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Some comments on τ-distance and existence theorems in complete metric spaces

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Abstract. Very recently, we have introduced the concept of τ'-distance, which is slightly weaker than that of τ-distance. We discuss the difference between both concepts, proving some existence theorems in complete metric spaces and giving an example of a τ'-distance which is not a τ-distance.

1. Introduction

Throughout this paper, we denote by $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$ the sets of all positive integers, all rational numbers and all real numbers, respectively.

In 2001, the concept of τ-distance was introduced in order to generalize results in [1, 4, 5, 19–21] and others.

Definition 1 ([12])**.** Let (X, d) be a metric space. Then a function p from $X \times X$ into $[0, \infty)$ is called a *τ-distance* on X if there exists a function η from $X \times [0, \infty)$ into $[0, \infty)$ and the following are satisfied:

(τ_d1) $p(x, z) \leq p(x, y) + p(y, z)$ for any $x, y, z \in X$.

(τ_d2) $\eta(x, 0) = 0$ and $\eta(x, t) \geq t$ for any $x \in X$ and $t \in [0, \infty)$, and η is concave and continuous in its second variable.

(τ_d3) $\lim_n x_n = x$ and $\lim_n \sup\Big\{\eta\big(z_n, p(z_n, x_m)\big) : m \geq n\Big\} = 0$ imply $p(w, x) \leq \liminf_n p(w, x_n)$ for any $w \in X$.

(τ_d4) $\lim_n \sup\{p(x_n, y_m) : m \geq n\} = 0$ and $\lim_n \eta(x_n, t_n) = 0$ imply $\lim_n \eta(y_n, t_n) = 0$.

(τ_d5) $\lim_n \eta\big(z_n, p(z_n, x_n)\big) = 0$ and $\lim_n \eta\big(z_n, p(z_n, y_n)\big) = 0$ imply $\lim_n d(x_n, y_n) = 0$.

We note that the metric d is one of τ-distances on X with $\eta = \big((x, t) \mapsto t\big)$. Every w-distance is also a τ-distance; see [8, 12]. See [8, 12–17] and references therein for many examples and theorems concerning τ-distance. For instance, using τ-distance, Suzuki [16] gave a simple proof of Zhong's theorem [20].

Very recently, strongly inspired by τ-function in Lin and Du [10], we introduced τ'-distance in [18].

Definition 2 ([18])**.** Let (X, d) be a metric space and let p be a function from $X \times X$ into $[0, \infty)$. Then p is called a *τ'-distance* on X if the following hold:

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($\tau'1$) $p(x, z) \leq p(x, y) + p(y, z)$ for any $x, y, z \in X$.
($\tau'2$) If $\lim_n \sup\{p(z_n, z_m) : m > n\} = 0$ and $\lim_n p(z_n, x_n) = 0$, then $\lim_n d(z_n, x_n) = 0$. Moreover if $\{x_n\}$ converges to some $x \in X$, then $p(w, x) \leq \liminf_n p(w, x_n)$ for any $w \in X$.
($\tau'3$) If $\lim_n p(z, x_n) = 0$, then $\lim_n d(x_n, x_{n+1}) = 0$ holds. Moreover if $\{x_n\}$ converges to some $x \in X$, then $p(w, x) \leq \liminf_n p(w, x_n)$ for any $w \in X$.

The concept of τ'-distance is 'slightly' weaker than that of τ-distance. The word 'slightly' means that we can prove τ'-distance versions of all the existence theorems in [12–16] with using the same proofs. So, we could tell that we 'redefine' the definition of τ-distance. In [18], we showed that τ'-distance is more natural than τ-distance.

In this paper, we find another merit of τ'-distance. While we cannot separate Conditions (τ_d1)–(τ_d5) on τ-distance, we can separate Conditions ($\tau'1$)–($\tau'3$) on τ'-distance. That is, we can discuss something mathematical more finely. Also, we give an example of a τ'-distance which is not a τ-distance.

2. Lemmas

In this section, paying attention to how ($\tau'2$) and ($\tau'3$) work separately, we discuss some lemmas proved in [18].

Definition 3 ([18]). Let p be a τ'-distance on a metric space (X, d). Let $\{x_\alpha : \alpha \in D\}$ be a net in X. Then $\{x_\alpha\}$ is said to satisfy *Condition (CL)* if the following hold:

(CL1) $\{x_\alpha\}$ is a Cauchy net in the usual sense.
(CL2) Either of the following hold:

- $\{x_\alpha\}$ does not converge.
- If $\{x_\alpha\}$ converges to x, then $p(w, x) \leq \liminf_\alpha p(w, x_\alpha)$ holds for any $w \in X$.

We first begin with ($\tau'3$).

Lemma 4. *Let (X, d) be a metric space and let p be a function from $X \times X$ into $[0, \infty)$ satisfying ($\tau'3$). Let $\{x_\alpha : \alpha \in D\}$ be a net in X satisfying $\lim_\alpha p(z, x_\alpha) = 0$ for some $z \in X$. Then the following hold:*

(i) *$\{x_\alpha\}$ satisfies Condition* (CL).
(ii) *If a net $\{y_\alpha : \alpha \in D\}$ in X also satisfies $\lim_\alpha p(z, y_\alpha) = 0$, then $\lim_\alpha d(x_\alpha, y_\alpha) = 0$ holds.*

Proof. The proof of Lemma 13 in [18] works. □

As corollaries of Lemma 4, we obtain the following.

