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Nonautonomous impulsive differential equations of alternately advanced and retarded type

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Abstract. A variation of parameters formula and Gronwall type integral inequality are proved for an impulsive differential equation involving piecewise alternately advanced and retarded argument. At the same time, we study its oscillation and asymptotic stability properties. Our results are new, extend and improve earlier ones. Several numerical examples and simulations are also given to show the feasibility of our results.

1. Introduction

Differential equations with piecewise continuous arguments (DEPCAs) are related to impulse and loaded equations, DEPCAs represent a hybrid of discrete and continuous dynamical systems and combine the properties of both differential and differential-difference equations and share the properties of certain models of vertically transmitted diseases [6]. We also refer the reader to the papers [21, 22, 29, 30] where several dissipation problems impulsive effects are also idealized by means of external and/or drift sources.

The study of the DEPCAs is initiated by S. M. Shah and J. Wiener [35] and the theory of DEPCAs has been developed by many authors [1, 2, 7, 8, 23, 24, 33, 36–38]. Others applications of DEPCA are discussed in [19].

The study of DEPCAs of alternately advanced and retarded type is initiated by Aftabizadeh and Wiener [1]. They observed that the change of sign in the argument deviation leads not only to interesting periodic properties but also to complications in the asymptotic and oscillatory behaviour of solutions. Oscillatory, stability and periodic properties of the DEPCAs have been investigated in [1, 2, 4, 5, 9–12, 37]. Also, Wiener's book [38] is a distinguished source with respect to this area. See also the papers [20, 31, 32] for the oscillation and asymptotic behavior of differential equations with deviating arguments.

But, there are only a few papers for impulsive differential equations with piecewise constant arguments. Moreover, in [3], H. Bereketoglu et al. considered first the impulsive differential equations with piecewise constant argument of mixed type

$$x'(t) + a(t)x(t) + b(t)x([t]) + c(t)x([t+1]) = f(t), \quad x(0) = x_0, \quad t \neq n,$$
$$\Delta x|_{t=n} = d_k x(n), \quad n \in \mathbb{Z},$$

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and the authors also studied the conditions of periodicity, oscillation, nonoscillation and global asymptotic stability for some special cases.

In [25], F. Karakoc et al. considered the first order linear scalar impulsive delay differential equation of the type

$$x'(t) + a(t)x(t) + b(t)x([t-1]) = f(t), \quad x(-1) = x_-, \quad x(0) = x_0, \quad t \neq n,$$
$$\Delta x|_{t=n} = d_k x(n), \quad n \in \mathbb{Z},$$

and the authors studied the sufficient conditions for the existence of periodic and oscillatory solution for some special cases.

In [34], G.S. Oztepe et al. considered the second order impulsive delay differential equation with a piecewise constant argument of the type

$$x''(t) - a^2 x(t) = bx([t-1]), \quad x(-1) = x_-, \quad x(0) = x_0, \quad x'(0) = y_0, \quad t \neq n,$$
$$\Delta x'|_{t=n} = dx'(n), \quad n \in \mathbb{Z},$$

and the authors studied the sufficient conditions for the existence and uniqueness of solutions and the existence of periodic and oscillatory solution.

In [27], M. Lafci et al. considered the impulsive differential system with piecewise constant argument

$$x_1'(t) = \lambda x_1(t) - g(x_2[t-1]), \quad x_2'(t) = \lambda x_2(t) - g(x_1[t-1]), \qquad t \neq n,$$
$$\Delta x_i|_{t=n} = dx_i(n), \quad i = 1, 2, \quad n \in \mathbb{Z},$$

using Carvalho's method, the authors studied the sufficient conditions for the existence of periodic solution.

In the present paper we shall consider the impulsive differential equations with piecewise alternately advanced and retarded argument (IDEPCAs):

$$x'(t) = a(t)x(t) + b(t)x\left(p\left[\tfrac{t+l}{p}\right]\right), \quad x(\tau) = c_0, \quad t \neq kp - l, \tag{1.1a}$$
$$\Delta x|_{t=kp-l} = d_k x(kp - l^-), \quad k \in \mathbb{Z}, \tag{1.1b}$$

and

$$y'(t) = a(t)y(t) + b(t)y\left(p\left[\tfrac{t+l}{p}\right]\right) + f(t), \quad y(\tau) = c_0, \quad t \neq kp - l, \tag{1.2a}$$
$$\Delta y|_{t=kp-l} = d_k y(kp - l^-), \quad k \in \mathbb{Z}, \tag{1.2b}$$

where $a(t) \neq 0$, $b(t)$ and $f(t)$ are real-valued continuous functions of t defined on $\mathbb{R}$, $p\left[\frac{t+l}{p}\right]$ is a piecewise constant function defined by

$$p\left[\tfrac{t+l}{p}\right] = kp \quad \text{for} \quad t \in [kp - l, (k+1)p - l), \quad k \in \mathbb{Z},$$

where p and l are positive constants satisfying $p > l$.
Moreover $d_k \in \mathbb{R} \setminus \{-1\}$, $\Delta y|_{t=kp-l} = y(kp - l^+) - y(kp - l^-)$,

$$y(kp - l^+) = \lim_{t \to kp - l^+} y(t) \quad \text{and} \quad y(kp - l^-) = \lim_{t \to kp - l^-} y(t), \quad k \in \mathbb{Z}.$$

Since the deviation argument of the IDEPCAs (1.1) and (1.2), namely

$$\wp(t) := t - p\left[\tfrac{t+l}{p}\right]$$

is negative in $[kp - l, kp)$ and positive in $[kp, (k+1)p - l)$, the IDEPCAs (1.1) and (1.2) are said to be of alternately advanced and retarded type.

In 1991, for a scalar DEPCA (1.1a) with the deviation argument $2\left[\frac{t+1}{2}\right]$, Jayasree and Deo [23] are the first in obtaining a variation of parameters formula and Gronwall type integral inequalities in terms of the advanced and delayed intervals. In 2001, Meng and Yan [33] obtained a variation of parameters formula of the DEPCA to study oscillation and asymptotic stability properties of the DEPCA (1.1a). We extend these results to the IDEPCAs (1.1) and (1.2).

To the best of our knowledge, there are only a few papers in the literature deal with IDEPCA, the theory of the IDEPCA has been discussed in [3, 13–18, 25–28, 34, 39]. All these and especially Wiener's 16th open problem [38] have made us motivated to investigate the qualitative properties of solutions of the IDEPCAs (1.1) and (1.2).

Our paper is organized in the following way: in the next section, we consider existence and uniqueness of a global solution of the linear IDEPCA defined on the real axis. In the sections 3 and 4, a variation of parameters formula and Gronwall type integral inequality are proved for an impulsive differential equation involving piecewise alternately advanced and retarded argument. In Section 5, we study its oscillation and asymptotic stability properties. Several numerical examples and simulations are also given to show the feasibility of our results.

