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A new approach for Hardy spaces with variable exponents on spaces of homogeneous type

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Abstract. In the paper, we establish and study Hardy spaces with variable exponents on spaces of homogeneous type (X, d, μ) in the sense of Coifman and Weiss, where d may have no any regularity property and μ fulfills the doubling property only. First we introduce the Hardy spaces with variable exponents $H^{p(\cdot)}(X)$ by using the wavelet Littlewood–Paley square functions and give their equivalent characterizations. Then we establish the atomic characterization theory for $H^{p(\cdot)}(X)$ via the new Calderón-type reproducing identity and the Littlewood–Paley–Stein theory. Finally, we give a unified method for defining these variable Hardy spaces $H^{p(\cdot)}(X)$ in terms of the same spaces of test functions and distributions. More precisely, we introduce the variable Carleson measure spaces $CMO^{p(\cdot)}_{L^2}(X)$ and characterize the variable Hardy spaces via the distributions of $CMO^{p(\cdot)}_{L^2}(X)$.

1. Introduction

The pioneer work on the real-variable theory of Hardy space $H^p(\mathbb{R}^n)$ was initiated by Stein and Weiss [33] and systematically developed by Fefferman and Stein [13]. Especially when $p \leq 1$, the spaces $H^p(\mathbb{R}^n)$ are better suited to a host of questions in various fields of analysis than the Lebesgue spaces $L^p(\mathbb{R}^n)$. The real-variable characterizations for Hardy space, such as the maximal, wavelet and Littlewood–Paley characterization, play an important role in Harmonic analysis. The Littlewood–Paley characterization of $H^p(\mathbb{R}^n)$ was due to Uchiyama [41]. For more details on $H^p(\mathbb{R}^n)$, see, for example, [19, 23, 27, 32, 38].

On the other hand, due to the fact that the variable exponent space was stimulated by the study of fluid dynamics, image processing and variational calculus, variable Lebesgue space, which is a generalization of the classical Lebesgue space, has been studied extensively since the early 1990s. See, for instance, [6, 8, 11, 12, 25]. The theory of the variable Hardy space was established independently by Nakai and Sawano [30], Cruz-Uribe and Wang [9]. Since then, the theory of real Hardy-type spaces with variable exponents has been attracting a lot of attention from many researchers (for instance, see [22, 31, 44, 45]). More recently, we further studied the variable Hardy and local Hardy spaces and gave some applications of these spaces in [34, 36, 40].

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To obtain the $H^p(\mathbb{R}^n)$−boundedness of the classical operators in harmonic analysis, one can appeal to the atomic decomposition theory of $H^p(\mathbb{R}^n)$, which is very useful in the study of harmonic analysis and related topics. The atomic characterization of $H^p(\mathbb{R}^n)$ was established by Coifman [3] for $n = 1$ and Latter [26] for $n \geq 2$. The atomic characterization provides more flexible approach to the study of general Hardy spaces on spaces of homogeneous type in the sense of Coifman and Weiss [4, 5]. Recently, the orthonormal wavelet basis with Hölder regularity and exponential decay for spaces of homogeneous type was constructed by Auscher and Hytönen [2]. The orthonormal wavelet basis play an important part in developing wavelet analysis on spaces of homogeneous type. Later, Han et al. [18] introduced and studied the theory of Hardy space on spaces of homogeneous type. Meanwhile, in [15] by establishing a new Calderón identity, the atomic decomposition result of the Hardy spaces on spaces of homogeneous type was obtained. Furthermore, they established the mapping properties of Calderón–Zygmund operators on Carleson measure spaces on spaces of homogeneous type. Also, the complete theories of real Hardy and local Hardy spaces on spaces of homogeneous type were obtained in [23, 24].

In the present paper, motivated by these studies, we will establish the theory of Hardy space with variable exponents on (X, d, μ), where d may have no any regularity property and μ fulfills the doubling property only. The first goal is to introduce the Hardy spaces with variable exponents $H^{p(\cdot)}(X)$ via the Littlewood–Paley square functions. Secondly, we aim to obtaining a new proof of the atomic characterization for $H^{p(\cdot)}(X)$. Very recently, as we were completing this paper we learned that the Hardy spaces associated with ball quasi-Banach function spaces on spaces of homogeneous type had been systematically developed by Yan et al. [42, 43]. When they are applied to variable Hardy spaces, they also define an atomic characterization. However, the approaches and results are slightly different from ours. Moreover, the convergence of the atomic decomposition in our article takes sense in both $L^q(X)$ and $H^{p(\cdot)}(X)$ norms whenver $f \in H^{p(\cdot)}(X) \cap L^q(X)$. Hence, the new atomic decomposition in our paper has many applications. Finally, we give a unified method for defining these variable Hardy spaces $H^{p(\cdot)}(X)$ by using the same test function spaces and the same distribution spaces. The new discrete Calderón-type reproducing formula is also a key tool through the paper.

The remainder of this paper is organized as follows. In Section 2, we first recall some basic definitions and necessary results about the wavelet analysis on spaces of homogeneous type and the test functions as well as the spaces of distributions. Some necessary results on variable Lebesgue spaces were also restated in this section. Section 3 concerns Hardy spaces defined by the wavelet and continuous Littlewood–Paley functions. Moreover, we obtain the equivalent Littlewood–Paley characterizations and the Plancherel–Pólya type inequalities. In Section 4, the main aim is to establish the atomic decomposition theory of $H^{p(\cdot)}(X)$ via the discrete Littlewood–Paley theory. Since the wavelets ψ^k_α have no compact supports, we can not obtain the atomic characterization theory for $H^{p(\cdot)}(X)$ by using the wavelet reproducing formula. To achieve it, a new discrete Calderón-type reproducing formula is needed. Meanwhile, the reconstruction theorem for the atomic decompositions of $H^{p(\cdot)}(X)$ is also established. To do so, we need the generalized Grafakos–Kalton lemma, which is very useful in our proofs. Finally, we give a unified method for defining these variable Hardy spaces $H^{p(\cdot)}(X)$ in terms of the same test function spaces and the distribution spaces in Section 5.

Throughout the paper, we need the following notations: The symbol $A \lesssim B$ denotes that there exists an absolute constant C such that $A \leq CB$ and the symbol $A \sim B$ means $A \lesssim B \lesssim A$ for some absolute constant which is independent of the main parameters, but may vary from line to line. For any $a,\ tb \in \mathbb{R}$, denote $a \wedge b := \min\{a, b\}$ and $a \wedge b := \max\{a, b\}$. For any set E of X, we use χ_E to denote its characteristic function and E^c the set $X \setminus E$.

2. Preliminaries

In this section, we will recall the necessary definitions and results about spaces of homogeneous type and the variable Lebesgue spaces.

2.1. The space of homogeneous type

In this subsection, we give some definitions and known results on spaces of homogeneous type in [2, 4, 18]. A quasi-metric d on a non-empty set X is a non-negative function defined on $X \times X$, fulfilling that,

(i) $d(x,y)=0$ if and only if $x=y$;
(ii) $d(x,y)=d(y,x)$ for all $x,\ y\in X$;
(iii) there exists a constant $A_0\in[1,\infty)$ such that for all $x,\ y,\ z\in X$, $d(x,z)\le A_0[d(x,y)+d(y,z)]$.

A quasi-metric space (X,d) is a non-empty set X together with a quasi-metric d. We say B is a quasi-metric ball B on a non-empty set X, where $x_0\in X$ is the center and $r>0$ is its radius, if

$$B:=B(x_0,r):=\{x\in X:\ d(x,x_0)<r\}.$$

For any ball B and some positive constant c, we write $cB:=B(x_0,cr)$, where x_0 is the center of B and r is its radius. A space of homogeneous type (X,d,μ) is a non-empty set X equipped with a quasi-metric d and a nonnegative measure μ fulfilling the following doubling condition: there exists a positive constant $C_{(\mu)}\in[1,\infty)$ such that, for any ball $B\subset X$,

$$\mu(2B)\le C_{(\mu)}\mu(B).$$

It is equivalent to that, for any ball B and $\lambda\in[1,\infty)$,

$$\mu(\lambda B)\le C_{(\mu)}\lambda^{\omega}\mu(B), \tag{1}$$

where $\omega:=\log_2 C_{(\mu)}$ is called the upper dimension of X. If $A_0=1$, we call (X,d,μ) a doubling metric measure space. Throughout the paper, we always let (X,d,μ) be a space of homogeneous type with $\mu(X)=\infty$.

Next, we recall the definitions of the space of test functions spaces and the distribution spaces in [18]. We also remark that the spaces of test function and the spaces of distribution were originally introduced by Han et al. [19, 20].

Definition 2.1. *Let $x_1\in X$, $r\in(0,\infty)$, $\beta\in(0,1]$ and $\gamma\in(0,\infty)$. A function f defined on X is said to be a test function of type (x_1,r,β,γ), denoted by $f\in\mathcal{G}(x_1,r,\beta,\gamma)$, if f fulfills the following conditions:*

(i) *for any $x\in X$,*

$$|f(x)|\le C\frac{1}{V_r(x_1)+V(x_1,x)}\left[\frac{r}{r+d(x_1,x)}\right]^{\gamma};$$

(ii) *for any $x,\ y\in X$ such that $d(x,y)\le(2A_0)^{-1}[r+d(x_1,x)]$,*

$$|f(x)-f(y)|\le C\left[\frac{d(x,y)}{r+d(x_1,x)}\right]^{\beta}\frac{1}{V_r(x_1)+V(x_1,x)}\left[\frac{r}{r+d(x_1,x)}\right]^{\gamma}.$$

For all $f\in\mathcal{G}(x_1,r,\beta,\gamma)$, we define the norm

$$\|f\|_{\mathcal{G}(x_1,r,\beta,\gamma)}:=\inf\{C\in(0,\infty):\ C\text{ satisfies (i) and (ii)}\}.$$

Define

$$\mathring{\mathcal{G}}(x_1,r,\beta,\gamma):=\left\{f\in\mathcal{G}(x_1,r,\beta,\gamma):\ \int_X f(x)\,d\mu(x)=0\right\}$$

equipped with the norm $\|\cdot\|_{\mathring{\mathcal{G}}(x_1,r,\beta,\gamma)}:=\|\cdot\|_{\mathcal{G}(x_1,r,\beta,\gamma)}$.

Fix $x_0\in X$. For any $x\in X$ and $r\in(0,\infty)$, we know that $\mathcal{G}(x,r,\beta,\gamma)=\mathcal{G}(x_0,1,\beta,\gamma)$ with equivalent norms, but the equivalent positive constants depend on x and r. Obviously, $\mathcal{G}(x_0,1,\beta,\gamma)$ is a Banach space. In what follows, we simply write $\mathcal{G}(\beta,\gamma):=\mathcal{G}(x_0,1,\beta,\gamma)$ and $\mathring{\mathcal{G}}(\beta,\gamma):=\mathring{\mathcal{G}}(x_0,1,\beta,\gamma)$. Fix $\epsilon\in(0,1]$ and $\beta,\ \gamma\in(0,\epsilon)$. Let $\mathring{\mathcal{G}}_0^{\epsilon}(\beta,\gamma)$ be the completion of the set $\mathring{\mathcal{G}}(\epsilon,\epsilon)$ in $\mathcal{G}(\beta,\gamma)$, that is, if $f\in\mathring{\mathcal{G}}_0^{\epsilon}(\beta,\gamma)$, then there exists $\{\phi_j\}_{j=1}^{\infty}\subset\mathring{\mathcal{G}}(\epsilon,\epsilon)$ such that $\|\phi_j-f\|_{\mathcal{G}(\beta,\gamma)}\to0$ as $j\to\infty$. If $f\in\mathring{\mathcal{G}}_0^{\epsilon}(\beta,\gamma)$, we then let $\|f\|_{\mathring{\mathcal{G}}_0^{\epsilon}(\beta,\gamma)}:=\|f\|_{\mathcal{G}(\beta,\gamma)}$. The dual space $(\mathring{\mathcal{G}}_0^{\epsilon}(\beta,\gamma))'$ is defined to be the collection of all continuous linear functionals on $\mathring{\mathcal{G}}_0^{\epsilon}(\beta,\gamma)$ and equipped with the weak-$*$ topology. Then we call $(\mathcal{G}_0^{\epsilon}(\beta,\gamma))'$ and $(\mathring{\mathcal{G}}_0^{\epsilon}(\beta,\gamma))'$ the distributions spaces.

