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Extension of the generalized n-strong Drazin inverse

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Abstract. The aim of this paper is to present an extension of the generalized n-strong Drazin inverse for Banach algebra elements using a g–Drazin invertible element rather than a quasinilpotent element in the definition of the generalized n-strong Drazin inverse. Thus, we introduce a new class of generalized inverses which is a wider class than the classes of the generalized n-strong Drazin inverse and the extended generalized strong Drazin inverses. We prove a number of characterizations for this new inverse and some of them are based on idempotents and tripotents. Several generalizations of Cline's formula are investigated for the extension of the generalized n-strong Drazin inverse.

1. Introduction

In this paper, $\mathcal{A}$ represents a complex Banach algebra with unit 1. For $a \in \mathcal{A}$, the symbols $\sigma(a)$, $r(a)$ and acc $\sigma(a)$, respectively, will denote the spectrum of a, the spectral radius of a and the set of all accumulation points of $\sigma(a)$. The sets of all invertible, nilpotent and quasinilpotent elements of $\mathcal{A}$, respectively, are denoted by $\mathcal{A}^{-1}$, $\mathcal{A}^{nil}$ and $\mathcal{A}^{qnil}$, respectively. Recall that $a \in \mathcal{A}^{qnil}$ if $\sigma(a) = \{0\}$. We use $\sigma_{\mathcal{B}}(a)$ for the spectrum of $a \in \mathcal{B}$ with respect to $\mathcal{B}$, where $\mathcal{B}$ is a subalgebra of $\mathcal{A}$, and also $a_{\mathcal{B}}^{-1}$ will be the inverse of a in $\mathcal{B}$. It is known that $a \in \mathcal{A}$ is tripotent (or idempotent) if $a^3 = a$ (or $a^2 = a$).

Koliha [9] presented the definition of the g–Drazin inverse for elements of Banach algebras, extending the notion of the Drazin inverse [7]. An element $a \in \mathcal{A}$ is g–Drazin invertible if there exists an element $x \in \mathcal{A}$ which satisfies

$$xax = x, \qquad ax = xa \qquad \text{and} \qquad a - axa \in \mathcal{A}^{qnil}.$$

In this case, x is called the g–Drazin inverse of a (or Koliha–Drazin inverse of a) [9]. The g–Drazin inverse of a is unique, if it exists, and denoted by a^d. Recall that a^d exists if and only if $0 \notin$ acc $\sigma(a)$. The g–Drazin inverse of a doubly commutes with a, that is, a^d commutes with every element of $\mathcal{A}$ that commutes with a (that is, $ab = ba$ implies $a^d b = b a^d$) [9]. We use $\mathcal{A}^d$ to denote the set of all g–Drazin invertible elements of $\mathcal{A}$.

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Since the g–Drazin inverse of a quasinilpotent element is equal to zero, we have that $\mathcal{A}^{qnil} \subseteq \mathcal{A}^d$. For $a \in \mathcal{A}^d$, $a^\pi = 1 - aa^d$ is the spectral idempotent of a corresponding to the set $\{0\}$. More properties of the g–Drazin inverse were given in [4–6].

When $a - axa \in \mathcal{A}^{nil}$ in the definition of the g–Drazin inverse, then $a^d = a^D$ is the Drazin inverse of a. The group inverse of a, denoted by $a^\#$, is a special case of the Drazin inverse for which $a = axa$ is satisfied. The sets of all Drazin invertible and group invertible elements of $\mathcal{A}$ are denoted by $\mathcal{A}^D$ and $\mathcal{A}^\#$, respectively.

One significant property of the Drazin inverse was presented by Cline [2] as: if $ab \in \mathcal{A}^D$, then $ba \in \mathcal{A}^D$ and $(ba)^D = b((ab)^D)^2a$. This so-called Cline's formula was generalized to many generalized inverses under different assumptions [10, 20].

The concept of a strong Drazin inverse was introduced by Wang [19]. As a generalization of the strong Drazin inverse, a generalized strong Drazin inverse was defined in [12] for Banach algebra elements. For $n \in \mathbb{N}$, the generalized n-strong Drazin inverse was presented in [13] for elements of rings as a new class of generalized inverses which extends the generalized strong Drazin inverse from [12] and the generalized Hirano inverse presented in [18].

Let $n \in \mathbb{N}$. An element $a \in \mathcal{A}$ is called generalized n-strongly Drazin invertible (or gns–Drazin invertible) if there exists an element $x \in \mathcal{A}$ such that

$$xax = x, \quad ax = xa \quad \text{and} \quad a^n - ax \in \mathcal{A}^{qnil}.$$

The gns–Drazin inverse x of a is unique if it exists [13]. If $a^n - ax \in \mathcal{A}^{nil}$ in the above definition, then x is the n-strong Drazin inverse (or ns–Drazin inverse) of a. For $n = 1$, the gns–Drazin inverse becomes the generalized strong Drazin inverse [12]. In the case that $n = 2$, the gns–Drazin inverse reduces to the generalized Hirano inverse [18]. Some interesting results about (generalized) strong Drazin inverse, (generalized) Hirano inverse and (generalized) n-strongly Drazin inverse can be found in [1, 8, 17, 21, 22].

