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Characterization of bi-slant submanifolds of paraSasakian manifold

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Abstract. This paper aims to present work on bi-slant submanifolds of paraSasakian manifold. The study includes the definitions and some results on $Type\,1$, $Type\,2$, and $Type\,3$ slant submanifolds. We define bi-slant submanifolds and derive characterization results with some illustrative examples. Further, we also study the particular cases of bi-slant submanifolds named semi-slant and pseudo-slant submanifolds and derive some results. An example for each case is also constructed.

1. Introduction

The theory of slant submanifolds is an important area of differential geometry as it generalizes the cases of both holomorphic(complex) and totally real submanifolds. This theory was first given by B.Y. Chen in [3] for the complex manifolds where author defined slant submanifolds as the submanifolds for which the *Wirtinger angle* $\theta(X)$, *i.e.* angle between the ϕX and tangent space is constant. This angle $\theta(X) = 0$ or $\frac{\pi}{2}$ corresponds to complex or totally real submanifolds, respectively and slant submanifolds otherwise. Slant submanifolds of an almost contact and K-contact manifolds were studied by A. Lotta [2] and J.L. Cabrerizo *et.al* [5], respectively. Further, N. Papaphuic gave the notion of semi-slant submanifolds which is the generalization of slant submanifolds [11]. In [4], J.L. Cabrerizo *et.al* generalized the slant and semi-slant submanifolds and defined the term bi-slant submanifolds and also studied some properties of semi-slant submanifolds. Afterthat, the term pseudo-slant submanifolds was initiated by A. Carriazo [1]. Thus, slant, semi-slant, pseudo-slant, CR, complex and totally real submanifolds are the classifications of bi-slant submanifolds and studied extensively by many geometers in different ambient spaces [7–10, 14, 20, 21]. Also, the bi-slant submersions and their generalization studied in [16–18].

The *Wirtinger angle* for the submanifolds is calculated using *Cauchy-Schwarz inequality* as $\cos\theta = \frac{\|tX\|}{\|\phi X\|}$, where θ denotes slant angle and tX is the tangential part of the vector field X on submanifold. For Riemannian geometry $\|\phi X\| = \|X\| > 0$ but it does not holds for semi-Riemannian geometry. So, authors P. Alegre and A. Carriazo in the article [13] defined slant submanifolds for para-Hermitian manifold with semi-Riemannian metric. They also mentioned the types of slant submanifolds out of which the case of

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Riemannian manifold is one. Same authors in [12] gave the notion of bi-slant submanifolds in the same ambient manifold. Then, the same study is carried out by S. K. Chanyal for almost paracontact metric manifold [15].

This paper contains the study of bi-slant submanifolds which is organized as: Section 2 consists the basic definition of paraSasakian semi-Riemannian manifold and its submanifolds. In section 3, slant distributions and slant submanifolds of paraSasakian manifold are studied. Section 4, includes the definition of bi-slant submanifolds with its particular cases. In section 5 and section 6, we derive integrability and totally geodesic condition of the distributions involved in semi-slant and pseudo-slant submanifolds. Some examples are also given.

2. Preliminaries

Definition 2.1. *An almost paracontact semi-Riemannian manifold* $\mathbf{M}^{2m+1}$ *is a smooth manifold enriched with* (ϕ, η, ξ, g) *structure having* ϕ *as a* $(1,1)$*-tensor field,* η *as a globally differential one-form,* ξ *as a characteristic vector field and* g *as a semi-Riemannian metric on* $\mathbf{M}^{2m+1}$ *satisfying*

$$\phi^2 = I - \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{1}$$

$$g(\cdot,\cdot) = -g(\phi\cdot,\phi\cdot) + \eta(\cdot)\eta(\cdot). \tag{2}$$

Here I *denotes an identity transformation of* $T\mathbf{M}^{2m+1}$ *and* $\otimes$ *denotes tensor product.*

With the eye on equations (1) and (2), an almost paracontact semi-Riemannian manifold $\mathbf{M}^{2m+1}$ also follows the given conditions

$$\phi\xi = 0, \quad \eta \circ \phi = 0, \tag{3}$$

$$g(\cdot,\xi) = \eta(\cdot), \tag{4}$$

$$g(\phi\,\cdot,\cdot) + g(\cdot,\phi\,\cdot) = 0. \tag{5}$$

In addition to the above properties, an almost paracontact semi-Riemannian manifold also holds

$$d\eta(X_1, X_2) = g(X_1, \phi X_2), \tag{6}$$

for any tangent vector fields X_1, X_2 on $\mathbf{M}^{2m+1}$. An almost paracontact manifold turns to be a *paracontact manifold* if the fundamental 2-form Φ on $\mathbf{M}^{2m+1}$ satisfies $d\eta = \Phi$. Moreover,

$$(\widehat{\nabla}_{X_3}\Phi)(X_1, X_2) = g((\widehat{\nabla}_{X_3}\phi)X_1, X_2) = (\widehat{\nabla}_{X_3}\Phi)(X_2, X_1), \tag{7}$$

for any tangent vector fields X_1, X_2, X_3 and Levi-Civita connection $\widehat{\nabla}$ on $\mathbf{M}^{2m+1}$.

Normality. A *normal almost paracontact manifold* is one on which the Nijenhuis tensor becomes zero identically. The Nijenhuis tensor N_ϕ is a $(1,2)$-tensor field on $\mathbf{M}^{2m+1}$ and denote $[\phi,\phi]$ as the Nijenhuis torsion of ϕ then N_ϕ is given as

$$N_\phi = [\phi,\phi] - 2d\eta \otimes \xi.$$

The normal almost paracontact semi-Riemannian manifold $\mathbf{M}^{2m+1}$ holds an integrable almost paracomplex structure J on $\mathbf{M}^{2m+1} \times \mathbb{R}$ given as

$$J\left(X_1, f\frac{d}{dt}\right) = \left(\phi X_1 + f\xi, \eta(X_1)\frac{d}{dt}\right),$$

for $X_1 \in \Gamma\left(T\mathbf{M}^{2m+1}\right)$, smooth function f on $\mathbf{M}^{2m+1} \times \mathbb{R}$ and coordinate t on $\mathbb{R}$ [6].

Definition 2.2. *An almost paracontact manifold* $\mathbf{M}^{2m+1}$ *is called paraSasakian manifold, if the following condition is satisfies*

$$\left(\widehat{\nabla}_{X_1}\phi\right)X_2 = g(X_1, X_2)\xi - \eta(X_2)X_1, \tag{8}$$

for any tangent vector fields X_1, X_2 *on* $\mathbf{M}^{2m+1}$.

Simply substituting X_2 by ξ in equation (8), the next result follows:

Lemma 2.3. *If the characteristic vector field* $\xi \in \Gamma(T\mathbf{M}^{2m+1})$*, then for any tangent vector fields* X_1 *on* $\mathbf{M}^{2m+1}$ *the paraSasakian manifold* $\mathbf{M}^{2m+1}$ *satisfies*

$$\widehat{\nabla}_{X_1}\xi = \phi X_1. \tag{9}$$

Example 2.4. *Considering* $\overline{M} = \mathbb{R}^{2m}\times\mathbb{R}_+ \subset \mathbb{R}^{2m+1}$ *to be a* $(2m+1)$*-dimensional manifold having standard Cartesian coordinates* $(x_1, x_2, \cdots, x_m, y_1, y_2, \cdots, y_m, z)$*. Define the paracontact structure as*

$$\begin{cases} \eta = \frac{1}{2}[dz - \sum y^i dx^i], \quad \xi = 2dz, \\ \phi(X_1^i\frac{\partial}{\partial x^i} + X_2^i\frac{\partial}{\partial y^i} + X_3^i\frac{\partial}{\partial z}) = X_1^i(\frac{\partial}{\partial y^i}) + X_2^i(\frac{\partial}{\partial x^i} + y^i\frac{\partial}{\partial z}), \\ g = \eta\otimes\eta + \frac{1}{4}\left\{\sum_{i=1}^{\frac{m}{2}} dx^i\otimes dx^i - \sum_{i=\frac{m}{2}}^{m} dy^i\otimes dy^i\right\}, \end{cases} \tag{10}$$

where $i \in \{1, 2, \cdots m\}$*. Thus, the above defined structure is a paraSasakian structure and basis for this is* $\{\frac{1}{2}\frac{\partial}{\partial y^i}, \frac{1}{2}\left(\frac{\partial}{\partial x^i} + y^i\frac{\partial}{\partial z}\right), \frac{\partial}{\partial z}\}$.