We also need the following orthonormal wavelet basis on spaces of homogeneous type in [2, Theorem 7.1].

Theorem 2.2. *Suppose that (X, d, μ) is a space of homogeneous type in the sense of Coifman and Weiss with quasi-triangle constant A_0, and*

$$a := (1 + 2\log_2 A_0)^{-1}. \tag{2}$$

There exists an orthonormal wavelet basis $\{\psi_\alpha^k\}$, $k \in \mathbb{Z}$, $y_\alpha^k \in \mathscr{Y}^k$, of $L^2(X)$, having exponential decay

$$|\psi_\alpha^k(x)| \le \frac{C}{\sqrt{\mu(B(y_\alpha^k, \delta^k))}} \exp\Big(-\nu\Big(\frac{d(y_\alpha^k, x)}{\delta^k}\Big)^a\Big), \tag{3}$$

Hölder regularity

$$|\psi_\alpha^k(x) - \psi_\alpha^k(y)| \le \frac{C}{\sqrt{\mu(B(y_\alpha^k, \delta^k))}} \Big(\frac{d(x,y)}{\delta^k}\Big)^\eta \exp\Big(-\nu\Big(\frac{d(y_\alpha^k, x)}{\delta^k}\Big)^a\Big) \tag{4}$$

for $d(x, y) \le \delta^k$, and the cancellation property

$$\int_X \psi_\alpha^k(x)\, d\mu(x) = 0, \qquad \text{for} \qquad k \in \mathbb{Z}. \tag{5}$$

Moreover, the wavelet expansion is given by

$$f(x) = \sum_{k\in\mathbb{Z}} \sum_{\alpha\in\mathscr{Y}^k} \langle f, \psi_\alpha^k\rangle \psi_\alpha^k(x) \tag{6}$$

in the sense of $L^2(X)$.

Notice that δ is a fixed small parameter, say $\delta \le 10^{-3}A_0^{-10}$, and $C < \infty$, $\nu > 0$ and $\eta \in (0, 1]$ are constants which is independent of k, α, x and y_α^k. Motivated by these, the wavelet representation for the test and distribution was obtained in [18].

Theorem 2.3. *Suppose that β, $\gamma \in (0, \eta)$. If $f \in (\mathring{\mathcal{G}}_0^\eta(\beta, \gamma))'$, then we have the following wavelet reproducing identity*

$$f(x) = \sum_{k\in\mathbb{Z}} \sum_{\alpha\in\mathscr{Y}^k} \langle f, \psi_\alpha^k\rangle \psi_\alpha^k(x),$$

which holds in $\mathring{\mathcal{G}}_0^\eta(\beta', \gamma')$ and in the distributions space $(\mathring{\mathcal{G}}_0^\eta(\beta', \gamma'))'$ for each $\beta' \in (0, \beta)$, $\gamma' \in (0, \gamma)$.

2.2. *Hardy–Littlewood maximal operator on variable Lebesgue spaces*

Let $L^1_{\rm loc}(X)$ be the space of all locally integrable functions on X. Denote by M the Hardy–Littlewood maximal function, that is, for all $f \in L^1_{\rm loc}(X)$,

$$M(f)(x) := \sup_{B\ni x} \frac{1}{\mu(B)} \int_B |f(y)|\, d\mu(y),$$

where the supremum is taken over all balls B of X that contain x. For any $p \in (0, \infty]$, the Lebesgue space $L^p(X)$ is defined to be the set of all μ-measurable functions f such that

$$\|f\|_{L^p(X)} := \Big[\int_X |f(x)|^p\, d\mu(x)\Big]^{\frac{1}{p}} < \infty.$$

From [5], we learn that M is of strong $(L^p(X) - L^p(X))$−type whenever $p \in (1, \infty]$. In what follows, let $p(\cdot) : X \to (0, \infty)$ be a μ−measurable function fulfilling

$$0 < p^- := \text{essinf}_{x\in X} p(x) \le p^- := \text{esssup}_{x\in X} p(x) =: p^+ < \infty.$$

Moreover, write $p_- := \min(1, p^-)$. Denote by $\mathcal{P}^0$ the set of all variable exponents on X with $0 < p^- \le p^+ < \infty$ and denote by $\mathcal{P}$ the set of all variable exponents on X with $1 < p^- \le p^+ < \infty$. The variable Lebesgue space $L^{p(\cdot)}(X)$ is defined to be the set of all $\mu-$measurable functions f on X such that $\int_X |f(x)|^{p(x)} d\mu(x) < \infty$ and equipped with the quasi-norm

$$\|f\|_{L^{p(\cdot)}(X)} := \inf\{\lambda \in (0,\infty) : \int_X \left|\frac{f(x)}{\lambda}\right|^{p(x)} d\mu(x) \le 1\}.$$

Then $L^{p(\cdot)}(X)$ is a quasi-Banach function spaces. Moreover, if $p^- \ge 1$, it is a Banach space. In the study of variable exponent function spaces it is common to assume that the exponent function $p(\cdot)$ satisfies the LH conditions. In what follows, we always fix the base point x_0, which plays the same role as the origin of $\mathbb{R}^n$. We say that $p(\cdot) \in LH$, if $p(\cdot)$ satisfies

$$|p(x) - p(y)| \le \frac{C}{\log(e + 1/d(x,y))}$$

for all $x, y \in X$ and if there exist $p_\infty \in \mathbb{R}$ satisfying that, for all $x \in X$,

$$|p(x) - p_\infty| \le \frac{C}{\log(e + d(x, x_0))}.$$

Next we recall the following lemma in [1, Corollary 1.8].

Lemma 2.4. *Suppose that $p(\cdot) \in \mathcal{P} \cap LH$ and B be a ball of X. Then there exists a positive constant C such that, for all measurable functions $f \in L^{p(\cdot)}(X)$,*

$$\|M(f)\|_{L^{p(\cdot)}(X)} \le C\|f\|_{L^{p(\cdot)}(X)}.$$

Moreover, for all $\lambda \in (1, \infty)$,

$$\|\chi_{\lambda B}\|_{L^{p(\cdot)}(X)} \le C\lambda^{n/\lambda}\|\chi_B\|_{L^{p(\cdot)}(X)}.$$

The known Fefferman–Stein vector-valued inequality on $L^{p(\cdot)}(X)$ in [46, Theorem 2.7] is also needed in our proofs.

Lemma 2.5. *Let $p(\cdot) \in \mathcal{P} \cap LH$ and $u \in (1, \infty)$. Then for any measurable functions sequence $\{f_i\}_{i=1}^\infty \subset L^{p(\cdot)}(X)$,*

$$\left\|\left\{\sum_{i=1}^\infty [M(f_i)]^u\right\}^{\frac{1}{u}}\right\|_{L^{p(\cdot)}(X)} \le C\left\|\left(\sum_{i=1}^\infty |f_i|^u\right)^{\frac{1}{u}}\right\|_{L^{p(\cdot)}(X)}.$$

3. Littlewood–Paley characterizations of $H^{p(\cdot)}(X)$

We first define the wavelet and continuous Littlewood–Paley functions in this section. Then the proof of the Littlewood–Paley square functions characterization is established by the help with the Plancherel–Pólya type inequalities. Finally, we introduce the Hardy spaces with variable exponents $H^{p(\cdot)}(X)$ and obtain some equivalent Littlewood–Paley characterizations.

Definition 3.1. *Suppose that $\beta, \gamma \in (0, \eta)$. Let $\{E_k\}_{k\in\mathbb{Z}}$ be the operators on $L^2(X)$ associated with integral kernels*

$$E_k(x,y) := \sum_{\alpha \in \mathscr{Y}_k} \psi_\alpha^k(x)\psi_\alpha^k(y), \qquad \forall\ x,\ y \in X.$$

Let $f \in (\mathring{\mathcal{G}}_0^\eta(\beta,\gamma))'$, the continuous Littlewood–Paley square function is defined by

$$S_c(f)(x) := \left[\sum_{k=-\infty}^{\infty} |E_k f(x)|^2\right]^{\frac{1}{2}}.$$

Also, for any $f \in (\mathring{\mathcal{G}}_0^\eta(\beta,\gamma))'$, the wavelet Littlewood–Paley function $S(f)$ is defined by

$$S(f)(x) := \left\{\sum_{k\in\mathbb{Z}} \sum_{\alpha\in\mathscr{Y}^k} \left|\langle \psi_\alpha^k, f\rangle \widetilde{\chi}_{Q_\alpha^k}(x)\right|^2\right\}^{1/2},$$

where $\widetilde{\chi}_{Q_\alpha^k}(x) := \chi_{Q_\alpha^k}(x)(\mu_i(Q_\alpha^k))^{-1/2}$ and $\chi_{Q_\alpha^k}(x)$ is the characteristic function of the dyadic cube Q_α^k.

Theorem 3.2. *Suppose that $0 < \beta, \gamma < \eta$ and $p(\cdot) \in \mathcal{P}^0 \cap LH$ with $\frac{\omega}{\omega+\eta} < p^- \le p^+ < \infty$. Then for any $f \in L^2(X)$,*

$$\|S_c(f)\|_{L^{p(\cdot)}(X)} \sim \|S(f)\|_{L^{p(\cdot)}(X)}.$$

To achieve our goal, we next need to obtain the Plancherel–Pólya type inequalities in the variable exponent setting as follows. The proof of the following Plancherel–Pólya type inequalities is nearly identical to that in [18, Theorem 4.4] (e.g., also see [10, 17]). For convenience, we will give the outline of the proof and show the differences.