Lemma 5. *Let (X, d) and p be as in Lemma 4. Let $\{x_n\}$ be a sequence in X satisfying $\lim_n p(z, x_n) = 0$ for some $z \in X$. Then the following hold:*

(i) *$\{x_n\}$ satisfies Condition* (CL).
(ii) *If a sequence $\{y_n\}$ in X also satisfies $\lim_\alpha p(z, y_n) = 0$, then $\lim_n d(x_n, y_n) = 0$ holds.*

Lemma 6. *Let (X, d) and p be as in Lemma 4. If $p(z, x) = p(z, y) = 0$ holds, then $x = y$ holds.*

We next pay attention to how ($\tau'2$) works.

Lemma 7. *Let (X, d) be a metric space and let p be a function from $X \times X$ into $[0, \infty)$ satisfying ($\tau'2$). Let D be a directed set such that for any $\alpha \in D$, there exists $\beta \in D$ with $\alpha \not\geq \beta$. Let $\{z_\alpha : \alpha \in D\}$ be a net in X satisfying $\lim_\alpha \sup\{p(z_\alpha, z_\beta) : \beta > \alpha\} = 0$. Then the following hold:*

(i) *If a net $\{x_\alpha : \alpha \in D\}$ in X satisfies $\lim_\alpha p(z_\alpha, x_\alpha) = 0$, then $\{x_\alpha\}$ satisfies Condition* (CL) *and $\lim_\alpha d(z_\alpha, x_\alpha) = 0$ holds.*
(ii) *$\{z_\alpha\}$ satisfies Condition* (CL).

Proof. We note that the assumption on D is the condition of the second case in the proof of Lemma 16 in [18]. We note the following:

- For any $\alpha \in D$ there exists $\beta \in D$ with $\beta > \alpha$.

Therefore the proof of Lemma 16 (the second case) in [18] works. □

As a corollary of Lemma 7, we obtain the following sequential version.

Lemma 8. *Let (X, d) and p be as in Lemma 7. Let $\{z_n\}$ be a sequence in X satisfying $\lim_n \sup\{p(z_n, z_m) : m > n\} = 0$. Then the following hold*:

(i) *If a sequence $\{x_n\}$ in X satisfies $\lim_n p(z_n, x_n) = 0$, then $\{x_n\}$ satisfies Condition* (CL) *and $\lim_n d(z_n, x_n) = 0$ holds.*
(ii) *$\{z_n\}$ satisfies Condition* (CL).

By Lemma 8, we obtain the following, which plays a very important role in this paper. Compare Lemma 9 with Lemmas 5 and 6.

Lemma 9. *Let (X, d) be a metric space and let p be a function from $X \times X$ into $[0, \infty)$ satisfying $(\tau'2)$. Let $z \in X$ satisfy $p(z, z) = 0$. Then the following hold*:

(i) *If a sequence $\{x_n\}$ in X satisfies $\lim_n p(z, x_n) = 0$, then $\{x_n\}$ satisfies Condition* (CL) *and $\lim_n d(z, x_n) = 0$ holds.*
(ii) *If $x \in X$ satisfies $p(z, x) = 0$, then $z = x$ holds.*

Proof. Define a sequence $\{z_n\}$ in X by $z_n = z$. Then $\lim_n \sup\{p(z_n, z_m) : m > n\} = 0$ holds. So, by Lemma 8, we obtain the desired result. □

Remark. The proof employs the method in the proof of Lemma 1.1 in [3].

3. Existence Theorems

In this section, we give proofs of four existence theorems. In Theorem 16, we need only $(\tau'2)$. In Corollary 11, we need $(\tau'1)$ and $(\tau'2)$. In Theorem 17, we need $(\tau'2)$ and $(\tau'3)$. On the other hand, in Theorem 13, we need $(\tau'1)$–$(\tau'3)$. It is interesting that $(\tau'2)$ is needed in all theorems, however, $(\tau'1)$ and $(\tau'3)$ are not always needed.

Theorem	$(\tau'1)$	$(\tau'2)$	$(\tau'3)$
Theorem 16	—	○	—
Corollary 11	○	○	—
Theorem 17	—	○	○
Theorem 13	○	○	○

The following is a generalization of Nadler's fixed point theorem [11]. See also Theorem 3.7 in [13]

Theorem 10. *Let (X, d) be a complete metric space and let p be a function from $X \times X$ into $[0, \infty)$ satisfying $(\tau'1)$ and $(\tau'2)$. Let T be a set-valued mapping on X satisfying the following*:

- *For any $x \in X$, Tx is a nonempty closed subset of X.*
- *There exists $r \in [0, 1)$ satisfying*
$$Q(Tx, Ty) \le r\, p(x, y)$$
for all $x, y \in X$, where
$$Q(A, B) = \sup_{a \in A} \inf_{b \in B} p(a, b).$$

Then there exists $z \in X$ satisfying $z \in Tz$ and $p(z, z) = 0$.

Proof. Replace the value of r by $r := (1 + r)/2 \in (0, 1)$. We note the following:

- For any $x, y \in X$, $u \in Tx$ and $\eta > p(x, y)$, there exists $v \in Ty$ satisfying $p(u, v) < r\,\eta$.