Using an adequate g–Drazin invertible element rather than a quasinilpotent element in the definition of g–Drazin inverse, the concept of the g–Drazin inverse was extended in [11]. An element $a \in \mathcal{A}$ is called extended g–Drazin invertible (or eg–Drazin invertible) if there exists an element $x \in \mathcal{A}$ such that

$$xax = x, \quad xa = ax \quad \text{and} \quad a - axa \in \mathcal{A}^d.$$

In this case, x is an extended g–Drazin inverse (or eg–Drazin inverse) of a and it is not uniquely determined. Notice that a is extended g–Drazin invertible if and only if a is g–Drazin invertible [11]. Replacing $a - axa \in \mathcal{A}^d$ with $a - axa \in \mathcal{A}^D$ in the definition of eg–Drazin inverse, x is an extended Drazin inverse (or e–Drazin inverse) of a. The sets of all eg–Drazin invertible and e–Drazin invertible elements of $\mathcal{A}$ will be denoted by $\mathcal{A}^{ed}$ and $\mathcal{A}^{eD}$, respectively.

The notion of the generalized strong Drazin inverse was generalized in [16] using the condition $a - ax \in \mathcal{A}^d$ instead of $a - ax \in \mathcal{A}^{qnil}$ in its definition. An element $a \in \mathcal{A}$ is called extended gs–Drazin invertible (or egs–Drazin invertible) if there exists an element $x \in \mathcal{A}$ such that

$$xax = x, \quad xa = ax \quad \text{and} \quad a - ax \in \mathcal{A}^d.$$

In this case, x is an extended gs–Drazin inverse (or egs–Drazin inverse) of a. If $a - ax \in \mathcal{A}^D$ in this definition, x is an extended s–Drazin inverse (or es–Drazin inverse) of a. The symbols $\mathcal{A}^{esd}$ and $\mathcal{A}^{esD}$, respectively, represent the sets of all egs–Drazin invertible and es–Drazin invertible elements of $\mathcal{A}$.

Motivated by previous research papers about g–Drazin inverse, generalized strong Drazin inverse and their extensions, our aim is to present a wider class of the gns–Drazin inverse and egs–Drazin inverse. Precisely, we introduce an extended gns–Drazin inverse replacing the condition $a^n - ax \in \mathcal{A}^{qnil}$ in the definition of the gns–Drazin inverse with $a^n - ax \in \mathcal{A}^d$. In this way, we define a new class of generalized inverses for elements of Banach algebra. We present different kinds of equivalent conditions for an element to be extended gns–Drazin invertible. Some of these characterizations contain idempotent, and some of them involve tripotents. We prove that an element $a \in \mathcal{A}$ is extended gns–Drazin invertible if and only if a is eg–Drazin invertible if and only if a is g–Drazin invertible. Several extensions of Cline's formula for extended gns–Drazin inverse are proposed. Applying these results, we can get new characterizations for eg–Drazin invertible and g–Drazin invertible elements. At the end, we define weighted extended gns–Drazin invertible and weighted extended ns–Drazin invertible Banach algebra elements.

2. Extended *gns*–Drazin inverse

The new class of generalized inverses in a Banach algebra is defined in this section by replacing the condition $a^n - ax \in \mathcal{A}^{qnil}$ in the definition of *gns*–Drazin inverse with $a^n - ax \in \mathcal{A}^d$. In this way, we propose an extension of *gns*–Drazin inverse, i.e. a wider class of generalized inverses.

Definition 2.1. *For $n \in \mathbb{N}$, an element $a \in \mathcal{A}$ is called extended gns–Drazin invertible (or egns–Drazin invertible) if there exists an element $x \in \mathcal{A}$ such that*

$$xax = x, \quad xa = ax \quad \text{and} \quad a^n - ax \in \mathcal{A}^d.$$

In this case, x is an extended gns–Drazin inverse (or egns–Drazin inverse) of a.

Obviously, for $n = 1$, the *egns*–Drazin inverse reduces to the *egs*–Drazin inverse.

In particular, when $a^n - ax \in \mathcal{A}^D$, an extended *gns*–Drazin inverse becomes an extended *ns*–Drazin inverse.

Definition 2.2. *For $n \in \mathbb{N}$, an element $a \in \mathcal{A}$ is called extended ns–Drazin invertible (or ens–Drazin invertible) if there exists an element $x \in \mathcal{A}$ such that*

$$xax = x, \quad xa = ax \quad \text{and} \quad a^n - ax \in \mathcal{A}^D.$$

In this case, x is an extended ns–Drazin inverse (or ens–Drazin inverse) of a.

Denote by $\mathcal{A}^{n,esd}$ (resp. $\mathcal{A}^{n,esD}$) the set of all *egns*–Drazin (resp. *ens*–Drazin) invertible elements of $\mathcal{A}$.