Proposition 3.3. *Let $0 < \beta, \gamma < \eta$ and $p(\cdot) \in \mathcal{P}^0 \cap LH$ with $\frac{\omega}{\omega+\eta} < p^- \le p^+ < \infty$. Then for all $f \in L^2(X)$,*

$$\left\|\left\{\sum_{k'} \sum_{\alpha'\in\mathscr{X}^{k'+N}} \left[\sup_{z\in Q_{\alpha'}^{k'+N}} |D_{k'}(f)(z)|^2\right] \chi_{Q_{\alpha'}^{k'+N}}\right\}^{\frac{1}{2}}\right\|_{L^{p(\cdot)}(X)} \le C \left\|\left\{\sum_{k} \sum_{\alpha\in\mathscr{Y}^k} \left|\left\langle \psi_\alpha^k, f\right\rangle \widetilde{\chi}_{Q_\alpha^k}\right|^2\right\}^{\frac{1}{2}}\right\|_{L^{p(\cdot)}(X)}.$$

Furthermore, for all $f \in L^2(X)$ with a sufficiently large $N \in \mathbb{N}$. , we have

$$\left\|\left\{\sum_{k} \sum_{\alpha\in\mathscr{Y}^k} \left|\left\langle \psi_\alpha^k, f\right\rangle \widetilde{\chi}_{Q_\alpha^k}\right|^2\right\}^{\frac{1}{2}}\right\|_{L^{p(\cdot)}(X)} \le C \left\|\left\{\sum_{k'} \sum_{\alpha'\in\mathscr{X}^{k'+N}} \left[\sup_{z\in Q_{\alpha'}^{k'+N}} |D_{k'}(f)(z)|^2\right] \chi_{Q_{\alpha'}^{k'+N}}\right\}^{\frac{1}{2}}\right\|_{L^{p(\cdot)}(X)}.$$

Proof. We begin with the following wavelet identity in (6):

$$f(x) = \sum_{k\in\mathbb{Z}} \sum_{\alpha\in\mathscr{Y}^k} \langle f, \psi_\alpha^k\rangle \psi_\alpha^k(x)$$

in $L^2(X)$. Then for each $z \in Q_{\alpha'}^{k'+N}$ we deduce that

$$D_{k'}(f)(z) = \sum_{k\in\mathbb{Z}} \sum_{\alpha\in\mathscr{Y}^k} \mu(Q_\alpha^k) \left\langle f, \frac{\psi_\alpha^k}{\sqrt{\mu(Q_\alpha^k)}}\right\rangle \left\langle \frac{\psi_\alpha^k}{\sqrt{\mu(Q_\alpha^k)}}, D_{k'}(\cdot, z)\right\rangle.$$

Furthermore, applying the standard technique in [18, pp.149-150] and the Hölder inequality yield that

$$\begin{aligned}
&\sum_{k'}\sum_{\alpha'\in\mathscr{X}^{k'+N}}\sup_{z\in Q_{\alpha'}^{k'+N}}|D_{k'}(f)(z)|^2\chi_{Q_{\alpha'}^{k'+N}}(x)\\
&\le C\sum_{k'}\left|\sum_{k}\delta^{|k-k'|\eta}\delta^{[k-(k\wedge k')]\omega(1-1/r)}\left\{M\left(\sum_{\alpha\in\mathscr{Y}}\left|\left\langle f,\frac{\psi_\alpha^k}{\sqrt{\mu(Q_\alpha^k)}}\right\rangle\right|^r\chi_{Q_\alpha^k}(\cdot)\right)(x)\right\}^{\frac1r}\right|^2\\
&\le C\sum_{k'}\left(\sum_{k}\delta^{|k-k'|\eta}\delta^{[k-(k\wedge k')]\omega(1-1/r)}\right)\\
&\times\left(\sum_{k}\delta^{|k-k'|\eta}\delta^{[k-(k\wedge k')]\omega(1-1/r)}\left\{M\left(\sum_{\alpha\in\mathscr{Y}}\left|\left\langle f,\frac{\psi_\alpha^k}{\sqrt{\mu(Q_\alpha^k)}}\right\rangle\right|^r\chi_{Q_\alpha^k}(\cdot)\right)(x)\right\}^{\frac2r}\right)\\
&\le C\sum_{k}\left\{M\left(\sum_{\alpha\in\mathscr{Y}}\left|\left\langle f,\frac{\psi_\alpha^k}{\sqrt{\mu(Q_\alpha^k)}}\right\rangle\right|^r\chi_{Q_\alpha^k}(\cdot)\right)(x)\right\}^{\frac2r},
\end{aligned}$$

where $\frac{\omega}{\omega+\eta}<r<p_-$. Therefore, by Lemma 2.5, we conclude that

$$\left\|\left\{\sum_{k'}\sum_{\alpha'\in\mathscr{X}^{k'+N}}\left[\sup_{z\in Q_{\alpha'}^{k'+N}}|D_{k'}(f)(z)|^2\right]\chi_{Q_{\alpha'}^{k'+N}}\right\}^{\frac12}\right\|_{L^{p(\cdot)}(X)}\le C\left\|\left\{\sum_k\sum_{\alpha\in\mathscr{Y}^k}\left|\left\langle\psi_\alpha^k,f\right\rangle\widetilde{\chi}_{Q_\alpha^k}\right|^2\right\}^{\frac12}\right\|_{L^{p(\cdot)}(X)}.$$

To end it, by repeating the similar argument in [18, Theorem 4.4], there exits an operator T_N such that T_N^{-1} is bounded on $L^2(X)$. Hence, we have that the $L^{p(\cdot)}(X)$ norm of $S(T_N^{-1}(f))$ is controlled by that of $S(f)$ with the help of Lemma 2.5. The rest of the proof is identical. □

Now we are ready to prove Theorem 3.2.

Proof of Theorem 3.2. Note that

$$\begin{aligned}
\sum_{k'}\sum_{\alpha'\in\mathscr{X}^{k'+N}}\inf_{z\in Q_{\alpha'}^{k'+N}}\left|D_{k'}(f)(z)\right|^2\chi_{Q_{\alpha'}^{k'+N}}(x)&\le\sum_k\left|D_k(f)(x)\right|^2\\
&\le\sum_{k'}\sum_{\alpha'\in\mathscr{X}^{k'+N}}\sup_{z\in Q_{\alpha'}^{k'+N}}\left|D_{k'}(f)(z)\right|^2\chi_{Q_{\alpha'}^{k'+N}}(x).
\end{aligned}$$

From the estimate with Proposition 3.3, then we obtain the equivalence of the $L^{p(\cdot)}(X)$-norm of two Littlewood–Paley functions. This ends the proof of this theorem. □

Below we give the definition of Hardy space with variable exponents $H^{p(\cdot)}(X)$ on space of homogeneous type in the sense of Coifman and Weiss in terms of the wavelet Littlewood–Paley function.

Definition 3.4. *Let $0<\beta,\gamma<\eta$ and $p(\cdot)\in\mathcal{P}^0$. The wavelet Hardy space with variable exponents $H^{p(\cdot)}(X)$, $\frac{\omega}{\omega+\eta}<p^-\le p^+<\infty$, is defined as the completion of the collection of all $f\in L^2(X)$ for which the quantity*

$$\|f\|_{H^{p(\cdot)}(X)}:=\|S(f)\|_{L^{p(\cdot)}(X)}<\infty.$$

Combining Propositions 3.3 with Theorem 3.2, we immediately obtain the corollary as follows.

Corollary 3.5. *Fix a sufficiently large integer N. Let $0<\beta,\gamma<\eta$ and $p(\cdot)\in\mathcal{P}^0\cap LH$ with $\frac{\omega}{\omega+\eta}<p^-\le p^+<\infty$. Then for any $f\in L^2(X)$, then*

$$\|f\|_{H^{p(\cdot)}(X)}\sim\|S_c(f)\|_{L^{p(\cdot)}(X)}\sim\|S_d(f)\|_{L^{p(\cdot)}(X)},$$

where

$$S_d(f)(x) := \left(\sum_k \sum_{\alpha \in \mathscr{Y}^{k+N}} |D_k(f)(x_\alpha^{k+N})|^2 \chi_{Q_\alpha^{k+N}}(x) \right)^{1/2}.$$

4. The atomic decomposition of $H^{p(\cdot)}(X)$

In this section, we will establish the atomic decomposition of $H^{p(\cdot)}(X)$ by the use of the discrete Littlewood–Paley–Stein theory. Atomic characterization for Hardy spaces with variable exponents on $\mathbb{R}^n$ was established in [9, 30]. Atomic decomposition characterization plays an very important part in the real-variable theory of function spaces and the boundedness of operators (e.g., also see [35, 37, 39]). We first introduce the atom a for $H^{p(\cdot)}(X)$.

Definition 4.1. *Suppose that $p(\cdot) \in LH, 0 < p^- \le p^+ < q < \infty$ and $q \ge 1$. We say a function a is a $(p(\cdot), q)$–atom of $H^{p(\cdot)}(X)$, if a is supported in a cube $Q \subset \mathbb{R}^n$,*

$$\|a\|_{L^q(X)} \le |Q|^{\frac{1}{q}} \|\chi_Q\|_{L^{p(\cdot)}}^{-1},$$

and

$$\int_Q a(x) d\mu(x) = 0.$$

The set of all such pairs (a, Q) will be denoted by $A(p(\cdot), q)$. To obtain the atomic decomposition, we need to establish a new discrete Calderón-type reproducing formula. We apply a fundamental result in [29, Theorem 2], and recall the space of homogeneous type (X, d', μ), where the quasi-metric d' fulfills the following condition:

$$|d'(x, y) - d'(x', y)| \le C_0 d'(x, x')^\theta \ [d'(x, y) + d'(x', y)]^{1-\theta}$$

for some $C_0 > 0$, $0 < \theta = \frac{ln2}{ln3+2lnA_0} < 1$, and any x, x' $y \in X$. By using Coifman's construction for an approximation to the identity, there exists a family of operators S_k fulfilling that:

(i) $S_k(x, y) = 0$ for $d'(x, y) \ge C\delta^k$, and $\|S_k\|_\infty \le C \frac{1}{V_{\delta^k}(x) + V_{\delta^k}(y)}$,

(ii) $|S_k(x, y) - S_k(x', y)| \le C\Big(\frac{d'(x, x')}{\delta^k}\Big)^\theta \frac{1}{V_{\delta^k}(x) + V_{\delta^k}(y)}$,

(iii) $|S_k(x, y) - S_k(x, y')| \le C\Big(\frac{d'(y, y')}{\delta^k}\Big)^\theta \frac{1}{V_{\delta^k}(x) + V_{\delta^k}(y)}$,

(iv) $|[S_k(x, y) - S_k(x, y')] - [S_k(x', y) - S_k(x', y')]|$
$\le C\Big(\frac{d'(x, x')}{\delta^k}\Big)^\theta \Big(\frac{d'(y, y')}{\delta^k}\Big)^\theta \frac{1}{V_{\delta^k}(x) + V_{\delta^k}(y)}$,

(v) $\int_X S_k(x, y) d\mu(y) = \int_X S_k(x, y) d\mu(x) = 1.$

Here and below, we denote $\tilde{\eta} = \eta \wedge \theta$.

Proposition 4.2. *Let $p(\cdot) \in \mathcal{P}^0 \cap LH$ with $\frac{\omega}{\omega+\tilde{\eta}} < p^- \le p^+ < \infty$ and $1 < q < \infty$. Set $D_k = S_{k+1} - S_k$. Then there is a unique function $g \in L^q(X) \cap H^{p(\cdot)}(X)$ satisfying $\|f\|_{L^q(X)} \sim \|g\|_{L^q(X)}$, $\|f\|_{H^{p(\cdot)}(X)} \sim \|g\|_{H^{p(\cdot)}(X)}$ with*

$$f(x) = \sum_k \sum_{\alpha \in \mathscr{Y}^{K+N}} \mu(Q_\alpha^{k+N}) D_k(x, x_\alpha^{k+N}) \tilde{D}_k(g)(x_\alpha^{k+N}),$$

where the series holds in the space $H^{p(\cdot)}(X)$ and the space $L^q(X)$ and where N is a large fixed integer and $\tilde{D}_k = \sum_{|j| \le N} D_{k+j}$.