Fix $u_0 \in X$ and $u_1 \in Tu_0$. Put $\alpha = 1/(1 - r)$ and $\beta = p(u_0, u_1) + 1$. Then there exists $u_2 \in Tu_1$ satisfying $p(u_1, u_2) < r\,\beta$. Then there exists $u_3 \in Tu_2$ satisfying $p(u_2, u_3) < r^2\,\beta$. Continuing this argument, we can obtain a sequence $\{u_n\}$ in X satisfying

$$u_{n+1} \in Tu_n \quad \text{and} \quad p(u_n, u_{n+1}) < r^n\,\beta$$

for $n \in \mathbb{N} \cup \{0\}$. For any $m, n \in \mathbb{N} \cup \{0\}$ with $m > n$, we have by $(\tau'1)$

$$p(u_n, u_m) \le \sum_{k=n}^{m-1} p(u_k, u_{k+1}) < \sum_{k=n}^{m-1} r^k\,\beta < r^n\,\alpha\,\beta.$$

By Lemma 8, $\{u_n\}$ satisfies Condition (CL). Since X is complete, $\{u_n\}$ converges to some $z \in X$. We have for $n \in \mathbb{N} \cup \{0\}$,

$$p(u_n, z) \le \liminf_{m\to\infty} p(u_n, u_m) \le r^n\,\alpha\,\beta < r^n\,\alpha\,\beta + r^n. \tag{1}$$

So, for $n \in \mathbb{N}$, there exists $v_n \in Tz$ satisfying $p(u_n, v_n) < r^n\,\alpha\,\beta + r^n$. By Lemma 8 again, $\{v_n\}$ also satisfies Condition (CL) and converges to z. Since Tz is closed, we obtain $z \in Tz$. Put $w_0 = z$ and $\gamma = p(z, w_0)+1$. There exists $w_1 \in Tw_0$ satisfying $p(z, w_1) < r\,\gamma$. Then there exists $w_2 \in Tw_1$ satisfying $p(z, w_2) < r^2\,\gamma$. Continuing this argument, we can choose a sequence $\{w_n\}$ in X satisfying

$$w_n \in Tw_{n-1} \quad \text{and} \quad p(z, w_n) < r^n\,\gamma$$

for $n \in \mathbb{N}$. Using this and (1), we have

$$\lim_{n\to\infty} p(u_n, w_n) \le \lim_{n\to\infty} \Big(p(u_n, z) + p(z, w_n)\Big) = 0.$$

By Lemma 8 again, $\{w_n\}$ also satisfies Condition (CL) and converges to z. We have

$$p(z, z) \le \liminf_{n\to\infty} p(z, w_n) \le \lim_{n\to\infty} r^n\,\gamma = 0.$$

We obtain the desired result. □

The following is a generalization of the Banach contraction principle [1, 2]. See also Theorem 2 in [12].

Corollary 11. *Let (X, d) be a complete metric space and let p be a function from $X \times X$ into $[0, \infty)$ satisfying $(\tau'1)$ and $(\tau'2)$. Let T be a mapping on X. Assume that there exists $r \in [0, 1)$ satisfying*

$$p(Tx, Ty) \le r\,p(x, y)$$

for all $x, y \in X$. Then T has a unique fixed point z. Moreover $p(z, z) = 0$ holds and $\{T^n x\}$ converges to z for any $x \in X$.

Proof. We note that all the assumptions of Theorem 10 are satisfied. Fix $u \in X$. Then from the proof of Theorem 10, $\{T^n u\}$ converges to a fixed point z of T and $p(z, z) = 0$ holds. In order to show the uniqueness of z, let w be a fixed point of T. Then we have

$$p(z, w) = p(Tz, Tw) \le r\,p(z, w).$$

and hence $p(z, w) = 0$. By Lemma 9, we obtain $z = w$. Therefore the fixed point z is unique. □

Since Corollary 11 is important, we give a direct proof of Corollary 11.

Proof. Fix $u \in X$. For $m, n \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned} p(T^n u, T^m u) &\le \sum_{j=n}^{m-1} p(T^j u, T^{j+1} u) \le \sum_{j=n}^{m-1} r^j \, p(u, Tu) \\ &\le \sum_{j=n}^{\infty} r^j \, p(u, Tu) = \frac{r^n}{1-r} \, p(u, Tu) \end{aligned}$$

and hence

$$\lim_{n\to\infty} \sup_{m>n} p(T^n u, T^m u) \le \lim_{n\to\infty} \frac{r^n}{1-r} \, p(u, Tu) = 0.$$

By Lemma 8, $\{T^n u\}$ satisfies Condition (CL). Since X is complete, $\{T^n u\}$ converges to some $z \in X$. We have

$$\begin{aligned} &\lim_{n\to\infty} p(T^n u, Tz) \le \lim_{n\to\infty} r \, p(T^{n-1} u, z) \le \lim_{n\to\infty} \liminf_{m\to\infty} r \, p(T^{n-1} u, T^m u) \\ &\le \lim_{n\to\infty} \frac{r^n}{1-r} \, p(u, Tu) = 0. \end{aligned}$$

By $(\tau'2)$, $\{T^n u\}$ converges to Tz. Hence $Tz = z$ holds. We also have

$$p(z, z) = \lim_{n\to\infty} p(T^n z, T^n z) \le \lim_{n\to\infty} r^n p(z, z) = 0.$$

We can prove the uniqueness of the fixed point z as in the above proof. □

The following example tells that we need $(\tau'1)$ in Corollary 11.

Example 12 (Example 2 in [7]). Put $X = \mathbb{N}$ and $d(x, y) = |x - y|$ for $x, y \in X$. Define a function p from $X \times X$ into $[0, \infty)$ by

$$p(x, y) = r^{\min\{x,y\}} \, |x - y|,$$

where $r \in (0, 1)$. Define a mapping T on X by $Tx = x + 1$. Then the following hold:

(i) p satisfies $(\tau'2)$ and $(\tau'3)$.
(ii) $p(Tx, Ty) \le r \, p(x, y)$ for all $x, y \in X$.
(iii) T does not have a fixed point.

