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Some approximation results on Chlodowsky type $q-$Bernstein-Schurer operators

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Abstract. The main concern of this article is to obtain several approximation features of the new Chlodowsky type q-Bernstein-Schurer operators. We prove the Korovkin type approximation theorem and discuss the order of convergence with regard to the ordinary modulus of continuity, an element of Lipschitz type and Peetre's K-functional, respectively. In addition, we derive the Voronovskaya type asymptotic theorem. Finally, using of Maple software, we present the comparison of the convergence of Chlodowsky type q-Bernstein-Schurer operators to the certain functions with some graphical illustrations and error estimation tables.

1. Introduction

In 1912, Bernstein [14] proposed the simplest and most excellent proof for the famous German mathematician Weierstrass's approximation theorem. The polynomial sequences discussed in Bernstein's proof have shed light on the approximation theory since they have various shape preservation properties and are easily integrable and differentiable, and it is a subject that many authors are still working on from past to present. In 1930, Kantorovich [25] suggested an integral modification of the Bernstein operators. In [20], Chlodowsky investigated a generalization of Bernstein operators on an unbounded set. Another generalization of the Bernstein operators were defined by Szász-Mirakjan [28, 47] on $[0,\infty)$. In [12], Baskakov presented a sequence of linear operators for the convenient functions defined on $[0,\infty)$. In 1962, a new modification of Bernstein operators established by Schurer [45]. The above-mentioned authors have made significant contributions to the development of approximation theory, and it is still aimed to obtain a better approximation by using many different generalizations and modifications of these studies. In recent years, it has been desired to obtain better approximation results by using the shape parameter λ because it provides flexibility in modeling. Ansari et al. [7] obtained some numerical and theoretical approximation results for Schurer–Stancu operators based on shape parameter λ. In [40], Özger et al. estimated the rate of weighted statistical convergence for generalized blending-type Bernstein-Kantorovich operators. In 2021, Aslan [9] attained some valuable approximation results on λ-Szász-Mirakjan-Kantorovich operators. Braha

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et al. [15] introduced the convergence of λ-Bernstein operators via power series summability method. We refer also for readers to [16, 17, 39, 42]. On the other hand the quantum calculus briefly $q-$calculus, which has many applications in various fields, has also attracted the attention of many researchers working on approximation theory. Firstly, Lupaş [27] presented the approximation properties of the generalizations of $q-$Bernstein operators. Later, Phillips [41] derived some convergence theorems and Voronovskaya type asymptotic formula for the most popular generalizations of the $q-$Bernstein operators. Agratini [2] studied a new type of $q-$Bernstein type operators. Karslı and Gupta [26] proposed and discussed the following $q-$analogue of Chlodowsky operators for a positive increasing sequence b_r with $\lim_{r\to\infty} b_r = \infty$ and $y \in [0, b_r]$, as

$$C_{m,q}(\mu; y) = \sum_{j=0}^{m} \mu\left(\frac{[j]_q}{[m]_q} b_m\right) \begin{bmatrix} m \\ j \end{bmatrix}_q \left(\frac{y}{b_m}\right)^j \prod_{r=0}^{m-j-1} (1 - q^r \frac{y}{b_m}). \tag{1}$$

They explored the order of convergence and monotonocity properties of operators (1). Muraru [30] introduced the Bernstein-Schurer polynomials related on $q-$calculus, for any $r \in \mathbb{N}$, fixed $p \in \mathbb{N} \cup \{0\}$ and function $\mu \in C[0, p+1]$ as:

$$M^p_{r,q}(\mu; y) = \sum_{j=0}^{r+p} \mu\left(\frac{[j]_q}{[r]_q}\right) \begin{bmatrix} m+p \\ j \end{bmatrix}_q y^j \prod_{k=0}^{r+p-j-1} (1 - q^k y), \tag{2}$$

where $0 < q < 1$ and $y \in [0, 1]$.

For the operators given by (2), she derived the Bohman-Korovkin type approximation theorem and evaluated the order of approximation with regard to the modulus of smoothness. Agrawal et al. [3] considered the $q-$analogue of Bernstein-Schurer-Stancu type operators and studied the global and local direct approximation consequences of these operators. Furthermore, several approximation features of Chlodowsky type $q-$Bernstein-Schurer-Stancu operators are demonstrated by Vedi and Özarslan [48]. Mursaleen and Khan [34] studied some statistical approximation features of generalized $q-$Bernstein-Schurer operators and established several direct theorems for these operators. Ren and Zeng [43] discussed the statistical convergence of the Korovkin and Voronovskaya type results of the modification of $q-$Bernstein-Schurer type operators. Özarslan et al. [38] investigated the rate of convergence for q-Bernstein-Schurer-Kantorovich operators by means of the first and the second modulus of continuity. For Chlodowsky variant of several operators, one can refer to [6], [29], [31] and [32]. Moreover, Acu et al. [1] presented a Durrmeyer variant of q-Bernstein-Schurer operators and studied uniform and statistical convergence for these operators. In [13] Baxhaku et al. introduced two kinds of Chlodowsky-type q-Bernstein-Schurer-Stancu-Kantorovich operators on the onbounded domain. One has some papers based on q-calculus with([4, 8, 10, 18, 21, 33, 36, 37, 44]).

Now, before proceeding further, we present some basic notations and definitions which depend on q-calculus as set out in [24]. Let $0 < q < 1$, for all integer $j > 0$, the $q-$integer $[j]_q$ is given as

$$[j]_q := \begin{cases} \frac{1-q^j}{1-q}, & q \neq 1 \\ j, & q = 1 \end{cases}.$$

The $q-$factorial $[j]_q!$ and for any integers j, l, $j \geq l \geq 0$, the $q-$binomial $\begin{bmatrix} j \\ l \end{bmatrix}_q$ are given respectively, as below:

$$[j]_q! := \begin{cases} [j]_q [j-1]_q .. [1]_q, & j = 1, 2, .. \\ 1, & j = 0 \end{cases}$$

and

$$\begin{bmatrix} j \\ l \end{bmatrix}_q = \frac{[j]_q!}{[l]_q! [j-l]_q!}.$$

Also, for $0 < q < 1$ and all integers $j, l,\ j \geq l \geq 0$, the following identities we have

$$[j]_q = 1 + q\,[j-1]_q \quad \text{and} \quad [j]_q + q^j\,[l-j]_q = [l]_q\,.$$