Proof. For a fixed integer N and $f \in L^2(X)$, by applying Coifman's decomposition

$$\begin{aligned} f(x) &= \sum_l D_l(f)(x) = \sum_l \sum_k D_l D_k(f)(x) \\ &= \sum_k \sum_{\alpha \in \mathscr{Y}^{K+N}} \mu(Q_\alpha^{k+N}) D_k(x, x_\alpha^{k+N}) \tilde{D}_k(g)(x_\alpha^{k+N}) \\ &\quad + \left(\sum_k D_k \tilde{D}_k(f)(x) - \sum_k \sum_{\alpha \in \mathscr{Y}^{K+N}} \mu(Q_\alpha^{k+N}) D_k(x, x_\alpha^{k+N}) \tilde{D}_k(g)(x_\alpha^{k+N}) \right) \\ &\quad + \sum_k \sum_{|k-l|>N} D_k D_l(f)(x) \\ &=: T_N(f)(x) + R_N^{(1)}(f)(x) + R_N^{(2)}(f)(x), \end{aligned}$$

where the series converges in $L^2(X)$ norm. From [15, pp.19], we know that for $f \in L^2(X)$, $\tilde{q} \in (\frac{\omega}{\omega+\eta}, \infty)$ and $i = 1, 2$,

$$\|S(R_N^{(i)}(f))\|_{L^{\tilde{q}}(X)} \le C\delta^{\theta N} \|f\|_{L^{\tilde{q}}(X)}.$$

Particularly, for $1 < q < \infty$ we have

$$\|R_N^{(i)}(f)\|_{L^q(X)} \le C\delta^{\theta N} \|f\|_{L^q(X)}.$$

On the other hand, from [15, pp. 22] we also have the key inequality as follows:

$$\begin{aligned} &\left\{ \sum_k \sum_{\alpha \in \mathscr{Y}^k} \left| \langle \psi_\alpha^k, R_N^{(i)}(f) \rangle \widetilde{\chi}_{Q_\alpha^k}(x) \right|^2 \right\}^{1/2}, \\ &\le C\delta^{\theta N} \Bigg\{ \sum_k \sum_{\alpha \in \mathscr{Y}^k} \Bigg| \sum_{k'} \sum_{\alpha' \in \mathscr{Y}^{k'}} \mu(Q_{\alpha'}^{k'}) \delta^{|k'-k|\eta''} \frac{1}{V_{\delta^{(k\wedge k')}}(x_{\alpha'}^{k'}) + V_{\delta^{(k\wedge k')}}(x_\alpha^k) + V(x_{\alpha'}^{k'}, x_\alpha^k)} \\ &\quad \times \left(\frac{\delta^{(k\wedge k')}}{\delta^{(k\wedge k')} + d(x_{\alpha'}^{k'}, x_\alpha^k)} \right)^\gamma \left| \left\langle \frac{\psi_{\alpha'}^{k'}}{\sqrt{\mu(Q_{\alpha'}^{k'})}}, f \right\rangle \right| \Bigg|^2 \chi_{Q_\alpha^k}(x) \Bigg\}^{1/2} \end{aligned}$$

where $\widetilde{\chi}_{Q_\alpha^k}(x) := \chi_{Q_\alpha^k}(x) \mu(Q_\alpha^k)^{-1/2}$. Then by repeating the same argument as in [14, pp. 147-148] and Lemma 2.5, we conclude that

$$\|S(R_N^{(i)}(f))\|_{L^{p(\cdot)}(X)} \le C\delta^{\theta N} \|f\|_{L^{p(\cdot)}(X)}.$$

Thus, we can choose N enough large such that $2C\delta^{\theta N} < 1$ and then T_N^{-1} is bounded on $L^2(X) \cap H^{p(\cdot)}(X)$. Let $g = T_N^{-1}(f)$. Then

$$\|f\|_{L^q(X)} \sim \|g\|_{L^q(X)}, \qquad \|f\|_{H^{p(\cdot)}(X)} \sim \|g\|_{H^{p(\cdot)}(X)}.$$

Moreover,

$$f(x) = \sum_k \sum_{\alpha \in \mathscr{Y}^{K+N}} \mu(Q_\alpha^{k+N}) D_k(x, x_\alpha^{k+N}) \tilde{D}_k(g)(x_\alpha^{k+N}),$$

where the series converges in both norms of $L^2(X)$.

Next, we shall prove that the series converges in $H^{p(\cdot)}(X)$ norm, we only need to check that $\|S(f_L)\|_{L^{p(\cdot)}(X)} \to 0$ as $L \to \infty$, where

$$f_L(x) = \sum_{|k|>L} \sum_{\alpha \in \mathscr{Y}^{K+N}} \mu(Q_\alpha^{k+N}) D_k(x, x_\alpha^{k+N}) \tilde{D}_k(g)(x_\alpha^{k+N}).$$

In fact, following the similar argument as above we obtain that

$$\|S(f_L)\|_{L^{p(\cdot)}(X)} \leq C\left\|\left\{\sum_{|k|>L}\sum_{\alpha\in\mathscr{Y}^{K+N}}|\tilde{D}_k(g)(x_{Q_k})|^2\chi_{Q_k}\right\}^{1/2}\right\|_{L^{p(\cdot)}}$$

and then $\|S(f_L)\|_{L^{p(\cdot)}}$ tends to zero as $L\to\infty$. Finally, we show that the series also converges in $L^q(X)$, $1<q<\infty$, we assume that $f\in L^2(X)\cap L^q(X)$. To end it, we only need to show that for each function $f\in L^2(X)\cap L^q(X)$,

$$\left\|\sum_{|k|>M}\sum_{\alpha\in\mathscr{Y}^{K+N}}\mu(Q_\alpha^{k+N})D_k(x,x_\alpha^{k+N})\tilde{D}_k(g)(x_\alpha^{k+N})\right\|_{L^q(X)}$$

tends to zero as $L\to\infty$. Applying duality and Hölder's inequality yield that

$$\left\|\sum_{|k|>M}\sum_{\alpha\in\mathscr{Y}^{K+N}}\mu(Q_\alpha^{k+N})D_k(x,x_\alpha^{k+N})\tilde{D}_k(g)(x_\alpha^{k+N})\right\|_{L^q(X)} \leq C\left\|\left\{\sum_{|k|>M}\sum_{\alpha\in\mathscr{Y}^{K+N}}|\tilde{D}_k(g)(x_{Q_k})|^2\chi_{Q_k}\right\}^{1/2}\right\|_{L^q(X)},$$

which tends to zero as $L\to\infty$. Since $L^2(X)\cap L^q(X)$ is dense in $L^q(X)$. Therefore, by a standard density argument, we conclude that the convergence of the series in $L^q(X)$. This proves the proposition. □

Now we state the atomic decompositions for $H^{p(\cdot)}(X)$.

Theorem 4.3. *Suppose that $p(\cdot)\in LH$, $\frac{\omega}{\omega+\tilde{\eta}}<p^-\leq p^+<\infty$ and $(1\vee p^+)<q<\infty$. If $f\in L^q(X)\cap H^{p(\cdot)}(X)$, there is a sequence of $(p(\cdot),q)$–atoms $\{a_j\}$ and a sequence of non-negative scalars $\{\lambda_j\}$ with*

$$\mathcal{A}(\{\lambda_j\}_{j=1}^\infty,\{Q_j\}_{j=1}^\infty)\leq C\|f\|_{H^{p(\cdot)}(X)},$$

such that $f=\sum_j\lambda_j a_j$, where the series converges to f in both $H^{p(\cdot)}(X)$ and $L^q(X)$ norms, and

$$\mathcal{A}(\{\lambda_j\}_{j=1}^\infty,\{Q_j\}_{j=1}^\infty)=\left\|\sum_j\frac{\lambda_j\chi_{Q_j}}{\|\chi_{Q_j}\|_{L^{p(\cdot)}}}\right\|_{L^{p(\cdot)}(X)}<\infty.$$

Proof. Suppose that $f\in H^{p(\cdot)}(X)\cap L^q(X)$. Then by Proposition 4.2,

$$f(x)=\sum_k\sum_{\alpha\in\mathscr{Y}^{K+N}}\mu(Q_\alpha^{k+N})D_k(x,x_\alpha^{k+N})\tilde{D}_k(g)(x_\alpha^{k+N}),$$

where $g=T_N^{-1}(f)$ and $\|f\|_{L^q(X)}\sim\|g\|_{L^q(X)}$, $\|f\|_{H^{p(\cdot)}(X)}\sim\|g\|_{H^{p(\cdot)}(X)}$. Denote

$$S_d(g)(x):=\left(\sum_k\sum_{\alpha\in\mathscr{Y}^{k+N}}|\tilde{D}_k(g)(x_\alpha^{k+N})|^2\chi_{Q_\alpha^{k+N}}(x)\right)^{1/2}.$$

By the variable Plancherel-Pólya type inequalities we have that

$$\|f\|_{H^{p(\cdot)}(X)}\sim\|S_d(g)\|_{L^{p(\cdot)}(X)}.$$

Next, we set

$$\Omega_\ell=\left\{x\in X: S_d(g)(x)>2^\ell\right\}, B_\ell=\left\{Q_\alpha^k:\mu(Q_\alpha^k\cap\Omega_\ell)>\frac{1}{2}\mu(Q_\alpha^k),\ \mu(Q_\alpha^k\cap\Omega_{\ell+1})\leq\frac{1}{2}\mu(Q_\alpha^k)\right\},$$

and

$$\widetilde{\Omega}_\ell=\left\{x\in X: M\chi_{\Omega_\ell}(x)>\frac{1}{100}\right\},$$

where M is the Hardy-Littlewood maximal operator on X. Then it is easy to see that $\mu(\widetilde{\Omega_\ell}) \leq C\mu(\Omega_\ell)$. Denoting $\tilde{Q}_\alpha^{j,\ell} \in B_\ell$ are maximal dyadic cubes, By the discrete Calderón identity we can rewrite

$$f(x) = \sum_\ell \sum_{j:\tilde{Q}_\alpha^{j,\ell} \in B_\ell} \sum_{Q_\alpha^{k+N} \subset \tilde{Q}_\alpha^{j,\ell}} \mu(Q_\alpha^{k+N}) D_k(x, x_\alpha^{k+N}) \tilde{D}_k(g)(x_\alpha^{k+N}) =: \sum_\ell \sum_{j:\tilde{Q}_\alpha^{j,\ell} \in B_\ell} \lambda_{\tilde{Q}_\alpha^{j,\ell}}^\ell a_{\tilde{Q}_\alpha^{j,\ell}}^\ell(x),$$

where

$$a_{\tilde{Q}_\alpha^{j,\ell}}^\ell := \frac{1}{\lambda_{\tilde{Q}_\alpha^{j,\ell}}^\ell} \sum_{Q_\alpha^{k+N} \subset \tilde{Q}_\alpha^{j,\ell}} \mu(Q_\alpha^{k+N}) D_k(x, x_\alpha^{k+N}) \tilde{D}_k(g)(x_\alpha^{k+N})$$

and

$$\lambda_{\tilde{Q}_\alpha^{j,\ell}}^\ell := C \left\| \chi_{c\tilde{Q}_\alpha^{j,\ell}} \right\|_{L^{p(\cdot)}(X)} \mu(\tilde{Q}_\alpha^{j,\ell})^{-1/q} \left\| \left\{ \sum_{Q \subset \tilde{Q}_\alpha^{j,\ell}} |\tilde{D}_k(g)(x_{Q_\alpha^{k+N}})|^2 \chi_{Q_\alpha^{k+N}} \right\}^{1/2} \right\|_{L^q(X)}.$$