2. Alternately advanced and retarded impulsive differential equations

Definition 2.1. *A function x is a solution of the IDEPCA (1.1) (or (1.2)) in $\mathbb{R}$ if*

i) $x : \mathbb{R} \to \mathbb{R}$ *is continuous for $\mathbb{R}$ with the possible exception of the points $kp - l$, $k \in \mathbb{Z}$.*

ii) $x(t)$ *is right continuous and has left-hand limits at the points $kp - l$, $k \in \mathbb{Z}$.*

iii) The derivative $x'(t)$ exists at each point $t \in \mathbb{R}$ with the possible exception of the points $kp - l$, $k \in \mathbb{Z}$, where the one-side derivatives exist.

iv) $x(t)$ *satisfies (1.1a) (or (1.2a)) with the possible exception of the points $kp - l$, $k \in \mathbb{Z}$.*

v) $x(kp - l)$ *satisfies (1.1b) (or (1.2b)) for $k \in \mathbb{Z}$.*

The following assumption will be needed throughout the paper:

($\mathcal{N}$) For every $\tau \le t \in \mathbb{R}$, let $i = i(t) \in \mathbb{Z}$ be the unique integer such that $t \in I_i = [ip - l, (i+1)p - l)$, $\lambda\left(\tau, p\left[\frac{\tau+l}{p}\right]\right) \neq 0$, $\lambda(ip - l, ip) \neq 0$ for all $i \in \{i(\tau) + j\}_{j\in\mathbb{N}}$ and $\lambda((i+1)p - l, ip) \neq 0$ for all $i \in \{i(\tau) - j\}_{j\in\mathbb{N}}$, where

$$\lambda(t, s) := e^{\int_s^t a(\kappa)d\kappa} + \int_s^t e^{\int_u^t a(\kappa)d\kappa} b(u)du, \; s \in I_{i(s)}, \; t \in I_{i(t)}. \tag{2.1}$$

Moreover the impulsive effects $d_i \neq -1$ for all $i \in \mathbb{Z}$.

For the sake of convenience, we adopt the following notation:

$$\lambda_0(t, s) = \lambda\left(t, p\left[\frac{t+l}{p}\right]\right) \cdot \lambda\left(s, p\left[\frac{s+l}{p}\right]\right)^{-1}, \tag{2.2}$$

and

$$\varrho(t, s) = \lambda\left(t, p\left[\frac{s+l}{p}\right]\right) \cdot \lambda\left(t, p\left[\frac{t+l}{p}\right]\right)^{-1}. \tag{2.3}$$

The next theorem establishes a representation formula for solutions of the linear IDEPCA (1.1) on $t \in \mathbb{R}$.

Theorem 2.2. *Suppose that* **(N)** *holds. Then the initial value problem (1.1) has the unique solution* $x(t)$ *on* $\mathbb{R}$ *with the initial condition* $x(\tau) = c_0$ *and it is found by*

$$x(t) = \begin{cases} \lambda_0(t,\tau) \prod_{j=i(\tau)+1}^{i(t)} (1+d_j) \cdot \varrho\Big(jp-l,(j-1)p-l\Big)c_0, & i(t) > i(\tau), \\ \lambda_0(t,\tau)c_0, & i(t) = i(\tau), \\ \lambda_0(t,\tau) \prod_{j=i(t)+1}^{i(\tau)} \frac{1}{1+d_j} \cdot \varrho^{-1}\Big(jp-l,(j-1)p-l\Big)c_0, & i(t) < i(\tau). \end{cases} \tag{2.4}$$

Proof. Let $x_n(t) \equiv x(t)$ be a solution of the linear IDEPCA (1.1) on $np - l \le t < (n+1)p - l$. Then

$$x'(t) = a(t)x(t) + b(t)x(np).$$

So, integrating both sides from np to t

$$x(t) = \left(e^{\int_{np}^{t} a(s)ds} + \int_{np}^{t} e^{\int_{s}^{t} a(u)du} b(s)ds\right)x(np) = \lambda(t,np)x(np). \tag{2.5}$$

For $t = np - l$ and for $t \to (n+1)p - l^-$ in (2.5), we have

$$x_n((n+1)p - l^-) = \frac{\lambda((n+1)p-l,np)}{\lambda(np-l,np)} x_n(np-l) \text{ for all } n > i(\tau).$$

Because of the impulsive condition (1.1b),

$$x((n+1)p-l) = (1+d_{n+1})x((n+1)p-l^-), \quad \text{for } n \ge i(\tau).$$

This equality leads to the difference equation

$$x((n+1)p-l) = (1+d_{n+1})\frac{\lambda((n+1)p-l,np)}{\lambda(np-l,np)} x_n(np-l), \quad \text{for } n \ge i(\tau), \tag{2.6}$$

where $y_n = x(np-l)$.
Similarly, for (2.5) and $t \in I_{i(\tau)}$, we give

$$x(t) = \lambda(t,i(\tau)p)x(i(\tau)p), \quad x((i(\tau)+1)p^-) = \frac{\lambda((i(\tau)+1)p-l,i(\tau)p)}{\lambda(\tau,i(\tau)p)} x(\tau).$$

By using the impulsive condition (1.1b), for $n = i(\tau)+1$, we have

$$x(t_{i(\tau)+1}) = (1+d_{i(\tau)+1})x((i(\tau)+1)p^-) = (1+d_{i(\tau)+1})\frac{\lambda((i(\tau)+1)p-l,i(\tau)p)}{\lambda(\tau,i(\tau)p)} x(\tau).$$

It is to be noted that the initial condition (1.1) takes the form

$$y_{i(\tau)} = x(\tau) = c_0. \tag{2.7}$$

Therefore, the unique solution of the initial value problem (2.6)–(2.7) can be represented by

$$y_n = (1+d_n)\frac{\lambda(np-l,(n-1)p)}{\lambda(\tau,i(\tau)p)}\left(\prod_{j=i(\tau)+1}^{n-1} (1+d_j)\frac{\lambda(jp-l,(j-1)p)}{\lambda(jp-l,jp)}\right)c_0. \tag{2.8}$$

Using (2.5), we have for $t \in I_n$,

$$x(t) = \frac{\lambda(t,np)}{\lambda(np-l,np)}\left\{(1+d_n)\frac{\lambda(np-l,(n-1)p)}{\lambda(\tau,i(\tau)p)}\left(\prod_{j=i(\tau)+1}^{n-1}(1+d_j)\frac{\lambda(jp-l,(j-1)p)}{\lambda(jp-l,jp)}\right)\right\}c_0$$

$$= \frac{\lambda(t,np)}{\lambda(\tau,i(\tau)p)}\prod_{j=i(\tau)+1}^{n}\left((1+d_j)\frac{\lambda(jp-l,(j-1)p)}{\lambda(jp-l,jp)}\right)c_0.$$

As the "n" is chosen arbitrarily, therefore,

$$x(t) = \lambda_0(t,\tau)\prod_{j=i(\tau)+1}^{i(t)}(1+d_j)\cdot\varrho\Big(jp-l,(j-1)p-l\Big)c_0$$

where λ_0 is defined in (2.2) and ϱ is defined in (2.3).
In a similar way we obtain the solution of the linear IDEPCA (1.1) on $t \in (-\infty,\tau]$. The proof is complete. □

Remark 2.3. *a. Consider the particular case $\tau = kp - l$, $k \in \mathbb{Z}$, it is not difficult to see that the unique solution of the linear IDEPCA (1.1) on $t \in [kp-l,\infty)$ is given by*

$$x(t) = \lambda_0(t,kp-l)\prod_{j=k+1}^{i(t)}(1+d_j)\cdot\varrho\Big(jp-l,(j-1)p-l\Big)x(kp-l). \tag{2.9}$$

b. If $d_k = 0$, for all $k \in \{i(\tau)+j\}_{j\in\mathbb{N}}$. Then the linear DEPCA (1.1a) without impulsive effects has a unique solution on $[\tau,\infty)$ given by

$$x(t) = \lambda_0(t,kp-l)\prod_{j=k+1}^{i(t)}\varrho\Big(jp-l,(j-1)p-l\Big)x(kp-l).$$

Note that the explicit solution (2.9) *of the homogeneous linear IDEPCA (1.1) generalizes corresponding linear DEPCA's results obtained by A. R. Aftabizadeh and J. Wiener in [1] with deviation argument $\left[t+\frac{1}{2}\right]$, K. L. Cooke and J. Wiener in [8] with deviation argument $2\left[\frac{t+1}{2}\right]$ and J. Wiener and A. R. Aftabizadeh in [37] with deviation argument $m\left[\frac{t+k}{m}\right]$, where m and k are positive constants satisfying $m > k$.*

In the same way of Theorem 2.2, we have the following corollaries.