For $0 < q < 1$, the $q-$analogue of $(y-s)^j$ is defined by

$$(y-s)_q^j = \begin{cases} (y-s)(y-qs)...(y-q^{j-1}s), & j \geq 1, \\ 1, & j = 0. \end{cases}$$

By the motivation of the all above mentioned works, we define the new Chlodowsky type of Bernstein-Schurer operators for $0 < q < 1$, any $r \in \mathbb{N}$ and fixed $p \in \mathbb{N} \cup \{0\}$ as:

$$R_{r,q}(\mu; y) = \left(\frac{[r+1]_q}{[r]_q}\right)^{r+p} \sum_{j=0}^{r+p} \mu\left(\frac{[j]_q}{[r+1]_q} b_r\right) \begin{bmatrix} r+p \\ j \end{bmatrix}_q \left(\frac{y}{b_r}\right)^j \prod_{k=0}^{r+p-j-1} \left(\frac{[r]_q}{[r+1]_q} - q^k \frac{y}{b_r}\right), \tag{3}$$

where $y \in [0, b_r]$, b_r is a positive increasing sequence which satisfy $\lim_{r\to\infty} b_r = \infty$, $\lim_{r\to\infty} \frac{b_r}{[r]_q} = 0$.

It is obvious that the operators given by (3) are positive and linear.

The structure of this work is organized as follows: For the operators given by (3), in section 2, we compute the moments up to fourth order and the central moments for the first, second and fourth degree. In section 3, we establish the Korovkin type convergence theorem. In section 4, we estimate the order of approximation with regard to of the usual modulus of continuity, an element of Lipschitz type and Peetre's K-functional, respectively. In section 5, we derive the Voronovskaya type asymptotic formula. Finally, with the aid of Maple software, for the different parameters of (m, p, q), we compare the convergence of operators (3) to the certain functions with some graphical illustrations and error estimation tables.

2. Main Results

Lemma 2.1. *Let $\mu(t) = t^u$, $u = 0, 1, 2, 3, 4$. Then, we have the following moments for the operators (3):*

$$(i)\ R_{r,q}(1; y) = 1,$$

$$(ii)\ R_{r,q}(t; y) = \frac{[r+p]_q}{[r]_q} y,$$

$$(iii)\ R_{r,q}(t^2; y) = \frac{[r+p]_q\,[r+p-1]_q}{[r]_q^2} q y^2 + \frac{[r+p]_q\, b_r}{[r]_q\,[r+1]_q} y,$$

$$(iv)\ R_{r,q}(t^3; y) = \frac{[r+p]_q\,[r+p-1]_q\,[r+p-2]_q}{[r]_q^3} q^3 y^3 + \frac{[r+p]_q\,[r+p-1]_q\,(2q+q^2) b_r}{[r]_q^2\,[r+1]_q} y^2 + \frac{[r+p]_q\, b_r^2}{[r]_q\,[r+1]_q^2} y,$$

$$(v)\ R_{r,q}(t^4; y) = \frac{[r+p]_q\,[r+p-1]_q\,[r+p-2]_q\,[r+p-3]_q}{[r]_q^4} q^6 y^4 + \frac{[r+p]_q\,[r+p-1]_q\,[r+p-2]_q\,(q^5+2q^4+3q^3) b_r}{[r]_q^3\,[r+1]_q} y^3 + \frac{[r+p]_q\,[r+p-1]_q\,(q^3+3q^2+3q) b_r^2}{[r]_q^2\,[r+1]_q^2} y^2 + \frac{[r+p]_q\, b_r^3}{[r]_q\,[r+1]_q^3} y.$$

Proof. Since all the proofs of the above moments can be obtained using a similar methods, we will only give the proof of the first three.

In view of the following relation:

$$(1-y)_q^{r+p-j} = \prod_{k=0}^{r+p-j-1}(1-q^k y),$$

then, we may write

(i) $R_{r,q}(1;y)$

$$= \left(\frac{[r+1]_q}{[r]_q}\right)^{r+p} \sum_{j=0}^{r+p} \begin{bmatrix} r+p \\ j \end{bmatrix}_q \left(\frac{y}{b_r}\right)^j \prod_{k=0}^{r+p-j-1} \left(\frac{[r]_q}{[r+1]_q} - q^k \frac{y}{b_r}\right)$$

$$= \left(\frac{[r+1]_q}{[r]_q}\right)^{r+p} \left(\frac{[r]_q}{[r+1]_q}\right)^{r+p} = 1,$$

(ii) $R_{r,q}(t;y)$

$$= \left(\frac{[r+1]_q}{[r]_q}\right)^{r+p} \sum_{j=0}^{r+p} \begin{bmatrix} r+p \\ j \end{bmatrix}_q \left(\frac{y}{b_r}\right)^j \prod_{k=0}^{r+p-j-1} \left(\frac{[r]_q}{[r+1]_q} - q^k \frac{y}{b_r}\right)\left(\frac{[j]_q}{[r+1]_q} b_r\right)$$

$$= \left(\frac{[r+1]_q}{[r]_q}\right)^{r+p} \frac{[r+p]_q b_r}{[r+1]_q} \sum_{j=1}^{r+p} \begin{bmatrix} r+p-1 \\ j-1 \end{bmatrix}_q \left(\frac{y}{b_r}\right)^j \prod_{k=0}^{r+p-j-1} \left(\frac{[r]_q}{[r+1]_q} - q^k \frac{y}{b_r}\right)$$

$$= \left(\frac{[r+1]_q}{[r]_q}\right)^{r+p} \frac{[r+p]_q b_r}{[r+1]_q} \sum_{j=0}^{r+p-1} \begin{bmatrix} r+p-1 \\ j \end{bmatrix}_q \left(\frac{y}{b_r}\right)^{j+1} \prod_{k=0}^{r+p-j-2} \left(\frac{[r]_q}{[r+1]_q} - q^k \frac{y}{b_r}\right)$$

$$= \left(\frac{[r+1]_q}{[r]_q}\right)^{r+p} \frac{[r+p]_q b_r}{[r+1]_q} \frac{y}{b_r} \left(\frac{[r]_q}{[r+1]_q}\right)^{r+p-1}$$

$$= \frac{[r+p]_q}{[r]_q} y.$$

In view of the relation $[j]_q = q\,[j-1]_q + 1$, it becomes

(iii) $R_{r,q}(t^2;y)$

$$= \left(\frac{[r+1]_q}{[r]_q}\right)^{r+p} \sum_{j=0}^{r+p} \begin{bmatrix} r+p \\ j \end{bmatrix}_q \left(\frac{y}{b_r}\right)^j \prod_{k=0}^{r+p-j-1} \left(\frac{[r]_q}{[r+1]_q} - q^k \frac{y}{b_r}\right)\left(\frac{[j]_q}{[r+1]_q} b_r\right)^2$$

