\}))$$
$$= l_A(x,y)D_a(\lambda(z)) + m_A(x,z)D_a(\lambda(y)) + r_A(y,z)D_a(\lambda(x)) - r_A(y,z)(\lambda(D_l(x)))$$
$$- l_A(x,y)(\lambda(D_l(z))) - m_A(x,z)(\lambda(D_l(y))) - D_a(\lambda(\{x,y,z\})) + \lambda(D_l(\{x,y,z\}))$$
$$= \delta^1(D_a \circ \lambda - \lambda \circ D_l)(x,y,z) = 0.$$

Next, we prove that Φ is a Lie homomorphism, for any $(D_{a_1}, D_{l_1}), (D_{a_2}, D_{l_2}) \in G_A$, $[\omega] \in H^3(\mathfrak{L}, A)$, we have

$$[\Phi(D_{a_1}, D_{l_1}), \Phi(D_{a_2}, D_{l_2})]([\omega]) = \Phi(D_{a_1}, D_{l_1})\Phi(D_{a_2}, D_{l_2})([\omega]) - \Phi(D_{a_2}, D_{l_2})\Phi(D_{a_1}, D_{l_1})([\omega]).$$

Combining Eqs. (8) and (21), we see that

$$\Phi(D_{a_1}, D_{l_1})\Phi(D_{a_2}, D_{l_2})([\omega])$$
$$= \Phi(D_{a_1}, D_{l_1})\Big(D_{a_2}\omega - \omega(D_{l_2} \otimes \mathrm{id} \otimes \mathrm{id}) - \omega(\mathrm{id} \otimes D_{l_2} \otimes \mathrm{id}) - \omega(\mathrm{id} \otimes \mathrm{id} \otimes D_{l_2})\Big)$$
$$= \Big[D_{a_1}\Big(D_{a_2}\omega - \omega(D_{l_2} \otimes \mathrm{id} \otimes \mathrm{id}) - \omega(\mathrm{id} \otimes D_{l_2} \otimes \mathrm{id}) - \omega(\mathrm{id} \otimes \mathrm{id} \otimes D_{l_2})\Big)$$
$$- \Big(D_{a_2}\omega - \omega(D_{l_2} \otimes \mathrm{id} \otimes \mathrm{id}) - \omega(\mathrm{id} \otimes D_{l_2} \otimes \mathrm{id}) - \omega(\mathrm{id} \otimes \mathrm{id} \otimes D_{l_2})\Big)(D_{l_1} \otimes \mathrm{id} \otimes \mathrm{id})$$
$$- \Big(D_{a_2}\omega - \omega(D_{l_2} \otimes \mathrm{id} \otimes \mathrm{id}) - \omega(\mathrm{id} \otimes D_{l_2} \otimes \mathrm{id}) - \omega(\mathrm{id} \otimes \mathrm{id} \otimes D_{l_2})\Big)(\mathrm{id} \otimes D_{l_1} \otimes \mathrm{id})$$
$$- \Big(D_{a_2}\omega - \omega(D_{l_2} \otimes \mathrm{id} \otimes \mathrm{id}) - \omega(\mathrm{id} \otimes D_{l_2} \otimes \mathrm{id}) - \omega(\mathrm{id} \otimes \mathrm{id} \otimes D_{l_2})\Big)(\mathrm{id} \otimes \mathrm{id} \otimes D_{l_1})\Big]$$
$$= \Big[D_{a_1}D_{a_2}\omega - D_{a_1}\omega(D_{l_2} \otimes \mathrm{id} \otimes \mathrm{id}) - D_{a_1}\omega(\mathrm{id} \otimes D_{l_2} \otimes \mathrm{id}) - D_{a_1}\omega(\mathrm{id} \otimes \mathrm{id} \otimes D_{l_2})$$
$$- \Big(D_{a_2}\omega(D_{l_1} \otimes \mathrm{id} \otimes \mathrm{id}) - \omega(D_{l_2}D_{l_1} \otimes \mathrm{id} \otimes \mathrm{id}) - \omega(D_{l_1} \otimes D_{l_2} \otimes \mathrm{id}) - \omega(D_{l_1} \otimes \mathrm{id} \otimes D_{l_2})\Big)$$
$$- \Big(D_{a_2}\omega(\mathrm{id} \otimes D_{l_1} \otimes \mathrm{id}) - \omega(D_{l_2} \otimes D_{l_1} \otimes \mathrm{id}) - \omega(\mathrm{id} \otimes D_{l_2}D_{l_1} \otimes \mathrm{id}) - \omega(\mathrm{id} \otimes D_{l_1} \otimes D_{l_2})\Big)$$
$$- \Big(D_{a_2}\omega(\mathrm{id} \otimes \mathrm{id} \otimes D_{l_1}) - \omega(D_{l_2} \otimes \mathrm{id} \otimes D_{l_1}) - \omega(\mathrm{id} \otimes D_{l_2} \otimes D_{l_1}) - \omega(\mathrm{id} \otimes \mathrm{id} \otimes D_{l_2}D_{l_1})\Big)\Big].$$