Proof. In order to show $(\tau'2)$, we assume $\lim_n \sup\{p(z_n, z_m) : m > n\} = 0$ and $\lim_n p(z_n, x_n) = 0$. Then there exists $x \in X$ such that $z_n = x_n = x$ holds for sufficiently large $n \in \mathbb{N}$. Thus $(\tau'2)$ holds. In order to show $(\tau'3)$, we assume $\lim_n p(z, x_n) = 0$. Then $x_n = z$ holds for sufficiently large $n \in \mathbb{N}$. Thus $(\tau'3)$ holds. (ii) and (iii) are obvious. □

The following is connected with the strong Ekeland variational principle. See [4–6].

Theorem 13 ([15]). *Let X be a complete metric space and let p be a τ'-distance on X. Let f be a function from X into $(-\infty, +\infty]$ which is proper lower semicontinuous and bounded from below. Then for $u \in X$, there exists $v \in X$ satisfying the following*:

(i) *$f(v) \le f(u)$.*
(ii) *$f(w) > f(v) - p(v, w)$ for all $w \in X \setminus \{v\}$.*
(iii) *If a sequence $\{x_n\}$ in X satisfies $\lim_n \big(f(x_n) + p(v, x_n)\big) = f(v)$, then $\{x_n\}$ satisfies Condition* (CL); *and $\lim_n x_n = v$ and $p(v, v) = \lim_n p(v, x_n) = 0$ hold.*

Proof. The proof of Theorem 7 in [15] works. □

The following examples tell that we need $(\tau'1)$ and $(\tau'3)$ in Theorem 13.

Example 14. Let $X = [1, \infty)$ and $d(x, y) = |x - y|$ for $x, y \in X$. Define a function p from $X \times X$ into $[0, \infty)$ by

$$p(x, y) = \begin{cases} 1/\big(x\,(x+1)\big) & \text{if } y = x + 1 \\ 1 & \text{otherwise.} \end{cases}$$

Define a continuous function f from X into $[0, \infty)$ by $f(x) = 1/x$ and put $u = 1$. Then the following hold:

(j) p satisfies $(\tau'2)$ and $(\tau'3)$.
(jj) There does not exist $v \in X$ satisfying (i)–(iii) of Theorem 13.

Proof. Since the assumptions of $(\tau'2)$ and $(\tau'3)$ always do not hold, (j) holds. For any $x \in X$, we have

$$f(x + 1) \le f(x) - p(x, x + 1).$$

So (ii) of Theorem 13 always does not hold. Thus (jj) holds. □

Example 15. Let X and d be as in Example 12. Define a function p from $X \times X$ into $[0, \infty)$ by

$$p(x, y) = \begin{cases} 1/y & \text{if } x = 1 \\ 1 & \text{if } x \neq 1. \end{cases}$$

Define a continuous function f from X into $[0, \infty)$ by

$$f(x) = \begin{cases} 0 & \text{if } x = 1 \\ 1/x & \text{if } x \neq 1 \end{cases}$$

and put $u = 1$. Then the following hold:

(j) p satisfies $(\tau'1)$ and $(\tau'2)$.
(jj) There does not exist $v \in X$ satisfying (i)–(iii) of Theorem 13.

Proof. We have

$$p(x, z) \le 1 \le p(x, y) + p(y, z)$$

for any $x, y, z \in X$, thus $(\tau'1)$ holds. The assumption of $(\tau'2)$ always does not hold, thus, $(\tau'2)$ holds. Let us prove (jj). If $v \neq 1$, then v does not satisfy (i) of Theorem 13. Therefore we assume $v = 1$. We will show that v does not satisfy (iii) of Theorem 13. Define a sequence $\{x_n\}$ in X by $x_n = n$. Then

$$\lim_{n\to\infty} \big(f(x_n) + p(v, x_n)\big) = \lim_{n\to\infty} (1/n + 1/n) = 0 = f(v)$$

holds but $\{x_n\}$ does not converge to v. Therefore v does not satisfy (iii) of Theorem 13. □

The following are generalizations of Kannan's fixed point theorem [9]. See also Theorem 3.3 in [13].

Theorem 16. *Let (X, d) be a complete metric space and let p be a function from $X \times X$ into $[0, \infty)$ satisfying $(\tau'2)$. Let T be a mapping on X. Assume that there exists $\alpha \in [0, 1/2)$ satisfying*

$$p(Tx, Ty) \le \alpha\, p(Tx, x) + \alpha\, p(Ty, y)$$

for all $x, y \in X$. Then T has a unique fixed point z. Moreover $p(z, z) = 0$ holds and $\{T^n x\}$ converges to z for any $x \in X$.