$$= \left(\frac{[r+1]_q}{[r]_q}\right)^{r+p} \frac{[r+p]_q b_r^2}{[r+1]_q^2} \sum_{j=1}^{r+p} \begin{bmatrix} r+p-1 \\ j-1 \end{bmatrix}_q \left(\frac{y}{b_r}\right)^j$$

$$\times \prod_{k=0}^{r+p-j-1} \left(\frac{[r]_q}{[r+1]_q} - q^k \frac{y}{b_r}\right)\left(q\,[j-1]_q + 1\right)$$

$$= \left(\frac{[r+1]_q}{[r]_q}\right)^{r+p} \frac{q[r+p]_q b_r^2}{[r+1]_q^2} \sum_{j=1}^{r+p} \begin{bmatrix} r+p-1 \\ j-1 \end{bmatrix}_q \left(\frac{y}{b_r}\right)^j$$

$$\times \prod_{k=0}^{r+p-j-1} \left(\frac{[r]_q}{[r+1]_q} - q^k \frac{y}{b_r}\right)[j-1]_q$$

$$+\left(\frac{[r+1]_q}{[r]_q}\right)^{r+p}\frac{[r+p]_q b_r^2}{[r+1]_q^2}\sum_{j=0}^{r+p-1}\begin{bmatrix} r+p-1 \\ j \end{bmatrix}_q\left(\frac{y}{b_r}\right)^{j+1}$$

$$\times\prod_{k=0}^{r+p-j-2}\left(\frac{[r]_q}{[r+1]_q}-q^k\frac{y}{b_r}\right)$$

$$=\left(\frac{[r+1]_q}{[r]_q}\right)^{r+p}\frac{q[r+p]_q[r+p-1]_q b_r^2}{[r+1]_q^2}\frac{y^2}{b_r^2}\left(\frac{[r]_q}{[r+1]_q}\right)^{r+p-2}$$

$$+\left(\frac{[r+1]_q}{[r]_q}\right)^{r+p}\frac{[r+p]_q b_r^2}{[r+1]_q^2}\frac{y}{b_r}\left(\frac{[r]_q}{[r+1]_q}\right)^{r+p-1}$$

$$=\frac{[r+p]_q[r+p-1]_q}{[r]_q^2}qy^2+\frac{[r+p]_q b_r}{[r]_q[r+1]_q}y,$$

which gives the proof of this Lemma. □

Lemma 2.2. *Let* $0<q<1$, $r\in\mathbb{N}$, $y\in[0,b_r]$. *Then, the following central moments satisfies*

$$(i)\ R_{r,q}(t-y;y)=\left(\frac{[r+p]_q}{[r]_q}-1\right)y,$$

$$(ii)\ R_{r,q}((t-y)^2;y)=\left(\frac{[r+p]_q[r+p-1]_q}{[r]_q^2}q-\frac{2[r+p]_q}{[r]_q}+1\right)y^2+\frac{[r+p]_q b_r}{[r]_q[r+1]_q}y:=\gamma_{r,q}(y),$$

$$\begin{aligned}(iii)\ R_{r,q}((t-y)^4;y)=&\left(\frac{q^6[r+p]_q[r+p-1]_q[r+p-2]_q[r+p-3]_q}{[r]_q^4}\right.\\&-\frac{4q^3[r+p]_q[r+p-1]_q[r+p-2]_q}{[r]_q^3}\\&\left.+\frac{6q[r+p]_q[r+p-1]_q}{[r]_q^2}-\frac{4[r+p]_q}{[r]_q}+1\right)y^4\\&+\left(\frac{b_r(q^5+2q^4+3q^3)[r+p]_q[r+p-1]_q[r+p-2]_q}{[r]_q^3[r+1]_q}\right.\\&\left.-\frac{4b_r(2q+q^2)[r+p]_q[r+p-1]_q}{[r]_q^2[r+1]_q}+\frac{6b_r[r+p]_q}{[r]_q[r+1]_q}\right)y^3\\&+\left(\frac{b_r^2(q^3+3q^2+3q)[r+p]_q[r+p-1]_q}{[r]_q^2[r+1]_q^2}-\frac{4b_r^2[r+p]_q}{[r]_q[r+1]_q^2}\right)y^2\\&+\frac{b_r^3[r+p]_q}{[r]_q[r+1]_q^3}y.\end{aligned}$$

Lemma 2.3. *From Lemma 2.2-(ii), one has*

$$\sup_{y\in[0,b_r]}R_{r,q}((t-y)^2;y)\le b_r^2\left(\frac{q^r[p][r+1]_q+[r+p]_q}{[r]_q[r+1]_q}\right).$$

Proof. Using the inequality $q\,[r+p]_q\,[r+p-1]_q \leq [r+p]_q^2$, hence

$$\left(\frac{[r+p]_q\,[r+p-1]_q}{[r]_q^2}q - \frac{2\,[r+p]_q}{[r]_q} + 1\right)y^2 \leq \left(\frac{[r+p]_q}{[r]_q} - 1\right)^2 y^2.$$

Since $[r+p]_q = [r]_q + q^r[p]$, we derive

$$\begin{aligned} R_{r,q}((t-y)^2;y) &= \left(\frac{[r+p]_q\,[r+p-1]_q}{[r]_q^2}q - \frac{2\,[r+p]_q}{[r]_q} + 1\right)y^2 + \frac{[r+p]_q\,b_r}{[r]_q\,[r+1]_q}y \\ &\leq \left(\frac{[r+p]_q}{[r]_q} - 1\right)^2 y^2 + \frac{[r+p]_q\,b_r}{[r]_q\,[r+1]_q}y \\ &= \left(\frac{q^r[p]}{[r]_q}y\right)^2 + \frac{[r+p]_q\,b_r}{[r]_q\,[r+1]_q}y. \end{aligned}$$

Taking the supremum on $[0,b_r]$, which gives the required result as:

$$\sup_{y\in[0,b_r]} R_{r,q}((t-y)^2;y) \leq b_r^2\left(\frac{q^r[p]\,[r+1]_q + [r+p]_q}{[r]_q\,[r+1]_q}\right).$$

□

3. Korovkin type approximation

In this section, for the operators given by (3), we will prove the Korovkin type approximation theorem. Suppose that C_{1+y^2} denotes the set of all continuous functions of μ, verifying the condition

$$\left|\mu(y)\right| \leq N_\mu(1+y^2), \quad y\in(-\infty,\infty).$$

It is equipped with the norm as below:

$$\left\|\mu\right\|_{1+y^2} = \sup_{y\in(-\infty,\infty)} \frac{\left|\mu(y)\right|}{1+y^2}.$$

In order to prove our main theorem, it is important to discuss the following theorems.