2.1. Submanifold

Let M denotes an immersed submanifold of a paraSasakian manifold $\mathbf{M}^{2m+1}$. Considering the non degenerate metric induced on M by the same symbol g as on $\mathbf{M}^{2m+1}$. Further, the Gauss and Weingarten formulas are respectively given as

$$\widehat{\nabla}_{X_1}X_2 = \nabla_{X_1}X_2 + h(X_1, X_2), \tag{11}$$
$$\widehat{\nabla}_{X_1}V = -A_V X_1 + \nabla^{\perp}_{X_1}V, \tag{12}$$

for

- $X_1, X_2 \in \Gamma(TM)$(tangent bundle) and $V \in \Gamma(TM^{\perp})$(normal bundle),
- induced Levi-civita connection ∇ on M,
- normal connection $\nabla^{\perp}$ on $\Gamma(TM^{\perp})$,
- second fundamental form h on M,
- shape operator A_V associated with the normal section V.

Also, h is related to A_V as

$$g\left(h(X_1, X_2), V\right) = g\left(A_V X_1, X_2\right). \tag{13}$$

For any tangent vector fields X_1 and normal vector fields V on M, we write

$$\phi X_1 = tX_1 + nX_1, \tag{14}$$
$$\phi V = t'V + n'V, \tag{15}$$

where tX_1 and $t'V$ are the respective tangential parts of ϕX_1 and ϕV, nX_1 and $n'V$ are the respective normal parts of ϕX_1 and ϕV. Based on equation (14), the submanifold M is *anti-invariant* or *invariant* if tX_1 (tangent part) or nX_1 (normal part) is identically zero on M, respectively. After using equation (14) in equation (5), we get

$$g(X_1, tX_2) + g(tX_1, X_2) = 0, \qquad X_1, X_2 \in \Gamma(TM). \tag{16}$$

The covariant differentiation of t, n and $Q = t^2$ is given as

$$(\nabla_{X_1} t)\, X_2 = \nabla_{X_1} tX_2 - t\nabla_{X_1} X_2, \tag{17}$$
$$(\nabla_{X_1} n)\, X_2 = \nabla^{\perp}_{X_1} nX_2 - n\nabla_{X_1} X_2, \tag{18}$$
$$(\nabla_{X_1} Q)\, X_2 = \nabla_{X_1} QX_2 - Q\nabla_{X_1} X_2, \tag{19}$$

Further, using equations (11), (12) and (8), we get

$$(\nabla_{X_1} t)\, X_2 = A_{nX_2} X_1 + t'h(X_1, X_2) + g(X_1, X_2)\xi - \eta(X_2)X_1 \tag{20}$$
$$(\nabla_{X_1} n)\, X_2 = n'h(X_1, X_2) - h(X_1, tX_2). \tag{21}$$

Now, from Lemma 2.3 and equation (13), we have our next result.

Lemma 2.5. *If M is an immersed submanifold of a paraSasakian manifold $\mathbf{M}^{2m+1}$ with $\xi \in \Gamma(TM)$, then*

$$\nabla_{X_1}\xi = tX_1 \quad \text{and} \quad h(X_1, \xi) = nX_1, \tag{22}$$
$$g(A_V X_1, \xi) = g(nX_1, V), \tag{23}$$

$X_1 \in \Gamma(TM)$ and $V \in \Gamma(TM^{\perp})$.

3. Slant submanifold

This section deals with the notion of slant submanifolds of paraSasakian manifolds which are defined in the same way as in [13] for the semi-Riemannian metric manifold.

Definition 3.1. *The submanifold M of an almost paracontact semi-Riemannian manifold $\mathbf{M}^{2m+1}$ is called slant submanifold if the quotient $\frac{g(tX_1, tX_1)}{g(\phi X_1 . \phi X_1)}$ is constant for every spacelike(timelike) tangent vector field X_1 linearly independent to ξ.*

Further, on the basis of this definition we can say M is anti-invariant submanifold if the above quotient equals zero or $t \equiv 0$ and invariant submanifold if the above quotient equals 1 or $t \equiv \phi$. These submanifolds are called improper submanifolds. Thus, submanifolds other than these are considered as proper submanifolds.

Now, classifying the slant submanifold M into three types as in [13]:
The slant submanifold M of paraSasakian manifold $\mathbf{M}^{2m+1}$ is

1. *Type* 1 *slant* if tX_1 is timelike(spacelike) and $\frac{|tX_1|}{|\varphi X_1|} > 1$ for spacelike(timelike) tangent vector field X_1.
2. *Type* 2 *slant* if tX_1 is timelike(spacelike) and $\frac{|tX_1|}{|\varphi X_1|} \leq 1$ for spacelike(timelike) tangent vector field X_1.
3. *Type* 3 *slant* if tX_1 is spacelike(timelike) for spacelike(timelike) tangent vector field X_1.

Here, the tangent vector field X_1 is always linearly independent to ξ for either ξ can be normal or tangent to M [2]. Next result shows that we have the only proper slant submanifolds left to work with is the submanifolds with $\xi \in \Gamma(TM)$.

Theorem 3.2. [4] *The slant submanifold M of a paraSasakian manifold $\mathbf{M}^{2m+1}$ with ξ to be normal vector field is anti-invariant submanifold.*

Theorem 3.3. *The submanifold M of a paraSasakian manifold $\mathbf{M}^{2m+1}$ with ξ to be tangent vector field is slant submanifold if and only if there exists $\beta \in (-\infty, \infty)$ such that $t^2 = \beta\,(I - \eta \otimes \xi)$ and the submanifold M is*

1. *$Type\,1$ slant if* $\beta = cosh^2\theta \in (1, \infty)$ *for* $\theta > 0$.
2. *$Type\,2$ slant if* $\beta = cos^2\theta \in [0, 1]$ *for* $0 \le \theta \le 2\pi$.
3. *$Type\,3$ slant if* $\beta = -sinh^2\theta \in (-\infty, 0)$ *for* $\theta > 0$.

Here, β *is known as slant coefficient.*

Proof. This can be prove by following similar steps as in [15]. □

Clearly, for *invariant submanifolds*, the slant coefficient equals 1 and for *anti-invariant submanifolds*, the slant coefficient equals 0. These submanifolds are the special cases of slant submanifold particularly of $Type\,2$ *slant* submanifolds.

Proposition 3.4. *The submanifold M of a paraSasakian manifold $\mathbf{M}^{2m+1}$ with ξ to be tangent vector field is slant submanifold if and only if there exists $t'nX_1 = \beta^\star\,(X_1 - \eta(X_1)\xi)$ for timelike or spacelike tangent vector field X_1. For*

1. *$Type\,1$ slant submanifold,* $\beta^\star = -sinh^2\theta$ *for* $\theta > 0$.
2. *$Type\,2$ slant submanifold,* $\beta^\star = sin^2\theta$ *for* $0 \le \theta \le 2\pi$.
3. *$Type\,3$ slant submanifold,* $\beta^\star = cosh^2\theta$ *for* $\theta > 0$.