Lemma 2.3. *If $a \in \mathcal{A}^{n,esd}$, then $a \in \mathcal{A}^{ed}$. Furthermore, an egns–Drazin inverse of a is an eg–Drazin inverse of a.*

Proof. Assume that x is an *egns*–Drazin inverse of a. Then $1 - ax$ is an idempotent and so $1 - ax \in \mathcal{A}^{\#} \subseteq \mathcal{A}^d$. Notice that $a^n - ax \in \mathcal{A}^d$ and, applying [9, Theorem 5.5], $(a - a^2x)^n = a^n(1 - ax) = (a^n - ax)(1 - ax) \in \mathcal{A}^d$. By [10, Corollary 2.2], we deduce that $a - a^2x \in \mathcal{A}^d$ and x is an *eg*–Drazin inverse of a. □

Using Lemma 2.3, we can note that the similar result holds for *ens*–Drazin invertible elements.

Corollary 2.4. *If $a \in \mathcal{A}^{n,esD}$, then $a \in \mathcal{A}^{eD}$. Furthermore, an ens–Drazin inverse of a is an e–Drazin inverse of a.*

According to Lemma 2.3 and [11, Theorem 2.2], we conclude that $\mathcal{A}^{n,esd} \subseteq \mathcal{A}^{ed} = \mathcal{A}^d$. In the following theorem, we show that $\mathcal{A}^{n,esd} = \mathcal{A}^{ed} = \mathcal{A}^d$ and give more characterizations of *egns*–Drazin invertible elements.

Theorem 2.5. *Let $a \in \mathcal{A}$ and $n, m \in \mathbb{N}$. The following statements are equivalent:*

(i) *a is egns–Drazin invertible;*

(ii) *a is eg–Drazin invertible;*

(iii) *a is g–Drazin invertible;*

(iv) *there exists an idempotent $p \in \mathcal{A}$ commuting with a such that $ap \in (p\mathcal{A}p)^{-1}$ and $a^n - p \in \mathcal{A}^d$;*

(v) *there exists an idempotent $p \in \mathcal{A}$ commuting with a such that $ap + 1 - p \in \mathcal{A}^{-1}$ and $a^n - p \in \mathcal{A}^d$;*

(vi) *there exists an idempotent $p \in \mathcal{A}$ commuting with a such that $ap \in (p\mathcal{A}p)^{-1}$ and $a - a^m p \in \mathcal{A}^d$.*

In this case, we have that 0 and $(ap)^{-1}_{p\mathcal{A}p} = (ap + 1 - p)^{-1}p$ are egns–Drazin inverses of a.

Proof. (i) ⇒ (ii): It follows by Lemma 2.3.

(ii) ⇔ (iii): Using [11, Theorem 2], this equivalence is evident.

(iii) ⇒ (i): If $a \in \mathcal{A}^d$, by [10, Corollary 2.2], notice that $a^n \in \mathcal{A}^d$ and so 0 is an *egns*–Drazin inverse of a.

(i) ⇒ (iv) ∧ (v): For an *egns*–Drazin inverse x of a and $p = ax$, we observe that $p^2 = p$, $pa = ap$ and $a^n - p = a^n - ax \in \mathcal{A}^d$. Applying $apx = a^2x^2 = ax = p = xap$, we deduce that ap is invertible in the Banach algebra $p\mathcal{A}p$ and $x = (ap)^{-1}_{p\mathcal{A}p}$. Similarly, we get $(ap + 1 - p)^{-1} = (ap)^{-1}_{p\mathcal{A}p} + 1 - p$.

(iv) ⇒ (i): Let (iv) hold and $x = (ap)^{-1}_{p\mathcal{A}p}$. Then $x = xp = px$ gives $xa = xpa = (ap)^{-1}_{p\mathcal{A}p}ap = p = ap(ap)^{-1}_{p\mathcal{A}p} = ax$, $xax = xp = x$ and $a^n - ax = a^n - p \in \mathcal{A}^d$, i.e. x is an *egns*–Drazin inverse of a.

(v) ⇒ (i): Set $x = (ap + 1 - p)^{-1}p$. The equality $(ap + 1 - p)p = ap$ yields $p = (ap + 1 - p)^{-1}ap = xa = ax$. Now, $xax = px = x$ and $a^n - ax = a^n - p \in \mathcal{A}^d$, that is, x is an *egns*–Drazin inverse of a.

(i) ⇒ (vi): Case 1. $m \geq 2$: By the hypotheses and the proof of (i) ⇒ (iv), there exists an idempotent $p \in \mathcal{A}$ commuting with a such that $ap \in (p\mathcal{A}p)^{-1}$ and $a^{m-1} - p \in \mathcal{A}^d$. Hence, $(a - ap)^{m-1} = (a^{m-1} - p)(1 - p) \in \mathcal{A}^d$, which implies $a - ap \in \mathcal{A}^d$. Note that $ap \in \mathcal{A}^d$. So, $a - a^m p = (a - ap) - (a^{m-1} - p)ap \in \mathcal{A}^d$.

Case 2: $m = 1$. This is clear by the implication of (i) ⇒ (iv).

(vi) ⇒ (ii). Suppose that (vi) holds. Then, $a - ap = (a - a^m p)(1 - p) \in \mathcal{A}^d$. By [11, Theorem 1], we get that (ii) holds. □

By Theorem 2.5, we observe that the *egns*–Drazin inverse is not unique in general. The symbols $a^{n,esd}$ and $a^{n,esD}$ stand for an *egns*–Drazin inverse and *ens*–Drazin inverse of a, respectively. The set of all *egns*–Drazin (or *ens*–Drazin) inverses of a will be denoted by $a\{n, esd\}$ (or $a\{n, esD\}$).

