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The weak group-star matrix

Jiaxuan Yao[a], Hongwei Jin[a,*], Xiaoji Liu[a]

[a] *School of Mathematics and physics, Guangxi Minzu University, 530006, Nanning, PR China.*

Abstract. In this paper, we introduce one type of matrix, called the weak group-star matrix. We investigate the characterizations, representations, and properties of the matrix. A variant of the successive matrix squaring computational iterative scheme is given for calculating the weak group-star matrix. Moreover, the Cramer's rule for the solution of a singular equation $(A^{\dagger})^{*}x = b$ is presented. Then, the perturbation is also given for the weak group-star matrix. In the final, the weak group-star matrix being used in solving appropriate systems of linear equations is established.

1. Introduction

Throughout this paper, we denote the set of all $m \times n$ complex matrices by $\mathbb{C}^{m\times n}$. For $A \in \mathbb{C}^{n\times n}$, the symbols A^{*}, $rank(A), N(A),$ *and* $R(A)$ stand for the conjugate transpose, the rank, the null space and the range space of A, respectively. Moreover, I_n will refer to the $n \times n$ identity matrix. Let $A \in \mathbb{C}^{n\times n}$, the smallest positive integer k for which $rank(A^k) = rank(A^{k+1})$ is called the index of A and is denoted by $\mathrm{Ind}(A)$. Then $\mathbb{C}_k^{n\times n}$ represents all $n \times n$ complex matrices sets with index k. $P_{E,F}$ represents the projector on the subspace E along the subspace F. For $A \in \mathbb{C}^{n\times n}$, P_A stands for the orthogonal projection onto $R(A)$. The symbol $\mathbb{C}_n^{CM}$ represents the subset of all $n \times n$ complex matrices sets with index 1.

Next, let's review the definitions of some generalized inverses. For $A \in \mathbb{C}^{m\times n}$, the Moore-Penrose inverse $A^{\dagger}$ of A is the unique matrix $X \in \mathbb{C}^{n\times m}$ satisfying the following four Penrose equations [1]:

$$AXA = A, \quad XAX = X, \quad (AX)^{*} = AX, \quad (XA)^{*} = XA.$$

The Moore-Penrose inverse can be used to represent orthogonal projectors $P_A := AA^{\dagger}$ onto $R(A)$ and $Q_A := A^{\dagger}A$ onto $R(A^{*})$, respectively. A matrix $X \in \mathbb{C}^{n\times m}$ that satisfies the equality $AXA = A$ is called an inner inverse or $\{1\}$-inverse of A, and a matrix $X \in \mathbb{C}^{n\times m}$ that satisfies the equality $XAX = X$ is called an outer inverse or $\{2\}$-inverse of A.

The Drazin inverse is a kind of outer inverse defined for square matrices. For $A \in \mathbb{C}^{n\times n}$ and $\mathrm{Ind}(A) = k$, the Drazin inverse A^D of A is the unique matrix $X \in \mathbb{C}^{n\times n}$ satisfying the following three equations [13]:

$$A^{k+1}X = A^k, \qquad XAX = X, \qquad AX = XA.$$

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* Corresponding author: Hongwei Jin
Email addresses: jiaxuanYao@126.com (Jiaxuan Yao), jhw_math@126.com (Hongwei Jin), xiaojiliu72@126.com (Xiaoji Liu)

In particular, if $\text{Ind}(A) = 1$, $A^D = A^\#$ is the group inverse of A.

For $A \in \mathbb{C}^{n\times n}$ with $\text{Ind}(A) = k$, the core-EP inverse $A^{\textcircled{\dagger}}$ of A is the unique matrix $X \in \mathbb{C}^{n\times n}$ satisfying the following conditions [12]:

$$XAX = X, \qquad R(A^k) = R(X) = R(X^*).$$

Obviously, the core-EP inverse is an outer inverse of A. Recall that, by [6], the core-EP inverse can be expressed as $A^{\textcircled{\dagger}} = A^D A^k (A^k)^\dagger$.

The weak group inverse is proposed by Wang and Chen [15] for square matrices of an arbitrary index as an extension of the group inverse. For $A \in \mathbb{C}^{m\times n}$, the weak group inverse $A^{\textcircled{w}}$ of A is the uniquely determined matrix that satisfying:

$$AX^2 = X, \qquad AX = A^{\textcircled{\dagger}}A.$$

Notice that, by [15], we have$A^{\textcircled{w}} = (A^{\textcircled{\dagger}})^2 A$. Two new generalized inverses have emerged by combining Moore-Penrose inverse and the weak group inverse, which are the weak core inverse (WCI) $A^{\textcircled{w},\dagger}$ and the dual weak core inverse (d-WCI) $A^{\dagger,\textcircled{w}}$, respectively [2]. Precisely, the weak core inverse of $A \in \mathbb{C}^{n\times n}$ presents a unique solution to the matrix system [2]:

$$XAX = X, \quad AX = CA^\dagger, \quad XA = A^D C,$$

where C is the weak core part of A with $C = AA^{\textcircled{w}}A$. Notice that $A^{\textcircled{w},\dagger} = A^{\textcircled{w}}AA^\dagger$ and $A^{\dagger,\textcircled{w}} = A^\dagger AA^{\textcircled{w}}$.

In [2], let $A\in\mathbb{C}^{n\times n}$ and $\text{Ind}(A) = k$. The weak core part C of A satisfies the following equations:

$$CA^k = A^{k+1}, \qquad C = A^{\textcircled{\dagger}}A^2, \qquad (I - AA^D)C = 0, \tag{1}$$

$$(I - AA^{\textcircled{\dagger}})C = (I - AA^{\textcircled{w}})C = 0, \qquad C(I - Q_A) = 0. \tag{2}$$

The DMP-inverse of $A \in \mathbb{C}^{n\times n}_k$, written by $A^{D,\dagger}$, was defined in [8] as the unique matrix $X \in \mathbb{C}^{n\times n}_k$ satisfying

$$XAX = X, \qquad XA = A^D A, \qquad A^k X = A^k A^\dagger.$$

Moreover, it was proved that $A^{D,\dagger} = A^D AA^\dagger$. Also, the dual DMP-inverse of A was introduced in [8], namely $A^{\dagger,D} = A^\dagger AA^D$.

D. Mosić in [9] introduced the Drazin-Star and the Star-Drazin matrices of a square matrix. Let $A \in \mathbb{C}^{n\times n}$ with $\text{Ind}(A) = k$. The Drazin-Star matrix of A (or Drazin-Star inverse of $(A^\dagger)^*$) is

$$A^{D,*} = A^D AA^*$$

which is the unique solution of the following equations:

$$X(A^\dagger)^* X = X, \qquad A^k X = A^k A^*, \qquad X(A^\dagger)^* = A^D A.$$

Recall that the Star-Drazin matrix of A (or Star-Drazin inverse of $(A^\dagger)^*$) is also defined in [9] as $A^{*,D} = A^*AA^D$. Inspired by this types of matrices, we will introduce the weak group-star matrix in this article.

First of all, let us review the core-EP decomposition. Wang gave the core-EP decomposition in the document [14]. Let $A\in\mathbb{C}^{n\times n}$ with $Ind(A) = k$, $rank(A^k) = p$. Then, one has $A = A_1 + A_2$, $A_1\in\mathbb{C}^{CM}_n$, where $A_2^k = 0$, $A_1^* A_2 = A_2 A_1 = 0$. Furthermore, there exists an unitary matrix $U \in \mathbb{C}^{n\times n}$ such that

$$A = U\begin{pmatrix} T & S \\ 0 & N \end{pmatrix}U^*, \; A_1 = U\begin{pmatrix} T & S \\ 0 & 0 \end{pmatrix}U^*, \; A_2 = U\begin{pmatrix} 0 & 0 \\ 0 & N \end{pmatrix}U^*, \tag{3}$$

where $T\in\mathbb{C}^{p\times p}$ is nonsingular and $S\in\mathbb{C}^{p\times(n-p)}$, $N\in\mathbb{C}^{(n-p)\times(n-p)}$ is nilpotent of index k, i.e., $N^k = 0$.