Proof. For $X_1 \in \Gamma\,(TM)$, $\phi^2 X_1 = X_1 - \eta(X_1)\xi$ and

$$\phi^2 X_1 = (t^2 X_1 + t'nX_1) + (ntX_1 + n'nX_1). \tag{24}$$

After comparing the tangential parts and using Theorem 3.3, we directly have the stated result. □

Theorem 3.5. *The submanifold M of a paraSasakian manifold $\mathbf{M}^{2m+1}$ with ξ to be tangent vector field is slant submanifold if ${n'}^2 V = \beta V$ for timelike or spacelike vector field $V \in \Gamma\,(TM^\perp)$.*

(i) *$Type\,1$ slant submanifold if* $\beta = cosh^2\theta \in (1, \infty)$ *for* $\theta > 0$.

(ii) *$Type\,2$ slant submanifold if* $\beta = cos^2\theta \in [0, 1]$ *for* $0 \le \theta \le 2\pi$.

Proof. For $V \in \Gamma\,(TM^\perp)$, there exists $X_1 \in \Gamma\,(TM)$ with $nX_1 = V$. Now, using equation (24) we get the required result. □

Proposition 3.6. *If submanifold M of a paraSasakian manifold $\mathbf{M}^{2m+1}$ is slant submanifold then for $X_1, X_2 \in \Gamma\,(TM)$,*

$$g(tX_1, tX_2) = -\beta\,\{g(X_1, X_2) - \eta(X_1)\eta(X_2)\}, \tag{25}$$

$$g(nX_1, nX_2) = (1 - \beta)\,\{g(X_1, X_2) - \eta(X_1)\eta(X_2)\}. \tag{26}$$

Proof. For slant submanifold M,

$$t^2 X_1 = \beta\{X_1 - \eta(X_1)\xi\}.$$

for $X_1 \in \Gamma\,(TM)$. Now, using equation (14), (16) and (2) we get the required result. □

Theorem 3.7. *Let M be slant submanifold of a paraSasakian manifold $\mathbf{M}^{2m+1}$. Then the endomorphism $Q = t^2$ is parallel if and only if the submanifold M is of $Type\,2$ slant with zero slant coefficient or anti-invariant submanifold.*

Proof. With respect to Theorem 3.3, we can deduce that

$$t^2\nabla_{X_1}X_2 = \beta\left(\nabla_{X_1}X_2 - \eta\left(\nabla_{X_1}X_2\right)\xi\right). \tag{27}$$

Similarly,

$$\nabla_{X_1} t^2X_2 = \beta\left(\nabla_{X_1}X_2 - \eta\left(X_2\right)\nabla_{X_1}\xi\right) - \beta\left(\eta(X_2)\phi X_1 + g(X_2, \phi X_1)\xi\right). \tag{28}$$

Using equation (27) and (28), we get

$$\left(\nabla_{X_1}t^2\right)X_2 = -\beta\left(\eta(X_2)\phi X_1 + g(X_2, \phi X_1)\xi\right). \tag{29}$$

This equation implies that Q is parallel if and only if $\beta = 0$, which means M is of $Type\,2$ slant submanifold with zero slant coefficient. □

Theorem 3.8. *The submanifold M of a paraSasakian manifold $\mathbf{M}^{2m+1}$ is slant if and only if*

- *the endomorphism $Q|_{\mathfrak{D}}$ has a single eigenvalue at each point of M.*
- *$\exists$ a function $\beta : M \to (-\infty, \infty)$ satisfying*

$$\left(\nabla_{X_1}Q\right)X_2 = -\beta\left\{g(X_2, \phi X_1)\xi + \eta(X_2)\phi X_1\right\}. \tag{30}$$

for $X_1, X_2 \in \Gamma(TM)$ and β can be $cosh^2\theta$, $cos^2\theta$ or $-sinh^2\theta$.

Proof. For slant submanifold M, the given two conditions are directly follows from [4] and Theorem 3.7. Conversely, suppose that the submanifold M holds the given two conditions. Let the distribution $\mathfrak{D} =< \xi >^{\perp}$ and β be eigenvalue of $Q|_{\mathfrak{D}}$ then for any $X_2 \in \mathfrak{D}$, we have $t^2X_2 = QX_2 = \beta X_2$. Thus, from equation (30)

$$\left(X_1\beta\right)X_2 + \beta\nabla_{X_1}X_2 = Q\nabla_{X_1}X_2 + \beta g(X_2, \phi X_1)\xi. \tag{31}$$

Now, taking inner product with X_2, we have $X_1\beta = 0 \implies \beta$ is constant. According to distribution setting for any tangent vector field X_1, one can write $X_1 = \bar{X_1} + \eta(X_1)\xi$ for $\bar{X_1} \in \mathfrak{D}$ and reaches to $Q\bar{X_1} = \beta\bar{X_1} = \beta(X_1 - \eta(X_1)\xi)$. Using Theorem 3.3, M becomes slant with slant coefficient β. □

Example 3.9. *Let $\overline{M} = \mathbb{R}^4 \times \mathbb{R}_+ \subset \mathbb{R}^5$ be a 5-dimensional manifold having standard Cartesian coordinates (x_1, x_2, y_1, y_2, z). This manifold with paraSasakian structure defined in Example 2.4 is a paraSasakian manifold.*

Now, consider isometrically immersed submanifolds M_1, M_2 and M_3 with semi-Riemannian metric, and are defined by

$$M_1\left(u, v, z\right) = \left(-8u, 2v, 2u, 8u, z\right),$$
$$M_2\left(u, v, z\right) = \left(-3u, 2v, 4u, 8u, z\right),$$
$$M_3\left(u, v, z\right) = \left(2(u + v), 0, v, v, z\right).$$

Then M_1, M_2 and M_3 defines $Type\,1$, $Type\,2$ and $Type\,3$ slant submanifold, respectively.

3.1. Slant distributions

Definition 3.10. *A slant distribution $\mathfrak{D}$ is differentiable on M for which the quotient $g\left(t_{\mathfrak{D}}X_1, t_{\mathfrak{D}}X_1\right)/g\left(\varphi X_1, \varphi X_1\right)$ is constant for every spacelike(timelike) vector field $X_1 \in \mathfrak{D}$. Here,*

(i) $t_{\mathfrak{D}}X_1$ is the projection of φX_1 on the distribution $\mathfrak{D}$.

(ii) The slant coefficient corresponding to the distribution $\mathfrak{D}$ on M is $\lambda_{\mathfrak{D}}$.

Remark 3.11. *(i) A slant distribution $\mathfrak{D}$ is invariant if $t_{\mathfrak{D}}X_1 \equiv \varphi X_1$ or anti-invariant if $t_{\mathfrak{D}}X_1 \equiv 0$. Other than these two cases, the distribution is proper slant distribution [13].*

(ii) [10] For every vector fields X_1, X_2 on $\mathfrak{D}$, distribution $\mathfrak{D}$ on M is called

- *Integrable if $[X_1, X_2] \in \mathfrak{D}$ and*
- *totally geodesic if $h(X_1, X_2) = 0$.*

Proposition 3.12. *The distribution $\mathfrak{D}$ is invariant if and only if it is slant of Type 2 with slant coefficient 1.*

Proof. From Remark 3.11, the distribution $\mathfrak{D}$ is invariant if $t_{\mathfrak{D}}X_1 \equiv \varphi X_1$. Now, using Definition 3, we have slant coefficient $\lambda_{\mathfrak{D}} = 1$, which implies $\mathfrak{D}$ is slant of $Type\,2$. □

Similarly, we can prove the next result for the anti-invariant distribution.

Proposition 3.13. *The distribution $\mathfrak{D}$ is anti-invariant if and only if it is slant of Type 2 with slant coefficient 0.*

Proposition 3.14. *Let $\mathfrak{D}$ be the distribution on a paraSasakian manifold $\mathbf{M}^{2m+1}$. Then $\mathfrak{D}$ is slant distribution if and only if there exists $\beta \in (-\infty, \infty)$ such that $(t_{\mathfrak{D}})^2 = \beta\,(I - \eta \otimes \xi)$ and the distribution is*

(i) $Type\,1$ slant if $\beta = cosh^2\theta \in (1, \infty)$ for $\theta > 0$.