By Proposition 4.2, we know that the series converges in $H^{p(\cdot)}(X) \cap L^q(X)$. Observe that d and d' are geometrically equivalent. From the definition of $a_{\tilde{Q}}^\ell$ and the fact that $D_k(x, x_\alpha^{k+N})$ have compact supports, we conclude that $a_{\tilde{Q}_\alpha^{j,\ell}}^\ell$ is supported in $c\tilde{Q}_\alpha^{j,\ell}$. The cancellation conditions of $a_{\tilde{Q}_\alpha^{j,\ell}}^\ell$ follows from the vanishing moment of $D_k(x, x_\alpha^k)$. For $1 < q, q' < \infty$, by the duality argument together with Cauchy-Schwarz's and Hölder's inequalities, we have

$$\begin{aligned}
&\left\| \sum_{Q_\alpha^{k+N} \subset \tilde{Q}_\alpha^{j,\ell}} \mu(Q_\alpha^{k+N}) D_k(\cdot, x_\alpha^{k+N}) \tilde{D}_k(g)(x_\alpha^{k+N}) \right\|_{L^q} \\
= &\sup_{\|h\|_{L^{q'}} \leq 1} \left| \left\langle \sum_{Q_\alpha^{k+N} \subset \tilde{Q}_\alpha^{j,\ell}} \mu(Q_\alpha^{k+N}) D_k(\cdot, x_\alpha^{k+N}) \tilde{D}_k(g)(x_\alpha^{k+N}), h \right\rangle \right| \\
= &\sup_{\|h\|_{L^{q'}} \leq 1} \left| \int_X \sum_{Q_\alpha^{k+N} \subset \tilde{Q}_\alpha^{j,\ell}} D_k(h)(x_\alpha^{k+N}) \tilde{D}_k(g)(x_\alpha^{k+N}) \chi_{Q_\alpha^{k+N}}(x) dx \right| \\
\leq &C \left\| \left\{ \sum_{Q \subset \tilde{Q}_\alpha^{j,\ell}} |\tilde{D}_k(g)(x_{Q_\alpha^{k+N}})|^2 \chi_{Q_\alpha^{k+N}} \right\}^{1/2} \right\|_{L^q(X)}.
\end{aligned}$$

Then it implies that

$$\left\| a_{\tilde{Q}_\alpha^{j,\ell}}^\ell \right\|_{L^q(X)} \leq C \frac{\mu(\tilde{Q}_\alpha^{j,\ell})^{1/q}}{\|\chi_{c\tilde{Q}_\alpha^{j,\ell}}\|_{L^{p(\cdot)}}(X)}.$$

Thus, each $a_{\tilde{Q}_\alpha^{j,\ell}}^\ell$ is a $(p(\cdot), q)-$atom of $H^{p(\cdot)}(X)$. Finally, we need to show that

$$\mathcal{A}(\{\lambda_{\tilde{Q}_\alpha^{j,\ell}}^\ell\}_{\ell=1}^\infty, \{c\tilde{Q}_\alpha^{j,\ell}\}_{\ell=1}^\infty) \leq C\|f\|_{H^{p(\cdot)}(X)}.$$

If $x \in Q_\alpha^{k+N} \in B_\ell$, then

$$\chi_{Q_\alpha^{k+N}}(y) \leq CM^2(\chi_{Q_\alpha^{k+N} \cap \tilde{\Omega}_\ell \backslash \Omega_{\ell+1}})(y).$$

By the Fefferman-Stein vector valued maximal inequality in [13], for all $1 < q < \infty$ we conclude that

$$\begin{aligned}
&\left\|\left\{\sum_{Q\subset\tilde{Q}_\alpha^{j,\ell}} |\tilde{D}_k(g)(x_{Q_\alpha^{k+N}})|^2\chi_{Q_\alpha^{k+N}}\right\}^{1/2}\right\|_{L^q(X)}^q \\
&\leq C\int_X\left(\sum_{Q\subset\tilde{Q}_\alpha^{j,\ell}}\left|\tilde{D}_k(g)(x_{Q_\alpha^{k+N}})M(\chi_{Q_\alpha^{k+N}\cap\tilde{\Omega}_\ell\setminus\Omega_{\ell+1}})(y)\right|^2\right)^{q/2} dy \\
&\leq C\int_{\tilde{Q}\cap\tilde{\Omega}_\ell\setminus\Omega_{\ell+1}}\left(\sum_{Q\subset\tilde{Q}_\alpha^{j,\ell}}\left|\tilde{D}_k(g)(x_{Q_\alpha^{k+N}})\chi_{Q_\alpha^{k+N}}\right|^2\right)^{q/2} dx \\
&\leq C2^{\ell q}\mu(\tilde{Q}_\alpha^{j,\ell}).
\end{aligned}$$

As a consequence, we conclude

$$\left\|\left\{\sum_{Q\subset\tilde{Q}_\alpha^{j,\ell}} |\tilde{D}_Q(bf)(x_Q)|^2\chi_Q\right\}^{1/2}\right\|_{L^q} \leq C2^{\ell q}\mu(\tilde{Q}_\alpha^{j,\ell}),$$

which implies that

$$\left\|\sum_\ell\sum_{\tilde{Q}_\alpha^{j,\ell}\in B_\ell}\frac{\left|\lambda^\ell_{\tilde{Q}_\alpha^{j,\ell}}\right|\chi_{c\tilde{Q}_\alpha^{j,\ell}}}{\|\chi_{c\tilde{Q}_\alpha^{j,\ell}}\|_{L^{p(\cdot)}(X)}}\right\|_{L^{p(\cdot)}(X)} \leq C\left\|\sum_\ell\sum_{\tilde{Q}_\alpha^{j,\ell}\in B_\ell}2^\ell\chi_{c\tilde{Q}_\alpha^{j,\ell}}\right\|_{L^{p(\cdot)}(X)}.$$

Since $\Omega_\ell \subset \tilde{\Omega}_\ell$ for each $\ell \in \mathbb{Z}$ and $\mu(\tilde{\Omega}_\ell) \leq C\mu(\Omega_i)$, for all $x \in X$. From the definition of $\mathcal{D}_k$, we get that $\mu(Q_\alpha^{j,\ell}) \leq C\mu(\Omega_\ell \cap \tilde{Q}_\alpha^{j,\ell})$, and that

$$\chi_{\tilde{Q}_\alpha^{j,\ell}}(x) \leq CM^h\chi_{\Omega_\ell\cap\tilde{Q}_\alpha^{j,\ell}}(x),$$

where h is a fixed constant such that and $h > 1$ and $hp_- > 1$.
Combining this with Lemma 2.5, we deduce that

$$\begin{aligned}
&\mathcal{A}(\{\lambda^\ell_{\tilde{Q}_\alpha^{j,\ell}}\}_{\ell=1}^\infty, \{c\tilde{Q}_\alpha^{j,\ell}\}_{\ell=1}^\infty) \leq C\left\|\sum_\ell\sum_{j:\tilde{Q}_\alpha^{j,\ell}\in B_\ell}2^\ell\chi_{c\tilde{Q}_\alpha^{j,\ell}}\right\|_{L^{p(\cdot)}(X)} \\
&\leq C\left\|\sum_\ell\sum_{j:\tilde{Q}_\alpha^{j,\ell}\in B_\ell}2^\ell M^h\chi_{\Omega_\ell\cap\tilde{Q}_\alpha^{j,\ell}}\right\|_{L^{p(\cdot)}(X)} = C\left\|\left\{\sum_\ell\sum_{j:\tilde{Q}_\alpha^{j,\ell}\in B_\ell}2^\ell M^h\chi_{\Omega_\ell\cap\tilde{Q}_\alpha^{j,\ell}}\right\}^{\frac{1}{h}}\right\|_{L^{hp_-}(X)}^h \\
&\leq C\left\|\left\{\sum_\ell 2^\ell\chi_{\Omega_\ell}\right\}^{\frac{1}{h}}\right\|_{L^{hp_-}(X)}^h = C\left\|\sum_\ell 2^\ell\chi_{\Omega_\ell}\right\|_{L^{p(\cdot)}(X)}.
\end{aligned}$$

Note that $\Omega_{\ell+1} \subset \Omega_\ell$ and $\mu(\bigcap_{\ell=1}^\infty \Omega_\ell) = 0$, then for $a.e\ x \in X$ we conclude that

$$\sum_{\ell=-\infty}^\infty 2^\ell\chi_{\Omega_\ell}(x) = \sum_{\ell=-\infty}^\infty 2^\ell\sum_{m=\ell}^\infty\chi_{\Omega_m\setminus\Omega_{m+1}}(x) = 2\sum_{m=-\infty}^\infty 2^m\chi_{\Omega_m\setminus\Omega_{m+1}}(x),$$

and it implies that

$$\left\|\sum_{\ell} 2^{\ell}\chi_{\Omega_{\ell}}\right\|_{L^{p(\cdot)}(X)} \leq C\left\|\sum_{m} 2^{m}\chi_{\Omega_m\setminus\Omega_{m+1}}\right\|_{L^{p(\cdot)}(X)} \leq \left\|S_d(g)\sum_{m}\chi_{\Omega_m\setminus\Omega_{m+1}}\right\|_{L^{p(\cdot)}(X)}$$
$$\leq \left\|S_d(g)\right\|_{L^{p(\cdot)}(X)} \leq C\|f\|_{H^{p(\cdot)}(X)}.$$

This finishes the proof of Theorem 4.3. □

Next, we obtain the reconstruction theorem for the atomic decompositions of $H^{p(\cdot)}(X)$. The following generalized Grafakos–Kalton lemma is needed for our proofs. Applying nearly identical argument to [7, Section 4], we could establish this lemma. For brevity, we omit the details.