Proof. Since

$$p(T^2x, Tx) \le \alpha\, p(T^2x, Tx) + \alpha\, p(Tx, x),$$

we have

$$p(T^2x, Tx) \le r\, p(Tx, x)$$

for any $x \in X$, where $r := \alpha/(1-\alpha) \in [0,1)$. Fix $u \in X$. We have

$$\begin{aligned}&\lim_{n\to\infty}\sup_{m>n} p(T^n u, T^m u) \le \lim_{n\to\infty}\sup_{m>n}\Big(\alpha\, p(T^n u, T^{n-1}u) + \alpha\, p(T^m u, T^{m-1}u)\Big)\\ &\le \lim_{n\to\infty}\sup_{m>n} \alpha\,(r^{n-1}+r^{m-1})\, p(Tu,u) = \lim_{n\to\infty} \alpha\,(r^{n-1}+r^n)\, p(Tu,u) = 0.\end{aligned}$$

By Lemma 8, $\{T^n u\}$ satisfies Condition (CL). Since X is complete, $\{T^n u\}$ converges to some $z \in X$. We have

$$\begin{aligned}&p(Tz,z) \le \liminf_{n\to\infty} p(Tz, T^{n+1}u)\\ &\le \liminf_{n\to\infty}\Big(\alpha\, p(Tz,z) + \alpha\, p(T^{n+1}u, T^n u)\Big) = \alpha\, p(Tz,z).\end{aligned}$$

Since $\alpha < 1$, we obtain $p(Tz,z) = 0$. We also have

$$p(Tz,Tz) \le 2\,\alpha\, p(Tz,z) = 0.$$

So by Lemma 9, we obtain $Tz = z$. In order to show the uniqueness of z, let w be a fixed point of T. Then we have

$$p(w,w) = p(Tw,Tw) \le 2\,\alpha\, p(Tw,w) = 2\,\alpha\, p(w,w).$$

Since $2\,\alpha < 1$, we have $p(w,w) = 0$. So we have

$$p(z,w) = p(Tz,Tw) \le \alpha\, p(Tz,z) + \alpha\, p(Tw,w) = 0.$$

By Lemma 9, we obtain $z = w$. Therefore the fixed point z is unique. □

Theorem 17. *Let (X,d) be a complete metric space and let p be a function from $X \times X$ into $[0,\infty)$ satisfying $(\tau'2)$ and $(\tau'3)$. Let T be a mapping on X. Assume that there exists $\alpha \in [0,1/2)$ satisfying*

$$p(Tx,Ty) \le \alpha\, p(Tx,x) + \alpha\, p(y,Ty)$$

for all $x, y \in X$. Then T has a unique fixed point z. Moreover $p(z,z) = 0$ holds and $\{T^n x\}$ converges to z for any $x \in X$.

Proof. Since

$$p(T^2x, Tx) \le \alpha\, p(T^2x,Tx) + \alpha\, p(x,Tx)$$

and

$$p(Tx, T^2x) \le \alpha\, p(Tx,x) + \alpha\, p(Tx,T^2x),$$

we have

$$p(T^2x,Tx) \le r\, p(x,Tx) \quad \text{and} \quad p(Tx,T^2x) \le r\, p(Tx,x)$$

for any $x \in X$, where $r := \alpha/(1-\alpha) \in [0,1)$. Hence

$$\max\{p(T^2x,Tx), p(Tx,T^2x)\} \le r\, \max\{p(Tx,x), p(x,Tx)\}$$

for any $x \in X$. Fix $u \in X$. We have

$$\begin{aligned}&\lim_{n\to\infty}\sup_{m>n} p(T^n u, T^m u) \le \lim_{n\to\infty}\sup_{m>n}\Big(\alpha\, p(T^n u, T^{n-1}u) + \alpha\, p(T^{m-1}u, T^m u)\Big)\\ &\le \lim_{n\to\infty}\sup_{m>n} \alpha\,(r^{n-1}+r^{m-1})\, \max\{p(Tu,u), p(u,Tu)\}\\ &= \lim_{n\to\infty} \alpha\,(r^{n-1}+r^n)\, \max\{p(Tu,u), p(u,Tu)\} = 0.\end{aligned}$$

By Lemma 8, $\{T^n u\}$ satisfies Condition (CL). Since X is complete, $\{T^n u\}$ converges to some $z \in X$. We have

$$\begin{aligned}&p(Tz,z) \le \liminf_{n\to\infty} p(Tz, T^{n+1}u)\\ &\le \liminf_{n\to\infty}\Big(\alpha\, p(Tz,z) + \alpha\, p(T^n u, T^{n+1}u)\Big) = \alpha\, p(Tz,z).\end{aligned}$$

Since $\alpha < 1$, we obtain $p(Tz, z) = 0$. We also have

$$p(Tz, T^2 z) \le r\, p(Tz, z) = 0.$$

So by Lemma 6, we obtain $T^2 z = z$. Then we note that $\left\{T^n z : n \in \mathbb{N} \cup \{0\}\right\}$ consists of at most two elements. Since $\{T^n z\}$ is a Cauchy sequence, we obtain $Tz = z$. Hence $p(z, z) = 0$ holds. We can prove the uniqueness of a fixed point z as in the proof of Theorem 16. □

The following example tells that we need $(\tau'3)$ in Theorem 17.

Example 18. Let $\alpha \in (0, 1/2)$ and put $X = \mathbb{N} \cup \{0\}$. Define a mapping S from X into $[0, 1)$ by

$$Sx = \begin{cases} 0 & \text{if } x = 0 \\ \alpha^x & \text{if } x \neq 0 \end{cases}$$

and a function d from $X \times X$ into $[0, \infty)$ by $d(x, y) = |Sx - Sy|$. Define a function p from $X \times X$ into $[0, \infty)$ by

$$p(x, y) = \begin{cases} 0 & \text{if } x \text{ is odd and } y \text{ is even} \\ \alpha^y & \text{if } x \text{ is odd and } y \text{ is odd} \\ \alpha^x & \text{if } x \text{ is even and } y \text{ is even} \\ \alpha^x + \alpha^y & \text{if } x \text{ is even and } y \text{ is odd} \end{cases}$$

and a mapping T on X by $Tx = x + 1$. Then the following hold:

(i) p satisfies $(\tau'1)$ and $(\tau'2)$.