Applying Theorem 2.5, new characterizations for *ens*–Drazin invertible elements can be given.

Corollary 2.6. *Let $a \in \mathcal{A}$ and $n, m \in \mathbb{N}$. The following statements are equivalent:*

(i) *a is ens–Drazin invertible;*

(ii) *a is e–Drazin invertible;*

(iii) *a is Drazin invertible;*

(iv) *there exists an idempotent $p \in \mathcal{A}$ commuting with a such that $ap \in (p\mathcal{A}p)^{-1}$ and $a^n - p \in \mathcal{A}^D$;*

(v) *there exists an idempotent $p \in \mathcal{A}$ commuting with a such that $ap + 1 - p \in \mathcal{A}^{-1}$ and $a^n - p \in \mathcal{A}^D$;*

(iv) *there exists an idempotent $p \in \mathcal{A}$ commuting with a such that $ap \in (p\mathcal{A}p)^{-1}$ and $a - a^m p \in \mathcal{A}^D$.*

In this case, we have that 0 *and* $(ap)^{-1}_{p\mathcal{A}p} = (ap + 1 - p)^{-1}p$ *are ens–Drazin inverses of a.*

We now establish some characterizations of *egns*–Drazin invertible elements by means of tripotents.

Theorem 2.7. *Let $a \in \mathcal{A}$ and $n \in \mathbb{N}$. The following statements are equivalent:*

(i) *a is egns–Drazin invertible;*

(ii) *there exists a tripotent $p \in \mathcal{A}$ commuting with a such that $ap \in (p^2\mathcal{A}p^2)^{-1}$ and $a^n - p^2 \in \mathcal{A}^d$;*

(iii) *there exists a tripotent $p \in \mathcal{A}$ commuting with a such that $ap + 1 - p^2 \in \mathcal{A}^{-1}$ and $a^n - p^2 \in \mathcal{A}^d$.*

In this case, we have that $(ap)^{-1}_{p^2\mathcal{A}p^2}p = (ap + 1 - p^2)^{-1}p$ is the egns–Drazin inverse of a.

Proof. (i) ⇒ (ii): According to Theorem 2.5(iv), there exists $p^2 = p \in \mathcal{A}$ commuting with a such that $ap \in (p^2\mathcal{A}p^2)^{-1}$ and $a^n - p^2 \in \mathcal{A}^d$. Hence, $p^3 = p$.

(ii) ⇒ (i): Let $p \in \mathcal{A}$ be a tripotent commuting with a, $ap \in (p^2\mathcal{A}p^2)^{-1}$ and $a^n - p^2 \in \mathcal{A}^d$. For $x = (ap)^{-1}_{p^2\mathcal{A}p^2}p$, we have $xa = (ap)^{-1}_{p^2\mathcal{A}p^2}ap = p^2$ and $ax = ap(ap)^{-1}_{p^2\mathcal{A}p^2} = p^2$. Thus, $ax = xa$, $xax = p^2x = x$ and $a^n - ax = a^n - p^2 \in \mathcal{A}^d$, i.e. x is an *egns*–Drazin inverse of a.

(i) ⇒ (iii): This implication follows similarly as (i) ⇒ (ii) by Theorem 2.5(v).

(iii) ⇒ (ii): Suppose that there exists $p^3 = p \in \mathcal{A}$, $pa = ap$, $ap + 1 - p^2 \in \mathcal{A}^{-1}$ and $a^n - p^2 \in \mathcal{A}^d$. Since $(ap + 1 - p^2)p^2 = ap$, we obtain $p^2 = (ap + 1 - p^2)^{-1}ap = ap(ap + 1 - p^2)^{-1}$, which implies $ap \in (p^2\mathcal{A}p^2)^{-1}$ and $(ap)^{-1}_{p\mathcal{A}p} = (ap + 1 - p^2)^{-1}$. □

According to Theorem 2.7, we characterize *ens*–Drazin invertible elements by tripotents.

Corollary 2.8. *Let $a \in \mathcal{A}$ and $n \in \mathbb{N}$. The following statements are equivalent:*

(i) *a is ens–Drazin invertible;*

(ii) *there exists a tripotent $p \in \mathcal{A}$ commuting with a such that $ap \in (p^2\mathcal{A}p^2)^{-1}$ and $a^n - p^2 \in \mathcal{A}^D$;*

(iii) *there exists a tripotent $p \in \mathcal{A}$ commuting with a such that $ap + 1 - p^2 \in \mathcal{A}^{-1}$ and $a^n - p^2 \in \mathcal{A}^D$.*

In this case, we have that $(ap)^{-1}_{p\mathcal{A}p}p = (ap + 1 - p^2)^{-1}p$ is the ens–Drazin inverse of a.

Applying Theorem 2.5, notice that statements (ii) and (iii) of Theorem 2.7 present new characterizations of *eg*–Drazin and *g*–Drazin invertible elements. Also, for $n = 1$ in Theorem 2.7, we recover [16, Theorem 2.2] for *egs*–Drazin invertible elements.