Theorem 3.1. *[23]. For a linear positive operators U_m, acting from C_ρ to C_ρ, verifying the assumption*

$$\lim_{m\to\infty} \|U_m(1;.)-1\|_\rho = 0, \tag{4}$$

$$\lim_{m\to\infty} \left\|U_m(\varphi;.)-\varphi\right\|_\rho = 0, \tag{5}$$

$$\lim_{m\to\infty} \left\|U_m(\varphi^2;.)-\varphi^2\right\|_\rho = 0, \tag{6}$$

where φ is a increasing and continuous function on $(-\infty,\infty)$ such that $\lim_{y\to\pm\infty}\varphi(y) = \pm\infty$ and there consist a function $\mu^* \in C_\rho$ which $\lim_{m\to\infty}\left\|U_m(\mu^*;.)-\mu^*\right\|_\rho > 0$.

Theorem 3.2. *[23]. For a function $\mu \in C_\rho^0 \subset C_\rho$, the conditions which are given by (4), (5) and (6) require $\lim_{m\to\infty}\left\|U_m(\mu;.)-\mu\right\|_\rho = 0$. If*

$$\lim_{|y|\to\infty} \frac{\mu(y)}{\rho(y)} < \infty.$$

Theorem 3.3. *Suppose that* $q := (q_r)$ *such that* $0 < q_r < 1, \lim\limits_{r\to\infty} q_r = 1, \lim\limits_{r\to\infty} \frac{b_r}{[r]_{q_r}} = 0$ *as* $r \to \infty$. *Then, for all* $\mu \in C^0_{1+y^2}$ *the following relation verify*

$$\lim_{r\to\infty} \sup_{y\in[0,b_r]} \frac{\left|R_{r,q_r}(\mu; y) - \mu(y)\right|}{1 + y^2} = 0.$$

Proof. Taking into account the operators given by [23]. Then,

$$U_{r,q_r}(\mu; y) = \begin{cases} R_{r,q_r}(\mu; y), & \text{for } 0 \le y \le b_r \\ \mu(y), & \text{for } y > b_r \end{cases},$$

$$\lim_{r\to\infty} \left\|U_{r,q_r}(t^s; .) - y^s\right\|_{1+y^2} = \lim_{r\to\infty} \sup_{y\in[0,b_r]} \frac{\left|R_{r,q_r}(t^s; y) - y^s\right|}{1 + y^2} \quad s = 0, 1, 2.$$

Considering to Lemma 2.1-(i), it is easy to check $R_{r,q_r}(1; y) = 1$.
By Lemma refLem:1-(ii), yields

$$\begin{aligned} \sup_{y\in[0,b_r]} \frac{\left|R_{r,q_r}(t; y) - y\right|}{1 + y^2} &\le \sup_{y\in[0,b_r]} \frac{\left|\frac{[r+p]_{q_r}}{[r]_{q_r}} - 1\right| y}{1 + y^2} \\ &\le \left|\frac{[r + p]_{q_r}}{[r]_{q_r}} - 1\right| \to 0. \end{aligned}$$

Proceeding similarly, by Lemma 2.1-(iii)

$$\sup_{y\in[0,b_r]} \frac{\left|R_{r,q_r}(t; y) - y\right|}{1 + y^2} \le \sup_{y\in[0,b_r]} \frac{\left|q_r \frac{[r+p]_{q_r}[r+p-1]_{q_r}}{[r]^2_{q_r}} - 1\right| y^2 + \frac{[r+p]_{q_r} b_r}{[r]_{q_r}[r+1]_{q_r}} y}{1 + y^2}$$

$$\le \left|q_r \frac{[r + p]_{q_r}[r + p - 1]_{q_r}}{[r]^2_{q_r}} - 1\right| + \frac{[r + p]_{q_r} b_r}{[r]_{q_r}[r + 1]_{q_r}} \to 0,$$

since $\lim\limits_{r\to\infty} q_r = 1, \lim\limits_{r\to\infty} \frac{b_r}{[r]_{q_r}} = 0$ as $r \to \infty$.
Thus, we arrive at the desired result. □

4. Direct Theorems

In this section, we compute the order of convergence with regard to the ordinary modulus of continuity, class of Lipschitz functions and Peetre's K-functional. Let $C_B[0, \infty)$ denotes the space of all real-valued continuous and bounded functions κ on $[0, \infty)$. On $C_B[0, \infty)$, the norm is given as:

$$\|\kappa\| = \sup_{y\in[0,\infty)} \left|\kappa(y)\right|.$$

The Peetre's K-functional is given as

$$K_2(\kappa, \eta) = \inf_{\lambda\in C^2_B} \{\|\kappa - \lambda\| + \eta \|\lambda''\|\}$$

where $\eta > 0$ and $C^2_B = \{\lambda \in C_B[0, \infty) : \lambda', \lambda'' \in C_B[0, \infty)\}$.
Taking into account [22], there consist an absolute constant $C > 0$ such that

$$K_2(\kappa; \eta) \le C\omega_2(\kappa; \sqrt{\eta}), \qquad \eta > 0 \tag{7}$$

where

$$\omega_2(\kappa;\eta) = \sup_{0<a\leq\eta}\sup_{y\in[0,\infty)}\left|\kappa(y+2a)-2\kappa(y+a)+\kappa(y)\right|$$

is the second-order modulus of smoothness of the function $\kappa \in C_B[0,\infty)$. In addition, by

$$\omega(\kappa;\eta) := \sup_{0<a\leq\eta}\sup_{y\in[0,\infty)}\left|\kappa(y+a)-\kappa(y)\right|$$

we state the ordinary modulus of continuity of $\kappa \in C_B[0,\infty)$. Since $\eta > 0$, $\omega(\kappa;\eta)$ has some useful properties see: [5].

Furthermore, we give the elements of Lipschitz type with $Lip_L(\zeta)$, where $L > 0$ and $0 < \zeta \leq 1$. If the following inequality

$$\left|\kappa(t)-\kappa(y)\right| \leq L\left|t-y\right|^{\zeta}, \qquad (t,y\in\mathbb{R})$$

holds, then one can say a function κ is belong to $Lip_L(\zeta)$.