Corollary 2.4. *Let $\hat{\lambda}(t) = e^{at} + \frac{b}{a}(e^{at}-1)$, $\hat{\lambda}\left(\tau - p\left[\frac{\tau+l}{p}\right]\right) \neq 0$ and $\hat{\lambda}(-l) \neq 0$. For $a(t) \equiv a \neq 0$, $b(t) \equiv b$ constants, the unique solution of the linear IDEPCA (1.1) with constant coefficients on $[\tau,\infty)$ is given by*

$$x(t) = \frac{\hat{\lambda}\left(t-p\left[\frac{t+l}{p}\right]\right)}{\hat{\lambda}\left(\tau-p\left[\frac{\tau+l}{p}\right]\right)}\left(\frac{\hat{\lambda}(p-l)}{\hat{\lambda}(-l)}\right)^{i(t)-i(\tau)}\left(\prod_{j=i(\tau)+1}^{i(t)}(1+d_j)\right)c_0$$

where $x(\tau) = c_0$.

Corollary 2.5. *Let $\beta(t) := \int_{p\left[\frac{t+l}{p}\right]}^{t} b(s)ds$ and $\beta_j^- := \int_{jp}^{(j+1)p-l} b(s)ds$, $\beta_j^+ := \int_{jp}^{jp-l} b(s)ds$ for all $j \in \{i(\tau)+k\}_{k\in\mathbb{N}}$. Then the following linear IDEPCA with $a(t) \equiv 0$*

$$u'(t) = b(t)u\left(p\left[\tfrac{t+l}{p}\right]\right),\ u(\tau) = c_0, \quad t \neq t_k,$$
$$\Delta u|_{t=kp-l} = d_k u(kp-l^-), \quad k \in \mathbb{Z},$$

has a unique solution on $[\tau, \infty)$ *given by*

$$u(t) = \frac{1+\beta(t)}{1+\beta(\tau)}\left(\prod_{j=i(\tau)+1}^{i(t)}(1+d_j)\frac{1+\beta_{j-1}^-}{1+\beta_j^+}\right)c_0.$$

3. Variation of parameters method

This section is to establish a representation formula for solutions of the IDEPCA (1.2) in alternately advanced and delayed impulsive differential equations which corresponds to the variation of parameters formula in the theory of the IDEPCA.

For every $\tau < t \in \mathbb{R}$, let $i = i(t) \in \mathbb{Z}$ be the unique integer such that $t \in I_i = [ip - l, (i+1)p - l)$. Let

$$\Im(t,s) = \begin{cases} \lambda_0(t,s) \prod_{j=i(s)+1}^{i(t)} (1+d_j)\cdot \varrho\Big(jp-l,(j-1)p-l\Big), & i(t) > i(s), \\ \varrho(t,s), & i(t) = i(s). \end{cases} \tag{3.1}$$

Theorem 3.1. *Suppose that* **(N)** *holds. Then* $y(t)$ *is the unique solution of the IDEPCA (1.2) for* $\tau \le t$ *if and only if* $y(t)$ *is given by*

$$\begin{aligned} y(t) =& \Im(t,\tau)\, c_0 + \int_\tau^{p\left[\frac{\tau+l}{p}\right]} \Im(t,\tau)\, e^{\int_s^\tau a(\kappa)d\kappa} f(s)ds \\ &+ \sum_{k=i(\tau)}^{k=i(t)-1}\left(\int_{kp}^{(k+1)\cdot p} \Im(t,(k+1)\cdot p - l)\, e^{\int_s^{(k+1)\cdot p - l} a(\kappa)d\kappa} f(s)ds\right) + \int_{p\left[\frac{t+l}{p}\right]}^t e^{\int_s^t a(\kappa)d\kappa} f(s)ds. \end{aligned} \tag{3.2}$$

In particular, $x(t)$ *given by* $x(t) = \Im(t,\tau)\, c_0$ *is the unique solution of the linear IDEPCA (1.1) for* $\tau \le t$, $\tau, t \in \mathbb{R}$.

Proof. First, we prove that the function $y(t)$ given by (3.2) is a solution of the IDEPCA (1.2). This follows at once from $\frac{d\lambda(t,s)}{dt} = a(t)\lambda(t,s) + b(t)$, for s fixed, and using the notation (3.1):

$$\begin{aligned} \frac{d\Im(t,s)}{dt} &= \lambda_0'(t,s)\prod_{j=i(s)+1}^{i(t)} \varrho\Big(jp-l,(j-1)p-l\Big) = \frac{\lambda'(t,i(t)p)}{\lambda(s,i(s)p)}\prod_{j=i(s)+1}^{i(t)} \varrho\Big(jp-l,(j-1)p-l\Big) \\ &= \frac{a(t)\lambda(t,i(t)p)+b(t)}{\lambda(s,i(s)p)}\prod_{j=i(s)+1}^{i(t)} \varrho\Big(jp-l,(j-1)p-l\Big) = a(t)\Im(t,s) + b(t)\left(\frac{1}{\lambda(s,i(s)p)}\right)\Im(i(t)p-l,s) \\ &= a(t)\Im(t,s) + b(t)\Im(i(t)p,s), \ s < t. \end{aligned}$$

Reciprocally, assuming that $y_i(t)$ is a solution of the IDEPCA (1.2) on the interval $ip - l \le t < (i+1)p - l$, we have

$$y_i'(t) = a(t)y_i(t) + b(t)y_i(ip) + f(t).$$

Taking $F(t,u) = \int_u^t e^{\int_s^t a(\kappa)d\kappa} f(s)ds$, the solution of this equation on $I_i = [ip - l, (i+1)p - l)$ is given by:

$$\begin{aligned} y_i(t) &= \left(e^{\int_{\xi_i}^t a(s)ds} + \int_{ip}^t e^{\int_s^t a(\kappa)d\kappa} b(s)ds\right) y_i(ip) + \int_{ip}^t e^{\int_s^t a(\kappa)d\kappa} f(s)ds \\ &= \lambda(t,ip)\, y_i(ip) + F(t,ip). \end{aligned} \tag{3.3}$$

From (3.3), with $t = ip - l$ and $t \to (i+1)p - l^-$, respectively we have

$$y_i(ip) = \left(\frac{y_i(ip - l) - F(ip - l, ip)}{\lambda\,(ip - l, ip)} \right) \tag{3.4}$$

and

$$y_i((i+1)p - l^-) \; = \lambda\,((i+1)p - l, ip)\, y_i(ip) + F((i+1)p - l^-, ip). \tag{3.5}$$

Hence, for (3.4), (3.5) and the impulsive condition (1.2b), we have

$$y_i((i+1)p - l) = (1 + d_{i+1}) \left\{ \left(\frac{\lambda((i+1)p - l, ip)}{\lambda(ip - l, ip)} \right) (y_i(ip - l) - F(ip - l, ip)) + F((i+1)p - l, ip) \right\}.$$