Basic properties of *egns*–Drazin invertible elements are developed too.

Lemma 2.9. *If $a \in \mathcal{A}^{n,esd}$, then, for arbitrary $a^{n,esd} \in a\{n, esd\}$,*

(i) *$a^{n,esd} \in \mathcal{A}^{\#}$ and $(a^{n,esd})^{\#} = a^2a^{n,esd}$;*

(ii) *$a^{n,esd} \in \mathcal{A}^{n,esd}$ and $a^2a^{n,esd} \in a^{n,esd}\{n, esd\}$.*

Proof. (i) It is clear that $a^{n,esd}$ commutes with $a^2a^{n,esd}$. Further, from $(a^2a^{n,esd})a^{n,esd}(a^2a^{n,esd}) = a^2a^{n,esd}$ and $a^{n,esd}(a^2a^{n,esd})a^{n,esd} = a^{n,esd}$, we observe that $a^{n,esd} \in \mathcal{A}^{\#}$ and $(a^{n,esd})^{\#} = a^2a^{n,esd}$.

(ii) We know that $a^n - aa^{n,esd} \in \mathcal{A}^d$ and $a^{n,esd}$ commutes with $a^n - aa^{n,esd}$. Since $a^{n,esd} \in \mathcal{A}^{\#}$ by part (i), then $(a^{n,esd})^n \in \mathcal{A}^{\#}$. Applying [9, Theorem 5.5], we have that

$$\begin{aligned}(a^{n,esd})^n - a^{n,esd}(a^2a^{n,esd}) &= (a^{n,esd})^n - aa^{n,esd} \\ &= -(a^{n,esd})^n(a^n - aa^{n,esd}) \in \mathcal{A}^d.\end{aligned}$$

□

Lemma 2.9 yields the next properties of a *ens*–Drazin inverse.

Corollary 2.10. *If $a \in \mathcal{A}^{n,esD}$, then, for arbitrary $a^{n,esD} \in a\{n, esD\}$,*

(i) *$a^{n,esD} \in \mathcal{A}^{\#}$ and $(a^{n,esD})^{\#} = a^2a^{n,esD}$;*

(ii) *$a^{n,esD} \in \mathcal{A}^{n,esD}$ and $a^2a^{n,esD} \in a^{n,esD}\{n, esD\}$.*

Recall that an arbitrary element $a \in \mathcal{A}$ can be represented by the following matrix form relative to an idempotent $p \in \mathcal{A}$:

$$a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_p,$$

where $a_{11} = pap$, $a_{12} = pa(1 - p)$, $a_{21} = (1 - p)ap$, $a_{22} = (1 - p)a(1 - p)$. The matrix form of an *egns*–Drazin inverse of $a \in \mathcal{A}^d$ can be developed relative to idempotent aa^d.

Lemma 2.11. *If $a \in \mathcal{A}^d$, then*

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_{aa^d} \quad \text{and} \quad a^{n,esd} = \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix}_{aa^d},$$

where $a_1 \in (aa^d\mathcal{A}aa^d)^{-1}$, $a_2 \in (a^{\pi}\mathcal{A}a^{\pi})^{qnil}$ and $x_i \in a_i\{n, esd\}$ for $i = 1, 2$.

Proof. We have the next representation of $a \in \mathcal{A}^d$:

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_p,$$

where $p = aa^d$, $a_1 \in (p\mathcal{A}p)^{-1}$ and $a_2 \in ((1-p)\mathcal{A}(1-p))^{qnil}$. Also,

$$a^d = \begin{bmatrix} a_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}_p.$$

Let $x \in a\{n, esd\}$. Since a^d double commutes with a, then x commutes with p and so

$$x = \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix}_p.$$

The equalities $ax = xa$ and $xax = x$ imply $a_i x_i = x_i a_i$ and $x_i a_i x_i = x_i$, for $i = 1, 2$. Because

$$a^n - ax = \begin{bmatrix} a_1^n - a_1 x_1 & 0 \\ 0 & a_2^n - a_2 x_2 \end{bmatrix}_p \in \mathcal{A}^d$$

and $\sigma(a^n - ax) = \sigma_{p\mathcal{A}p}(a_1^n - a_1 x_1) \cup \sigma_{(1-p)\mathcal{A}(1-p)}(a_2^n - a_2 x_2)$, we deduce that $a_1^n - a_1 x_1 \in (p\mathcal{A}p)^d$ and $a_2^n - a_2 x_2 \in ((1-p)\mathcal{A}(1-p))^d$. Therefore, $x_i \in a_i\{n, esd\}$, for $i = 1, 2$. □

Lemma 2.11 gives the next matrix form of an *ens*–Drazin inverse of $a \in \mathcal{A}^D$.

Corollary 2.12. *If $a \in \mathcal{A}^D$, then*

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_{aa^D} \quad \text{and} \quad a^{n,esD} = \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix}_{aa^D},$$

where $a_1 \in (aa^D\mathcal{A}aa^D)^{-1}$, $a_2 \in (a^\pi \mathcal{A} a^\pi)^{nil}$ and $x_i \in a_i\{n, esD\}$ for $i = 1, 2$.