Theorem 4.1. *Let $q := (q_r)$ such that $0 < q_r < 1$, $\lim\limits_{r\to\infty} q_r = 1$, $\lim\limits_{r\to\infty}\frac{b_r}{[r]_{q_r}} = 0$ as $r\to\infty$. Then, for all $\mu \in C_B[0,\infty)$ we obtain*

$$\left|R_{r,q_r}(\mu;y)-\mu(y)\right| \leq 2\omega(\mu;\sqrt{\gamma_{r,q_r}(y)}),$$

where $\gamma_{r,q_r}(y) = R_{r,q_r}((t-y)^2;y)$.

Proof. From the definition of operators (3) and applying the triangular inequality, then we may write

$$\begin{aligned}
&\left|R_{r,q_r}(\mu;y)-\mu(y)\right| \\
&\leq \left|\left(\frac{[r+1]_{q_r}}{[r]_{q_r}}\right)^{r+p}\sum_{j=0}^{r+p}\begin{bmatrix} r+p \\ j\end{bmatrix}_{q_r}\left(\frac{y}{b_r}\right)^j\left\{\mu\left(\frac{[j]_{q_r}}{[r+1]_{q_r}}b_r\right)-\mu(y)\right\}\right. \\
&\left.\times\prod_{k=0}^{r+p-j-1}\left(\frac{[r]_{q_r}}{[r+1]_{q_r}}-q^k\frac{y}{b_r}\right)\right| \\
&\leq \left(\frac{[r+1]_{q_r}}{[r]_{q_r}}\right)^{r+p}\sum_{j=0}^{r+p}\begin{bmatrix} r+p \\ j\end{bmatrix}_{q_r}\left(\frac{y}{b_r}\right)^j\left|\mu\left(\frac{[j]_{q_r}}{[r+1]_{q_r}}b_r\right)-\mu(y)\right| \\
&\times\prod_{k=0}^{r+p-j-1}\left(\frac{[r]_{q_r}}{[r+1]_{q_r}}-q^k\frac{y}{b_r}\right).
\end{aligned}$$

Using the common property of modulus of continuity as below:

$$\left|\mu(t)-\mu(y)\right| \leq \omega(\mu;\gamma)\left(\frac{\left|t-y\right|}{\gamma}+1\right), \ \ \gamma > 0.$$

Then,

$$
\begin{aligned}
&\left|R_{r,q_r}(\mu;y)-\mu(y)\right| \\
&\leq \left(\frac{[r+1]_{q_r}}{[r]_{q_r}}\right)^{r+p}\sum_{j=0}^{r+p}\begin{bmatrix} r+p \\ j \end{bmatrix}_{q_r}\left(\frac{y}{b_r}\right)^j\left(\frac{\left|\frac{[j]_{q_r}}{[r+1]_{q_r}}b_r-y\right|}{\gamma}+1\right)\omega(\mu;\gamma) \\
&\times\prod_{k=0}^{r+p-j-1}\left(\frac{[r]_{q_r}}{[r+1]_{q_r}}-q^k\frac{y}{b_r}\right) \\
&=\omega(\mu;\gamma)+\frac{\omega(\mu;\gamma)}{\gamma}\left(\left(\frac{[r+1]_{q_r}}{[r]_{q_r}}\right)^{r+p}\sum_{j=0}^{r+p}\begin{bmatrix} r+p \\ j \end{bmatrix}_{q_r}\left(\frac{y}{b_r}\right)^j\left|\frac{[j]_{q_r}}{[r+1]_{q_r}}b_r-y\right|\right. \\
&\left.\times\prod_{k=0}^{r+p-j-1}\left(\frac{[r]_{q_r}}{[r+1]_{q_r}}-q^k\frac{y}{b_r}\right)\right).
\end{aligned}
$$

Utilizing the Hölder's inequality, thus

$$
\begin{aligned}
&\left|R_{r,q_r}(\mu;y)-\mu(y)\right| \\
&\leq \omega(\mu;\gamma)+\frac{\omega(\mu;\gamma)}{\gamma}\left\{\left(\frac{[r+1]_{q_r}}{[r]_{q_r}}\right)^{r+p}\sum_{j=0}^{r+p}\begin{bmatrix} r+p \\ j \end{bmatrix}_{q_r}\left(\frac{y}{b_r}\right)^j\left(\frac{[j]_{q_r}}{[r+1]_{q_r}}b_r-y\right)^2\right. \\
&\left.\times\prod_{k=0}^{r+p-j-1}\left(\frac{[r]_{q_r}}{[r+1]_{q_r}}-q^k\frac{y}{b_r}\right)\right\}^{\frac{1}{2}} \\
&=\omega(\mu;\gamma)+\frac{\omega(\mu;\gamma)}{\gamma}\left\{R_{r,q_r}((t-y)^2;y)\right\}^{\frac{1}{2}}.
\end{aligned}
$$

Taking $\gamma = \sqrt{\gamma_{r,q_r}(y)} = \sqrt{R_{r,q_r}((t-y)^2;y)}$, which gives the required result as:

$$
\left|R_{r,q_r}(\mu;y)-\mu(y)\right| \leq 2\omega(\mu;\sqrt{\gamma_{r,q_r}(y)}).
$$

□

Theorem 4.2. *Suppose that* $q := (q_r)$ *such that* $0 < q_r < 1, \lim\limits_{r\to\infty} q_r = 1, \lim\limits_{r\to\infty}\frac{b_r}{[r]_{q_r}} = 0$ *as* $r \to \infty$. *Then, for* $\mu \in Lip_L(\zeta)$ *we derive*

$$
\left|R_{r,q_r}(\mu;y)-\mu(y)\right| \leq L(\gamma_{r,q_r}(y))^{\frac{\zeta}{2}},
$$

where $\gamma_{r,q_r}(y)$ *is given by Theorem 4.1.*

Proof. Let $\mu \in Lip_L(\zeta)$. From the linearity and monotonicity of the operators (3), then we arrive