(ii) $Type\,2$ slant if $\beta = cos^2\theta \in [0, 1]$ for $0 \leq \theta \leq 2\pi$.

(iii) $Type\,3$ slant if $\beta = -sinh^2\theta \in (-\infty, 0)$ for $\theta > 0$.

Here, $t_{\mathfrak{D}}$ is the projection on the distribution $\mathfrak{D}$.

4. Bi-slant submanifold

Definition 4.1. *[12] The submanifold M of a paraSasakian manifold $\mathbf{M}^{2m+1}$ is called bi-slant submanifold if $\exists$ two orthogonal distributions $\mathfrak{D}_\gamma$ and $\mathfrak{D}_\beta$ such that*

1. *The tangent space decomposes into $TM = \mathfrak{D}_\gamma \oplus \mathfrak{D}_\beta \oplus < \xi >$.*
2. *The distributions $\mathfrak{D}_\gamma$ and $\mathfrak{D}_\beta$ are both slant distributions with slant coefficients γ and β, respectively.*

Further, the slant distributions $\mathfrak{D}_\gamma$ and $\mathfrak{D}_\beta$ can be of $Type\,1\,slant$, $Type\,2\,slant$ or $Type\,3\,slant$ with values of γ and β be $cosh^2\theta$, $cos^2\theta$ or $-sinh^2\theta$. We denote the dimension of these slant distributions $\mathfrak{D}_\gamma$ and $\mathfrak{D}_\beta$ by d_γ and d_β, respectively. It is easy to analyse from Table 1(a) and Table 1(b) that bi-slant submanifolds are generalization of slant, semi-slant, pseudo-slant, anti-invariant, invariant, semi-invariant submanifolds.

Table 1(a). Particular cases of bi-slant submanifold when either $d_\gamma \neq 0$ or $d_\beta \neq 0$:

If the following conditions are satisfied then	submanifold M is
$d_\gamma = 0, d_\beta \neq 0$ and $\beta \neq 0, 1$	proper slant
$d_\gamma \neq 0, d_\beta = 0$ and $\gamma \neq 0, 1$	
$d_\gamma = 0, d_\beta \neq 0$ and $\beta = 0$	anti-invariant
$d_\gamma \neq 0, d_\beta = 0$ and $\gamma = 0$	
$d_\gamma = 0, d_\beta \neq 0$ and $\beta = 1$	invariant
$d_\gamma \neq 0, d_\beta = 0$ and $\gamma = 1$	

Table 1(b). Particular cases of bi-slant submanifold when both $d_\gamma \neq 0$ and $d_\beta \neq 0$ (depending only on slant coefficients γ and β):

If the following conditions are satisfied then	submanifold M is
$\gamma \neq 0,1$ and $\beta = 1$	semi-slant
$\gamma = 1$ and $\beta \neq 0,1$	
$\gamma \neq 0,1$ and $\beta = 0$	pseudo-slant
$\gamma = 0$ and $\beta \neq 0,1$	
$\gamma = 0$ and $\beta = 1$	semi-invariant
$\gamma = 1$ and $\beta = 0$	
$\gamma \neq 0,1$ and $\beta \neq 0,1$	proper bi-slant

For bi-slant submanifold M, if P_γ and P_β denotes the projections on distributions $\mathfrak{D}_\gamma$ and $\mathfrak{D}_\beta$, respectively. Then

$$X_1 = P_\gamma X_1 + P_\beta X_1 + \eta(X_1)\xi. \tag{32}$$

for any tangent vector field X_1. After operating ϕ in above equation, we get

$$\phi X_1 = tP_\gamma X_1 + nP_\gamma X_1 + tP_\beta X_1 + nP_\beta X_1. \tag{33}$$

Thus, we have

$$TM = tP_\gamma X_1 + tP_\beta X_1, \quad TM^\perp = nP_\gamma X_1 + nP_\beta X_1. \tag{34}$$

Following this, orthogonal decomposition of $TM^\perp$ is given as

$$TM^\perp = n\mathfrak{D}_\gamma \oplus n\mathfrak{D}_\beta \oplus N. \tag{35}$$

Here, N is invariant subbundle of $TM^\perp$.

Proposition 4.2. *Let M be bi-slant submanifold of a paraSasakian manifold $\mathbf{M}^{2m+1}$. If slant coefficient $\gamma = \beta$ and $g\left(\phi X_1, X_3\right) = 0$ for $X_1 \in \mathfrak{D}_\gamma$, $X_3 \in \mathfrak{D}_\beta$ then M is slant submanifold. Moreover, the Type of slant submanifold depends on value of slant coefficient.*

Proof. As $g\left(\phi X_1, X_3\right) = 0$ for $X_1 \in \mathfrak{D}_\gamma$, $X_3 \in \mathfrak{D}_\beta$. This concludes that $g(tX_1, X_3) = 0$ and also using (5), $g(X_1, tX_3) = 0$. Then from equation (32), we have

$$\begin{aligned} t^2 X_1 &= t^2 P_\gamma X_1 + t^2 P_\beta X_1 \\ &= \gamma P_\gamma X_1 + \beta P_\beta X_1 \\ &= \gamma\left(P_\gamma X_1 + P_\beta X_1\right) \\ &= \gamma\left(X_1 - \eta(X_1)\xi\right). \end{aligned}$$

Thus, Theorem 3.3 implies that M is slant submanifold. □

Example 4.3. *Consider $\overline{M} = \mathbb{R}^6 \times \mathbb{R}_+ \subset \mathbb{R}^7$ to be a 7-dimensional manifold having standard Cartesian coordinates as $(x_1, x_2, x_3, y_1, y_2, y_3, z)$. This manifold with paraSasakian structure defined in Example 2.4 is a paraSasakian manifold.*
Now, consider an isometrically immersed submanifold M with semi-Riemannian metric, and is defined by

$$M(u_1, u_2, v_1, v_2, z) = (bu_1 + cv_2, u_1 + du_2, av_1 + u_2, v_1, v_2, 0, z),$$

where $a, b \in \mathbb{R}$ and $\{a, c\} \neq 1$. Following are the vector fields which spans the tangent space TM

$$\begin{aligned} X_{u_1} &= b\,\partial x^1 + \partial x^2 + b\,y^1 \partial z + y^2 \partial z \\ X_{v_1} &= a\,\partial x^3 + \partial y^1 + a\,y^3 \partial z \\ X_{u_2} &= d\,\partial x^2 + \partial x^3 + d\,y^2 \partial z + y^3 \partial z \\ X_{v_2} &= c\,\partial x^1 + \partial y^2 + c\,y^1 \partial z \\ X_z &= 2\partial z. \end{aligned}$$

Thus, M defines a Bi-slant submanifold having slant distributions

$$\mathfrak{D}_\gamma = \text{span}\{X_{u_1}, X_{v_1}\} \quad \text{and} \quad \mathfrak{D}_\beta = \text{span}\{X_{u_2}, X_{v_2}\},$$

where, slant coefficient $\gamma = \frac{b^2}{(1-a^2)(1+b^2)}$ *and* $\beta = \frac{d^2}{(1-c^2)(1+d^2)}$.

5. Semi-slant submanifolds

We define semi-slant submanifolds of paraSasakian manifold as they are subcases of bi-slant submanifolds and derive totally geodesic foliations and integrability conditions of the included distributions.

Definition 5.1. *[10] The bi-slant submanifold M becomes semi-slant submanifold if the decomposition of its tangent space can be given as*

$$TM = \mathfrak{D}_1 \oplus \mathfrak{D}_\beta \oplus < \xi >,$$

where distribution $\mathfrak{D}_1$ *is an invariant distribution and* $\mathfrak{D}_\beta$ *is a slant distribution.*

Further, $\mathfrak{D}_\beta$ is proper slant distribution with slant coefficient $\beta \neq \{1, 0\}$ and $\mathfrak{D}_1$ is invariant distribution with slant coefficient $\gamma = 1$. We denote the dimension of this distribution $\mathfrak{D}_1$ by d_1.