Lemma 1.1. *[4, 14, 16] Let* $A\in\mathbb{C}^{n\times n}_k$ *as in* (3). *Then*

(i) $A^{\dagger} = U\begin{pmatrix} T^{*}\triangle & -T^{*}\triangle SN^{\dagger} \\ (I_{n-p}-N^{\dagger}N)S^{*}\triangle & N^{\dagger}-(I_{n-p}-N^{\dagger}N)S^{*}\triangle SN^{\dagger} \end{pmatrix}U^{*}$,

(ii) $A^{\textcircled{\dagger}} = U\begin{pmatrix} T^{-1} & 0 \\ 0 & 0 \end{pmatrix}U^{*}$,

(iii) $A^{\textcircled{w}} = (A^{\textcircled{\dagger}})^{2}A = U\begin{pmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{pmatrix}U^{*}$,

(iv) $AA^{\dagger} = U\begin{pmatrix} I_p & 0 \\ 0 & NN^{\dagger} \end{pmatrix}U^{*}$,

(v) $A^{\dagger}A = \begin{pmatrix} T^{*}\triangle T & T^{*}\triangle S(I-NN^{\dagger}) \\ (I-NN^{\dagger})S^{*}\triangle T & (I-NN^{\dagger})S^{*}\triangle S(I-NN^{\dagger})+N^{\dagger}N \end{pmatrix}$,

where $\triangle = [TT^{*}+S(I_{n-p}-N^{\dagger}N)S^{*}]^{-1}$.

Lemma 1.2. *[7] Let* $A\in\mathbb{C}^{n\times n}$ *with rank* $r>0$. *Then there exists a unitary matrix* $U\in\mathbb{C}^{n\times n}$ *such that*

$$A = U\begin{pmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{pmatrix}U^{*}, \tag{4}$$

where $\Sigma = diag(\sigma_1 I_{r1}, \sigma_2 I_{r2}, \ldots, \sigma_t I_{rt})$ *is the diagonal matrix of singular values of A,* $\sigma_1 > \sigma_2 > \ldots, > \sigma_t > 0$, $r_1 + r_2 + \ldots + r_t = r$, *and* $K\in\mathbb{C}^{n\times n}$, $L\in\mathbb{C}^{n\times(n-r)}$ *satisfy* $KK^{*}+LL^{*}=I_r$.

Lemma 1.3. *Let* $A\in\mathbb{C}^{n\times n}$ *be a matrix written as in* (4). *Then,*

(i)*[3] the core-EP inverse of A is*

$$A^{\textcircled{\dagger}} = U\begin{pmatrix} (\Sigma K)^{\textcircled{\dagger}} & 0 \\ 0 & 0 \end{pmatrix}U^{*}.$$

(ii)*[2] the weak group inverse of A is*

$$A^{\textcircled{w}} = U\begin{pmatrix} ((\Sigma K)^{\textcircled{\dagger}})^{2}\Sigma K & ((\Sigma K)^{\textcircled{\dagger}})^{2}\Sigma L \\ 0 & 0 \end{pmatrix}U^{*} = U\begin{pmatrix} (\Sigma K)^{\textcircled{w}} & ((\Sigma K)^{\textcircled{\dagger}})^{2}\Sigma L \\ 0 & 0 \end{pmatrix}U^{*}.$$

The main structure of this paper is as follows. In Sect. 2, we introduce the weak group-star matrix. Then, we give some representations and characterizations of this type of the matrix. In Sect. 3, we develop the SMS method for finding the weak group-star matrix. In Sect. 4, the Cramer's rule for the solution of a singular equation $(A^{\dagger})^{*}x = b$ is presented. In Sect. 5, we study the perturbation of the weak group-star matrix. In Sect. 6, we give the application of the weak group-star matrix in solving linear equations.

2. Definition, characterizations and representations of the weak group-star Matrix

Theorem 2.1. *Let $A\in\mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A)=k$, C is the weak core part of A. Then, the system of equations*

$$X(A^{\dagger})^{*}X = X,\ AX = CA^{*},\ X(A^{\dagger})^{*} = A^{D}C \tag{5}$$

is consistent and its unique solution is $X = A^{D}CA^{*}$.

Proof. For $X = A^{D}CA^{*}$. In fact, (1) implies $AX = AA^{D}CA^{*} = CA^{*}$. On the other hand, (2) implies $X(A^{\dagger})^{*} = A^{D}CA^{*}(A^{\dagger})^{*} = A^{D}CA^{\dagger}A = A^{D}C$. Finally,

$$X(A^{\dagger})^{*}X = A^{D}CX = A^{D}AA^{\textcircled{W}}CA^{*} = A^{D}CA^{*} = X,$$

where the last equality follows by (2). Hence, $X = A^{D}CA^{*}$ satisfies the system of (5).

In order to show that system (5) has a unique solution, assume that both two matrices X_1 and X_2 satisfy (5), then

$$AX_1 = CA^{*} = AX_2,\ X_1(A^{\dagger})^{*} = A^{D}C = X_2(A^{\dagger})^{*}.$$

Thus, we can obtain

$$\begin{aligned} X_2 &= X_2(A^{\dagger})^{*}X_2 = A^{D}CX_2 = A^{D}AA^{\textcircled{W}}AX_2 \\ &= A^{D}AA^{\textcircled{W}}AX_1 = A^{D}CX_1 = X_1(A^{\dagger})^{*}X_1 = X_1, \end{aligned}$$

which implies that system (5) has the unique solution. □

Definition 2.2. *Let $A\in\mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A)=k$, and C be the weak core part of A. The weak group-star matrix of A (or the weak group-star inverse of $(A^{\dagger})^{*}$) denoted as $A^{\textcircled{W},*}$, is defined to be the solution of the system* (5).

Theorem 2.3. *Let $A\in\mathbb{C}^{n\times n}$ and $\mathrm{Ind}(A)=k$. Then,*

$$A^{\textcircled{W},*} = A^{\textcircled{W}}AA^{*}.$$

Proof. Since $R(A^{\textcircled{W}}) = R(A^{k})$, then $A^{\textcircled{W}} = A^{k}Z$, for some $Z\in\mathbb{C}^{n\times n}$. Thus, we have

$$A^{\textcircled{W},*} = A^{D}CA^{*} = A^{D}AA^{\textcircled{W}}AA^{*} = A^{D}AA^{k}ZAA^{*} = A^{k}ZAA^{*} = A^{\textcircled{W}}AA^{*}.$$

□

Remark 2.4. *Obviously, the weak group-star matrix is named based on the expressions whom are defined. In general, the weak group-star matrix are not generalized inverses of a given matrix A, but they are outer inverses of $(A^{\dagger})^{*}$.*

We observe that the weak group-star matrix provide new classes of square matrices by the following example, because they are different from each of the Moore-Penrose inverse, the weak group inverse, the weak core inverse and the dual weak core inverse.

Example 2.5. *Let*

$$A = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then,

$$A^{\dagger} = \begin{pmatrix} 2/3 & -1/3 & 2/3 & 0 \\ -1/3 & 2/3 & -1/3 & 0 \\ 1/3 & 1/3 & 1/3 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},\ A^{\textcircled{W}} = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A^{\textcircled{W},\dagger} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ A^{\dagger,\textcircled{W}} = \begin{pmatrix} 2/3 & -1/3 & 1/3 & -2/3 \\ -1/3 & 2/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 2/3 & -1/3 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ A^{\textcircled{W},*} = \begin{pmatrix} 2 & 2 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

In the next example, we show that the weak group-star inverse of $(A^{\dagger})^*$ is different from each of the Moore-Penrose inverse, the weak group inverse, the weak core inverse and the dual weak core inverse of $(A^{\dagger})^*$. Note that the weak group-star inverse present new classes of generalized inverse.