Then, we deduce that

$$[\Phi(D_{a_1}, D_{l_1}), \Phi(D_{a_2}, D_{l_2})]([\omega])$$
$$= \Big(\Phi(D_{a_1}, D_{l_1})\Phi(D_{a_2}, D_{l_2}) - \Phi(D_{a_2}, D_{l_2})\Phi(D_{a_1}, D_{l_1})\Big)([\omega])$$
$$= [[D_{a_1}, D_{a_2}]\omega - \omega([D_{l_1}, D_{l_2}] \otimes \mathrm{id} \otimes \mathrm{id}) - \omega(\mathrm{id} \otimes [D_{l_1}, D_{l_2}] \otimes \mathrm{id}) - \omega(\mathrm{id} \otimes \mathrm{id} \otimes [D_{l_1}, D_{l_2}])]$$
$$= [\mathrm{Ob}^{\omega}_{([D_{a_1}, D_{a_2}], [D_{l_1}, D_{l_2}])}]$$
$$= \Phi([D_{a_1}, D_{a_2}], [D_{l_1}, D_{l_2}])([\omega])$$
$$= \Phi[(D_{a_1}, D_{l_1}), (D_{a_2}, D_{l_2})]([\omega]),$$

as desired. □

Theorem 3.11. *The pair $(D_a, D_l) \in G_A$ is extensible if and only if $\Phi(D_a, D_l) = 0$.*

Proof. (⇒) For any $[\varphi] \in H^3(\mathfrak{L}, A)$, by Corollary 2.12, there exists an abelian extension

$$0 \longrightarrow A \hookrightarrow \tilde{\mathfrak{L}} \xrightarrow{\pi} \mathfrak{L} \longrightarrow 0$$

where π is the canonical projection and the bracket on $\tilde{\mathfrak{L}} := \mathfrak{L} \dotplus A$ is given by

$$\{x+u, y+v, z+w\}_{\tilde{\mathfrak{L}}} = \{x,y,z\}_{\mathfrak{L}} + \varphi(x,y,z) + l_A(x,y)(w) + r_A(y,z)(u) + m_A(x,z)(v),$$

for any $x, y, z \in \mathfrak{L}$, $u, v, w \in A$. Choose the section s of π defined by $s(x) = x$, for any $x \in \mathfrak{L}$. The associated representation (r_A, m_A, l_A) is given by Eq. (4). Let $H^3_A(\mathfrak{L}, A)$ denote the cohomology group with respect to (r_A, m_A, l_A). Since we defined ω by Eq. (5) (resp. φ) is a 3-cocycle in $H^3_A(\mathfrak{L}, A)$ (resp. in $H^3(\mathfrak{L}, A)$), we have $[\omega] = [\varphi]$. Then it follows that

$$\begin{aligned}\Phi(D_a, D_l)([\varphi]) &= \Phi(D_a, D_l)([\omega]) \quad \text{(by Lemma 3.10)}\\ &= [\mathrm{Ob}^{\tilde{\mathfrak{L}}}_{(D_a,D_l)}]\\ &= 0. \quad \text{(by Theorem 3.6)}\end{aligned}$$

($\Leftarrow$) Suppose $\Phi(D_a, D_l) = 0$. For any abelian extension

$$0 \longrightarrow A \hookrightarrow \tilde{\mathfrak{L}} \xrightarrow{\pi} \mathfrak{L} \longrightarrow 0$$

there exists a section s of π and the associated representation (r_A, m_A, l_A), ω defined by Eq. (5) is a 3-cocycle in $H^3(\mathfrak{L}, A)$. Then we have

$$[\mathrm{Ob}^{\tilde{\mathfrak{L}}}_{(D_a,D_l)}] = \Phi(D_a, D_l)([\omega]) = 0.$$

By Theorem 3.6 again, (D_a, D_l) is extensible. This complete the proof. □

The following corollary is straightforward.

Corollary 3.12. *Any pair $(D_a, D_l) \in G_A$ is extensible if and only if $\Phi \equiv 0$.*

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