Similarly,

$$\begin{aligned} y_{i-1}(ip - l) = (1 + d_i) &\left\{ \left(\frac{\lambda\,(ip - l, (i-1)p)}{\lambda\,(ip - l, (i-1)p)} \right) (y_{i-1}((i-1)p - l) - F((i-1)p - l, (i-1)p)) \right. \\ &\left. + F(ip - l, (i-1)p) \right\}, \; i \geq i(\tau) + 2, \end{aligned}$$

and

$$\begin{aligned} y_{i(\tau)}((i(\tau)+1)p - l) = (1 + d_{i(\tau)+1}) &\left\{ \left(\frac{\lambda\,((i(\tau)+1)p - l, i(\tau)p)}{\lambda\,(\tau, i(\tau)p)} \right) \big(y_{i(\tau)}(\tau) - F(\tau, i(\tau)p)\big) \right. \\ &\left. + F((i(\tau)+1)p - l, i(\tau)p) \right\}. \end{aligned}$$

Replacing the two previous relationships gives us:

$$\begin{aligned} y_i((i+1)p - l) = &\left((1 + d_{i+1}) \frac{\lambda\,((i+1)p - l, ip)}{\lambda\,(\tau, i(\tau)p)} \right) \left(\prod_{j=i(\tau)+1}^{i} (1 + d_j) \frac{\lambda(jp - l, (j-1)p)}{\lambda(jp - l, jp)} \right) c_0 \\ &+ \sum_{k=i(\tau)+1}^{i} \left\{ \left((1 + d_{i+1}) \frac{\lambda\,((i+1)p - l, ip)}{\lambda\,((k+1)p - l, (k+1)p)} \right) \left(\prod_{j=k+2}^{i} (1 + d_j) \frac{\lambda(jp - l, (j-1)p)}{\lambda(jp - l, jp)} \right) \times \right. \\ &\left. \left((1 + d_{k+1}) \frac{\lambda\,((k+1)p - l, kp)}{\lambda\,(kp - l, kp)} \, (-F(kp - l, kp)) + F((k+1)p - l, kp) \right) \right\} \\ &+ \left\{ \left((1 + d_{i+1}) \frac{\lambda\,((i+1)p - l, ip)}{\lambda\,((k+1)p - l, (k+1)p)} \right) \left(\prod_{j=i(\tau)+2}^{i} (1 + d_j) \frac{\lambda(jp - l, (j-1)p)}{\lambda(jp - l, jp)} \right) \times \right. \\ &\left((1 + d_{i(\tau)+1}) \frac{\lambda\,((i(\tau)+1)p - l, i(\tau)p)}{\lambda\,(\tau, i(\tau)p)} \, (-F(\tau, i(\tau)p)) + F((i(\tau)+1)p - l, i(\tau)p) \right) \end{aligned}$$

$$
\begin{aligned}
&= (1+d_{i+1})\,\lambda_0((i+1)p-l,\tau)\prod_{j=i(\tau)+1}^{i}(1+d_j)\cdot\varrho(jp-l,(j-1)p-l)c_0\\
&\quad+\sum_{k=i(\tau)+1}^{i}\Big\{\Big[(1+d_{i+1})\lambda_0((i+1)p-l,kp-l)\times\\
&\qquad\qquad\prod_{j=k+1}^{i}(1+d_j)\cdot\varrho(jp-l,(j-1)p-l)\,(-F(kp-l,kp))]\Big\}\\
&\quad+\sum_{k=i(\tau)+1}^{i}\Big\{\Big[(1+d_{i+1})\lambda_0((i+1)p-l,(k+1)p-l)\times\\
&\qquad\qquad\prod_{j=k+2}^{i}(1+d_j)\cdot\varrho(jp-l,(j-1)p-l)(F((k+1)p-l,kp))]\Big\}\\
&\quad+(1+d_{i+1})\lambda_0((i+1)p-l,\tau)\prod_{j=i(\tau)+1}^{i}(1+d_j)\cdot\varrho(jp-l,(j-1)p-l)\,(-F(kp-l,kp))\\
&\quad+(1+d_{i+1})\lambda_0((i+1)p-l,(k+1)p-l)\prod_{j=i(\tau)+2}^{i}(1+d_j)\cdot\varrho(jp-l,(j-1)p-l)(F((k+1)p-l,kp)).
\end{aligned}
$$

Using the notation (3.1), we have for all $i \geq i(\tau)+1$

$$
\begin{aligned}
y_i((i+1)p-l) = (1+d_{i+1})\Big\{&\Im\,((i+1)p-l,\tau)\,c_0+\sum_{k=i(\tau)+1}^{i}\Big[\Im\,((i+1)p-l,kp-l)\cdot(-F(kp-l,kp))\\
&+\Im\,((i+1)p-l,(k+1)p-l)\cdot F((k+1)p-l,kp)\Big]\\
&+\Im\,((i+1)p-l,\tau)\cdot(-F(\tau,i(\tau)p))\\
&+\Im\,((i+1)p-l,(i(\tau)+1)p-l)\cdot F((i(\tau)+1)p-l,i(\tau)p)\Big\}.
\end{aligned}
$$

In particular,

$$
\begin{aligned}
y_i(ip-l) = (1+d_i)\Big\{&\Im\,(ip-l,\tau)\,c_0+\sum_{k=i(\tau)+1}^{i-1}\Big[\Im\,(ip-l,kp-l)\,(-F(kp-l,kp))\\
&+\Im\,(ip-l,(k+1)p-l)\cdot F((k+1)p-l,kp)\Big]+\Im\,(ip-l,\tau)\cdot(-F(\tau,i(\tau)p))\\
&+\Im\,(ip-l,(i(\tau)+1)p-l)\,F((i(\tau)+1)p-l,i(\tau)p)\Big\}.
\end{aligned}
\tag{3.6}
$$