Lemma 4.4. *Given $p(\cdot) \in \mathcal{P}_0 \cap LH$. Fix $q > 1$. Suppose that $0 < p^+ < q$, then for given countable collections of cubes $\{Q_j\}_{j=1}^{\infty}$ and of nonnegative measurable functions $\{g_j\}_{j=1}^{\infty}$ such that* $\operatorname{supp}(g_j) \subset Q_j$,

$$\left\|\sum_{j=1}^{\infty} g_j\right\|_{L^{p(\cdot)}(X)} \leq C\left\|\sum_{j=1}^{\infty}\left(\frac{1}{\mu(Q_j)}\int_{Q_j} g_j^q\right)^{\frac{1}{q}}\chi_{Q_j}\right\|_{L^{p(\cdot)}(X)}.$$

Theorem 4.5. *Let $p(\cdot) \in \mathcal{P}^0 \cap LH$ with $\frac{\omega}{\omega+\eta} < p^- \leq p^+ < \infty$ and $(1 \vee p^+) < q < \infty$. For any $\{a_j, Q_j\} \subset A(p(\cdot), q)$ satisfying*

$$\mathcal{A}(\{\lambda_j\}_{j=1}^{\infty}, \{Q_j\}_{j=1}^{\infty}) < \infty,$$

then the series $f = \sum_j \lambda_j a_j$ converges in $H^{p(\cdot)}(X)$ and satisfies

$$\|f\|_{H^{p(\cdot)}(X)} \leq C\mathcal{A}(\{\lambda_j\}_{j=1}^{\infty}, \{Q_j\}_{j=1}^{\infty}).$$

Proof. For and fixed non-negative integer N and $\tilde{Q}_j := 2A_0Q_j(x_j, r_j)$,

$$\begin{aligned} S_c(f)(x) &\leq \sum_{j=1}^{N}|\lambda_j|S_c(a_j)(x) \\ &\leq \sum_{j=1}^{N}|\lambda_j|S_c(a_j)(x)\chi_{\tilde{Q}_j}(x) + \sum_{j=1}^{N}|\lambda_j|S_c(a_j)(x)\chi_{(\tilde{Q}_j)^c}(x) \\ &= I_1 + I_2. \end{aligned}$$

For any $(p(\cdot), q)$−atoms $\{a_j\}_{j=1}^{\infty}$ with supp $a_j \subset Q_j$ and $x \in (\tilde{Q}_j)^c$, following the same argument in [15, pp. 3436-3437], we conclude that

$$S_c(a_j)(x) \leq C\frac{\mu(Q_j)}{\|\chi_{Q_j}\|_{L^{p(\cdot)}(X)}}\left(\frac{r_j}{d(x, x_j)|}\right)^{\eta}\frac{1}{V(x, x_j)}.$$

It implies

$$I_2 \leq C\sum_{j=1}^{N}|\lambda_j|\frac{1}{|x - c_{Q_j}|^{n+\epsilon}}\frac{(l(Q_j))^{n+\epsilon}}{\|\chi_{Q_j}\|_{L^{p(\cdot)}}}\chi_{(2Q)^c}(x).$$

Denote $\gamma = \frac{\omega+\eta}{\omega}$. Then we further conclude that

$$\begin{aligned} I_2 &\leq C\sum_{j=1}^{N}|\lambda_j|\frac{1}{\|\chi_{Q_j}\|_{L^{p(\cdot)}(X)}}\left(\frac{\mu(Q_j)}{\mu(x, d(x, x_j)}\right)^{\gamma}\chi_{(\tilde{Q}_j)^c}(x) \\ &\leq C\sum_{j=1}^{N}\frac{|\lambda_j|}{\|\chi_{Q_j}\|_{L^{p(\cdot)}(X)}}[M(\chi_{Q_j}))(x)]^{\gamma}. \end{aligned}$$

From Lemma 2.5 and $p^- \in (\frac{\omega}{\omega+\eta}, \infty)$, we deduce that

$$\begin{aligned}
\|I_2\|_{L^{p(\cdot)}(X)} &\le C \left\| \sum_{j=1}^{N} \frac{|\lambda_j|}{\|\chi_{Q_j}\|_{L^{p(\cdot)}(X)}} (M\chi_{Q_j})^{\gamma} \right\|_{L^{p(\cdot)}(X)} \\
&= C \left\| \left(\sum_{j=1}^{N} \frac{|\lambda_j|}{\|\chi_{Q_j}\|_{L^{p(\cdot)}(X)}} (M\chi_{Q_j})^{\gamma} \right)^{1/\gamma} \right\|_{L^{\gamma p(\cdot)}(X)}^{\gamma} \le \left\| \left(\sum_{j=1}^{N} \frac{|\lambda_j|}{\|\chi_{Q_j}\|_{L^{p(\cdot)}}} \chi_{Q_j} \right)^{1/\gamma} \right\|_{L^{\gamma p(\cdot)}}^{\gamma} \\
&\le \left\| \sum_{j=1}^{N} \frac{|\lambda_j|}{\|\chi_{Q_j}\|_{L^{p(\cdot)}(X)}} \chi_{Q_j} \right\|_{L^{p(\cdot)}(X)} = C\mathcal{A}(\{\lambda_j\}_{j=1}^{\infty}, \{Q_j\}_{j=1}^{\infty}).
\end{aligned}$$

For the term I, applying the $L^q(X)$ boundedness of S_c gives that

$$\|S_c(a_j)\|_{L^q(X)} \le C\|a_j\|_{L^q(X)} \le C \frac{\mu(Q_j)^{\frac{1}{q}}}{\|\chi_{Q_j}\|_{L^{p(\cdot)}(X)}}.$$

Combining the fact $\operatorname{supp}(a_j) \subset \tilde{Q}_j$ with Lemma 4.4,

$$\begin{aligned}
\|I_1\|_{L^{p(\cdot)}(X)} &\le \left\| \left(\sum_{j=1}^{\infty} \left(\frac{|\lambda_j|}{\|\chi_{Q_j}\|_{L^{p(\cdot)}}} h_j(x) \right)^{p_-} \right)^{\frac{1}{p_-}} \right\|_{L^{p(\cdot)}} \\
&\le \left\| \sum_{j=1}^{N} \frac{|\lambda_j|}{\|\chi_{Q_j}\|_{L^{p(\cdot)}(X)}} \chi_{Q_j} \right\|_{L^{p(\cdot)}(X)} = C\mathcal{A}(\{\lambda_j\}_{j=1}^{\infty}, \{Q_j\}_{j=1}^{\infty}).
\end{aligned}$$

From the above estimates, then we have

$$\left\| \sum_{j=1}^{N} \lambda_j a_j \right\|_{H_b^{p(\cdot)}} \le C\mathcal{A}(\{\lambda_j\}_{j=1}^{\infty}, \{Q_j\}_{j=1}^{\infty}).$$

Then, for any $1 \le N_1 \le N_2 < \infty$,

$$\left\| \sum_{j=N_1}^{N_2} \lambda_j a_j \right\|_{H_b^{p(\cdot)}} \le \left\| \sum_{j=N_1}^{N_2} \frac{|\lambda_j|}{\|\chi_{Q_j}\|_{L^{p(\cdot)}(X)}} \chi_{Q_j} \right\|_{L^{p(\cdot)}(X)}.$$

Note that

$$\mathcal{A}(\{\lambda_j\}_{j=1}^{\infty}, \{Q_j\}_{j=1}^{\infty}) < \infty.$$

It implies that

$$\lim_{N\to\infty} \left\| \sum_{j=N}^{\infty} \frac{|\lambda_j|}{\|\chi_{Q_j}\|_{L^{p(\cdot)}}} \chi_{Q_j} \right\|_{L^{p(\cdot)}(X)} = 0.$$

Therefore, $\left\{ \sum_{j=1}^{N} \lambda_j a_j \right\}_{j=1}^{\infty}$ is a Cauchy in $H^{p(\cdot)}(X)$ and converges to an element $f \in H^{p(\cdot)}(X)$ and

$$\|f\|_{H^{p(\cdot)}(X)} \le C \lim_{N\to\infty} \left\| \sum_{j=1}^{N} \lambda_j a_j \right\|_{H^{p(\cdot)}(X)} \le C\mathcal{A}(\{\lambda_j\}_{j=1}^{\infty}, \{Q_j\}_{j=1}^{\infty}).$$

The proof of Theorem 4.5 is now complete. □

5. A unified approach for variable Hardy and Carleson measure spaces

In this section, we will give a unified method for defining these Hardy spaces with variable exponents $H^{p(\cdot)}(X)$ in terms of the same test function spaces and the distribution spaces. First, let us define the variable Carleson measure spaces on $L^2(X)$.

Definition 5.1. *For $p(\cdot) \in \mathcal{P}^0$ we define the quasi norm for $f \in L^2(X)$ by*

$$\|f\|_{C_{p(\cdot)}}(X) := \sup_P \left\{ \frac{\mu(P)}{\|\chi_P\|^2_{L^{p(\cdot)}(X)}} \sum_{Q^k_\alpha \subset P, k \in \mathbb{Z}, \alpha \in \mathscr{Y}^k,} \left|\langle \psi^k_\alpha, f\rangle\right|^2 \right\}^{1/2},$$

where P runs over all dyadic cubes in X. Furthermore, we denote

$$CMO^{p(\cdot)}_{L^2}(X) = \{g \in L^2(X) : \|g\|_{C_{p(\cdot)}}(X) < \infty\}.$$

The fundamental duality argument on L^2 is given as follows, which is a generalization of [16, Theorem 2.17].

Proposition 5.2. *Let $p(\cdot) \in \mathcal{P}^0 \cap LH$ with $\frac{\omega}{\omega+\eta} < p^- \le p^+ \le 1$ and $f, g \in L^2(X)$. Then*

$$\left|\left\langle f, g\right\rangle\right| \le C\|S(f)\|_{L^{p(\cdot)}(X)}\|g\|_{C_{p(\cdot)}(X)},$$

where $S(f)(x)$ is the wavelet Littlewood–Paley function of f in Section 2.

Proof. Let $f, g \in L^2(X)$. By using the wavelet reproducing identity, we conclude that

$$\left\langle f, g\right\rangle = \left\langle \sum_{k\in\mathbb{Z}} \sum_{\alpha\in\mathscr{Y}^k} \langle f, \psi^k_\alpha\rangle \psi^k_\alpha(x), g\right\rangle = \sum_{k\in\mathbb{Z}} \sum_{\alpha\in\mathscr{Y}^k} \langle f, \psi^k_\alpha\rangle\langle g, \psi^k_\alpha\rangle.$$

Now we set

$$\Omega_\ell = \left\{x \in X : S(f)(x) > 2^\ell\right\},$$

$$B_\ell = \left\{Q^k_\alpha : \mu\left(Q^k_\alpha \cap \Omega_\ell\right) > \frac{1}{2}\mu(Q^k_\alpha) \text{ and } \mu\left(Q^k_\alpha \cap \Omega_{\ell+1}\right) \le \frac{1}{2}\mu(Q^k_\alpha)\right\},$$

and

$$\widetilde{\Omega}_\ell = \{x \in X : M(\chi_{\Omega_\ell})(x) > \frac{1}{2}\}.$$