New equivalent conditions for an element to be *egns*–Drazin invertible are proposed now.

Theorem 2.13. *Let $a \in \mathcal{A}$ and $n, k \in \mathbb{N}$. The following statements are equivalent:*

(i) *a is egns–Drazin invertible;*

(ii) *there exists an element $y \in \mathcal{A}$ such that $ya^k y = y$, $ya = ay$ and $a^n - a^k y \in \mathcal{A}^d$;*

(iii) *a^k is egns–Drazin invertible;*

In this case, $a^{k-1}y \in a\{n, esd\}$.

Proof. (i) ⇒ (ii): For $x \in a\{n, esd\}$, set $y = x^k$. Then $ya = x^k a = ax^k = ay$, $ya^k y = x^k a^k x^k = (xax)^k = x^k = y$ and $a^n - a^k y = a^n - a^k x^k = a^n - ax \in \mathcal{A}^d$.

(ii) ⇒ (i): Assume that there exists an element $y \in \mathcal{A}$ such that $ya^k y = y$, $ya = ay$ and $a^n - a^k y \in \mathcal{A}^d$. Set $x = a^{k-1}y$. Because $ax = a^k y = a^{k-1}ya = xa$, $xax = a^{k-1}(ya^k y) = a^{k-1}y = x$ and $a^n - ax = a^n - a^k y \in \mathcal{A}^d$, we deduce that $x \in a\{n, esd\}$.

(i) ⇔ (iii): Using Theorem 2.5 and [10, Corollary 2.2], $a \in \mathcal{A}^{n,esd}$ if and only if $a \in \mathcal{A}^d$ if and only if $a^k \in \mathcal{A}^d$ if and only if $a^k \in \mathcal{A}^{n,esd}$. □

Using Theorem 2.13, we obtain the next result.

Corollary 2.14. *Let $a \in \mathcal{A}$ and $n, k \in \mathbb{N}$. The following statements are equivalent:*

(i) *a is ens–Drazin invertible;*

(ii) *there exists an element $y \in \mathcal{A}$ such that $ya^k y = y$, $ya = ay$ and $a^n - a^k y \in \mathcal{A}^D$;*

(iii) *a^k is ens–Drazin invertible;*

In this case, $a^{k-1}y \in a\{n, esD\}$.

3. Cline's formula for the *egns*–Drazin inverse

In this section, a generalization of Cline's formula is considered for the *egns*–Drazin inverse. The next useful result for elements of an associative ring $\mathcal{R}$ with the unit 1, was proposed in [20].

Lemma 3.1. [20, Theorem 2.7] *Let $a, b, c, d \in \mathcal{R}$ satisfy $acd = dbd$ and $dba = aca$. Then $bd \in \mathcal{R}^d \Leftrightarrow ac \in \mathcal{R}^d$. In this case, $(bd)^d = b((ac)^d)^2 d$ and $(ac)^d = d((bd)^d)^3 bac$.*

Under the restrictions $acd = dbd$ and $dba = aca$, an extension of Cline's formula is proved for *egns*–Drazin inverse.

Theorem 3.2. *Let $a, b, c, d \in \mathcal{A}$ satisfy $acd = dbd$ and $dba = aca$. Then*

$$bd \in \mathcal{A}^{n,esd} \quad \Leftrightarrow \quad ac \in \mathcal{A}^{n,esd}.$$

In this case, for arbitrary $(bd)^{n,esd}$ and $(ac)^{n,esd}$, we have $b((ac)^{n,esd})^2 d \in (bd)\{n, esd\}$ and $d((bd)^{n,esd})^3 bac \in (ac)\{n, esd\}$.

Proof. $\Rightarrow$: Let $bd \in \mathcal{A}^{n,esd}$. For arbitrary $(bd)^{n,esd} \in (bd)\{n, esd\}$, $x = d((bd)^{n,esd})^3 bac$ satisfies

$$\begin{aligned} acx &= acd((bd)^{n,esd})^3 bac = dbd((bd)^{n,esd})^3 bac = d((bd)^{n,esd})^3 bdbac \\ &= d((bd)^{n,esd})^3 bacac = xac \end{aligned}$$

and

$$\begin{aligned} xacx &= d((bd)^{n,esd})^2 bacx = d((bd)^{n,esd})^2 bacd((bd)^{n,esd})^3 bac \\ &= d((bd)^{n,esd})^2 bdbd((bd)^{n,esd})^3 bac = d((bd)^{n,esd})^3 bac = x. \end{aligned}$$