For (3.3), (3.4) and (3.6), it follows that

$$
\begin{aligned}
y_i(t) &= \lambda\,(t,ip)\,y_i(ip)+F(t,ip)\\
&= \frac{\lambda\,(t,ip)}{\lambda\,(ip-l,ip)}(1+d_i)\cdot\Im\,(ip-l,\tau)\,c_0+\frac{\lambda\,(t,ip)}{\lambda\,(ip-l,ip)}(1+d_i)\cdot\Big[\Im\,(ip-l,\tau)\cdot(-F(\tau,i(\tau)p))\\
&\qquad+\Im\,(ip-l,(i(\tau)+1)p-l)\cdot F((i(\tau)+1)p-l,i(\tau)p)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\lambda\,(t,ip)}{\lambda\,(ip-l,ip)}(1+d_i)\cdot\sum_{k=i(\tau)+1}^{i-1}\Big[\Im\,(ip-l,kp-l)\cdot(-F(kp-l,kp))\\
&\qquad\qquad\qquad+\Im\,(ip-l,(k+1)p-l)\cdot F((k+1)p-l,kp)\Big]\\
&-\frac{\lambda\,(t,ip)}{\lambda\,(ip-l,ip)}F(ip-l,ip)+F(t,ip)\\
=&\ \Im\,(t,\tau)\,c_0+\Im\,(t,\tau)\,(-F(\tau,i(\tau)p))+\Im(t,(i(\tau)+1)p-l)F((i(\tau)+1)p-l,i(\tau)p)\\
&+\sum_{k=i(\tau)+1}^{i-1}\left[\Im\,(t,kp-l)\,(-F(kp-l,kp))+\Im\,(t,(k+1)p-l)\,F((k+1)p-l,kp)\right]\\
&-\frac{\lambda\,(t,ip)}{\lambda\,(ip-l,ip)}F(ip-l,ip)+F(t,ip)\\
=&\ \Im\,(t,\tau)\,c_0+F(t,ip)+\Im\,(t,\tau)\,(-F(\tau,i(\tau)p))+\Im\,(t,(i(\tau)+1)p-l)\,F((i(\tau)+1)p-l,i(\tau)p)\\
&+\sum_{k=i(\tau)+1}^{i-1}\left[\Im\,(t,kp-l)\,(-F(kp-l,kp))+\Im\,(t,(k+1)p-l)\,F((k+1)p-l,kp)\right]\\
&-\frac{\lambda\,(t,ip)}{\lambda\,(ip-l,ip)}F(ip-l,ip)\\
=&\ \Im\,(t,\tau)\,c_0+\int_{\tau}^{i(\tau)p}\Im\,(t,\tau)\,e^{\int_s^{\tau}a(\kappa)d\kappa}f(s)ds+\int_{i(\tau)p}^{(i(\tau)+1)p-l}\Im\,(t,(i(\tau)+1)p-l)\,e^{\int_s^{(i(\tau)+1)p-l}a(\kappa)d\kappa}f(s)ds\\
&\qquad+\sum\nolimits_{k=i(\tau)+1}^{k=i}\left(\int_{kp-l}^{kp}\Im\,(t,kp-l)\,e^{\int_s^{kp-l}a(\kappa)d\kappa}f(s)ds\right)\\
&\qquad+\sum\nolimits_{k=i(\tau)+1}^{k=i-1}\left(\int_{kp}^{(k+1)p-l}\Im\,(t,(k+1)p-l)\,e^{\int_s^{(k+1)p-l}a(\kappa)d\kappa}f(s)ds\right)\\
&\qquad+\int_{ip}^{t}e^{\int_s^{t}a(\kappa)d\kappa}f(s)ds\\
=&\ \Im\,(t,\tau)\,c_0+\int_{\tau}^{i(\tau)p}\Im\,(t,\tau)\,e^{\int_s^{\tau}a(\kappa)d\kappa}f(s)ds\\
&\qquad+\sum\nolimits_{k=i(\tau)}^{k=i(t)-1}\left(\int_{kp}^{(k+1)\cdot p}\Im\,(t,(k+1)\cdot p-l)\,e^{\int_s^{(k+1)\cdot p-l}a(\kappa)d\kappa}f(s)ds\right)\\
&\qquad+\int_{ip}^{t}e^{\int_s^{t}a(\kappa)d\kappa}f(s)ds
\end{aligned}
$$

from where (3.2) follows.

If $f(t)=0$ in (3.2), we obtain the solution of the linear IDEPCA (1.1), $x(t)=\Im\,(t,\tau)\,c_0$. The proof is complete. □

Remark 3.2. *The representation formula (3.2) extends DEPCA's results of Jayasree and Deo [23] and Meng et al. [33] to the IDEPCA's formula.*

Let

$$
\psi\,(t,s)=\begin{cases}\lambda_0\,(t,s)\displaystyle\prod_{j=i(t)+1}^{i(s)}\frac{1}{1+d_j}\cdot\varrho^{-1}\Big(jp-l,(j-1)p-l\Big), & i(t)<i(s),\\ \varrho(t,s), & i(t)=i(s).\end{cases}
$$

In a similar way we obtain

Theorem 3.3. *Suppose that ($\mathcal{N}$) holds. Then the solution of problem (1.2) has a unique backward continuation on* $(-\infty, \tau]$ *given by*

$$\begin{aligned} z(t) &= \psi(t,\tau)c_0 + \int_{\tau}^{p\left[\frac{\tau+l}{p}\right]} \psi(t,\tau) e^{\int_s^{\tau} a(\kappa)d\kappa} f(s)ds \\ &+ \sum_{j=i(t)+1}^{j=i(\tau)} \left(\int_{jp}^{(j-1)\cdot p} \psi(t, jp-l) e^{\int_s^{jp-l} a(\kappa)d\kappa} f(s)ds \right) + \int_{p\left[\frac{t+l}{p}\right]}^{t} e^{\int_s^t a(\kappa)d\kappa} f(s)ds. \end{aligned} \tag{3.7}$$

In particular, $x(t)$ *given by* $x(t) = \psi(t,\tau)c_0$ *is the unique solution of the linear IDEPCA (1.1) for* $\tau \geq t$.

Note that Jayasree and Deo [23] do not study the unique solution of the DEPCA (1.1a) for $t \in (-\infty, \tau]$. Theorem 3.3 is the first result of the IDEPCA for (1.2) on $(-\infty, \tau]$ and it extends the result of Meng and Yan [33, pp.602] to the IDEPCA's result.

The next results are particular cases of Theorem 3.1.

Corollary 3.4. *Let* $\hat{\lambda}(t) = e^{at} + \frac{b}{a}(e^{at} - 1)$, $\hat{\lambda}\left(\tau - p\left[\frac{\tau+l}{p}\right]\right) \neq 0$, $\hat{\lambda}(-l) \neq 0$. *For* $a(t) \equiv a \neq 0$, $b(t) \equiv b$ *constants, the unique solution of the IDEPCA (1.2) on* $[\tau, \infty)$ *is given by*

$$\begin{aligned} y(t) &= \frac{\hat{\lambda}\left(t - p\left[\frac{t+l}{p}\right]\right)}{\hat{\lambda}\left(\tau - p\left[\frac{\tau+l}{p}\right]\right)} \left(\frac{\hat{\lambda}(p-l)}{\hat{\lambda}(-l)}\right)^{i(t)-i(\tau)} \left(\prod_{j=i(\tau)+1}^{i(t)} (1+d_j)\right) c_0 \\ &+ \frac{\hat{\lambda}\left(t - p\left[\frac{t+l}{p}\right]\right)}{\hat{\lambda}\left(\tau - p\left[\frac{\tau+l}{p}\right]\right)} \left(\frac{\hat{\lambda}(p-l)}{\hat{\lambda}(-l)}\right)^{i(t)-i(\tau)} \left(\prod_{j=i(\tau)+1}^{i(t)} (1+d_j)\right) \int_{\tau}^{i(\tau)p} e^{a(\tau-s)} f(s)ds \\ &+ \sum_{k=i(\tau)}^{k=i(t)-1} \frac{\hat{\lambda}\left(t - p\left[\frac{t+l}{p}\right]\right)}{\hat{\lambda}\left(\tau - p\left[\frac{\tau+l}{p}\right]\right)} \left(\frac{\hat{\lambda}(p-l)}{\hat{\lambda}(-l)}\right)^{i(t)-k} \prod_{j=k+1}^{i(t)} (1+d_j) \int_{kp}^{(k+1)\cdot p} e^{a\cdot((k+1)\cdot p-l-s)} f(s)ds \\ &+ \int_{p\left[\frac{t+l}{p}\right]}^{t} e^{a(t-s)} f(s)ds. \end{aligned}$$

Corollary 3.5. *Let* $\beta(t) := \int_{p\left[\frac{t+l}{p}\right]}^{t} b(s)ds$, $\beta_i^- := \int_{ip-l}^{(i+1)p-l} b(s)ds$, $\beta(s) \neq -1$ *and* $\beta(ip-l) \neq -1$ *for all* $i \in \{i(s)+j\}_{j\in\mathbb{N}}$:

$$\varphi(t,s) := \frac{1+\beta(t)}{1+\beta(s)} \left(\prod_{j=i(s)+1}^{i(t)} (1+d_j) \frac{1+\beta_{j-1}^-}{1+\beta(jp-l)}\right), s < t.$$

Then $u'(t) = b(t)u\left(p\left[\frac{t+l}{p}\right]\right) + f(t)$ *has a unique solution on* $[\tau, \infty)$ *given by*

$$u(t) = \varphi(t,\tau)u(\tau) + \int_{\tau}^{p\left[\frac{\tau+l}{p}\right]} \varphi(t,\tau) f(s)ds + \sum_{k=i(\tau)}^{k=i(t)-1} \left(\int_{kp}^{(k+1)p} \varphi(t,(k+1)p-l) f(s)ds\right) + \int_{p\left[\frac{t+l}{p}\right]}^{t} f(s)ds.$$

4. Integral inequality of Gronwall type

Integral inequalities play a useful role in the study of the qualitative behavior of solutions of linear differential equations. We extend the well-known Gronwall inequality to IDEPCA case.