Example 2.6. *Let*

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is easy to check that $\mathrm{Ind}(A) = 2$. *We can obtain that the Moore-Penrose inverse, the Weak group inverse and the core EP inverse are*

$$A^{\dagger} = \begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},\ A^{ⓦ} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ A^{\textcircled{\dagger}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We also have

$$(A^{\dagger})^* = \begin{pmatrix} 1/2 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ [(A^{\dagger})^*]^{\dagger} = A^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix},$$

$$[(A^{\dagger})^*]^{ⓦ} = \begin{pmatrix} 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ [(A^{\dagger})^*]^{\textcircled{\dagger}} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$[(A^{\dagger})^*]^{ⓦ,\dagger} = \begin{pmatrix} 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ [(A^{\dagger})^*]^{\dagger,ⓦ} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ [(A^{\dagger})^*]^{ⓦ,*} = \begin{pmatrix} 1 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Theorem 2.7. *Let* $A \in \mathbb{C}^{n\times n}$ *be a matrix written as in* (3). *Then*

$$A^{ⓦ,*} = U\begin{pmatrix} T^* + (T^{-1}S + T^{-2}SN)S^* & (T^{-1}S + T^{-2}SN)N^* \\ 0 & 0 \end{pmatrix}U^*. \tag{6}$$

Proof. From Lemma 1.1, we can obtain

$$\begin{aligned} A^{ⓦ,*} = A^{ⓦ}AA^* &= U\begin{pmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{pmatrix}\begin{pmatrix} T & S \\ 0 & N \end{pmatrix}\begin{pmatrix} T^* & 0 \\ S^* & N^* \end{pmatrix}U^* \\ &= U\begin{pmatrix} T^* + (T^{-1}S + T^{-2}SN)S^* & (T^{-1}S + T^{-2}SN)N^* \\ 0 & 0 \end{pmatrix}U^*. \end{aligned}$$

□

Corollary 2.8. *Let* $A \in \mathbb{C}^{n\times n}$ *be a matrix written as in* (3). *Then*

$$AA^{ⓦ,*} = U\begin{pmatrix} F_1 & F_2 \\ 0 & 0 \end{pmatrix}U^*, \tag{7}$$

where $F_1 = TT^* + (S + T^{-1}SN)S^*$, $F_2 = (S + T^{-1}SN)N^*$. *Besides,*

$$A^{ⓦ,*}A = U\begin{pmatrix} F_3 & F_4 \\ 0 & 0 \end{pmatrix}U^*,$$

where $F_3 = T^*T + (T^{-1}S + T^{-2}SN)S^*T$, $F_4 = T^*S + (T^{-1}S + T^{-2}SN)S^*S + (T^{-1}S + T^{-2}SN)N^*N$.

Remark 2.9. *Let* $A\in\mathbb{C}^{n\times n}$ *as in* (3) *and with* $\mathrm{Ind}(A) = k$. *We can obtain* $A^{\textcircled{W}} = A^{\#}$ *if and only if* $A \in \mathbb{C}_n^{CM}$, *i.e.,* $N = 0$.

Lemma 2.10. *If* $A\in\mathbb{C}^{n\times n}$ *with* $\mathrm{Ind}(A) = k$. *Then* $R(A^{\textcircled{W},*}) = R(A^k)$.

Proof. In fact, according to Theorem 2.1, we have

$$R(A^{\textcircled{W},*}) \subseteq R(A^{\textcircled{W},*}(A^{\dagger})^*) = R(A^D C) \subseteq R(A^D) = R(A^k).$$

On the other hand, $R(A^k) \subseteq R(A^{\textcircled{W},*})$. By Theorem 2.1, we can see

$$R(A^k) \subseteq R(A^D) \subseteq R(A^D C) = R(A^{\textcircled{W},*}(A^{\dagger})^*) \subseteq R(A^{\textcircled{W},*}).$$

Hence, $R(A^k) = R(A^{\textcircled{W},*})$. □

Lemma 2.11. *[5] Let* $A\in\mathbb{C}^{n\times n}$. *Then, the following statements hold.*

(*i*) $AA^{\textcircled{W}} = P_{R(A^k),N((A^k)^*A)}$,

(*ii*) $A^{\textcircled{W}}A = P_{R(A^k),N((A^k)^*A^2)}$.

According Theorem 2.1 and Lemma 2.10, we can obtain Lemma 2.12.

Lemma 2.12. *[11] Let* $A\in\mathbb{C}^{n\times n}$ *be such that* $\mathrm{Ind}(A) = k$. *Then*

(*i*) $A^{\textcircled{W},*} = [(A^{\dagger})^*]^{(2)}_{R(A^k),N(A^k)^*}$,

(*ii*) $(A^{\dagger})^*)A^{\textcircled{W},*}$ *is a projector on* $R((A^{\dagger})^*)A^{\textcircled{W}})$ *along* $N((A^k)^*A^2A^*)$,

(*iii*) $A^{\textcircled{W},*}(A^{\dagger})^*$ *is a projector on* $R(A^k)$ *along* $N((A^k)^*A^2)$.

Corollary 2.13. *Let* $A\in\mathbb{C}^{n\times n}$ *with* $\mathrm{Ind}(A) = k$. *For* $l \geq k$,

$$A^{\textcircled{W},*} = A^l(A^{l+2})^{\dagger}A^2A^*. \tag{8}$$

Proof. According to [10], it follows $A^{\textcircled{W}} = A^l(A^{l+2})^{\dagger}A$. By the corresponding Theorem 2.3, we get the equality (8).

Theorem 2.14. *Let* $A\in\mathbb{C}^{n\times n}$ *be a matrix written as in* (4). *Then*

$$A^{\textcircled{W},*} = U\begin{pmatrix} (\Sigma K)^{\textcircled{W}}\Sigma\Sigma^* & 0 \\ 0 & 0 \end{pmatrix}U^*.$$

Proof. From Lemma 1.3, we can obtain

$$\begin{aligned} A^{\textcircled{W},*} = A^{\textcircled{W}}AA^* = (A^{\textcircled{\dagger}})^2A^2A^* &= U\begin{pmatrix} (\Sigma K)^{\textcircled{W}}\Sigma KK^*\Sigma^* + (\Sigma K)^{\textcircled{W}}\Sigma LL^*\Sigma^* & 0 \\ 0 & 0 \end{pmatrix}U^* \\ &= U\begin{pmatrix} (\Sigma K)^{\textcircled{W}}\Sigma(KK^* + LL^*)\Sigma^* & 0 \\ 0 & 0 \end{pmatrix}U^*. \end{aligned}$$

□

Theorem 2.15. *Let* $A\in\mathbb{C}^{n\times n}$ *with* $\mathrm{Ind}(A) = k$ *and C is the weak core part of A. Then, the following statements are equivalent:*

(*i*) $X\in\mathbb{C}^{n\times n}$ *is the weak group-star matrix of A.*

(*ii*) *X satisfies equations*

$$X(A^{\dagger})^{*}X = X,\ AX = CA^{*},\ X(A^{\dagger})^{*} = A^{D}C.$$

(*iii*) *X satisfies equations*

$$A^{\textcircled{W}}AX = X,\ AX = CA^{*}.$$

(*iv*) *X satisfies equations*

$$AX(A^{\dagger})^{*} = C,\ A^{\textcircled{W}}AXAA^{\dagger} = X,\ (A^{\dagger})^{*}X(A^{\dagger})^{*} = (A^{\dagger})^{*}A^{\textcircled{W}}A.$$

(*v*) *X satisfies equations*

$$XAA^{\dagger} = X,\ X(A^{\dagger})^{*} = A^{\textcircled{W}}A,\ X(A^{\dagger})^{*}A^{\dagger} = A^{\textcircled{W},\dagger},\ XA = A^{\textcircled{W}}AA^{*}A.$$

(*vi*) *X satisfies equations*

$$X(A^{\dagger})^{*}A^{\textcircled{W}}AA^{*} = X,\ X(A^{\dagger})^{*}A^{\textcircled{W}}AX = X,\ (A^{\dagger})^{*}A^{\textcircled{W}}AX = (A^{\dagger})^{*}A^{\textcircled{W}}AA^{*}.$$

(*vii*) *X satisfies equations*

$$(A^{\dagger})^{*}A^{\textcircled{W}}AX(A^{\dagger})^{*}A^{\textcircled{W}}A = (A^{\dagger})^{*}A^{\textcircled{W}}A,\ X(A^{\dagger})^{*}A^{\textcircled{W}}A = A^{\textcircled{W}}A.$$

Proof. (*i*) ⇒ (*ii*): By Theorem 2.1, the proof is clear.