Table 2. Particular cases of semi-slant submanifold for $d_\beta \neq 0$:

If the following conditions are satisfied then	submanifold M is
$d_1 = 0$ and $\beta \neq 0, 1$	proper slant
$d_1 = 0$ and $\beta = 0$	anti-invariant
$d_1 = 0$ or $d_1 \neq 0$ and $\beta = 1$	invariant
$d_1 \neq 0$ and $\beta = 0$	semi-invariant
$d_1 \neq 0$ and $\beta \neq 0, 1$	proper semi-slant

For semi-slant submanifold M, if P_1 and P_β denotes the projections on distributions $\mathfrak{D}_1$ and $\mathfrak{D}_\beta$, respectively. Then,

$$X_1 = P_1 X_1 + P_\beta X_1 + \eta(X_1)\,\xi, \tag{36}$$

for any tangent vector field X_1. After operating ϕ in above equation, we get

$$\phi X_1 = tP_1 X_1 + tP_\beta X_1 + nP_\beta X_1. \tag{37}$$

Thus, we have

$$TM = tP_1 X_1 + tP_\beta X_1, \quad TM^\perp = nP_\beta X_1. \tag{38}$$

Following this, orthogonal decomposition of $TM^\perp$ is given as

$$TM^\perp = n\mathfrak{D}_\beta \oplus N. \tag{39}$$

Here, N is invariant subbundle of $TM^\perp$ orthogonal to $n\mathfrak{D}_\beta$. Next, we provide some basic results of proper semi-slant submanifold.

Lemma 5.2. *If M be proper semi-slant submanifold of paraSasakian manifold* $\mathbf{M}^{2m+1}$ *then slant distribution satisfies* $t\mathfrak{D}_\beta \subseteq \mathfrak{D}_\beta$.

Proof. Taking inner product of equation (37) with $X_2 \in \mathfrak{D}_1$, we obtain

$$g\left(\phi X_1, X_2\right) = g\left(tP_1X_1, X_2\right) + g\left(tP_\beta X_1, X_2\right). \tag{40}$$

For $X_1 \in \mathfrak{D}_\beta$, above equation gives

$$g\left(tP_1X_1, X_2\right) = 0.$$

Also, for every $X_3 \in \mathfrak{D}_\beta$

$$g\left(P_1X_1, X_3\right) = 0, \tag{41}$$

which implies $P_1X_1 = 0$ or $t\mathfrak{D}_\beta \subseteq \mathfrak{D}_\beta$. □

Theorem 5.3. *The necessary and sufficient condition for submanifold M of paraSasakian manifold $\mathbf{M}^{2m+1}$ to be semi-slant submanifold is that $\exists$ a distribution $\mathfrak{D}$ and a constant $\beta \in (-\infty, \infty)$ satisfying*

(i) $\mathfrak{D} = \{X_1 \in \Gamma(TM) \mid t^2X_1 = \beta X_1\}$

(ii) $nX_1 = 0$, *for tangent vector field X_1 orthogonal to $\mathfrak{D}$.*

Further, β can be $cosh^2\theta$, $cos^2\theta$ or $-sinh^2\theta$.

Proof. Suppose M as a proper semi-slant submanifold. From Definition 5.1, condition (ii) follows for any $X_1 \in \mathfrak{D}_1$ and condition (i) follows for $X_1 \in \mathfrak{D}_\beta$.
Conversely, by assuming (i) and (ii) we can write $TM = \mathfrak{D} \oplus \mathfrak{D}' \oplus < \xi >$. Now for $X_1 \in T(\mathfrak{D}_\perp)$, we have $nX_1 = 0$ implies $t\mathfrak{D}' \subseteq \mathfrak{D}'$ giving $\mathfrak{D}'$ as an invariant distribution and applying Proposition 3.14 on first assertion $\mathfrak{D}$ becomes a slant distribution. □

Further, we study integrability and totally geodesic conditions for the distributions of semi-slant submanifold M. From [4] invariant distribution and slant distribution are not integrable so we work for both the distribution in combination with $< \xi >$.

Theorem 5.4. *The distribution $\mathfrak{D}_1 \oplus < \xi >$ of proper semi-slant submanifold M of a paraSasakian manifold $\mathbf{M}^{2m+1}$ is integrable if and only if*

$$h(t X_1, X_2) = h(X_1, t X_2), \qquad X_1, X_2 \in \mathfrak{D}_1 \oplus < \xi > . \tag{42}$$

Proof. Equation (21) gives

$$\nabla_{X_1} nX_2 - n\nabla_{X_1} X_2 = n'h(X_1, X_2) - h(X_2, tX_1). \tag{43}$$

for any vector fields X_1, X_2 in $\mathfrak{D}_1 \oplus < \xi >$. Now, another similar equation is formed by interchanging the roles of X_1 and X_2 and then subtracting from equation (43), we get

$$n[X_1, X_2] = h(X_1, tX_2) - h(X_2, tX_1), \tag{44}$$

which is the desired result. Also, for an invariant distribution, one can replace ϕX_1 by tX_1 as $nX_1 = 0$. □

Theorem 5.5. *The distribution $\mathfrak{D}_\beta \oplus < \xi >$ of proper semi-slant submanifold M of a paraSasakian manifold $\mathbf{M}^{2m+1}$ is integrable if and only if*

$$P_1\left(A_{nX_1}X_2 - A_{nX_2}X_1\right) + P_1\left(\nabla_{X_1} tX_2 - \nabla_{X_2} tX_1\right) = 0, \tag{45}$$

$X_1, X_2 \in \mathfrak{D}_\beta \oplus < \xi >$ and here P_1 denotes projection on distribution $\mathfrak{D}_1$.

Proof. For $X_1, X_2 \in \mathfrak{D}_\beta \oplus <\xi>$, equation (20) under the projection P_1 gives

$$P_1 t\nabla_{X_1} X_2 = -P_1 A_{nX_2} X_1 - P_1 t' h(X_1, X_2) + P_1 \nabla_{X_1} tX_2 - \eta(X_2) P_1 X_1. \quad (46)$$

Forming new equation by switching X_1 and X_2 and then subtracting from equation (46), we get

$$P_1 t[X_1, X_2] = P_1(A_{nX_1} X_2 - A_{nX_2} X_1) + P_1(\nabla_{X_1} tX_2 - \nabla_{X_2} tX_1), \quad (47)$$

which gives the required result. □

Theorem 5.6. *The distribution $\mathfrak{D}_1 \oplus <\xi>$ of proper semi-slant submanifold M of a paraSasakian manifold $\mathbf{M}^{2m+1}$ is totally geodesic if and only if $A_{nX_1} tX_3 = A_{ntX_1} X_3$ for any vector field X_1 in $\mathfrak{D}_1$ and X_3 in $\mathfrak{D}_\beta$.*

Proof. For any vector fields X_1, X_2 in $\mathfrak{D}_1$ and X_3 in $\mathfrak{D}_\beta$, equations (2), (9) and (11) gives

$$g(\nabla_{X_1} X_2, X_3) = -g\left(\nabla_{X_1} \phi X_2, nX_3\right) - g\left(\nabla_{X_1} X_2, t^2 X_3 + ntX_3\right). \quad (48)$$

Using equation (48) and (12) with the Theorem 3.3, we get

$$(1-\beta)\, g(\nabla_{X_1} X_2, X_3) = A_{nX_1} tX_3 - A_{ntX_3} X_1.$$

Since, $\beta \neq 1$ thus the result follows. □

Proposition 5.7. *Let M be a proper semi-slant submanifold of paraSasakian manifold $\mathbf{M}^{2m+1}$. If invariant distribution $\mathfrak{D}_1$ is totally geodesic then $\nabla_{X_1} tX_2 = g(X_1, X_2)\xi$ and $\nabla_{X_1} X_2 \in \mathfrak{D}_1$ for any vector fields X_1, X_2 in $\mathfrak{D}_1$.*