Then we rewrite

$$\left\langle f, g\right\rangle = \sum_{k\in\mathbb{Z}} \sum_{\alpha\in\mathscr{Y}^k} \langle f, \psi^k_\alpha\rangle\langle g, \psi^k_\alpha\rangle = \sum_\ell \sum_{j:\tilde{Q}^{j,\ell}_\alpha \in B_\ell} \sum_{Q^k_\alpha \subset \tilde{Q}^{j,\ell}_\alpha} \langle f, \psi^k_\alpha\rangle\langle g, \psi^k_\alpha\rangle,$$

where $\tilde{Q}^{j,\ell}_\alpha$ is the maximal dyadic cubes in B_ℓ. By applying the Cauchy-Schwartz inequality, we deduce that

$$\begin{aligned}
|\left\langle f, g\right\rangle| &\le \sum_\ell \sum_{j:\tilde{Q}^{j,\ell}_\alpha \in B_\ell} \left(\sum_{Q^k_\alpha \subset \tilde{Q}^{j,\ell}_\alpha} |\langle f, \psi^k_\alpha\rangle|^2\right)^{\frac{1}{2}} \left(\sum_{Q^k_\alpha \subset \tilde{Q}^{j,\ell}_\alpha} |\langle g, \psi^k_\alpha\rangle|^2\right)^{\frac{1}{2}} \\
&\le C\|g\|_{C_{p(\cdot)}} \sum_\ell \sum_{j:\tilde{Q}^{j,\ell}_\alpha \in B_\ell} \mu\left(\tilde{Q}^{j,\ell}_\alpha\right)^{-\frac{1}{2}} \|\chi_{\tilde{Q}^{j,\ell}_\alpha}\|_{L^{p(\cdot)}(X)} \left(\sum_{j:Q^k_\alpha \subset \tilde{Q}^{j,\ell}_\alpha} |\langle f, \psi^k_\alpha\rangle|^2\right)^{\frac{1}{2}} \\
&\le C\|g\|_{C_{p(\cdot)}} \sum_\ell \left(\sum_{j:\tilde{Q}^{j,\ell}_\alpha \in B_\ell} \mu\left(\tilde{Q}^{j,\ell}_\alpha\right)^{-1} \|\chi_{\tilde{Q}^{j,\ell}_\alpha}\|^2_{L^{p(\cdot)}(X)}\right)^{\frac{1}{2}} \left(\sum_{j:\tilde{Q}^{j,\ell}_\alpha \in B_\ell} \sum_{j:Q^k_\alpha \subset \tilde{Q}^{j,\ell}_\alpha} |\langle f, \psi^k_\alpha\rangle|^2\right)^{\frac{1}{2}} \\
&\le C\|g\|_{C_{p(\cdot)}} \sum_\ell \left(\sum_{j:\tilde{Q}^{j,\ell}_\alpha \in B_\ell} \mu\left(\tilde{Q}^{j,\ell}_\alpha\right)^{-\frac{1}{2}} \|\chi_{\tilde{Q}^{j,\ell}_\alpha}\|_{L^{p(\cdot)}(X)}\right) \left(\sum_{j:\tilde{Q}^{j,\ell}_\alpha \in B_\ell} \sum_{Q^k_\alpha \subset \tilde{Q}^{j,\ell}_\alpha} |\langle f, \psi^k_\alpha\rangle|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

When $\mu(\tilde{Q}_\alpha^{j,\ell}) \leq 1$ and $x \in \tilde{Q}_\alpha^{j,\ell}$, by [28, Lemma 3.3] and the fact that all $\tilde{Q}_\alpha^{j,\ell}$ are disjoint we have

$$\sum_{j:\tilde{Q}_\alpha^{j,\ell}\in B_\ell} \mu(\tilde{Q}_\alpha^{j,\ell})^{-\frac{1}{2}} \|\chi_{\tilde{Q}_\alpha^{j,\ell}}\|_{L^{p(\cdot)}(X)} \sim \sum_{j:\tilde{Q}_\alpha^{j,\ell}\in B_\ell} \mu(\tilde{Q}_\alpha^{j,\ell})^{\frac{1}{p(x)}-\frac{1}{2}} \leq \mu(\tilde{\Omega}_\ell)^{\frac{1}{p(x)}-\frac{1}{2}} \sim \mu(\tilde{\Omega}_\ell)^{-\frac{1}{2}} \|\chi_{\tilde{\Omega}_\ell}\|_{L^{p(\cdot)}(X)}.$$

Similarly, when $\mu(\tilde{Q}_\alpha^{j,\ell}) > 1$ we also have

$$\sum_{j:\tilde{Q}_\alpha^{j,\ell}\in B_\ell} \mu(\tilde{Q}_\alpha^{j,\ell})^{-\frac{1}{2}} \|\chi_{\tilde{Q}_\alpha^{j,\ell}}\|_{L^{p(\cdot)}(X)} \leq C\mu(\tilde{\Omega}_\ell)^{-\frac{1}{2}} \|\chi_{\tilde{\Omega}_\ell}\|_{L^{p(\cdot)}(X)}.$$

Applying [28, Lemma 3.3] again yields that

$$\sum_{j:\tilde{Q}_\alpha^{j,\ell}\in B_\ell} \mu(\tilde{Q}_\alpha^{j,\ell})^{-\frac{1}{2}} \|\chi_{\tilde{Q}_\alpha^{j,\ell}}\|_{L^{p(\cdot)}(X)} \leq C\mu(\Omega_\ell)^{-\frac{1}{2}} \|\chi_{\Omega_\ell}\|_{L^{p(\cdot)}(X)}.$$

On the other hand,

$$\sum_{j:\tilde{Q}_\alpha^{j,\ell}\in B_\ell} \sum_{Q_\alpha^k\subset\tilde{Q}_\alpha^{j,\ell}} |\langle f, \psi_\alpha^k\rangle|^2 = \sum_{Q_\alpha^k\in B_\ell} |\langle f, \psi_\alpha^k\rangle|^2,$$

and

$$\sum_{Q_\alpha^k\in B_\ell} |\langle f, \psi_\alpha^k\rangle|^2 \leq C2^{2\ell}\mu(\Omega_\ell).$$

Indeed, note that for $Q_k^\alpha \in B_\ell$, $\mu(\widetilde{\Omega}_\ell/\Omega_{\ell+1} \cap Q_k^\alpha) \geq \frac{1}{2}\mu(Q_k^\alpha)$. It follows that

$$\int_{\widetilde{\Omega}_\ell/\Omega_{\ell+1}} (\mu(Q_k^\alpha))^{-1}\chi_{Q_k^\alpha}(x)d\mu(x) \geq \frac{1}{2}.$$

Then

$$\begin{aligned}&\int_{\widetilde{\Omega}_\ell/\Omega_{\ell+1}} \sum_{Q_k^\alpha\in B_\ell} |\langle f, \psi_\alpha^k\rangle|^2 (\mu(Q_k^\alpha))^{-1}\chi_{Q_k^\alpha}(x)d\mu(x)\\ &\leq \int_{\widetilde{\Omega}_\ell/\Omega_{\ell+1}} |S(f)(x)|^2 d\mu(x) \leq C2^{2\ell}\mu\big(\widetilde{\Omega}_\ell\big) \leq C2^{2\ell}\mu\big(\Omega_\ell\big).\end{aligned}$$

Combining with these above estimates, we conclude that

$$\begin{aligned}
|\langle f, g\rangle| &\leq C\|g\|_{C_{p(\cdot)}} \sum_\ell \mu(\Omega_\ell)^{-\frac{1}{2}} \|\chi_{\Omega_\ell}\|_{L^{p(\cdot)}(X)} \Big(2^{2\ell}\mu\big(\Omega_\ell\big)\Big)^{\frac{1}{2}}\\
&\leq C\|g\|_{C_{p(\cdot)}} \sum_\ell 2^\ell \|\chi_{\Omega_\ell}\|_{L^{p(\cdot)}(X)}\\
&\leq C\|g\|_{C_{p(\cdot)}} \left\| \Big(\sum_\ell (2^\ell\chi_{\Omega_\ell})^{p_-}\Big)^{\frac{1}{p_-}} \right\|_{L^{p(\cdot)}(X)}\\
&\leq C\|g\|_{C_{p(\cdot)}} \left\| \Big(\sum_\ell (2^\ell\chi_{\Omega_\ell\setminus\Omega\ell+1})^{p_-}\Big)^{\frac{1}{p_-}} \right\|_{L^{p(\cdot)}(X)}\\
&\leq C\|g\|_{C_{p(\cdot)}} \left\| \sum_\ell 2^\ell\chi_{\Omega_\ell\setminus\Omega\ell+1} \right\|_{L^{p(\cdot)}(X)}\\
&\leq C\|g\|_{C_{p(\cdot)}} \left\| S(f)\sum_\ell \chi_{\Omega_\ell\setminus\Omega\ell+1} \right\|_{L^{p(\cdot)}(X)} \leq C\|g\|_{C_{p(\cdot)}} \|S(f)\|_{L^{p(\cdot)}(X)}.
\end{aligned}$$

Therefore, we have completed the proof of Proposition 5.2. □

Remark 5.3. *By Proposition 5.2, each $f \in H^{p(\cdot)}(X) \cap L^2(X)$ can be considered as a linear functional on $CMO_{L^2}^{p(\cdot)}(X)$ and conversely, each $g \in CMO_{L^2}^{p(\cdot)}(X)$ can be considered as a linear functional on $H^{p(\cdot)}(X) \cap L^2(X)$. This naturally leads to study the closure of $H^{p(\cdot)}(X) \cap L^2(X)$ and the closure of $CMO_{L^2}^{p(\cdot)}(X)$.*

Next we will consider $CMO_{L^2}^{p(\cdot)}(X)$ as a space of test function for establishing $H^p(X)$ by the set of some linear functionals on $CMO_{L^2}^{p(\cdot)}(X)$. So we need the following discrete Calderón reproducing formula in the distribution sense.

Proposition 5.4. *Suppose that $p(\cdot) \in \mathcal{P}^0 \cap LH$ with $\frac{\omega}{\omega+\eta} < p^- \le p^+ \le 1$. Let $\{f_n\}$ be a Cauchy sequence in $L^2(X)$ with respect to the norm of $H^{p(\cdot)}(X)$. Then for each $g \in CMO_{L^2}^{p(\cdot)}(X)$,*

$$\lim_{n\to\infty} \left\langle f_n, g \right\rangle = \left\langle f, g \right\rangle$$

and f has a representation of wavelet

$$f(x) = \sum_{k\in\mathbb{Z}} \sum_{\alpha\in\mathscr{Y}^k} \langle f, \psi_\alpha^k \rangle \psi_\alpha^k(x),$$

where the series converges in $(CMO_{L^2}^{p(\cdot)}(X))'$. Moreover, $\|S(f - f_n)\|_{L^{p(\cdot)}(X)}$ tends to zero as $n \to \infty$.

Proof. Let f_n be a Cauchy sequence in $L^2(X)$ with respect to the norm of $H^{p(\cdot)}(X)$. By applying Proposition 5.2, we deduce that for each $g \in CMO_{L^2}^{p(\cdot)}(X)$,

$$\left|\left\langle f_n - f_m, g \right\rangle\right| \le C\|S(f_n - f_m)\|_{L^{p(\cdot)}(X)}\|g\|_{C_{p(\cdot)}(X)}$$

which implies that for each $g \in CMO_{L^2}^{p(\cdot)}(X)$,

$$\lim_{n\to\infty} \left\langle f_n, g \right\rangle = \left\langle f, g \right\rangle.$$

By Fatou's lemma we obtain that

$$\begin{aligned} \lim_{n\to\infty} \|S(f_n - f)\|_{L^{p(\cdot)}(X)} &= \lim_{n\to\infty} \|S(\lim_{m\to\infty}(f_n - f_m))\|_{L^{p(\cdot)}(X)} \\ &\le \lim_{n\to\infty} \lim_{m\to\infty} \|S(f_n - f_m)\|_{L^{p(\cdot)}(X)} = 0, \end{aligned}$$

and hence

$$\|S(f)\|_{L^{p(\cdot)}(X)} = \lim_{n\to\infty} \|f_n\|_{H^{p(\cdot)}(X)}.$$

Therefore, for each for each $g \in CMO_{L^2}^{p(\cdot)}(X)$, we get that

$$\begin{aligned} \left|\left\langle f, g \right\rangle\right| &= \lim_{n\to\infty} \left|\left\langle f_n, g \right\rangle\right| \le C \lim_{n\to\infty} \|f_n\|_{H^{p(\cdot)}(X)}\|g\|_{C_{p(\cdot)}(X)} \\ &= C\|S(f)\|_{L^{p(\cdot)}(X)}\|g\|_{C_{p(\cdot)}(X)}. \end{aligned} \tag{7}$$