(*ii*) ⇒ (*iii*): Using $AX = CA^{*}$, we can obtain

$$A^{\textcircled{W}}AX = ACA^{*} = X.$$

(*iii*) ⇒ (*i*): The hypothesis $A^{\textcircled{W}}AX = X$, $AX = CA^{*}$ imply

$$X = A^{\textcircled{W}}AX = A^{\textcircled{W}}CA^{*} = A^{\textcircled{W}}AA^{\textcircled{W}}AA^{*} = A^{\textcircled{W}}AA^{*} = X.$$

(*i*) ⇒ (*iv*): Since $X = A^{\textcircled{W}}AA^{*}$ and by (2), it follows that

$$AX(A^{\dagger})^{*} = AA^{\textcircled{W}}AA^{*}(A^{\dagger})^{*} = CAA^{\dagger} = C,$$

$$A^{\textcircled{W}}AXAA^{\dagger} = (A^{\textcircled{W}}AA^{\textcircled{W}})A(A^{*}AA^{\dagger}) = A^{\textcircled{W}}AA^{*} = X,$$

and

$$(A^{\dagger})^{*}X(A^{\dagger})^{*} = (A^{\dagger})^{*}A^{\textcircled{W}}AA^{*}(A^{\dagger})^{*} = (A^{\dagger})^{*}A^{\textcircled{W}}A(A^{\dagger}A)^{*} = (A^{\dagger})^{*}A^{\textcircled{W}}AA^{\dagger}A = (A^{\dagger})^{*}A^{\textcircled{W}}A.$$

(*iv*) ⇒ (*i*): By $A^{\textcircled{W}}AXAA^{\dagger} = X, AX = AA^{\textcircled{W}}AA^{*}$,we have

$$X = A^{\textcircled{W}}AXAA^{\dagger} = A^{\textcircled{W}}AA^{\textcircled{W}}AA^{*}AA^{\dagger} = A^{\textcircled{W}}AA^{*}AA^{\dagger} = A^{\textcircled{W}}AA^{*} = X.$$

The rest can be proved similarly according to the above method. □

By Lemma 2.12 and $A^{\textcircled{W},*} = A^{\textcircled{W}}AA^{*}$, we obtain

$$(A^{\dagger})^{*}A^{\textcircled{W},*} = P_{R((A^{\dagger})^{*}A^{\textcircled{W}},N((A^{k})^{*}A^{2}A^{*})},\ R(A^{\textcircled{W},*}) \subseteq R(A^{\textcircled{W}}) = R(A^{k}).$$

Then we can get Theorem 2.16.

Theorem 2.16. *[11] Let $A \in \mathbb{C}^{n\times n}$ with $Ind(A) = k$. Then, the matrix equation*

$$(A^{\dagger})^{*}X = P_{R((A^{\dagger})^{*}A^{\textcircled{W}}),N((A^{k})^{*}A^{2}A^{*})},\ R(X) \subseteq R(A^{k}) \tag{9}$$

is consistent and it has the unique solution $X = A^{\textcircled{W},*}$.

Lemma 2.17 can be checked by using the same method of [11]. Therefore, we omit the proof.

Lemma 2.17. *[11] Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{Ind}(A) = k$. Then,*

(*i*) $(A^\dagger)^* A^{Ⓦ,*}(A^\dagger)^* = (A^\dagger)^* \Leftrightarrow A^\dagger A A^{Ⓦ} A = A^\dagger A \Leftrightarrow AA^{Ⓦ}A = A \Leftrightarrow AA^{Ⓦ}AA^\dagger = AA^\dagger$;

(*ii*) $A^k A^{Ⓦ,*} A^k = A^k \Leftrightarrow A^k A^* A^k = A^k$;

(*iii*) $AA^{Ⓦ,*} = AA^{Ⓦ} \Leftrightarrow A^{Ⓦ,*} = A^{Ⓦ}$;

(*iv*) $A^{Ⓦ,*}A = AA^{Ⓦ} \Leftrightarrow A^{Ⓦ,*} = A^{Ⓦ,\dagger}$;

(*v*) $A^{Ⓦ,*}A = A^\dagger A \Leftrightarrow A^{Ⓦ,*} = A^\dagger$;

(*vi*) $AA^{Ⓦ,*} = AA^\dagger \Leftrightarrow AA^{Ⓦ,*}A = A$;

(*vii*) $A^{Ⓦ,*} = A^* \Leftrightarrow A^{Ⓦ,\dagger} = A^\dagger$.

Let $A \in \mathbb{C}^{n\times n}$ with $ind(A) = k$, then

$$A^{Ⓦ,*} = A^{Ⓦ}AA^* = U\begin{pmatrix} G_1 & G_2 \\ 0 & 0 \end{pmatrix}U^*,$$

where $G_1 = T^* + (T^{-1}S + T^{-2}SN)S^*$, $G_2 = (T^{-1}S + T^{-2}SN)N^*$.

Theorem 2.18. *Let $A \in \mathbb{C}^{n\times n}$ be a matrix with $\operatorname{Ind}(A) = k$ written as in* (3)*. Then*

(*i*) $A^{Ⓦ,*}A = A^*A \Leftrightarrow$ *A is a symmetrical and EP matrix.*

(*ii*) $AA^{Ⓦ,*} = AA^{*,Ⓦ} \Leftrightarrow S + T^{-1}SN = (TT^* + SS^*)T^{-1}S,\ NS^* = 0$.

Proof.

(i)

$$\begin{aligned} A^{Ⓦ,*}A = A^*A &\Leftrightarrow \begin{pmatrix} G_1T & G_1S + G_2N \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} T^*T & T^*S \\ S^*T & S^*S + N^*N \end{pmatrix} \\ &\Leftrightarrow T^*T + (T^{-1}S + T^{-2}SN)S^*T = T^*T, \\ &T^*S + (T^{-1}S + T^{-2}SN)S^*S + (T^{-1}S + T^{-2}SN)N^*N = T^*S,\ S^*T = 0,\ S^*S + N^*N = 0. \\ &\Leftrightarrow S = 0,\ N = 0. \\ &\Leftrightarrow\ A \text{ is a symmetrical and EP matrix.} \end{aligned}$$

(ii)

$$\begin{aligned} AA^{Ⓦ,*} = AA^{*,Ⓦ} &\Leftrightarrow \begin{pmatrix} TG_1 & TG_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} TT^* + SS^* & TT^*T^{-1}S + SS^*T^{-1}S \\ NS^* & NS^*T^{-1}S \end{pmatrix} \\ &\Leftrightarrow TT^* + (S + T^{-1}SN)S^* = TT^* + SS^*,\ NS^* = 0,\ T(T^{-1}S + T^{-2}SN) = TT^*T^{-1}S + SS^*T^{-1}S. \\ &\Leftrightarrow S + T^{-1}SN = (TT^* + SS^*)T^{-1}S,\ NS^* = 0. \ \square \end{aligned}$$

Theorem 2.19. *Let $A \in \mathbb{C}^{n\times n}$ be a matrix with $\operatorname{Ind}(A) = k$ written as in* (3)*. Then*

(*i*) $A^{Ⓦ,*} = A \Leftrightarrow$ *A is a symmetrical and EP matrix.*

(*ii*) $A^{Ⓦ,*} = A^* \Leftrightarrow$ *A is an EP matrix.*

(*iii*) $A^{Ⓦ,*} = AA^\dagger \Leftrightarrow TT^* + SS^* = T,\ N = 0$.