Proof. Using invariance and totally geodesic condition of $\mathfrak{D}_1$ in equation (20), we obtain $\nabla_{X_1} tX_2 = g(X_1, X_2)\xi$, for any vector fields X_1, X_2 in $\mathfrak{D}_1$. Also, from equation (21) we get $n\nabla_{X_1} X_2 = 0$, which is the desired result. □

Lemma 5.8. *For proper semi-slant submanifold M of a paraSasakian manifold $\mathbf{M}^{2m+1}$ if $\nabla n = 0$ then $A_{n'V} X_1 + A_V tX_1 = 0$ for any tangent vector field X_1 and normal vector field V.*

Proof. Taking inner product of equation (21) with normal vector field $V \in \Gamma(TM^\perp)$, we have

$$g(n'h(X_1, X_2), V) = g(h(X_1, tX_2), V),$$
$$g\left(h(X_1, X_2), \phi V\right) = -g(h(X_1, tX_2), V),$$

which gives $A_{n'V} X_1 = -A_V tX_1$. □

Lemma 5.9. *For proper semi-slant submanifold M of a paraSasakian manifold $\mathbf{M}^{2m+1}$ if $\nabla t = 0$ then $A_{nX_1} X_3 - A_{nX_3} X_1 = 0$ for any tangent vector fields X_1, X_3.*

Proof. For $X_1, X_2, X_3 \in \Gamma(TM)$ equation (20) gives

$$g(A_{nX_1} X_2, X_3) + g(X_1, X_2)\,\eta(X_3) - \eta(X_2)\, g(X_1, X_3) - g(t'h(X_1, X_2), X_3) = 0.$$

In this expression using equations (9) and (3.1), we get

$$A_{nX_1} X_3 - A_{nX_3} X_1 - \left(\nabla_{X_1} \phi\right) X_3 = 0. \quad (49)$$

After expanding $\left(\nabla_{X_1} \phi\right) X_3$ and using given assertion and equation (21) result holds. □

Definition 5.10. *[12] A semi-slant submanifold M of a paraSasakian manifold $\mathbf{M}^{2m+1}$ is called*

- *mixed totally geodesic if $h(X_1, X_2) = 0$ for any vector field X_1 in $\mathfrak{D}_1$ and X_2 in $\mathfrak{D}_\beta$.*

- $\mathfrak{D}_1$*-totally geodesic if* $h(X_1, X_2) = 0$ *for any vector fields* X_1, X_2 *in* $\mathfrak{D}_1$.
- $\mathfrak{D}_\beta$*-totally geodesic if* $h(X_1, X_2) = 0$ *for any vector fields* X_1, X_2 *in* $\mathfrak{D}_\beta$.

Lemma 5.11. *Let* M *be proper semi-slant submanifold of a paraSasakian manifold* $\mathbf{M}^{2m+1}$ *then* $A_V X_1 \in \mathfrak{D}_1$ *and* $A_V X_2 \in \mathfrak{D}_\beta$ *for* $X_1 \in \mathfrak{D}_1$, $X_2 \in \mathfrak{D}_\beta$, $V \in \Gamma(TM^\perp)$.

Proof. In equation (13) using the condition of mixed geodesic, required result follows. □

Theorem 5.12. *Let* M *be mixed totally geodesic proper semi-slant submanifold of a paraSasakian manifold* $\mathbf{M}^{2m+1}$ *then the distribution* $\mathfrak{D}_1 \oplus < \xi >$ *is integrable if and only if* $\phi A_V X_1 = A_V \phi X_1$ *for* $X_1 \in \mathfrak{D}_1$ *and normal vector field* V.

Proof. For $X_1 \in \mathfrak{D}_1$ and $X_2 \in \mathfrak{D}_\beta$, Lemma 5.11 gives that no component of $A_V X_1$ lies in the distribution $\mathfrak{D}_\beta$. Thus,

$$g\left(h(\phi X_1, X_2), V\right) = g\left(A_V \phi X_1, X_2\right) \text{ and } g\left(h(X_1, \phi X_2), V\right) = -g\left(\phi A_V X_1, X_2\right) \tag{50}$$

The integrability of the distribution $\mathfrak{D}_1 \oplus < \xi >$ and above expressions implies the result. □

Theorem 5.13. *The proper semi-slant submanifold* M *of a paraSasakian manifold* $\mathbf{M}^{2m+1}$ *is mixed totally geodesic semi-slant submanifold if and only if* $(\nabla_{X_1} t) X_2 = A_{nX_1} X_2$ *and* $(\nabla_{X_1} n) X_2 = 0$ *for* $X_1 \in \mathfrak{D}_1$ *and* $X_2 \in \mathfrak{D}_\beta$.

Proof. Suppose the given two conditions satisfied. For any tangent vector fields X_1, X_2 equation (21) with $\nabla n = 0$ gives

$$h(X_1, tX_2) = n' h(X_1, X_2). \tag{51}$$

Particularly, for $X_2 \in \mathfrak{D}_\beta$ we have

$$(n')^2 h(X_1, X_2) = n' h(X_1, tX_2) = h\left(X_1, t^2 X_2\right) = \beta h(X_1, X_2). \tag{52}$$

Similarly, for $X_1 \in \mathfrak{D}_1$ we have

$$(n')^2 h(X_2, X_1) = n' h(X_2, tX_1) = h\left(X_2, t^2 X_1\right) = h(X_2, X_1). \tag{53}$$

Above equations gives $(1-\beta) h(X_1, X_2) = 0$, which implies $h(X_1, X_2) = 0$ as M is proper semi-slant submanifold and the condition $(\nabla_{X_1} t) X_2 = A_{nX_1} X_2$ gives $t' h(X_1, X_2) = 0$. Converse part can be prove with the help of equations (20) and (21). □

Theorem 5.14. *Let* M *be proper semi-slant submanifold of a paraSasakian manifold* $\mathbf{M}^{2m+1}$. *If* $\nabla n = 0$ *then*

(i) invariant distribution $\mathfrak{D}_1$ *is either* $\mathfrak{D}_1$*-totally geodesic or the endomorphism* $(n')^2$ *has the eigenvalue* 1 *for associated eigenvector* h.

(ii) slant distribution $\mathfrak{D}_\beta$ *is either* $\mathfrak{D}_\beta$*-totally geodesic or the endomorphism* $(n')^2$ *has the eigenvalue* β *for associated eigenvector* h.

here h *is the second fundamental form on* M.

Proof. From equations (52) and (53), we have

$$\left((n')^2 - 1\right) h(X_1, X_2) = 0, \quad \text{for} \quad X_1, X_2 \in \mathfrak{D}_1$$
$$\left((n')^2 - \beta\right) h(X_1, X_2) = 0, \quad \text{for} \quad X_1, X_2 \in \mathfrak{D}_\beta$$

these two expressions proves the assertions (i) and (ii). □

Example 5.15. *Let $\overline{M} = \mathbb{R}^6 \times \mathbb{R}_+ \subset \mathbb{R}^7$ be a 7-dimensional manifold having standard Cartesian coordinates as $(x_1, x_2, x_3, y_1, y_2, y_3, z)$. This manifold with paraSasakian structure defined in Example 2.4 is a paraSasakian manifold.*
Now, consider an isometrically immersed submanifold M with semi-Riemannian metric, and is defined by

$$M(u_1, u_2, v_1, v_2, z) = (u_1 \cosh\theta + v_1, 0, a \sinh v_2 + u_2, v_1 \cosh\theta + u_1, a \cosh v_2, v_2, z),$$

where $a \in \mathbb{R}$ and $a \neq 1$. Following are the vector fields which spans the tangent space TM

$$X_{u_1} = \cosh\theta\, \partial x^1 + \partial y^1 + y^1 \cosh\theta\, \partial z$$
$$X_{v_1} = \partial x^1 + \cosh\theta\, \partial y^1 + y^1 \partial z$$
$$X_{u_2} = \partial x^3 + y^3 \partial z$$
$$X_{v_2} = a \cosh v_2\, \partial x^3 + \partial y^3 + a \sinh v_2\, \partial y^2 + a\, y^3 \cosh v_2\, \partial z$$
$$X_z = 2\partial z.$$

Thus, M defines the semi-slant submanifold having slant distribution

$$\mathfrak{D}_1 = \mathrm{span}\{X_{u_1}, X_{v_1}\} \quad \text{and} \quad \mathfrak{D}_\beta = \mathrm{span}\{X_{u_2}, X_{v_2}\},$$

where, slant coefficient $\beta = \frac{1}{(1-a^2)}$.