(ii) $p(Tx, Ty) \le \alpha\, p(Tx, x) + \alpha\, p(y, Ty)$ for all $x, y \in X$.

(iii) T does not have a fixed point.

Proof. Let $I, J, K \in X$ be odd numbers, let $\iota, j, \kappa \in X$ be even numbers and let $y \in X$. We have

$$\begin{aligned} &p(I, \kappa) = 0 \le p(I, y) + p(y, \kappa), \\ &p(I, K) = \alpha^K \le p(y, K) \le p(I, y) + p(y, K), \\ &p(\iota, \kappa) = \alpha^\iota \le p(\iota, y) \le p(\iota, y) + p(y, \kappa), \\ &p(\iota, K) = \alpha^\iota + \alpha^K \le \alpha^\iota + \alpha^J + \alpha^K = p(\iota, J) + p(J, K), \\ &p(\iota, K) = \alpha^\iota + \alpha^K \le \alpha^\iota + \alpha^j + \alpha^K = p(\iota, j) + p(j, K). \end{aligned}$$

Thus $(\tau'1)$ holds. In order to show $(\tau'2)$, we assume $\lim_n \sup\{p(z_n, z_m) : m > n\} = 0$ and $\lim_n p(z_n, x_n) = 0$. Then it is obvious that $\lim_n z_n = \lim_n x_n = \infty$ holds. Thus, $\{z_n\}$ and $\{x_n\}$ converge to 0 in (X, d). So $\lim_n d(z_n, x_n) = 0$ holds. We have

$$\begin{aligned} &p(I, 0) = 0 \le \liminf_{n\to\infty} p(I, x_n), \\ &p(\iota, 0) = \alpha^\iota \le \liminf_{n\to\infty} p(\iota, x_n). \end{aligned}$$

Thus $(\tau'2)$ holds. We have

$$\begin{aligned} &p(\iota + 1, J + 1) = 0 = \alpha\, p(\iota + 1, \iota) + \alpha\, p(J, J + 1), \\ &p(\iota + 1, j + 1) = \alpha^{j+1} \le \alpha\, (\alpha^j + \alpha^{j+1}) \\ &\qquad = \alpha\, p(\iota + 1, \iota) + \alpha\, p(j, j + 1), \\ &p(I + 1, J + 1) = \alpha^{I+1} \le \alpha\, (\alpha^{I+1} + \alpha^I) \\ &\qquad = \alpha\, p(I + 1, I) + \alpha\, p(J, J + 1), \\ &p(I + 1, j + 1) = \alpha^{I+1} + \alpha^{j+1} \le \alpha\, (\alpha^{I+1} + \alpha^I + \alpha^j + \alpha^{j+1}) \\ &\qquad = \alpha\, p(I + 1, I) + \alpha\, p(j, j + 1). \end{aligned}$$

Thus (ii) holds. (iii) is obvious. □

4. Example

In this section, we give an example of a τ'-distance which is not a τ-distance.

Lemma 19. *Let A, B and C be subsets of $[0,1]$ defined by*

$$A = \Big\{ \sum_{j=1}^{\infty} a_j \, 10^{-j} : a_j \in \{0,1\} \Big\},$$
$$B = A \cap \mathbb{Q} \quad \text{and} \quad C = A \setminus \mathbb{Q}.$$

For $a \in A$, we write a_j for $[a\,10^j] \bmod 10$, where $[x]$ is the maximum integer not exceeding x. That is, $a = \sum_{j=1}^{\infty} a_j \, 10^{-j}$ holds for any $a \in A$. Then the following hold:

(i) *For $c \in C$, $\varepsilon > 0$ and $k \in \mathbb{N}$, there exists $b \in B$ satisfying the following:*
- $|b - c| < \varepsilon$.
- $b_j = c_j$ *for* $j \in \{1, \cdots, k\}$.

(ii) *For $b \in B$, $\varepsilon > 0$ and $k, \ell \in \mathbb{N}$, there exist $c \in C$ and $n \in \mathbb{N}$ satisfying the following:*
- $|b - c| < \varepsilon$.
- $b_j = c_j$ *for* $j \in \{1, \cdots, k\}$.
- *For any $i, j \in \{1, \cdots, \ell\}$, there exists $h \in \mathbb{N}$ such that $i + jh \le n$ and $c_i \ne c_{i+jh}$.*

Proof. We first show (i). Fix $c \in C$, $\varepsilon > 0$ and $k \in \mathbb{N}$. Then we can choose $\ell \in \mathbb{N}$ satisfying $10^{-\ell} < \varepsilon$ and $\ell \ge k$. Then define a sequence $\{b_j\}$ in $\{0,1\}$ by

- $b_j = c_j$ for $j \in \mathbb{N}$ with $j \le \ell$ and
- $b_j = 0$ for $j \in \mathbb{N}$ with $j > \ell$.

Then

$$b := \sum_{j=1}^{\infty} b_j \, 10^{-j} \in B \quad \text{and} \quad |b - c| < 2 \cdot 10^{-\ell-1} < 10^{-\ell} < \varepsilon$$

hold. We next show (ii). Fix $b \in B$, $\varepsilon > 0$ and $k, \ell \in \mathbb{N}$. Then we can choose $c \in C$ satisfying $|b - c| < \varepsilon$ and $b_j = c_j$ for $j \in \{1, \cdots, k\}$. It is obvious that for sufficiently large $n \in \mathbb{N}$, n satisfies the conclusion because c is irrational. □

From now on, we write $b(c, \varepsilon, k)$ for b in (i) of Lemma 19. Also we write $c(b, \varepsilon, k, \ell)$ for c and $n(b, \varepsilon, k, \ell)$ for n in (ii) of Lemma 19, respectively. Though $b(c, \varepsilon, k)$, $c(b, \varepsilon, k, \ell)$ and $n(b, \varepsilon, k, \ell)$ above are not functions, we will use these notations in making examples. because there is no room for ambiguity.