(*iv*) $A^{Ⓦ,*} = A^{*,Ⓦ} \Leftrightarrow S = 0$.

Proof.
(i)

$$\begin{aligned} A^{\textcircled{w},*} = A &\Leftrightarrow \begin{pmatrix} G_1 & G_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} T & S \\ 0 & N \end{pmatrix} \\ &\Leftrightarrow T^* + (T^{-1}S + T^{-2}SN)S^* = T,\ (T^{-1}S + T^{-2}SN)N^* = S \text{ and } N = 0. \\ &\Leftrightarrow T = T^*,\ S = 0,\ N = 0. \\ &\Leftrightarrow A \text{ is a symmetrical and EP matrix}. \end{aligned}$$

(ii)

$$\begin{aligned} A^{\textcircled{w},*} = A^* &\Leftrightarrow \begin{pmatrix} G_1 & G_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} T^* & 0 \\ S^* & N^* \end{pmatrix} \\ &\Leftrightarrow T^* + (T^{-1}S + T^{-2}SN)S^* = T^*,\ (T^{-1}S + T^{-2}SN)N^* = 0,\ S^* = 0 \text{ and } N^* = 0. \\ &\Leftrightarrow S = 0,\ N = 0. \\ &\Leftrightarrow A \text{ is an EP matrix}. \end{aligned}$$

(iii)

$$\begin{aligned} A^{\textcircled{w},*} = AA^{\dagger} &\Leftrightarrow \begin{pmatrix} G_1 & G_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & NN^{\dagger} \end{pmatrix} \\ &\Leftrightarrow T^* + (T^{-1}S + T^{-2}SN)S^* = I,\ (T^{-1}S + T^{-2}SN)N^* = 0 \text{ and } NN^{\dagger} = 0. \\ &\Leftrightarrow TT^* + SS^* = T,\ N = 0. \end{aligned}$$

(iv)

$$\begin{aligned} A^{\textcircled{w},*} = A^{*,\textcircled{w}} &\Leftrightarrow \begin{pmatrix} G_1 & G_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} T^* & T^*T^{-1}S \\ S^* & S^*T^{-1}S \end{pmatrix} \\ &\Leftrightarrow T^* + (T^{-1}S + T^{-2}SN)S^* = T^*,\ (T^{-1}S + T^{-2}SN)N^* = T^*T^{-1}S,\ S^* = 0, \text{ and } S^*T^{-1}S = 0. \\ &\Leftrightarrow S = 0. \ \square \end{aligned}$$

Theorem 2.20. *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A) = 1$. Then, the following statements are equivalent:*

(*i*) *A is a partial isometry and A is an EP matrix.*

(*ii*) $AA^{\textcircled{w},*} = AA^{\dagger}$.

(*iii*) $A^{\textcircled{w},*}A = AA^{\dagger}$.

(*iv*) $AA^{\textcircled{w},*} = A^{\dagger}A$.

(*v*) $A^{\textcircled{w},*}A = A^{\dagger}A$.

Proof.
(*i*) ⇔ (*ii*)

$$\begin{aligned} AA^{\textcircled{w},*} = AA^{\dagger} &\Leftrightarrow \begin{pmatrix} TG_1 & TG_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & NN^{\dagger} \end{pmatrix} \\ &\Leftrightarrow TT^* + (S + T^{-1}SN)S^* = I,\ (S + T^{-1}SN)N^* = S \text{ and } NN^{\dagger} = 0. \\ &\Leftrightarrow TT^* = I,\ N = 0,\ (S + T^{-1}SN)N^* = S = 0. \\ &\Leftrightarrow TT^* = I,\ S = 0,\ N = 0. \end{aligned} \tag{10}$$

$(i) \Leftrightarrow (iii)$

$$A^{\textcircled{w},*}A = AA^{\dagger} \Leftrightarrow \begin{pmatrix} G_1T & G_1S + G_2N \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & NN^{\dagger} \end{pmatrix}$$
$$\Leftrightarrow T^*T + (T^{-1}S + T^{-2}SN)S^*T = I,\ T^*S + (T^{-1}S + T^{-2}SN)S^*S + (T^{-1}S + T^{-2}SN)N^*N = 0 \text{ and } NN^{\dagger} = 0.$$
$$\Leftrightarrow N = 0,\ (TT^* + SS^*)S = 0 \text{ and } (TT^* + SS^*)T = T.$$
$$\Leftrightarrow TT^* = I,\ S = 0,\ N = 0.$$

$(i) \Leftrightarrow (iv)$

$$AA^{\textcircled{w},*} = A^{\dagger}A \Leftrightarrow \begin{pmatrix} TG_1 & TG_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} T^* \triangle T & T^* \triangle S(I - NN^{\dagger}) \\ (I - NN^{\dagger})S^* \triangle T & (I - NN^{\dagger})S^* \triangle S(I - NN^{\dagger}) + N^{\dagger}N \end{pmatrix}.$$
$$\Leftrightarrow T^* \triangle T = TT^* + (S + T^{-1}SN)S^*,\ T^* \triangle S(I - NN^{\dagger}) = (S + T^{-1}SN)N^*,\ S = SNN^{\dagger},\ N^{\dagger}N = 0.$$
$$\Leftrightarrow T^* \triangle T = TT^*,\ S = 0, \text{ and } N = 0.$$
$$\Leftrightarrow TT^* = I,\ S = 0 \text{ and } N = 0.$$

$(i) \Leftrightarrow (v)$

$$A^{\textcircled{w},*}A = A^{\dagger}A \Leftrightarrow \begin{pmatrix} G_1T & G_1S + G_2N \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} T^* \triangle T & T^* \triangle S(I - NN^{\dagger}) \\ (I - NN^{\dagger})S^* \triangle T & (I - NN^{\dagger})S^* \triangle S(I - NN^{\dagger}) + N^{\dagger}N \end{pmatrix}$$
$$\Leftrightarrow T^* \triangle T = T^*T + (T^{-1}S + T^{-2}SN)S^*T,\ S = SNN^{\dagger},\ N^{\dagger}N = 0,$$
$$T^* \triangle S(I - NN^{\dagger}) = T^*S + (T^{-1}S + T^{-2}SN)S^*S + (T^{-1}S + T^{-2}SN)N^*N.$$
$$\Leftrightarrow T^* \triangle T = T^*T,\ N = 0 \text{ and } S = SNN^{\dagger} = 0.$$
$$\Leftrightarrow T^*T = I,\ S = 0 \text{ and } N = 0.$$

Therefore, the above conditions are equivalent. □

Definition 2.21. *Let $A, B \in \mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A) = k$. We call A is below B under the relation $\leqslant^{\textcircled{w},*}$ if*

$$AA^{\textcircled{w},*} = BA^{\textcircled{w},*} \text{ and } A^{\textcircled{w},*}A = A^{\textcircled{w},*}B.$$

Naturally, we will consider whether this binary relationship can become a partial order. The answer to this question is No. A binary relation is called a partial order if it is reflexive, transitive, and anti-symmetric on a non-empty set. Next, we give a concrete example to prove that this relationship is not satisfied antisymmetry.