In order to check that

$$(ac)^n - acx = (ac)^n - d((bd)^{n,esd})^2 bac = ((db)^{n-1} - d((bd)^{n,esd})^2 b)ac \in \mathcal{A}^d,$$

set $u = ((db)^{n-1} - d((bd)^{n,esd})^2 b)a$ and $v = ((bd)^{n-1} - (bd)^{n,esd})b$. We observe that $vd = (bd)^n - (bd)^{n,esd} bd \in \mathcal{A}^d$,

$$\begin{aligned} ucd &= ((db)^{n-1} - d((bd)^{n,esd})^2 b)acd = ((db)^{n-1} - d((bd)^{n,esd})^2 b)dbd \\ &= d((bd)^{n-1} - (bd)^{n,esd})bd = dvd \end{aligned}$$

and

$$\begin{aligned} dvu &= d((bd)^{n-1} - (bd)^{n,esd})b((db)^{n-1} - d((bd)^{n,esd})^2 b)a \\ &= (d(bd)^{n-1} - d((bd)^{n,esd})^2 bd)(b(db)^{n-1} - bd((bd)^{n,esd})^2 b)a \\ &= ((db)^{n-1} - d((bd)^{n,esd})^2 b)d(b(db)^{n-1} - bd((bd)^{n,esd})^2 b)a \\ &= ((db)^{n-1} - d((bd)^{n,esd})^2 b)(db(db)^{n-1} a - dbd((bd)^{n,esd})^2 ba) \\ &= ((db)^{n-1} - d((bd)^{n,esd})^2 b)ac((db)^{n-1} - d((bd)^{n,esd})^2 b)a \\ &= ucu. \end{aligned}$$

Applying Lemma 3.1, we deduce that $((db)^{n-1} - d((bd)^{n,esd})^2 b)ac = uc \in \mathcal{A}^d$. So, $ac \in \mathcal{A}^{n,esd}$ and $d((bd)^{n,esd})^3 bac \in (ac)\{n, esd\}$.

$\Leftarrow$: Analogously, this implication can be verified. □

Consequently, Theorem 3.2 implies the next extension of Cline's formula for the *ens*–Drazin inverse.

Corollary 3.3. *Let $a, b, c, d \in \mathcal{A}$ satisfy $acd = dbd$ and $dba = aca$. Then*

$$bd \in \mathcal{A}^{n,esD} \quad \Leftrightarrow \quad ac \in \mathcal{A}^{n,esD}.$$

In this case, for arbitrary $(bd)^{n,esD}$ and $(ac)^{n,esD}$, we have $b((ac)^{n,esD})^2 d \in (bd)\{n, esD\}$ and $d((bd)^{n,esD})^3 bac \in (ac)\{n, esD\}$.

In the case that $d = a$ in Theorem 3.2, we obtain a generalization of Cline's formula for the *egns*–Drazin inverse under the assumption $aca = aba$.

Corollary 3.4. *Let $a, b, c \in \mathcal{A}$ satisfy $aca = aba$. Then*

$$ba \in \mathcal{A}^{n,esd} \quad \Leftrightarrow \quad ac \in \mathcal{A}^{n,esd}.$$

In this case, for arbitrary $(ba)^{n,esd}$ and $(ac)^{n,esd}$, $b((ac)^{n,esd})^2 a \in (ba)\{n, esd\}$ and $a((ba)^{n,esd})^2 c \in (ac)\{n, esd\}$.

When $c = b$ in Corollary 3.4, we get Cline's formula for the *egns*–Drazin inverse.

Corollary 3.5. *Let $a, b \in \mathcal{A}$. Then $ba \in \mathcal{A}^{n,esd} \Leftrightarrow ab \in \mathcal{A}^{n,esd}$. In this case, for arbitrary $(ab)^{n,esd}$, $b((ab)^{n,esd})^2 a \in (ba)\{n, esd\}$.*

Applying Corollary 3.4 and Corollary 3.5, we get the following Cline's formula for the *ens*–Drazin inverse as consequences.

Corollary 3.6. *Let $a, b, c \in \mathcal{A}$ satisfy $aca = aba$. Then*

$$ba \in \mathcal{A}^{n,esD} \quad \Leftrightarrow \quad ac \in \mathcal{A}^{n,esD}.$$

In this case, for arbitrary $(ba)^{n,esD}$ and $(ac)^{n,esD}$, $b((ac)^{n,esD})^2 a \in (ba)\{n, esD\}$ and $a((ba)^{n,esD})^2 c \in (ac)\{n, esD\}$.

Corollary 3.7. *Let $a, b \in \mathcal{A}$. Then $ba \in \mathcal{A}^{n,esD} \Leftrightarrow ab \in \mathcal{A}^{n,esD}$. In addition, for arbitrary $(ab)^{n,esD}$, $b((ab)^{esD})^2 a \in (ba)\{n, esD\}$.*

4. Weighted *egns*–Drazin inverse

For $w \in \mathcal{A}\backslash\{0\}$, let $\mathcal{A}_w$ be the complex Banach algebra $\mathcal{A}$ equipped with the w-product $a * b = awb$ and the w-norm $\|a\|_w = \|a\|\|w\|$, where $a, b \in \mathcal{A}$. Also, we denote by $a^{*n} = a * a * \cdots * a$ (n factors), for $n \in N$ and $a \in \mathcal{A}$.

Lemma 4.1. [3, 14] *Let $\mathcal{A}$ be a complex Banach algebra, and let $w \in \mathcal{A}\backslash\{0\}$. For $a \in \mathcal{A}$, $a \in \mathcal{A}_w^d$ if and only if $aw \in \mathcal{A}^d$ if and only if $wa \in \mathcal{A}^d$.*

We define weighted extended *gns*–Drazin invertible and weighted extended *ns*–Drazin invertible Banach algebra elements.