$$\begin{aligned}
&\left|R_{r,q_r}(\mu;y)-\mu(y)\right| \leq R_{r,q_r}(\left|\mu(t)-\mu(y)\right|;y)\\
&=\left(\frac{[r+1]_{q_r}}{[r]_{q_r}}\right)^{r+p}\sum_{j=0}^{r+p}\begin{bmatrix} r+p\\ j\end{bmatrix}_{q_r}\left(\frac{y}{b_r}\right)^j\\
&\times\prod_{k=0}^{r+p-j-1}\left(\frac{[r]_{q_r}}{[r+1]_{q_r}}-q^k\frac{y}{b_r}\right)\left|\mu\left(\frac{[j]_{q_r}}{[r+1]_{q_r}}b_r\right)-\mu(y)\right|\\
&\leq L\left(\frac{[r+1]_{q_r}}{[r]_{q_r}}\right)^{r+p}\sum_{j=0}^{r+p}\begin{bmatrix} r+p\\ j\end{bmatrix}_{q_r}\left(\frac{y}{b_r}\right)^j\\
&\times\prod_{k=0}^{r+p-j-1}\left(\frac{[r]_{q_r}}{[r+1]_{q_r}}-q^k\frac{y}{b_r}\right)\left|\frac{[j]_{q_r}}{[r+1]_{q_r}}b_r-y\right|^{\zeta}.
\end{aligned}$$

Using the Hölder's inequality and choosing $p_1=\frac{2}{\zeta}$ and $p_2=\frac{2}{2-\zeta}$, one has $\frac{1}{p_1}+\frac{1}{p_2}=1$. Hence, we may write

$$\begin{aligned}
&\left|R_{r,q_r}(\mu;y)-\mu(y)\right|\\
&\leq L\left(\frac{[r+1]_{q_r}}{[r]_{q_r}}\right)^{r+p}\sum_{j=0}^{r+p}\left\{\left[\begin{bmatrix} r+p\\ j\end{bmatrix}_{q_r}\left(\frac{y}{b_r}\right)^j\prod_{k=0}^{r+p-j-1}\left(\frac{[r]_{q_r}}{[r+1]_{q_r}}-q^k\frac{y}{b_r}\right)\right]^{\frac{2-\zeta}{2}}\right.\\
&\left.\times\left[\left(\frac{[j]_{q_r}}{[r+1]_{q_r}}b_r-y\right)^2\begin{bmatrix} r+p\\ j\end{bmatrix}_{q_r}\left(\frac{y}{b_r}\right)^j\prod_{k=0}^{r+p-j-1}\left(\frac{[r]_{q_r}}{[r+1]_{q_r}}-q^k\frac{y}{b_r}\right)\right]^{\frac{\zeta}{2}}\right\}\\
&\leq L\left[\left\{\left(\frac{[r+1]_{q_r}}{[r]_{q_r}}\right)^{r+p}\sum_{j=0}^{r+p}\begin{bmatrix} r+p\\ j\end{bmatrix}_{q_r}\left(\frac{y}{b_r}\right)^j\prod_{k=0}^{r+p-j-1}\left(\frac{[r]_{q_r}}{[r+1]_{q_r}}-q^k\frac{y}{b_r}\right)\right\}^{\frac{2-\zeta}{2}}\right.\\
&\times\left\{\left(\frac{[r+1]_{q_r}}{[r]_{q_r}}\right)^{r+p}\sum_{j=0}^{r+p}\left(\frac{[j]_{q_r}}{[r+1]_{q_r}}b_r-y\right)^2\begin{bmatrix} r+p\\ j\end{bmatrix}_{q_r}\left(\frac{y}{b_r}\right)^j\right.\\
&\left.\left.\times\prod_{k=0}^{r+p-j-1}\left(\frac{[r]_{q_r}}{[r+1]_{q_r}}-q^k\frac{y}{b_r}\right)\right\}^{\frac{\zeta}{2}}\right]\\
&=L\left\{R_{r,q_r}((t-y)^2;y)\right\}^{\frac{\zeta}{2}}\\
&\leq L(\gamma_{r,q_r}(y))^{\frac{\zeta}{2}}.
\end{aligned}$$

Thus, the proof is completed. □

Theorem 4.3. *For all $\mu \in C_B[0,\infty)$, the following inequality satisfy*

$$\left|R_{r,q}(\mu;y)-\mu(y)\right| \leq C\omega_2(\mu;\sqrt{\chi_{r,q}(y)})+\omega(\mu;\beta_{r,q}(y)),$$

where $C>0$ is a constant, $\gamma_{r,q}(y)=R_{r,q}((t-y)^2;y)$, $\beta_{r,q}(y)=y\left(\frac{[r+p]_q}{[r]_q}-1\right)$ and $\chi_{r,q}(y)=\gamma_{r,q}(y)+(\beta_{r,q}(y))^2$.

Proof. Using the following auxiliary operators:

$$\overset{*}{R}_{r,q}(\mu; y) = R_{r,q}(\mu; y) - \mu(\frac{[r+p]_q}{[r]_q} y) + \mu(y) \tag{8}$$

By taking into account Lemma 2.2-(ii), it follows

$$\overset{*}{R}_{r,q}(t-y; y) = 0.$$

From Taylor formula, then

$$\lambda(t) = \lambda(y) + (t-y)\lambda'(y) + \int_y^t (t-u)\lambda''(u)du, \quad \lambda \in C_B^2[0,\infty) \tag{9}$$

Operating $\overset{*}{R}_{r,q}(.; y)$ to (9), we derive

$$\overset{*}{R}_{r,q}(\lambda; y) - \lambda(y) = \overset{*}{R}_{r,q}((t-y)\lambda'(y); y) + \overset{*}{R}_{r,q}(\int_y^t (t-u)\lambda''(u)du; y)$$

$$= \lambda'(y)\overset{*}{R}_{r,q}(t-y; y) + R_{r,q}(\int_y^t (t-u)\lambda''(u)du; y) - \int_y^{\frac{[r+p]_q}{[r]_q}y} (\frac{[r+p]_q}{[r]_q} y - u)\lambda''(u)du$$

$$= R_{r,q}(\int_y^t (t-u)\lambda''(u)du; y) - \int_y^{\frac{[r+p]_q}{[r]_q}y} (\frac{[r+p]_q}{[r]_q} y - u)\lambda''(u)du.$$

Considering to Lemma 2.2-(ii) and by (8),

$$\left|\overset{*}{R}_{r,q}(\lambda; y) - \lambda(y)\right|$$

$$\leq \left|R_{r,q}(\int_y^t (t-u)\lambda''(u)du; y)\right| + \left|\int_y^{\frac{[r+p]_q}{[r]_q}y} (\frac{[r+p]_q}{[r]_q} y - u)\lambda''(u)du\right|$$

$$\leq R_{r,q}(\left|\int_y^t (t-u)\right| |\lambda''(u)|\, |du|; y) + \int_y^{\frac{[r+p]_q}{[r]_q}y} \left|\frac{[r+p]_q}{[r]_q} y - u\right| |\lambda''(u)|\, |du|$$

$$\leq \|\lambda''(u)\| \left\{ R_{r,q}((t-y)^2; y) + \left(\frac{[r+p]_q}{[r]_q} y - y\right)^2 \right\}.$$