Example 2.22. *Consider the matrices*

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since

$$A^{\textcircled{w}} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ B^{\textcircled{w}} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

we can get

$$A^{\textcircled{w},*}A = \begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ A^{\textcircled{w},*}B = \begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$AA^{ⓦ,*} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ BA^{ⓦ,*} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$BB^{ⓦ,*} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ AB^{ⓦ,*} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B^{ⓦ,*}B = \begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ B^{ⓦ,*}A = \begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus,

$$AA^{ⓦ,*} = BA^{ⓦ,*}, A^{ⓦ,*}A = A^{ⓦ,*}B,$$
$$AB^{ⓦ,*} = BB^{ⓦ,*}, B^{ⓦ,*}B = B^{ⓦ,*}A.$$

Clearly, $A \leqslant^{ⓦ,*} B$ and $B \leqslant^{ⓦ,*} A$ hold, but $A \neq B$. Hence, the weak group-star relation can not be a partial order.

3. Successive matrix squaring algorithm for the weak group-star matrix

In this section, we give successive matrix squaring algorithms for computing the weak group-star matrix. The development of the SMS iterations start from the transformations.

Since

$$\begin{aligned}(A^{k+2})^{\dagger}A(AA^{ⓦ,*}) &= (A^{k+2})^{\dagger}A^2A^k(A^{k+2})^{\dagger}A^2A^* \\ &= (A^{k+2})^{\dagger}A^{k+2}(A^{k+2})^{\dagger}A^2A^* = (A^{k+2})^{\dagger}A^2A^*,\end{aligned}$$

we have

$$\begin{aligned}A^{ⓦ,*} &= A^{ⓦ,*} - \beta((A^{k+2})^{\dagger}A(AA^{ⓦ,*}) - (A^{k+2})^{\dagger}A^2A^*) \\ &= (I - \beta(A^{k+2})^{\dagger}A^2)A^{ⓦ,*} + \beta(A^{k+2})^{\dagger}A^2A^*.\end{aligned}$$

Observe the following matrices

$$P = I - \beta(A^{k+2})^{\dagger}A^2, \ \ Q = \beta(A^{k+2})^{\dagger}A^2A^*, \ \ \beta > 0.$$

It is obvious that $A^{ⓦ,*}$ is the unique solution of $X = PX + Q$. Then an iterative procedure for computing the weak group-star matrix $A^{ⓦ,*}$ can be defined as follows

$$X_1 = Q, \ X_{m+1} = PX_m + Q. \tag{11}$$

This algorithm can be implemented in parallel by considering the block matrix

$$T = \begin{pmatrix} P & Q \\ 0 & I \end{pmatrix}, \quad T^m = \begin{pmatrix} P^m & \Sigma_{i=0}^{m-1}P^iQ \\ 0 & I \end{pmatrix}.$$

The top right block of T^m is X^m, the mth approximation to $A^{ⓦ,*}$. The matrix power T^m can be computed by the successive squaring, i.e.

$$T_0 = T, \ T_{i+1} = T_i^2, \ i = 0, 1, \ldots, j,$$

where the integer j is such that $2^j \geq m$. The following theorem gives the sufficient condition for the convergence of the iterative process (11).

Theorem 3.1. *Let $A\in\mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A) = k$ and rank$(A^k) = r$. Then the approximation*

$$X_{2^m} = \sum_{i=0}^{2^m-1} (I - \beta(A^{k+2})^{\dagger}A^2)^i \beta(A^{k+2})^{\dagger}A^2A^*,$$

defined by the iterative process (11) *converges to the weak group-star matrix $A^{Ⓦ,*}$ if the spectral radius $\rho(I-X_1(A^{\dagger})^*) \leq 1$. Moreover, the following error estimation holds:*

$$\left\|A^{Ⓦ,*} - X_{2^m}\right\| \leq \left\|(I - X_1(A^{\dagger})^*)^{2^m}\right\|.$$

As a result,

$$\lim_{m\to\infty} sup \sqrt[2^m]{\left\|A^{Ⓦ,*} - X_{2^m}\right\|} \leq (I - X_1(A^{\dagger})^*).$$

Proof. We know that

$$A^{Ⓦ,*}(A^{\dagger})^*A^{Ⓦ,*} = A^{Ⓦ,*}, \quad X_{2^m}(A^{\dagger})^*A^{Ⓦ,*} = X_{2^m}.$$

By the mathematical induction, we can get

$$I - X_{2^m}(A^{\dagger})^* = (I - X_1(A^{\dagger})^*)^{2^m}.$$

Therefore,

$$\begin{aligned} \left\|A^{Ⓦ,*} - X_{2^m}\right\| &= \left\|A^{Ⓦ,*} - X_{2^m}(A^{\dagger})^*A^{Ⓦ,*}\right\| \\ &= \left\|(I - X_{2^m}(A^{\dagger})^*)A^{Ⓦ,*}\right\| \\ &\leq \left\|A^{Ⓦ,*}\right\| \left\|I - X_{2^m}(A^{\dagger})^*\right\| \\ &= \left\|A^{Ⓦ,*}\right\| \left\|(I - X_1(A^{\dagger})^*)^{2^m}\right\|, \end{aligned}$$

and

$$\begin{aligned} \lim_{m\to\infty} sup \sqrt[2^m]{\left\|A^{Ⓦ,*} - X_{2^m}\right\|} &\leq \lim_{m\to\infty} sup \sqrt[2^m]{\left\|A^{Ⓦ,*}\right\| \left\|(I - X_1(A^{\dagger})^*)^{2^m}\right\|} \\ &= \rho(I - X_1(A^{\dagger})^*). \end{aligned}$$

In the last equality, we use the fact that $\lim_{m\to\infty} \|B^n\|^{1/n} = \rho(B)$, for any square matrix B.

If β is a real parameter such that $\max\limits_{1\leq i\leq t} \left|1 - \beta\lambda_i\right| < 1$, where λ_i $(i = 1, 2, \ldots, s)$ are the nonzero eigenvalues of $(A^{k+2})^{\dagger}A^2A^*$, then

$$\rho(I - X_1(A^{\dagger})^*) = \rho(I - \beta(A^{k+2})^{\dagger}A^2) \leq 1.$$

It completes the proof. □

Example 3.2. *Consider the following matrix:*

$$A = \begin{pmatrix} 0 & 4/3 & -1/3 \\ -1/3 & 1 & -1/3 \\ -2/3 & -2/3 & 0 \end{pmatrix}, \mathrm{Ind}(A) = 2.$$

Let

$$P = I - \beta(A^4)^{\dagger}A^2, \;\; Q = \beta(A^4)^{\dagger}A^2A^*, \beta = 0.6.$$

The eigenvalues λ_i of QA are included in the set $\{0, 0, 0.5\}$. The nonzero eigenvalues λ_i satisfy

$$\max_i |1 - \lambda_i| = |1 - 0.5| = 0.5 < 1.$$

Then we obtain the satisfactory approximation for $A^{\textcircled{w},*}$ after the 6th iteration of the successive matrix squaring algorithm.

$$(T^2)^6 \approx \begin{pmatrix} 0.982 & 0.130 & -0.037 & -0.185 & -0.148 & 0.074 \\ 0.130 & 0.093 & 0.026 & 1.300 & 1.037 & -0.519 \\ -0.031 & 0.218 & 0.938 & -0.311 & -0.249 & 0.125 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The upper right corner of $(T^2)^6$ is an approximation of the weak group-star matrix, that is

$$A^{\textcircled{w},*} = \begin{pmatrix} -0.185 & -0.148 & 0.074 \\ 1.300 & 1.037 & -0.519 \\ -0.311 & -0.249 & 0.125 \end{pmatrix}.$$

4. The Cramer's rule for the solution of a singular equation $(A^\dagger)^* x = b$

Since $R(A^{\textcircled{w},*}) = R(A^k) \subseteq N(V)$, we obtain $VA^{\textcircled{w},*} = 0$. By $R(I - AA^{\textcircled{w},*}) \subseteq R(U) = R(UU^\dagger) = N(I - UU^\dagger)$, we can obtain $I - AA^{\textcircled{w},*} = UU^\dagger(I - AA^{\textcircled{w},*})$. Then, we get Theorem 4.1.