Theorem 4.1. *Let $a(t), b(t)$ be two continuous positive functions and u is a non-negative piecewise continuous with possible discontinuity points of the first kind at $t = pk - l$, $k \in \mathbb{Z}$ and $u(\tau) \in \mathbb{R}$ for which the inequality satisfying*

$$u(t) \leq u(\tau) + \int_{\tau}^{t} \left[a(s)u(s) + b(s)u\left(p\left[\frac{s+l}{p}\right]\right)\right]ds + \sum_{k=i(\tau)+1}^{i(t)} d_k u(pk - l^-), \quad t \in [\tau, \infty), \tag{4.1}$$

holds and

$$\int_{\tau}^{p\left[\frac{\tau+l}{p}\right]} b(s)e^{\int_s^{p\left[\frac{\tau+l}{p}\right]} a(\kappa)d\kappa}ds < 1, \quad \int_{ip-l}^{ip} b(s)e^{\int_s^{ip} a(\kappa)d\kappa}ds < 1, \quad -1 < d_i \tag{4.2}$$

for all $i \in \{i(\tau) + j\}_{j \in \mathbb{N}}$. Then, for $t \geq \tau$, we have

$$u(t) \leq u(\tau)\frac{\lambda\left(t, p\left[\frac{t+l}{p}\right]\right)}{\lambda\left(\tau, p\left[\frac{\tau+l}{p}\right]\right)} \prod_{j=i(\tau)+1}^{i(t)} (1 + d_j)\frac{\lambda(jp - l, (j-1)p)}{\lambda(jp - l, jp)} \tag{4.3}$$

where $\lambda(t, s)$ is defined in (2.1). The case $a \equiv 0$ is also included.

Proof. Call $v(t)$ the right member of (4.1). So $v(\tau) = u(\tau)$, $u \leq v$, v is a piecewise differentiable function and by (4.1), it satisfies

$$\begin{cases} v'(t) \leq a(t)v(t) + b(t)v\left(p\left[\frac{t+l}{p}\right]\right), & t \neq kp - l, \\ v(kp - l) \leq (1 + d_k) \cdot v(kp - l^-), & k \in \mathbb{Z}. \end{cases} \tag{4.4}$$

So,

$$(v'(t) - a(t)v(t))\, e^{-\int_r^t a(s)ds} \leq b(t)v\left(p\left[\frac{t+l}{p}\right]\right)e^{-\int_r^t a(s)ds} \tag{4.5}$$

and integrating (4.5) for r and t in $I_n = [np - l, (n+1)p - l)$, if $r \leq t$ we have:

$$v_n(t) \leq v_n(r)e^{\int_r^t a(\kappa)d\kappa} + v_n(np)\int_r^t b(s)e^{\int_s^t a(\kappa)d\kappa}ds. \tag{4.6}$$

With $r = np - l$ and $t = np$ in (4.6), we have:

$$v_n(np) \leq v_n(np - l)e^{\int_{np-l}^{np} a(\kappa)d\kappa} + v_n(np)\int_{np-l}^{np} b(s)e^{\int_s^{np} a(\kappa)d\kappa}ds$$

or

$$v_n(np)\left(e^{\int_{np}^{np-l} a(\kappa)d\kappa} + \int_{np}^{np-l} b(s)e^{\int_s^{np-l} a(\kappa)d\kappa}ds\right) \leq v_n(np - l).$$

Then

$$v_n(np)\lambda\,(np - l, np) \leq v_n(np - l). \tag{4.7}$$

With $r = np$ in (4.6), we have:

$$v_n(t) \leq v_n(np)\left[e^{\int_{np}^t a(\kappa)d\kappa} + \int_{np}^t e^{\int_s^t a(\kappa)d\kappa}b(s)ds\right] = v_n(np)\lambda\,(t, np)\,. \tag{4.8}$$

If $t \to (n+1)p - l^-$ implies

$$v_n((n+1)p - l^-) \leq \lambda\left((n+1)p - l, np\right) \cdot v_n(np). \tag{4.9}$$

For (4.2), we have $\lambda(np - l, np) > 0$ and by (4.7) and (4.8) implies

$$v_n(t) \leq \frac{\lambda(t, np)}{\lambda(np - l, np)} \cdot v_n(np - l). \tag{4.10}$$

By (4.2), (4.9), (4.10) and the impulsive conditions (4.2) and (4.4), we obtain

$$v_n((n+1)p - l) \leq (1 + d_{n+1})\frac{\lambda((n+1)p - l, np)}{\lambda(np - l, np)} \cdot v_n(np - l). \tag{4.11}$$

By (4.10) and (4.11), we obtain

$$\begin{aligned}
v_n(t) &\leq \frac{\lambda(t, np)}{\lambda(np - l, np)} v_n(np - l) \\
&\leq \frac{\lambda(t, np)}{\lambda(np - l, np)} \left\{ (1 + d_n) \frac{\lambda(np - l, (n-1)p)}{\lambda(\tau, i(\tau)p)} \left(\prod_{j=i(\tau)+1}^{n-1} (1 + d_j) \frac{\lambda(jp - l, (j-1)p)}{\lambda(jp - l, jp)} \right) \right\} v(\tau) \\
&\leq \frac{\lambda(t, np)}{\lambda(\tau, i(\tau)p)} \cdot \prod_{j=i(\tau)+1}^{n} (1 + d_j) \frac{\lambda(jp - l, (j-1)p)}{\lambda(jp - l, jp)} \cdot v(\tau).
\end{aligned}$$

As we know $u_n(t) \leq v_n(t)$, so,

$$u_n(t) \leq u(\tau) \frac{\lambda(t, np)}{\lambda(\tau, i(\tau)p)} \prod_{j=i(\tau)+1}^{n} (1 + d_j) \frac{\lambda(jp - l, (j-1)p)}{\lambda(jp - l, jp)}.$$

As the "n" is chosen arbitrarily, therefore,

$$u(t) \leq u(\tau) \frac{\lambda\left(t, p\left[\frac{t+l}{p}\right]\right)}{\lambda\left(\tau, p\left[\frac{\tau+l}{p}\right]\right)} \prod_{j=i(\tau)+1}^{i(t)} (1 + d_j) \frac{\lambda(jp - l, (j-1)p)}{\lambda(jp - l, jp)}.$$