Also, by Lemma 2.1, we arrive

$$\begin{aligned} \left|\overset{*}{R}_{r,q}(\mu; y)\right| &\leq \left|R_{r,q}(\mu; y)\right| + 2\left\|\mu\right\| \\ &\leq \left\|\mu\right\| R_{r,q}(1; y) + 2\left\|\mu\right\| \leq 3\left\|\mu\right\| \end{aligned} \tag{10}$$

On the other hand, by (8) and (10), imply

$$\begin{aligned}\left|R_{r,q}(\mu;y)-\mu(y)\right| &\leq \left|\overset{*}{R}_{r,q}(\mu-\lambda;y)-(\mu-\lambda)(y)\right| \\ &+\left|\overset{*}{R}_{r,q}(\lambda;y)-\lambda(y)\right|+\left|\mu(y)-\mu(\frac{[r+p]_q}{[r]_q}y)\right| \\ &\leq 4\left\|\mu-\lambda\right\|+\chi_{r,q}(y)\left\|\lambda''\right\|+\omega\left(\mu;\frac{[r+p]_q}{[r]_q}y-y\right).\end{aligned}$$

On account of this, if we take the infimum on the right hand side over all $\lambda \in C_B^2[0,\infty)$ and use (7), hence

$$\begin{aligned}\left|R_{r,q}(\mu;y)-\mu(y)\right| &\leq 4K_2(\mu;\chi_{r,q}(y))+\omega(\mu;\beta_{r,q}(y)) \\ &\leq C\omega_2(\mu;\sqrt{\chi_{r,q}(y)})+\omega(\mu;\beta_{r,q}(y)),\end{aligned}$$

which gives the proof. □

5. Voronovskaya type asymptotic theorem

In this section, in order to proof the Voronovskaya type asymptotic theorem, firstly we need to give the following Lemma.

Lemma 5.1. *Let $q := (q_r)$ such that $0 < q_r < 1$, $\lim_{r\to\infty} q_r = 1$, $\lim_{r\to\infty} \frac{b_r}{[r]_{q_r}} = 0$ as $r \to \infty$. Then, for each $0 \leq y \leq b_r$, the following identities holds:*

$$(i)\lim_{r\to\infty}\frac{[r]_{q_r}}{b_r}R_{r,q_r}(t-y;y)=0,$$

$$(ii)\lim_{r\to\infty}\frac{[r]_{q_r}}{b_r}R_{r,q_r}((t-y)^2;y)=y,$$

$$(iii)\lim_{r\to\infty}\frac{[r]_{q_r}^2}{b_r^2}R_{r,q_r}((t-y)^4;y)=3y^2.$$

Proof. Considering the results computed in Lemma 2.2, so the proof of the above equalities can be obtained by simple calculations, thus we have omitted the details. □

Theorem 5.2. *Let $q := (q_r)$ such that $0 < q_r < 1$, $\lim_{r\to\infty} q_r = 1$, $\lim_{r\to\infty} \frac{b_r}{[r]_{q_r}} = 0$ as $r \to \infty$. Then, for any $\mu \in C_B^2[0,\infty)$ such that $\mu', \mu'' \in C_B^2[0,\infty)$ the following conclusion verify*

$$\lim_{r\to\infty}\frac{[r]_{q_r}}{b_r}(R_{r,q_r}(\mu;y)-\mu(y))=\frac{y}{2}\mu''(y).$$

Proof. Let $y \in [0, b_r]$. From Taylor formula of μ, then

$$\mu(t)=\mu(y)+(t-y)\mu'(y)+\frac{1}{2}(t-y)^2\mu''(y)+(t-y)^2\phi(t;y) \tag{11}$$

In (11), $\phi(t;y)$ is a Peano of the remainder term and since $\phi(.;y) \in C_B[0,\infty)$, we have $\lim_{t\to y}\phi(t;y) = 0$.

Operating $R_{r,q_r}(.;y)$ to (11), hence

$$\begin{aligned}R_{r,q_r}(\mu;y)-\mu(y) &= R_{r,q_r}((t-y);y)\mu'(y)+\frac{1}{2}R_{r,q_r}((t-y)^2;y)\mu''(y) \\ &+R_{r,q_r}((t-y)^2\phi(t;y);y)\end{aligned}$$

If we take the limit of the both sides of the above equality as $r \to \infty$, then

$$\lim_{r\to\infty}\frac{[r]_{q_r}}{b_r}(R_{r,q_r}(\mu;y)-\mu(y))$$
$$=\lim_{r\to\infty}\frac{[r]_{q_r}}{b_r}\left(\frac{1}{2}R_{r,q_r}((t-y)^2;y)\mu''(y)+R_{r,q_r}((t-y)^2\phi(t;y);y)\right).$$

Utilizing the Cauchy-Schwarz inequality to the last term on the right hand side of the above relation, it becomes

$$\lim_{r\to\infty}\frac{[r]_{q_r}}{b_r}R_{r,q_r}((t-y)^2\phi(t;y);y)$$
$$\leq\sqrt{\lim_{r\to\infty}R_{r,q_r}(\phi^2(t;y);y)}\sqrt{\lim_{r\to\infty}\frac{[r]_{q_r}^2}{b_r^2}R_{r,q_r}((t-y)^4;y)} \tag{12}$$

It is observed that as $\phi(t;y)\in C_B[0,\infty)$, thus by Theorem 3.3, $\lim_{t\to y}\phi(t;y)=0$. Then,

$$\lim_{r\to\infty}R_{r,q_r}(\phi^2(t;y);y)=\phi^2(y;y)=0 \tag{13}$$

Combining (12)-(13) and in view of Lemma 5.1-(iii), yields

$$\lim_{r\to\infty}\frac{[r]_{q_r}}{b_r}R_{r,q_r}((t-y)^2\phi(t;y);y)=0.$$

Thus, we attain the desired result as follows:

$$\lim_{r\to\infty}\frac{[r]_{q_r}}{b_r}(R_{r,q_r}(\mu;y)-\mu(y))=\frac{y}{2}\mu''(y).$$

□

6. Graphics and error of estimation tables

In this section, using the Maple software, the comparison of the convergence of Chlodowsky type q-Bernstein-Schurer operators to the certain functions is provided by some graphical illustrations and error estimation tables.