Next we will show f has a wavelet Calderón idendity in the distribution sense. Observe that for each $g \in CMO_{L^2}^{p(\cdot)}(X)$, we conclude that

$$\left\langle f, g \right\rangle = \lim_{n\to\infty} \left\langle f_n, g \right\rangle = \lim_{n\to\infty} \left\langle \sum_{k,\alpha} \psi_\alpha^k \left\langle f_n, \psi_\alpha^k \right\rangle, g \right\rangle.$$

To end this, it suffices to show that

$$\lim_{n\to\infty}\Big\langle\sum_{k,\alpha}\psi_\alpha^k\Big\langle f_n,\psi_\alpha^k\Big\rangle,g\Big\rangle=\Big\langle\sum_{k,\alpha}\psi_\alpha^k\Big\langle f,\psi_\alpha^k\Big\rangle,g\Big\rangle$$

as $n\to\infty$. Indeed, we write $B_L=\{(k,\alpha):|k|\le L,Q_\alpha^k\subset B(x_0,L)\}$, where x_0 is any fixed point in X and $B(x_0,L)$ is ball centered at x_0 with radius L. Consider the partial sum given by

$$S_L(f)=\sum_{(k,\alpha)\in B_L}\psi_\alpha^k\Big\langle f,\psi_\alpha^k\Big\rangle.$$

Choose that M such that $L\le M\to\infty$. By using (7), we obtain that

$$\Big|\Big\langle S_M(f)-S_L(f),g\Big\rangle\Big|\le C\Big\|S\Big(\sum_{(k,\alpha)\in B_M/B_L}\psi_\alpha^k\Big\langle f,\psi_\alpha^k\Big\rangle\Big)\Big\|_{L^{p(\cdot)}(X)}\|g\|_{C_{p(\cdot)}(X)}$$

tends to zero as $L,M\to\infty$. Thus,

$$\Big\langle\sum_{k,\alpha}\psi_\alpha^k\Big\langle f,\psi_\alpha^k\Big\rangle,g\Big\rangle$$

is well defined. Furthermore, by applying again (7), we obtain that

$$\begin{aligned}&\Big|\Big\langle\sum_{k,\alpha}\psi_\alpha^k\Big\langle f_n-f,\psi_\alpha^k\Big\rangle,g\Big\rangle\Big|\\ \le\ & C\Big\|S\Big(\sum_{k,\alpha}\psi_\alpha^k\Big\langle f_n-f,\psi_\alpha^k\Big\rangle\Big)\Big\|_{L^{p(\cdot)}(X)}\|g\|_{C_{p(\cdot)}(X)}\\ =\ & \|S(f_n-f)\|_{L^{p(\cdot)}(X)}\|g\|_{C_{p(\cdot)}(X)},\end{aligned}$$

where the last term tends to zero as $n\to\infty$. Therefore, we have completed the proof of Proposition 5.4. □

Now we are ready to introduce the Hardy spaces $\mathbb{H}^{p(\cdot)}(X)$ via using the subspace $CMO_{L^2}^{p(\cdot)}$ as a test function space.

Definition 5.5. *Suppose that $p(\cdot)\in\mathcal{P}^0\cap LH$ with $\frac{\omega}{\omega+\eta}<p^-\le p^+\le 1$. The Hardy space $\mathbb{H}^{p(\cdot)}(X)$ is defined by the collection of all distributions $f\in(CMO_{L^2}^{p(\cdot)}(X))'$ such that*

$$f(x)=\sum_{k\in\mathbb{Z}}\sum_{\alpha\in\mathscr{Y}^k}\langle f,\psi_\alpha^k\rangle\psi_\alpha^k(x)$$

in $(CMO_{L^2}^{p(\cdot)}(X))'$ with $\|S(f)\|_p<\infty$, where the series converges in the distribution sense.

If $f\in\mathbb{H}^{p(\cdot)}(X)$, the norm of f in $\mathbb{H}^{p(\cdot)}(X)$ is defined by $\|f\|_{\mathbb{H}^{p(\cdot)}(X)}=\|S(f)\|_{L^{p(\cdot)}(X)}$.

Definition 5.6. *Suppose that $p(\cdot)\in\mathcal{P}^0\cap LH$ with $\frac{\omega}{\omega+\bar\eta}<p^-\le p^+\le 1$. $f\in(CMO_{L^2}^{p(\cdot)}(X))'$ is said to be an element of the atomic Hardy space with variable exponents $\mathbb{H}_a^{p(\cdot)}(X)$ if f has an atomic decomposition*

$$f=\sum_{j=1}^{\infty}\lambda_ja_j, \tag{8}$$

where $\{a_j, Q_j\} \subset A(p(\cdot), q)$ *with the quantities*

$$\mathcal{A}(\{\lambda_j\}_{j=1}^{\infty}, \{Q_j\}_{j=1}^{\infty}) = \left\| \sum_j \frac{\lambda_j \chi_{Q_j}}{\|\chi_{Q_j}\|_{L^{p(\cdot)}}} \right\|_{L^{p(\cdot)}(X)} < \infty.$$

We define

$$\|f\|_{H^{p(\cdot),q}_{\text{atom}}(X)} \equiv \inf \mathcal{A}(\{\lambda_j\}_{j=1}^{\infty}, \{Q_j\}_{j=1}^{\infty}),$$

where the infimum is taken over all such atomic representations of f.

Theorem 5.7. *Suppose that* $p(\cdot) \in \mathcal{P}^0 \cap LH$ *with* $\frac{\omega}{\omega+\tilde{\eta}} < p^- \le p^+ \le 1$. *Then*

$$H^{p(\cdot)}(X) = \mathbb{H}^{p(\cdot)}(X) = \mathbb{H}_a^{p(\cdot)}(X).$$

Proof. First we prove that $H^{p(\cdot)}(X) = \mathbb{H}^{p(\cdot)}(X)$. Suppose that $f \in \mathbb{H}^{p(\cdot)}(X)$. Then $f \in (CMO_{L^2}^{p(\cdot)}(X))'$ such that

$$f(x) = \sum_{k\in\mathbb{Z}} \sum_{\alpha\in\mathscr{Y}^k} \langle f, \psi_\alpha^k \rangle \psi_\alpha^k(x)$$

in $(CMO_{L^2}^{p(\cdot)}(X))'$ with $\|S(f)\|_p < \infty$. Note that the partial sum by

$$S_L(f) = \sum_{(k,\alpha)\in B_L} \psi_\alpha^k \left\langle f, \psi_\alpha^k \right\rangle,$$

where $B_L = \{(k,\alpha) : |k| \le L, Q_\alpha^k \subset B(x_0, L)\}$. Then $S_L(f) \in L^2(X) \cap H^{p(\cdot)}(X)$ and $S_L(f)$ converges to f in $(CMO_{L^2}^{p(\cdot)}(X))'$ as n tends to ∞. To end it, it suffices to show that

$$\|S_L(f) - S_M(f)\|_{H^{p(\cdot)}(X)} \to 0$$

as L and M tend to ∞. Indeed, if let $B_{L,M}^c = B_M \setminus B_L$ with $M \ge L$,

$$\begin{aligned}\|S_L(f) - S_M(f)\|_{H^{p(\cdot)}(X)} &= C \left\| \left\{ \sum_{k'\in\mathbb{Z}} \sum_{\alpha'\in\mathscr{Y}^{k'}} \left| \left\langle \psi_{\alpha'}^{k'}, \sum_{(k,\alpha)\in B_{L,M}^c} \psi_\alpha^k \left\langle f, \psi_\alpha^k \right\rangle \right\rangle \widetilde{\chi}_{Q_{\alpha'}^{k'}}(x) \right|^2 \right\}^{1/2} \right\|_{L^{p(\cdot)}(X)} \\ &\le C \left\| \left\{ \sum_{(k,\alpha)\in B_{L,M}^c} \left| \left\langle \psi_\alpha^k, f \right\rangle \widetilde{\chi}_{Q_\alpha^k}(x) \right|^2 \right\}^{1/2} \right\|_{L^{p(\cdot)}(X)} \to 0,\end{aligned}$$

as L and M tend to ∞. Hence, it implies that f is in the completion of the space of $L^2(X) \cap H^{p(\cdot)}(X)$ and $\mathbb{H}^{p(\cdot)}(X) \subset H^{p(\cdot)}(X)$. Conversely, if $f \in H^{p(\cdot)}(X)$, then f is the completion of the collection of all $f \in L^2(X)$ for which the quantity

$$\|f\|_{H^{p(\cdot)}(X)} := \|S(f)\|_{L^{p(\cdot)}(X)} < \infty.$$

By applying Proposition 5.4, we conclude that for each $g \in CMO_{L^2}^{p(\cdot)}(X)$, f has the following representation

$$f(x) = \sum_{k\in\mathbb{Z}} \sum_{\alpha\in\mathscr{Y}^k} \langle f, \psi_\alpha^k \rangle \psi_\alpha^k(x)$$

which holds in $(CMO_{L^2}^{p(\cdot)}(X))'$. Thus, $H^{p(\cdot)}(X) \subset \mathbb{H}^{p(\cdot)}(X)$.

Moreover, if $f \in \mathbb{H}^{p(\cdot)}(X)$, from Theorem 4.3 we obtain that $f \in L^2(X) \cap H^{p(\cdot)}(X)$, there exist a sequence of non-negative numbers $\{\lambda_j\}$ and a sequence of $(p(\cdot), 2)-$atoms $\{a_j\}$ together with $\mathcal{A}(\{\lambda_j\}_{j=1}^{\infty}, \{Q_j\}_{j=1}^{\infty}) \leq C\|f\|_{H^{p(\cdot)}(X)}$, such that

$$f = \sum_j \lambda_j a_j.$$

It implies that $f \in \mathbb{H}_a^{p(\cdot)}(X)$ and $\mathbb{H}^{p(\cdot)}(X) \subset \mathbb{H}_a^{p(\cdot)}(X)$. On the other hand, applying Theorem 4.5 yields that $\mathbb{H}_a^{p(\cdot)}(X) \subset \mathbb{H}^{p(\cdot)}(X)$.

Therefore, the proof of Theorem 5.7 is complete. □

Remark 5.8. *We can similarly give the definition of the variable Carleson measure space $CMO^{p(\cdot)}(X)$ by using the subspace $H^{p(\cdot)}(X) \cap L^2(X)$ as the space of test function. Precisely, suppose that a sequence $f_n \in L^2(X)$ is a Cauchy in the sense of $CMO^{p(\cdot)}(X)$. Thus, f_n has a limit in the distribution of $H^{p(\cdot)}(X) \cap L^2(X)$ as $n \to \infty$, and $f(x) = \sum_{k\in\mathbb{Z}} \sum_{\alpha\in\mathscr{Y}^k} \langle f, \psi_\alpha^k \rangle \psi_\alpha^k(x)$ in $(H^{p(\cdot)}(X) \cap L^2(X))'$. Let $CMO^{p(\cdot)}(X)$ be the variable Carleson measure space defined by the set of all $f \in (H^{p(\cdot)}(X) \cap L^2(X))'$ satisfying*

$$f(x) = \sum_{k\in\mathbb{Z}} \sum_{\alpha\in\mathscr{Y}^k} \langle f, \psi_\alpha^k \rangle \psi_\alpha^k(x)$$

in $(H^{p(\cdot)}(X) \cap L^2(X))'$ and $\|f\|_{C_{p(\cdot)}} < \infty$.

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