Lemma 20. *Let A, B and C be as in Lemma 19. Define sequences $\{b^{(n)}\}$ in B, $\{c^{(n)}\}$ in C, $\{s^{(n)}\}$ and $\{t^{(n)}\}$ in $(0, \infty)$ and $\{\nu^{(n)}\}$ in $\mathbb{N}$ as follows:*

(Step 1) $n = 1$, $b^{(1)} \in B$ *and* $s^{(1)} > 0$.
(Step 2) $c^{(1)} = c(b^{(1)}, s^{(1)}, 1, 1)$, $\nu^{(1)} = n(b^{(1)}, s^{(1)}, 1, 1)$ *and* $t^{(1)} > 0$.
(Step 3) $n := n + 1$, $b^{(n)} = b(c^{(n-1)}, t^{(n-1)}, \nu^{(n-1)})$ *and* $s^{(n)} > 0$.
(Step 4) $c^{(n)} = c(b^{(n)}, s^{(n)}, \nu^{(n-1)}, n)$, $\nu^{(n)} = \max\{\nu^{(n-1)}, n(b^{(n)}, s^{(n)}, \nu^{(n-1)}, n)\}$ *and* $t^{(n)} > 0$.
(Step 5) *goto* (Step 3).

Then $\{b^{(n)}\}$ and $\{c^{(n)}\}$ converge to a same number γ, which belongs to C.

Remark. We can choose $s^{(n)}$, depending on $b^{(k)}$ ($k \le n$) and others. On the other hand, we cannot choose $s^{(n)}$, depending on $b^{(k)}$ ($k > n$) and others. Similarly for $t^{(n)}$.

Proof. As in Lemma 19, for $a \in A$, we write a_j for $[a\,10^j] \bmod 10$. Since $\nu(b, \varepsilon, k, \ell) \geq 2\,\ell$ holds, we first note $\lim_n \nu^{(n)} = \infty$. We next note that $\{\nu^{(n)}\}$ is nondecreasing. Hence

$$c_j^{(n)} = b_j^{(n+1)} = c_j^{(n+1)} = \cdots \quad \text{provided} \quad j \leq \nu^{(n)}.$$

Therefore $\{b^{(n)}\}$ and $\{c^{(n)}\}$ converge to a same number γ. Arguing by contradiction, we assume that γ is rational. Then there exist $i, j \in \mathbb{N}$ such that $\gamma_r = \gamma_{r+j}$ for any $r \geq i$. Put $n = \max\{i, j\}$. Then there exists $h \in \mathbb{N}$ such that $i + jh \leq \nu^{(n)}$ and

$$\gamma_i = c_i^{(n)} \neq c_{i+jh}^{(n)} = \gamma_{i+jh} = \gamma_i,$$

which is a contradiction. Therefore γ is irrational. □

Example 21. Let X be a subset of $\mathbb{R}^2$ defined by

$$X = \Big(\{-1\} \times ([0,1] \cap \mathbb{Q})\Big) \cup \Big(\{0\} \times [0,1]\Big) \cup \Big((0,1] \times ([0,1] \setminus \mathbb{Q})\Big).$$

Define functions d and p from $X \times X$ into $[0, \infty)$ by

$$d\Big((x_1, x_2), (y_1, y_2)\Big) = \begin{cases} 0 & \text{if } (x_1, x_2) = (y_1, y_2) \\ |x_1| + |y_1| + |x_2 - y_2| & \text{otherwise} \end{cases}$$

and

$$p\Big((x_1, x_2), (y_1, y_2)\Big) = \begin{cases} d\Big((0, x_2), (y_1, y_2)\Big) & \text{if } x_1 = -1, y_1 = 0, y_2 \in \mathbb{Q} \\ d\Big((0, x_2), (y_1, y_2)\Big) & \text{if } x_1 = -1, y_1 > 0 \\ 3 & \text{otherwise} \end{cases}$$

for any $(x_1, x_2), (y_1, y_2) \in X$. Then the following hold:

(i) (X, d) is a complete metric space.
(ii) p is a τ'-distance on X.
(iii) p is not a τ-distance on X.

Proof. For any $x \in X$, we write x_1 for the first element of x and we write x_2 for the second element of x. That is, $x = (x_1, x_2)$ holds. (i) is obvious. We note $\max\{d(x, y) : x, y \in X\} = 3$. So we have

$$p(x, z) \leq 3 \leq p(x, y) + p(y, z)$$

for any x, y, z. We have shown $(\tau'1)$. From the definition of p, there does not exist a sequence $\{z^{(n)}\}$ satisfying $\lim_n \sup\{p(z^{(n)}, z^{(m)}) : m > n\} = 0$. Thus, $(\tau'2)$ holds. In order to show $(\tau'3)$, we let $z \in X$ and a sequence $\{x^{(n)}\}$ satisfy $\lim_n p(z, x^{(n)}) = 0$. From the definition of p, $z_1 = -1$ obviously holds. From the definition of X, $z_2 \in \mathbb{Q}$ holds. For sufficiently large $n \in \mathbb{N}$, either of the following holds:

- $x_1^{(n)} = 0$ and $x_2^{(n)} \in \mathbb{Q}$.
- $x_1^{(n)} > 0$.