Theorem 4.1. *[11] Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{Ind}(A) = k$. Suppose $U \in \mathbb{C}^{n\times r}$ and $V^* \in \mathbb{C}^{n\times r}$ having full column rank such that*

$$R(I - AA^{\textcircled{w},*}) \subseteq R(U) \subseteq N(A^{\textcircled{w},*}), \text{ and } R(A^k) \subseteq N(V).$$

Then, the bordered matrix

$$X = \begin{pmatrix} A & U \\ V & 0 \end{pmatrix}$$

is nonsingular and

$$X^{-1} = \begin{pmatrix} A^{\textcircled{w},*} & (I - A^{\textcircled{w},*}A)V^\dagger \\ U^\dagger(I - AA^{\textcircled{w},*}) & -U^\dagger(A - AA^{\textcircled{w},*}A)V^\dagger \end{pmatrix}. \tag{12}$$

Similarly, we can get the following result.

Theorem 4.2. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{Ind}(A) = k$. Suppose $U \in \mathbb{C}^{n\times r}$ and $V^* \in \mathbb{C}^{n\times r}$ having full column rank such that*

$$R(A^k) = N(V),\ R(U) = N(A^k)^*.$$

Then the bordered matrix

$$X = \begin{pmatrix} (A^\dagger)^* & U \\ V & 0 \end{pmatrix}$$

is nonsingular and

$$X^{-1} = \begin{pmatrix} A^{\textcircled{w},*} & (I - A^{\textcircled{w}}A)V^\dagger \\ U^\dagger(I - (A^\dagger)^*A^{\textcircled{w},*}) & -U^\dagger((A^\dagger)^* - (A^\dagger)^*A^{\textcircled{w}}A)V^\dagger \end{pmatrix}. \tag{13}$$

Since $B \in R((A^\dagger)^*A^{\textcircled{w}})$, we have $B = (A^\dagger)^*A^{\textcircled{w}}Z$, for some $Z \in \mathbb{C}^{n\times n}$. If $X = A^{\textcircled{w},*}B$, we obtain

$$(A^\dagger)^*X = (A^\dagger)^*A^{\textcircled{w},*}B = (A^\dagger)^*A^{\textcircled{w}}AA^*(A^\dagger)^*A^{\textcircled{w}}Z = (A^\dagger)^*A^{\textcircled{w}}Z = B.$$

Then we can get the following theorem.

Theorem 4.3. *[11] Let* $A \in \mathbb{C}^{n\times n}$ *with* $\mathrm{Ind}(A) = k$*, and* $B \in R((A^{\dagger})^{*}A^{\textcircled{w}})$*. Then*

$$(A^{\dagger})^{*}X = B \tag{14}$$

in $R(A^{k})$ *has the unique solution* $X = A^{\textcircled{w},*}B$*.*

Similar to the Theorem 4.3, we can prove the following theorem.

Theorem 4.4. *Let* $A \in \mathbb{C}^{n\times n}$ *with* $\mathrm{Ind}(A) = k$ *and* $B \in R(AA^{\textcircled{w}})$*. Then* $A^{*}B$ *is the unique solution in* $R(A^{*}(A^{k})^{*}A^{2})$ *of* $(A^{\dagger})^{*}X = B$*.*

Using the relationship between the weak group-star inverse of $(A^{\dagger})^{*}$ and a nonsingular bordered matrix, we give the Cramer's rule for solving a singular linear equation $(A^{\dagger})^{*}x = B$. $(A^{\dagger})^{*}(ij \to b_j)$ denotes the matrix obtained by replacing ith column of $(A^{\dagger})^{*}$ with b_j, where b_j is the jth column of B.

Theorem 4.5. *Let* $A, B \in \mathbb{C}^{n\times n}$ *with* $\mathrm{Ind}(A) = k$*. Suppose* $U \in \mathbb{C}^{n\times r}$ *and* $V^{*} \in \mathbb{C}^{n\times r}$ *having full column rank such that*

$$R(A^{\textcircled{w},*}) = R(A^{k}) = N(V), \text{ and } R(U) = N(A^{\textcircled{w},*}).$$

If $R(B) \subseteq R((A^{\dagger})^{*}A^{\textcircled{w}})$*, then the unique solution* $X = A^{\textcircled{w},*}B$ *of the singular linear equation* (14) *is given by*

$$x_{ij} = \frac{det\begin{pmatrix} (A^{\dagger})^{*}(i \to b_j) & U \\ V(i \to 0) & 0 \end{pmatrix}}{det\begin{pmatrix} (A^{\dagger})^{*} & U \\ V & 0 \end{pmatrix}},\ i = 1, 2, \ldots n,\ j = 1, 2, \ldots n. \tag{15}$$

Proof. Since $X = A^{\textcircled{w},*}B \in R(A^{k}) = N(V)$ and $B \in R((A^{\dagger})^{*}A^{\textcircled{w}}) = AR(A^{k})$, we have

$$VX = 0,\ (I - AA^{\textcircled{w},*})B = 0. \tag{16}$$

It follows from (16) that the solution of $(A^{\dagger})^{*}X = B$ satisfies

$$\begin{pmatrix} (A^{\dagger})^{*} & U \\ V & 0 \end{pmatrix}\begin{pmatrix} X \\ 0 \end{pmatrix} = \begin{pmatrix} B \\ 0 \end{pmatrix}. \tag{17}$$

By Theorem 4.2, the coefficient matrix of (17) is nonsingular. Using (13) and (16), we can obtain

$$\begin{pmatrix} X \\ 0 \end{pmatrix} = \begin{pmatrix} A^{\textcircled{w},*} & (I - A^{\textcircled{w}}A)V^{\dagger} \\ U^{\dagger}(I - (A^{\dagger})^{*}A^{\textcircled{w}}) & -U^{\dagger}((A^{\dagger})^{*} - (A^{\dagger})^{*}A^{\textcircled{w}}A)V^{\dagger} \end{pmatrix}\begin{pmatrix} B \\ 0 \end{pmatrix} = \begin{pmatrix} A^{\textcircled{w},*}B \\ 0 \end{pmatrix}.$$

Therefore, $x = A^{\textcircled{w},*}B$ and (15) follows from the classical Cramer's rule [13]. □

5. Perturbations of the weak group-star matrix

Using the form of the core-EP decomposition of $A^{\textcircled{w},*}$, we can calculate the perturbation of $A^{\textcircled{w},*}$.