□

Remark 4.2. *Observe that the right-hand side of inequality (4.3) is in fact the solution of the related IDEPCA (1.2). In this sense, (4.3) is the best estimate. When in (4.1) $b(t) = 0$, (4.3) reduces to the classical impulsive Gronwall's inequality. The conditions $\lambda\left(\tau, p\left[\frac{\tau+l}{p}\right]\right) \neq 0$ and $\lambda(ip - l, ip) \neq 0$ can be reduced by (4.2), so such conditions are included implicitly in Theorem 4.1.*

Corollary 4.3. *For $a(t) \equiv a \neq 0$, $b(t) \equiv b$ positive constants, assume that the inequality (4.1) holds and*

$$b < a\left(e^{a\cdot\left(p\left[\frac{\tau+l}{p}\right]-\tau\right)} - 1\right)^{-1}, \quad b < a(e^{a\cdot l} - 1)^{-1} \quad \text{and} \quad -1 < d_i \tag{4.12}$$

are satisfied for all $i \in \{i(\tau) + j\}_{j\in\mathbb{N}}$. Then for $t \geq \tau$

$$u(t) \leq u(\tau) \frac{\hat{\lambda}\left(t - p\left[\frac{t+l}{p}\right]\right)}{\hat{\lambda}\left(\tau - p\left[\frac{\tau+l}{p}\right]\right)} \left(\frac{\hat{\lambda}(p - l)}{\hat{\lambda}(-l)}\right)^{i(t)-i(\tau)} \left(\prod_{j=i(\tau)+1}^{i(t)} (1 + d_j)\right) c_0 \tag{4.13}$$

where $\hat{\lambda}(t) = e^{at} + \frac{b}{a}(e^{at} - 1)$. The case $a \equiv 0$ is also included.

Remark 4.4. *Corollary 4.3 generalizes the corresponding inequality considered by Jayasree and Deo [23] without impulsive effect.*

5. Oscillatory behaviour

In [37], the authors have obtained a result on the oscillatory behaviour of solutions of the DEPCA (1.1a). In this section we extend this result for the linear IDEPCA (1.1). For this purpose we need the following definition.

Definition 5.1. *A function $x(t)$ defined on $[\tau,\infty)$ is said to be oscillatory if there exist two real valued sequences $\{\nu_n\}_{n\geq 0}$, $\{\nu'_n\}_{n\geq 0} \subset [\tau,\infty)$ such that $\nu_n \to \infty$, $\nu'_n \to \infty$ as $n \to \infty$ and $x(\nu_n) \leq 0 \leq x(\nu'_n)$ for $n \geq N$, where N is sufficiently large. Otherwise, the solution is called nonoscillatory.*

A solution $\{y_n\}_{n\geq i(\tau)}$ of the difference equation is called oscillatory if $y_n \cdot y_{n+1} \leq 0$ for $n \geq N$, where N is sufficiently large. Otherwise, the solution $\{y_n\}_{n\geq i(\tau)}$ is called nonoscillatory.

Theorem 5.2. *a) Let $x(t)$ be the unique solution of the linear IDEPCAG (1.1) on $[\tau,\infty)$. If the solution $\{y_n\}_{n\geq i(\tau)}$ of the difference equation (2.8) is oscillatory, then the solution $x(t)$ of the linear IDEPCA (1.1) is also oscillatory.*
b) Let $b(t)$ be locally integrable on $[\tau,\infty)$. Every solution of the linear IDEPCA (1.1) is oscillatory if the sequence

$$\left\{(1+d_j)\frac{\lambda(jp-l,(j-1)p)}{\lambda(jp-l,jp)}\right\}_{j\geq i(\tau)+1} \tag{5.1}$$

is not eventually positive.

Proof. a) From (2.4), $x(t)$ can be written on the interval $np-l \leq t < (n+1)p-l$, $n \in \{i(\tau)+j\}_{j\in\mathbb{N}}$ as

$$x(t) = \frac{\lambda(t,np)}{\lambda(np-l,np)}y_n.$$

This implies $x(t) = x(np-l) = y_n$ for $t = np-l$. From the theory of the difference equations it is well known that y_n is oscillatory if and only if $y_n \cdot y_{n+1} \leq 0$ for $n \geq N'$, where N' is a sufficiently large integer. Thus $x(t)$ is an oscillatory solution.
b) From (2.8), $\{y_n\}_{n\geq i(\tau)+1}$ can be written as

$$y_n = (1+d_n)\frac{\lambda(np-l,(n-1)p)}{\lambda(\tau,i(\tau)p)}\left(\prod_{j=i(\tau)+1}^{n-1}(1+d_j)\frac{\lambda(jp-l,(j-1)p)}{\lambda(jp-l,jp)}\right)y(\tau).$$

Then it is easy to see that the sequence $\{y_n\}_{n\geq i(\tau)+1}$ oscillates if

$$\left\{(1+d_j)\frac{\lambda(jp-l,(j-1)p)}{\lambda(jp-l,jp)}\right\}_{j\geq i(\tau)+1}$$

is not eventually positive. Therefore, by a), $x(t)$ oscillates if $\{y_n\}_{n\geq i(\tau)+1}$ oscillates. This completes the proof. □

Remark 5.3. *We note that the condition of the sequence $\left\{(1+d_j)\frac{\lambda(jp-l,(j-1)p)}{\lambda(jp-l,jp)}\right\}_{j\geq i(\tau)+1}$ is not eventually positive, which implies that if $1+d_j > \kappa_+ > 0$, $j \in \mathbb{Z}$ and*

$$\limsup_{j\to\infty}\int_{jp-l}^{jp} b(s)e^{\int_s^{jp} a(\kappa)d\kappa}ds > 1 \quad or \quad \liminf_{j\to\infty}\int_{(j-1)p}^{jp-l} b(s)e^{\int_s^{(j-1)p} a(\kappa)d\kappa}ds < -1. \tag{5.2}$$

If $1+d_j < \kappa_- < 0$, $j \in \mathbb{Z}$ and

$$\limsup_{j\to\infty}\int_{jp-l}^{jp} b(s)e^{\int_s^{jp} a(\kappa)d\kappa}ds < 1 \quad or \quad \liminf_{j\to\infty}\int_{(j-1)p}^{jp-l} b(s)e^{\int_s^{(j-1)p} a(\kappa)d\kappa}ds > -1 \tag{5.3}$$

hold true.
Condition (5.2) is the standard hypothesis to verify the existence of oscillatory solutions for the DEPCA. See [1], [8], [37] and [38].

Remark 5.4. *If the conditions*

$$1 + d_j > 0, \quad \limsup_{j\to\infty} \int_{jp-l}^{jp} b(s) e^{\int_s^{jp} a(\kappa)d\kappa} ds < 1 \quad \text{and} \quad \liminf_{j\to\infty} \int_{(j-1)p}^{jp-l} b(s) e^{\int_s^{(j-1)p} a(\kappa)d\kappa} ds > -1$$

hold true for $j \geq i(\tau) + 1$, *then the sequence* $\{y_j\}_{j\geq i(\tau)+1}$ *of the difference equation (2.8) is nonoscillatory.*

Theorem 5.5. *Let* $a(t)$ *and* $b(t)$ *be locally integrable on* $[\tau, \infty)$. *Assume that* $|a(t)| < K$ *for* $t \in [\tau, \infty)$ *and*

$$\left|(1 + d_j)\frac{\lambda(jp - l, (j-1)p)}{\lambda(jp - l, jp)}\right| < \kappa < 1, \quad \text{for all} \quad j \in \{i(\tau) + k\}_{k\in\mathbb{N}}. \tag{5.4}$$

Then

i) If $f(t) \equiv 0$, *the zero solution of the linear IDEPCA* (1.1) *is globally asymptotically stable.*

ii) If $\lim_{t\to\infty} f(t) = 0$, *every solution of the IDEPCA* (1.2) *tends to zero as* $t \to \infty$.