Example 6.1. *Let the function $\mu(y)=\sin(\pi y)+\frac{3}{4}y$ (black). In Figure 1, for $r=20$ (red), $r=50$ (blue), $r=150$ (green) and by taking $b_r=\ln(r+1)$, $p=2$ and $q=0.999$, we demonstrate the convergence of $R_{r,q}(\mu;y)$ operators to $\mu(y)$. In addition, in Table 1 for $r=50,100,300$ respectively, we estimate the error of approximation $R_{r,q}(\mu;y)$ operators to $\mu(y)$ for the certain values of $0\leq y\leq 2$. It is obvious from Table 1 that, the convergence of operators $R_{r,q}(\mu;y)$ to $\mu(y)$ becomes better, since the r value are increases.*

Example 6.2. *Let the function $\mu(y)=e^{-y}\sqrt{y^2-2y+3}$ (black). In Figure 2, for $q=0.89$ (red), $q=0.95$ (blue), $q=0.99$ (green) and by taking $b_r=\ln(r+1)$, $p=0.75$ and $r=60$, we demonstrate the convergence of $R_{r,q}(\mu;y)$ operators to $\mu(y)$. Also, in Table 2 for $r=250$ and $q=0.89, 0.95, 0.99$ respectively, we estimate the error of approximation $R_{r,q}(\mu;y)$ operators to $\mu(y)$ for the certain values of $0\leq y\leq 2$. It is clear from Table 2 that, as q approaches 1 than the error of approximation $R_{r,q}(\mu;y)$ operators to $\mu(y)$ is decreases.*

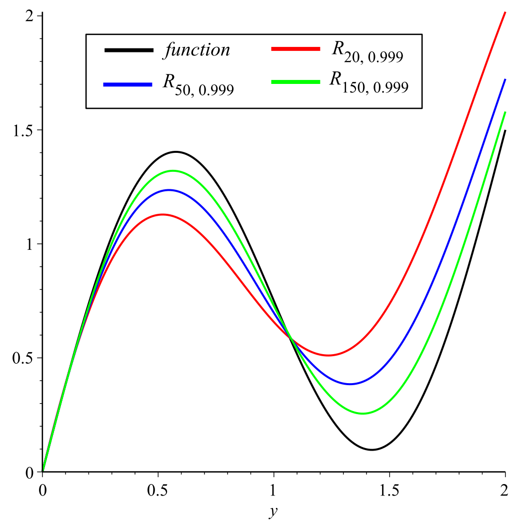

Figure 1: The convergence of $R_{r,q}(\mu; y)$ operators to $\mu(y) = \sin(\pi y) + \frac{3}{4}y$ (*black*) for $r = 20$ (*red*), $r = 50$ (*blue*), $r = 150$ (*green*), $b_r = \ln(r+1)$, $p = 2$ and $q = 0.999$

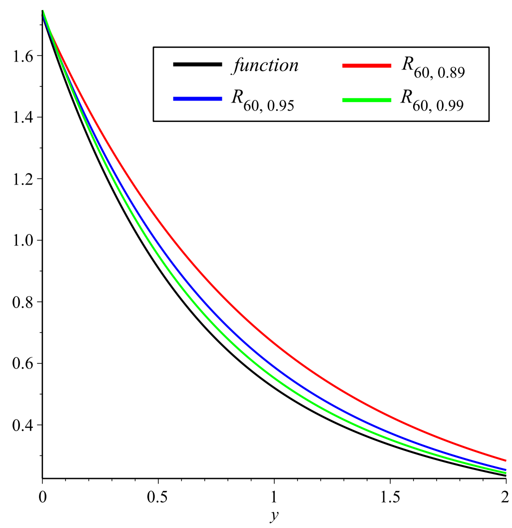

Figure 2: The convergence of $R_{r,q}(\mu; y)$ operators to $\mu(y) = e^{-y}\sqrt{y^2 - 2y + 3}$ (*black*) for $q = 0.89$ (*red*), $q = 0.95$ (*blue*), $q = 0.99$ (*green*), $b_r = \ln(r+1)$, $p = 0.75$ and $r = 60$

Table 1: Error of approximation $R_{r,q}(\mu; y)$ operators to $\mu(y) = \sin(\pi y) + \frac{3}{4}y$ for $r = 50, 100, 300$, $q = 0.999$, $b_r = \ln(r+1)$ and $p = 2$

y	$\lvert R_{50,0.999}(\mu; y) - \mu(y)\rvert$	$\lvert R_{100,0.999}(\mu; y) - \mu(y)\rvert$	$\lvert R_{300,0.999}(\mu; y) - \mu(y)\rvert$
0.2	0.024535523	0.016368259	0.008916649
0.4	0.109449379	0.069732888	0.034271290
0.6	0.179261554	0.113892382	0.054478809
0.8	0.167051059	0.105826846	0.049048034
1.0	0.051369425	0.030659408	0.010964036
1.2	0.132763215	0.089225389	0.048268139
1.4	0.311352924	0.204381903	0.103331236
1.6	0.406910171	0.261967136	0.127125632
1.8	0.374308021	0.231079940	0.104202776
2.0	0.220750247	0.117838335	0.039100348

Table 2: Error of approximation $R_{r,q}(\mu; y)$ operators to $\mu(y) = e^{-y}\sqrt{y^2 - 2y + 3}$ for $r = 250$, $q = 0.89, 0.95, 0.99$, $b_r = \ln(r+1)$ and $p = 0.75$

y	$\lvert R_{250,0.89}(\mu; y) - \mu(y)\rvert$	$\lvert R_{250,0.95}(\mu; y) - \mu(y)\rvert$	$\lvert R_{250,0.99}(\mu; y) - \mu(y)\rvert$
0.1	0.068495467	0.035152289	0.008291055
0.3	0.159791684	0.080015358	0.018491974
0.5	0.205323050	0.100099241	0.022653687
0.7	0.219467365	0.103800442	0.022948144
0.9	0.213207981	0.097356252	0.020925327
1.1	0.194890963	0.085427213	0.017732082
1.3	0.170698321	0.071413553	0.014216454
1.5	0.144956964	0.057617935	0.010950518
1.7	0.120416715	0.045386110	0.008240577
1.9	0.098570162	0.035299500	0.006172482

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