So

$$p(z, x^{(n)}) = d\Big((0, z_2), (x_1^{(n)}, x_2^{(n)})\Big) = |x_1^{(n)}| + |z_2 - x_2^{(n)}|$$

holds. Hence we have

$$\lim_{n\to\infty} x_1^{(n)} = 0 \quad \text{and} \quad \lim_{n\to\infty} x_2^{(n)} = z_2.$$

Therefore we obtain

$$\lim_{n\to\infty} d(x^{(n)}, x^{(n+1)}) = \lim_{n\to\infty} (|x_1^{(n)}| + |x_1^{(n+1)}| + |x_2^{(n)} - x_2^{(n+1)}|) = 0.$$

So $\{x^{(n)}\}$ converges to $x := (0, z_2)$. Fix $w \in X$. Then in the case where $w_1 = -1$, we have

$$p(w, x) = d\Big((0, w_2), x\Big) = \lim_{n\to\infty} d\Big((0, w_2), x^{(n)}\Big) = \lim_{n\to\infty} p(w, x^{(n)}).$$

In the other case, where $w_1 \neq -1$, we have $p(w, y) = 3$ for any $y \in X$. So in both cases, we obtain $p(w, x) \leq \liminf_n p(w, x^{(n)})$. We have shown $(\tau'3)$. Let us prove (iii). Arguing by contradiction, we assume that p is a τ-distance with η. Let A, B and C be as in Lemma 19. Define sequences $\{z^{(n)}\}$ and $\{x^{(n)}\}$ in X, $\{b^{(n)}\}$ in B, $\{c^{(n)}\}$ in C, $\{s^{(n)}\}$ and $\{t^{(n)}\}$ in $(0, \infty)$ and $\{\nu^{(n)}\}$ in $\mathbb{N}$ as follows:

(Step 1) $n = 1$, $b^{(1)} \in B$ and $z^{(1)} = (-1, b^{(1)})$.
(Step 2) Choose $s^{(1)}$ satisfying $\eta(z^{(1)}, s^{(1)}) < 2^{-1}$.
(Step 3) $c^{(1)} = c(b^{(1)}, s^{(1)}, 1, 1)$ and $\nu^{(1)} = n(b^{(1)}, s^{(1)}, 1, 1)$.
(Step 4) Choose $t^{(1)}$ satisfying $t^{(1)} + |b^{(1)} - c^{(1)}| < s^{(1)}$ and put $x^{(1)} = (t^{(1)}, c^{(1)})$.
(Step 5) $n := n + 1$, $b^{(n)} = b(c^{(n-1)}, t^{(n-1)}, \nu^{(n-1)})$ and $z^{(n)} = (-1, b^{(n)})$.
(Step 6) Choose $s^{(n)}$ satisfying $\eta(z^{(n)}, s^{(n)}) < 2^{-n}$ and $s^{(n)} + |b^{(n)} - c^{(n-1)}| < t^{(n-1)}$.
(Step 7) $c^{(n)} = c(b^{(n)}, s^{(n)}, \nu^{(n-1)}, n)$ and $\nu^{(n)} = \max\{\nu^{(n-1)}, n(b^{(n)}, s^{(n)}, \nu^{(n-1)}, n)\}$.
(Step 8) Choose $t^{(n)}$ satisfying $t^{(n)} + |b^{(n)} - c^{(n)}| < s^{(n)}$ and put $x^{(n)} = (t^{(n)}, c^{(n)})$.
(Step 9) goto (Step 5).

Then we have

$$\begin{aligned}
&p(z^{(n)}, x^{(m)}) = d\Big((0, b^{(n)}), (t^{(m)}, c^{(m)})\Big) \\
&\leq d\Big((0, b^{(n)}), (0, b^{(m)})\Big) + d\Big((0, b^{(m)}), (t^{(m)}, c^{(m)})\Big) \\
&= |b^{(n)} - b^{(m)}| + t^{(m)} + |b^{(m)} - c^{(m)}| \\
&< |b^{(n)} - b^{(m)}| + s^{(m)} \\
&\leq \sum_{k=n}^{m-1} \Big(|b^{(k)} - c^{(k)}| + |b^{(k+1)} - c^{(k)}|\Big) + s^{(m)} \\
&< \sum_{k=n}^{m-1} (s^{(k)} - t^{(k)} + t^{(k)} - s^{(k+1)}) + s^{(m)} \\
&= s^{(n)}
\end{aligned}$$

for $m, n \in \mathbb{N}$ with $m \geq n$ and hence

$$\lim_{n\to\infty} \sup_{m\geq n} \eta\Big(z^{(n)}, p(z^{(n)}, x^{(m)})\Big) \leq \lim_{n\to\infty} \eta(z^{(n)}, s^{(n)}) \leq \lim_{n\to\infty} 2^{-n} = 0.$$

By Lemma 20, $\{c^{(n)}\}$ converges to some irrational number γ. Also since $t^{(n)} < s^{(n)} < 2^{-n}$ holds by $(\tau_d 2)$, $\{t^{(n)}\}$ converges to 0. So $\{x^{(n)}\}$ converges to $(0, \gamma) \in X$. We have

$$p\Big((-1, 0), (0, \gamma)\Big) = 3 > \gamma = \lim_{n\to\infty} d\Big((0, 0), x^{(n)}\Big) = \lim_{n\to\infty} p\Big((-1, 0), x^{(n)}\Big),$$

which contradicts $(\tau_d 3)$. Therefore p is not a τ-distance. $\square$

Competing Interests

The author declares that he has no competing interests.

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