6. pseudo-slant submanifold

Definition 6.1. *[12] The bi-slant submanifold M becomes pseudo-slant submanifold if its tangent space can be decomposed as*

$$TM = \mathfrak{D}_0 \oplus \mathfrak{D}_\beta \oplus < \xi >,$$

here distribution $\mathfrak{D}_0$ is anti-invariant and distribution $\mathfrak{D}_\beta$ is slant.

Further, $\mathfrak{D}_\beta$ is proper slant distribution with slant coefficient $\beta \neq \{1, 0\}$ and $\mathfrak{D}_0$ is invariant distribution with slant coefficient $\gamma = 0$. We denote the dimension of this distribution $\mathfrak{D}_0$ by d_0.

Table 3. Particular cases of pseudo-slant submanifold for $d_\beta \neq 0$:

If the following conditions are satisfied then	submanifold M is
$d_0 = 0$ and $\beta \neq 0, 1$	proper slant
$d_0 = 0$ and $\beta = 1$	invariant
$d_0 = 0$ or $d_0 \neq 0$ and $\beta = 0$	anti-invariant
$d_0 \neq 0$ and $\beta = 1$	semi-invariant
$d_0 \neq 0$ and $\beta \neq 0, 1$	proper pseudo-slant

For pseudo-slant submanifold M, if P_0 and P_β denotes the projections on distributions $\mathfrak{D}_0$ and $\mathfrak{D}_\beta$, respectively. Then

$$X_1 = P_0 X_1 + P_\beta X_1 + \eta(X_1)\,\xi. \tag{54}$$

for any tangent vector field X_1. After operating ϕ in above equation, we get

$$\phi X_1 = n P_0 X_1 + t P_\beta X_1 + n P_\beta X_1. \tag{55}$$

Thus, we have

$$TM = t P_\beta X_1, \quad TM^\perp = n P_0 X_1 + n P_\beta X_1. \tag{56}$$

Following this, orthogonal decomposition of $TM^{\perp}$ is given as

$$TM^{\perp} = n\mathfrak{D}_0 \oplus n\mathfrak{D}_\beta \oplus N. \tag{57}$$

Here, N is the invariant subbundle orthogonal to both the distributions of $TM^{\perp}$. For $X_1 \in \mathfrak{D}_0$ and $X_2 \in \mathfrak{D}_\beta$, equations (9) and (5) results

$$g\left(nX_1, nX_2\right) = g\left(\phi X_1, \phi X_2\right) = -g\left(X_1, X_2\right) = 0. \tag{58}$$

Thus, the distributions $n\mathfrak{D}_0$ and $n\mathfrak{D}_\beta$ are orthogonal to each other which makes the decomposition of $TM^{\perp}$ given in equation (57), an orthogonal direct decomposition.

Theorem 6.2. *The necessary and sufficient condition for submanifold M of a paraSasakian manifold $\mathbf{M}^{2m+1}$ to be a pseuso-slant submanifold is that $\exists$ a distribution $\mathfrak{D}$ on M and a constant $\beta \in (-\infty, \infty)$ satisfying*

(i) $\mathfrak{D} = \{X_1 \in \Gamma(TM) \mid t^2 X_1 = \beta X_1\}$

(ii) $tX_1 = 0$, *for tangent vector field X_1 orthogonal to $\mathfrak{D}$.*

Further, β can be $cosh^2\theta$, $cos^2\theta$ or $-sinh^2\theta$.

Proof. Proof is similar to Theorem 5.3. □

Next, we work on the integrability and totally geodesic conditions of the distributions $\mathfrak{D}_0 \oplus \mathfrak{D}_\beta$, $\mathfrak{D}_0$ and $\mathfrak{D}_\beta$ of pseudo-slant submanifold M.

Theorem 6.3. *Let M be a proper pseudo-slant submanifold of paraSasakian manifold $\mathbf{M}^{2m+1}$, then*

(i) distribution $\mathfrak{D}_0 \oplus \mathfrak{D}_\beta$ is integrable if and only if $g(X_1, tX_2) = 0$ for $X_1, X_2 \in \mathfrak{D}_0 \oplus \mathfrak{D}_\beta$.

(ii) distribution $\mathfrak{D}_0$ is always integrable.

(iii) distribution $\mathfrak{D}_\beta \oplus < \xi >$ is integrable if and only if $A_{nX_1} tX_3 = A_{ntX_3} X_1$ for $X_1 \in \mathfrak{D}_0$ and $X_3 \in \mathfrak{D}_\beta$.

Proof. (i) After simple calculations and using equation (22) for $X_1, X_2 \in \mathfrak{D}_0 \oplus \mathfrak{D}_\beta$, we have

$$g([X_1, X_2], \xi) = 2g(X_1, tX_2).$$

As $g([X_1, X_2], \xi) = 0$ for the integrable distribution thus the result follows.

(ii) For vector fields $X_1, X_2 \in \mathfrak{D}_0$ and $X_3 \in \Gamma(TM)$, we attain

$$\begin{aligned} g\left(\phi[X_1, X_2], X_3\right) &= g\left(\phi\widehat{\nabla}_{X_1} X_2 - \phi\widehat{\nabla}_{X_2} X_1, X_3\right) \\ &= g\left(\widehat{\nabla}_{X_1}\phi X_2 - (\widehat{\nabla}_{X_1}\phi)X_2 - \widehat{\nabla}_{X_2}\phi X_1 + (\widehat{\nabla}_{X_2}\phi)X_1, X_3\right) \\ &= g\left(\widehat{\nabla}_{X_1} nX_2 - \widehat{\nabla}_{X_2} nX_1, X_3\right) \\ &= g\left(A_{nX_1} X_2 - A_{nX_2} X_1, X_3\right). \end{aligned}$$

Last expression implies $\mathfrak{D}_0$ is integrable distribution if and only if $A_{nX_1} X_2 = A_{nX_2} X_1$. But, this is always true for an anti-invariant distribution as

$$\begin{aligned} g\left(h(X_1, X_3), nX_2\right) = g\left(A_{nX_1} X_2, X_3\right) = g(\widehat{\nabla}_{X_1} X_3, nX_2) &= -\, g(X_3, \widehat{\nabla}_{X_1} nX_2) \\ &= g(A_{nX_2} X_1, X_3) \end{aligned}$$

Thus, distribution $\mathfrak{D}_0$ is always integrable.