Theorem 5.1. *Let* $A \in \mathbb{C}^{n\times n}$ *with* $\mathrm{Ind}(A) = k$*,* $B = A + E \in \mathbb{C}^{n\times n}$*. If*

$$EAA^{\textcircled{w}} = E, AA^{\textcircled{w}}E = E, \text{ and } \parallel A^{\textcircled{w}}E \parallel < 1,$$

then

$$B^{\textcircled{w},*} = (I_n + A^{\textcircled{w}}E)^{-1}A^{\textcircled{w}}(A + E)(A + E)^{*} = A^{\textcircled{w}}(I_n + EA^{\textcircled{w}})^{-1}(A + E)(A + E)^{*}.$$

Proof. Let A have the form of (3), and $E = U\begin{pmatrix} E_1 & E_2 \\ E_3 & E_4 \end{pmatrix}U^*$, where $E_1 \in \mathbb{C}^{r\times r}$. Since $AA^{ⓦ}E = E$, we get

$$\begin{aligned} AA^{ⓦ}E &= U\begin{pmatrix} T & S \\ 0 & N \end{pmatrix}\begin{pmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{pmatrix}\begin{pmatrix} E_1 & E_2 \\ E_3 & E_4 \end{pmatrix}U^* \\ &= U\begin{pmatrix} E_1 + T^{-1}SE_3 & E_2 + T^{-1}SE_4 \\ 0 & 0 \end{pmatrix}U^* \\ &= U\begin{pmatrix} E_1 & E_2 \\ E_3 & E_4 \end{pmatrix}U^*. \end{aligned} \tag{18}$$

Thus, we can get $E_3 = 0, E_4 = 0$. And applying $EAA^{ⓦ} = E$, we have

$$\begin{aligned} EAA^{ⓦ} &= U\begin{pmatrix} E_1 & E_2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} T & S \\ 0 & N \end{pmatrix}\begin{pmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{pmatrix}U^* \\ &= U\begin{pmatrix} E_1 & E_1T^{-1}S \\ 0 & 0 \end{pmatrix}U^* = U\begin{pmatrix} E_1 & E_2 \\ 0 & 0 \end{pmatrix}U^*. \end{aligned}$$

Hence, $E_2 = E_1T^{-1}S$.

Owing to $\rho(EA^{ⓦ}) = \rho(A^{ⓦ}E) \leq \| A^{ⓦ}E \| < 1$, we can get $I + A^{ⓦ}E$ is reversible and $T + E_1$ is nonsingular. Furthermore, notice that

$$E = U\begin{pmatrix} E_1 & E_2 \\ 0 & 0 \end{pmatrix}U^*,\ B = A + E = U\begin{pmatrix} T + E_1 & S + E_2 \\ 0 & N \end{pmatrix}U^*,$$

we can get

$$B^{ⓦ} = U\begin{pmatrix} (T + E_1)^{-1} & (T + E_1)^{-2}(S + E_2) \\ 0 & 0 \end{pmatrix}U^*.$$

Therefore,

$$B^{ⓦ,*} = U\begin{pmatrix} (T + E_1)^* + \triangle_1(S + E_2)^* & \triangle_1 N^* \\ 0 & 0 \end{pmatrix}U^*,$$

where $\triangle_1 = [(T + E_1)^{-1}(S + E_2) + (T + E_1)^{-2}(S + E_2)N]$. Thus,

$$B^{ⓦ,*} = (I_n + A^{ⓦ}E)^{-1}A^{ⓦ}(A + E)(A + E)^* = A^{ⓦ}(I_n + EA^{ⓦ})^{-1}(A + E)(A + E)^*. \square$$

Furthermore, we have the following result.

Theorem 5.2. *Let* $A \in \mathbb{C}^{n\times n}$ *with* $\mathrm{Ind}(A) = k$, $B = A + E \in \mathbb{C}^{n\times n}$. *If*

$$AA^{ⓦ,*}E = E, \text{ and } \| A^{ⓦ}E \| < 1,$$

then

$$\begin{aligned} B^{ⓦ,*} &= ((I_n + A^{ⓦ}E)^{-1}A^{ⓦ})^2AA^{†}(A + E)^2(A + E)^* \\ &= (I_n + A^{ⓦ}E)^{-1}A^{ⓦ}(I_n + A^{ⓦ}E)^{-1}A^{†}(A + E)^2(A + E)^*. \end{aligned}$$

6. Applications

In this section, we will give the application of the weak group-star matrix in solving linear equations.

Theorem 6.1. *Let $A\in\mathbb{C}^{n\times n}$ and $\mathrm{Ind}(A)=k$, the equation*

$$(A^{k+2})^*A^2x=(A^{k+2})^*A^2A^*b,\quad b\in\mathbb{C}^n, \tag{19}$$

is consistent and its general solution is

$$x=A^{\textcircled{W},*}b+(I_n-A^{\textcircled{W}}A)y, \tag{20}$$

for arbitrary $y\in\mathbb{C}^m$.

Proof. Suppose that x has the form (20). Applying $A^{\textcircled{W},*}=A^k(A^{k+2})^{\dagger}A^2A^*$, we have

$$\begin{aligned}(A^{k+2})^*A^2A^{\textcircled{W},*} &= (A^{k+2})^*A^2A^k(A^{k+2})^{\dagger}A^2A^* \\ &= (A^{k+2})^*A^{k+2}(A^{k+2})^{\dagger}A^2A^* \\ &= (A^{k+2})^*A^2A^*.\end{aligned}$$

Therefore $(A^{k+2})^*A^2A^{\textcircled{W},*}b=(A^{k+2})^*A^2A^*b$, which implies that (19) holds for x.

For a solution x to (19), we obtain

$$\begin{aligned}A^{\textcircled{W},*}b &= A^k(A^{k+2})^{\dagger}A^2A^*b \\ &= A^k(A^{k+2})^{\dagger}((A^{k+2})^{\dagger})^*(A^{k+2})^*A^2A^*b \\ &= A^k(A^{k+2})^{\dagger}((A^{k+2})^{\dagger})^*(A^{k+2})^*A^2x \\ &= A^{\textcircled{W}}Ax.\end{aligned}$$

Now, we get

$$x=A^{\textcircled{W},*}b+x-A^{\textcircled{W},*}Ax=A^{\textcircled{W},*}b+(I_n-A^{\textcircled{W}}A)x.$$

i.e., x possesses the form (20). □

Since $A^{\textcircled{W}}AX=A^{\textcircled{W}}AA^{\textcircled{W},*}b=A^{\textcircled{W},*}b$, we have $A^{\textcircled{W}}AX=A^{\textcircled{W}}AA^{\textcircled{W},*}b=A^{\textcircled{W},*}b$. Then we can obtain Theorem 6.2.

Theorem 6.2. *[11] Let $A\in\mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A)=k$, then the equation*

$$A^{\textcircled{W}}AX=A^{\textcircled{W},*}b \tag{21}$$

is consistent and its general solution is

$$x=A^{\textcircled{W},*}b+(I-A^{\textcircled{W}}A)y, \tag{22}$$

for arbitrary $y\in\mathbb{C}^{n\times n}$.

Similarly, the following theorem can be proved.

Theorem 6.3. *Let $A\in\mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A)=k$. Then the equation*

$$(A^{\dagger})^*x=AA^{\textcircled{W}}b$$

is consistent and its general solution is

$$x=A^{*,\textcircled{W}}b+(I-A^{\dagger}A)y,$$

for arbitrary $y\in\mathbb{C}^{n\times n}$.

Now, we can get the following consequence by the result of Theorem 6.3 in the case that $b\in R(A^k)$.

Corollary 6.4. *Let $A\in\mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A)=k$. Then the equation*

$$(A^{\dagger})^*x=b,b\in R(A^k)$$

is consistent and its general solution is

$$x=A^*b+(I-A^{\dagger}A)y,$$

for arbitrary $y\in\mathbb{C}^{n\times n}$.

7. Conclusion

In this paper, the definition and characterizations of the weak group-star matrix are given. The equivalence between various matrices and the weak group-star matrix are established. For Cramer's rule and the perturbation, we also give relevant theorems. Moreover, the weak group-star matrix can be applied to solving equations.

Moreover, dual weak group-star matrix can be called star-weak group matrix. Let $A \in \mathbb{C}^{n\times n}$ and $\mathrm{Ind}(A) = k$, C is the weak core part of A. Then

$$X(A^{\dagger})^{*}X = X,\ (A^{\dagger})^{*}X = CA^{D},\ XA = A^{*}C,$$

is consistent and its unique solution is $X = A^{*}CA^{D}$. The matrix satisfying the above equations is defined as $A^{*,Ⓦ} = A^{*}AA^{Ⓦ}$ and named the star-weak group matrix.

The star-weak group matrix also possesses similar properties of the weak group-star matrix.

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