Definition 4.2. *Let $w \in \mathcal{A}\backslash\{0\}$ and $n \in \mathbb{N}$. An element $a \in \mathcal{A}$ is called:*

(i) *w-weighted extended gns–Drazin invertible (or w-egns–Drazin invertible), if there exists a w-egns-Drazin inverse $a^{n,esd,w} = x \in \mathcal{A}$ such that*

$$x * a * x = x, \quad x * a = a * x \quad \text{and} \quad a^{*n} - a * x \in \mathcal{A}_w^d.$$

(ii) *w-weighted extended ns–Drazin invertible (or w-ens–Drazin invertible), if there exists a w-ens-Drazin inverse $a^{n,esD,w} = x \in \mathcal{A}$ such that*

$$x * a * x = x, \quad x * a = a * x \quad \text{and} \quad a^{*n} - a * x \in \mathcal{A}_w^D.$$

We use $\mathcal{A}^{n,esd,w}$ and $\mathcal{A}^{n,esD,w}$ to denote the sets of all *w-egns*–Drazin invertible and *w-ens*–Drazin invertible elements of $\mathcal{A}$, respectively. Notice that $a \in \mathcal{A}^{n,esd,w}$ if a is generalized n-strongly Drazin invertible in the algebra $\mathcal{A}_w$. When $w = 1$, a *w-egns*-Drazin inverse reduces to *egns*-Drazin inverse.

Some characterizations of *w-egns*-Drazin invertible elements are proved now.

Theorem 4.3. *Let $w \in \mathcal{A}\backslash\{0\}$. Then, for $a \in \mathcal{A}$, the following statements are equivalent:*

(i) $a \in \mathcal{A}^{n,esd,w}$;

(ii) $aw \in \mathcal{A}^{n,esd}$;

(iii) $wa \in \mathcal{A}^{n,esd}$.

In this case, for arbitrary $(aw)^{n,esd}$ *and* $(wa)^{n,esd}$*, we have that* $((aw)^{n,esd})^2 a$ *and* $a((wa)^{n,esd})^2$ *are w-egns-Drazin inverses of a.*

Proof. (i) ⇒ (ii): For $x = a^{n,esd,w}$, then $x * a * x = x$, $x * a = a * x$, and $a^{*n} - a * x \in \mathcal{A}^d_w$ which is equivalent to $xwawx = x$, $xwa = awx$ and $(aw)^{n-1}a - awx \in \mathcal{A}^d_w$. Hence, $xw(aw)xw = xw$ and $xw(aw) = (aw)xw$. Applying Lemma 4.1, we have $(aw)^n - (aw)xw \in \mathcal{A}^d$ and so $aw \in \mathcal{A}^{n,esd}$ with $(aw)^{n,esd} = xw$.

(ii) ⇒ (i): Assume that $z = (aw)^{n,esd}$ and $x = z^2 a$. Since $z(aw)z = z$ and $(aw)z = z(aw)$, then $a * x = awz^2 a = z^2 awa = x * a$ and $x * a * x = (z^2 aw)(awz^2)a = z^2 a = x$. From $((aw)^{n-1}a - za)w = (aw)^n - z(aw) \in \mathcal{A}^d$ and Lemma 4.1, one can see $a^{*n} - a * x = (aw)^{n-1}a - awz^2 a = (aw)^{n-1}a - za \in \mathcal{A}^d_w$. Thus, $a \in \mathcal{A}^{n,esd,w}$ and $a^{n,esd,w} = x = z^2 a$.

The equivalence (i) ⇔ (iii) can be verified analogously. □

As a consequence of Theorem 4.3, we characterize *w-ens*-Drazin invertible elements.

Corollary 4.4. *Let* $w \in \mathcal{A}\backslash\{0\}$*. Then, for* $a \in \mathcal{A}$*, the following statements are equivalent:*

(i) $a \in \mathcal{A}^{n,esD,w}$;

(ii) $aw \in \mathcal{A}^{n,esD}$;

(iii) $wa \in \mathcal{A}^{n,esD}$.

In this case, for arbitrary $(aw)^{n,esD}$ *and* $(wa)^{n,esD}$*, we have that* $((aw)^{n,esD})^2 a$ *and* $a((wa)^{n,esD})^2$ *are w-ens-Drazin inverses of a.*

By Theorem 2.5, Theorem 4.3 and Lemma 4.1, we obtain the following result.

Corollary 4.5. *Let* $w \in \mathcal{A}\backslash\{0\}$*. Then, for* $a \in \mathcal{A}$*, the following statements are equivalent:*

(i) $a \in \mathcal{A}^{n,esd,w}$;

(ii) $aw \in \mathcal{A}^{n,esd}$;

(iii) $wa \in \mathcal{A}^{n,esd}$;

(iv) $aw \in \mathcal{A}^d$;

(v) $wa \in \mathcal{A}^d$;

(v) $a \in \mathcal{A}^d_w$.

More characterizations of *w-egns*-Drazin invertible elements can be found using results proved in [15, 16].

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