(iii) For $X_1 \in \mathfrak{D}_0$ and $X_3, W \in \mathfrak{D}_\beta$, equations (2), (3.1) and (9) gives

$$g([X_3, W], X_1) = -g\left(\widehat{\nabla}_W X_3, X_1\right) - g\left(\widehat{\nabla}_{X_3} tW, nX_1\right) + g\left(\widehat{\nabla}_{X_3} t' nW - \widehat{\nabla}_{X_3} n' nW, X_1\right).$$

Using equation (13) and Proposition 3.4 in last equation, we get

$$
\begin{aligned}
g([X_3, W], X_1) &= -g\left(\widehat{\nabla}_W X_3, X_1\right) - g\left(A_{nX_1} tW, X_3\right) \\
&\quad + \beta^* g\left(\widehat{\nabla}_{X_3} W, X_1\right) - g\left(A_{ntW} X_1, X_3\right) \\
&= -g\left(A_{nX_3} tW + A_{ntW} X_1, X_3\right) + (\beta^* + 1)\, g([X_3, W], X_1) \\
\beta^* g([X_3, W], X_1) &= -g\left(A_{nX_1} tW + A_{ntW} X_1, X_3\right).
\end{aligned}
$$

Since $\beta^* \neq 0$, then the result follows. □

Theorem 6.4. *Let M be proper pseudo-slant submanifold of a paraSasakian manifold $\mathbf{M}^{2m+1}$ then both the distributions $\mathfrak{D}_0$ and $\mathfrak{D}_\beta \oplus < \xi >$ are totally geodesic if and only if $A_{nX_1} tX_3 = A_{ntX_3} X_1$ for $X_1 \in \mathfrak{D}_0$, $X_3 \in \mathfrak{D}_\beta \oplus < \xi >$.*

Proof. For $X_1, X_2 \in \mathfrak{D}_0$ and $X_3 \in \mathfrak{D}_\beta \oplus < \xi >$, we get

$$g(\nabla_{X_1} X_2 + h(X_1, X_2), X_3) = g(\widehat{\nabla}_{X_1} X_2, X_3) = -g(\phi \widehat{\nabla}_{X_1} X_2, \phi X_3),$$

using equations (8), (9), (11), (12) and (15), we get

$$g(\nabla_{X_1} X_2 + h(X_1, X_2), X_3) = g\left(A_{nX_2} tX_3, X_1\right) - g(X_2, \widehat{\nabla}_{X_1} X_3) - g(X_2, \widehat{\nabla}_{X_1} n' nX_3).$$

Applying Theorem 3.3 and 3.4, above equation turns as

$$(1 - \beta^*) g(\nabla_{X_1} X_2 + h(X_1, X_2), X_3) = g\left(A_{nX_2} tX_3 - A_{ntX_3} X_2, X_1\right).$$

Then from this expression it follows that $\mathfrak{D}_0$ is totally geodesic distribution if and only if the right hand side term vanishes. □

Remark 6.5. *The pseudo-slant submanifold M is locally product manifold if the distributions becomes totally geodesic. From above theorem it is clear that any pseudo-slant submanifold is locally product manifold if and only if $A_{nX_1} tX_3 = A_{ntX_3} X_1$ for $X_1 \in \mathfrak{D}_0$ and $X_3 \in \mathfrak{D}_\beta \oplus < \xi >$.*

Theorem 6.6. *Let M be proper pseudo-slant submanifold of a paraSasakian manifold $\mathbf{M}^{2m+1}$. If $\nabla n = 0$ on submanifold M then*

1. *anti-invariant distribution $\mathfrak{D}_0$ is either $\mathfrak{D}_0$-geodesic or the endomorphism $(n')^2$ has the eigenvalue zero for associated eigenvector h.*
2. *slant distribution $\mathfrak{D}_\beta$ is either $\mathfrak{D}_\beta$-geodesic or the endomorphism $(n')^2$ has the eigenvalue β for associated eigenvector h.*

here h is the second fundamental form on M.

Proof. Similar to Theorem 5.14. □

Theorem 6.7. *The proper pseudo-slant submanifold M of a paraSasakian manifold $\mathbf{M}^{2m+1}$ is anti-invariant submanifold if and only if $\nabla Q = 0$.*

Proof. Since, M is pseudo-slant submanifold then for $X_1, X_2 \in \mathfrak{D}_\beta$ by Theorem 3.3 we have

$$
\begin{aligned}
& t^2(\nabla_{X_1} X_2) = \beta\left(\nabla_{X_1} X_2 - \eta\left(\nabla_{X_1} X_2\right) \xi\right) \quad \text{and} \\
& \nabla_{X_1} t^2 X_2 = \beta\left(\nabla_{X_1} X_2 - \eta\left(\nabla_{X_1} X_2\right) \xi + g\left(X_2, \nabla_{X_1} \xi\right) \xi - \eta\left(X_2\right) \nabla_{X_1} \xi\right).
\end{aligned}
$$

Subtracting the above two equations and using equation (2.3), we have

$$\left(\nabla_{X_1} Q\right) X_2 = t^2(\nabla_{X_1} X_2) - \nabla_{X_1} t^2 X_2 = -\beta\left(g\left(X_2, \phi X_1\right) \xi - \eta\left(X_2\right) \phi X_1\right) = 0,$$

which is only possible if $\beta = 0$. Otherwise the metric condition does not satisfies as $g\left(\phi X_1, \phi X_2\right)$ becomes zero. Thus, M is an anti-invariant submanifold. □

Definition 6.8. *Any submanifold M is called totally umbilical if the condition $h(X_1, X_2) = g(X_1, X_2)H$ is satisfied for the tangent vector fields X_1, X_2 and mean curvature vector field H.*

Theorem 6.9. *A totally umbilical pseudo-slant submanifold M of a paraSasakian manifold $\mathbf{M}^{2m+1}$ is either anti-invariant or totally geodesic submanifold if H and $\nabla^{\perp}_{X_1} H \in \Gamma(\nu)$.*

Proof. By using Definition 3.1 for $X_1 \in \mathfrak{D}_\beta$, we have

$$(\widehat{\nabla}_{X_1}\phi)X_1 = g(X_1, X_1)\,\xi,$$
$$\widehat{\nabla}_{X_1}\phi X_1 - \phi(\widehat{\nabla}_{X_1}X_1) = g(X_1, X_1)\,\xi.$$

Using equations (9), (11), (12) and totally umbilical condition then taking inner product with ϕH, we are left with

$$g(\widehat{\nabla}_{X_1} n X_1, \phi H) = g(X_1, X_1)||H||^2. \tag{59}$$

On the other side, considering $\widehat{\nabla}_{X_1}\phi H = (\widehat{\nabla}_{X_1}\phi)H + \phi(\widehat{\nabla}_{X_1}H)$ and following the similar steps, then inner product with nX_1 gives

$$g(\phi H, \widehat{\nabla}_{X_1} n X_1) = (1-\beta)\, g(X_1, X_1)\, ||H||^2. \tag{60}$$

After comparing equations (59) and (60), we get $\beta g(X_1, X_1)\,||H||^2 = 0$. This gives the two conditions either $H = 0$ (totally geodesic) or $\beta = 0$ (anti-invariant). □

Example 6.10. *Let $\overline{M} = \mathbb{R}^6 \times \mathbb{R}_+ \subset \mathbb{R}^7$ be a 7-dimensional manifold having standard Cartesian coordinates as $(x_1, x_2, x_3, y_1, y_2, y_3, z)$. This manifold with paraSasakian structure defined in Example 2.4 is a paraSasakian manifold.*
Now, consider an isometrically immersed submanifold M with semi-Riemannian metric, and is defined by

$$M(u_1, u_2, v_1, v_2, z) = (a\,u_2, u_1 + v_2, v_1, v_2, u_2, v_1 + v_2, z)\,,$$

where $a \in \mathbb{R}$ and $a \neq -1$. Following are the vector fields which spans the tangent space TM

$$\begin{aligned} X_{u_1} &= \partial x^2 + y^2\partial z \\ X_{v_1} &= \partial x^3 + \partial y^3 + y^3\partial z \\ X_{u_2} &= a\,\partial x^1 + \partial y^2 + a\,y^1\partial z \\ X_{v_2} &= \partial x^2 + \partial y^1 + \partial y^3 + y^2\partial z \\ X_z &= 2\partial z. \end{aligned}$$

Thus, M defines the pseudo-slant submanifold with slant distribution

$$\mathfrak{D}_0 = \text{span}\{X_{u_2}, X_{v_2}\} \quad \text{and} \quad \mathfrak{D}_\beta = \text{span}\{X_{u_2}, X_{v_2}\},$$

where, slant coefficient $\beta = \frac{(a-1)}{(a+1)}$.

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