Proof. i) Since $t \in I_{i(t)} = [i(t)p - l, (i(t) + 1)p - l)$ and $\lambda_0(t, \tau)$ is continuous, the function $\lambda_0(t, \tau)$ is bounded for all $t \in I_{i(t)}$. Then we have

$$\left|\Im(t, \tau)\, c_0\right| = \left|\lambda_0(t, \tau) \prod_{j=i(\tau)+1}^{i(t)} \left((1 + d_j)\frac{\lambda(jp - l, (j-1)p)}{\lambda(jp - l, jp)}\right) c_0\right| \leq M \cdot \kappa^{i(t)-i(\tau)} |c_0| \tag{5.5}$$

where $M := \sup_{t\in I_{i(t)}} |\lambda_0(t, \tau)|$. The proof then follows easily from (2.4) and (5.4).

ii) Using Theorem 3.1, the variation of parameters formula (3.2), we have

$$\begin{aligned} y(t) =& \Im(t, \tau)\, c_0 + \int_\tau^{p\left[\frac{\tau+l}{p}\right]} \Im(t, \tau)\, e^{\int_s^\tau a(\kappa)d\kappa} f(s) ds \\ &+ \sum_{k=i(\tau)}^{k=i(t)-1} \left(\int_{kp}^{(k+1)\cdot p} \Im(t, (k+1)\cdot p - l)\, e^{\int_s^{(k+1)\cdot p - l} a(\kappa)d\kappa} f(s) ds \right) \\ &+ \int_{p\left[\frac{t+l}{p}\right]}^t e^{\int_s^t a(\kappa)d\kappa} f(s) ds. \end{aligned}$$

The condition $\lim_{t\to\infty} f(t) = 0$ yields for

a)

$$\left| \int_{p\left[\frac{t+l}{p}\right]}^t e^{\int_s^t a(\kappa)d\kappa} f(s) ds \right| \leq \max\{l, p - l\} \cdot e^{K \max\{l, p-l\}} \cdot \max_{s\in I_{i(t)}} \left|f(s)\right| \to 0, \quad \text{as} \quad t \to \infty.$$

b)

$$\begin{aligned} &\sum_{k=i(\tau)}^{i(t)-1} \left| \int_{kp}^{(k+1)p} \Im(t, (k+1)\cdot p - l)\, e^{\int_s^{(k+1)\cdot p - l} a(\kappa)d\kappa} f(s) ds \right| \\ \leq& \sum_{k=i(\tau)}^{i(t)-1} p \cdot M \cdot \kappa^{i(t)-k} \cdot e^{K \max\{l, p-l\}} \cdot \max_{s\in[kp,(k+1)p]} \left|f(s)\right| \\ \leq& \frac{pM}{1-\kappa} \cdot e^{K \max\{l, p-l\}} \cdot \max_{s\in[kp,(k+1)p]} \left|f(s)\right| \to 0 \quad \text{as} \quad k \to \infty. \end{aligned}$$

Similarly, we have

c)
$$\left|\int_{\tau}^{p\left[\frac{\tau+l}{p}\right]} \mathfrak{I}(t,\tau)\, e^{\int_s^{\tau} a(\kappa)d\kappa} f(s)ds\right| \leq M\cdot \kappa^{i(t)-i(\tau)} e^{K\max\{l,p-l\}} \max_{s\in I_{i(\tau)}} |f(s)| \to 0 \quad \text{as} \quad t\to\infty.$$

By (5.5), *a*), *b*) and *c*), we conclude that if $\lim_{t\to\infty} f(t) = 0$, $y(t)$ tends to zero as $t\to\infty$. The proof is complete. □

Now we will give two examples to illustrate the conclusions in the paper.

Example 5.6. *Let us consider the linear IDEPCA*

$$x'(t) = \ln 3\cdot x(t) - 2\cdot x\left(3\left[\frac{t+2}{3}\right]\right), \quad x(0) = 12, \quad t \neq 3k-2, \tag{5.6a}$$

$$\Delta x|_{t=3k-2} = \frac{5+\sqrt{3}}{8}\cdot x(3k-2^-), \quad k\in\mathbb{Z}. \tag{5.6b}$$

Eq. (5.6) is a special case of the linear IDEPCA (1.1) with $a = \ln 3$, $b = -2$, $d_k \equiv d = \frac{5+\sqrt{3}}{8}$, $k\in\mathbb{Z}$, $p = 3$ *and* $l = 2$. *It is easy to see that* $\hat{\lambda}(-l) = e^{-2} - \frac{2}{\ln 3}\left(e^{-2}-1\right) \neq 0$ *and* $\hat{\lambda}(p-l) = e - \frac{2}{\ln 3}(e-1)$.
We calculate

$$(1+d)\cdot\frac{\hat{\lambda}(p-l)}{\hat{\lambda}(-l)} = \frac{13+\sqrt{3}}{8}\cdot\frac{e-\frac{2}{\ln 3}(e-1)}{e^{-2}-\frac{2}{\ln 3}(e^{-2}-1)} \approx -0.441475.$$

In this case, the condition (5.1) *of Theorem 5.2 holds. So, every solution of the linear IDEPCA (5.6) is oscillatory. On the other hand, the hypotheses* (5.4) *of Theorem 5.5 i) is satisfied, we conclude that any solution of the linear IDEPCA (5.6) goes to zero as* $t\to\infty$ *by oscillating. The numerical simulation, showing the oscillatory solution of the linear IDEPCA* (5.6), *is given in Figure 1.*

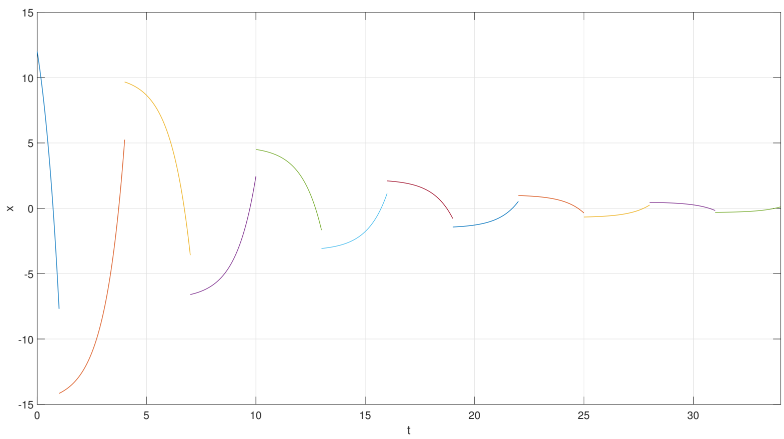

Figure 1. *The global asymptotic stability of the oscillatory solution for the linear IDEPCA* (5.6).

Example 5.7. *Let us consider the IDEPCA*

$$y'(t) = -\frac{5}{1+e^2}y(t) + \frac{1+\sqrt{5}}{2}\cos(t)y\left(2\pi\left[\frac{t+\pi}{2\pi}\right]\right) + \frac{100\cdot\sin(t)}{(t+1)^{1.8}}, \quad t\neq 2\pi k-\pi, \tag{5.7a}$$

$$\Delta y|_{t=2\pi k-\pi} = (-1)^k\cdot\frac{8+\sqrt{2}}{5}y(2\pi k-\pi^-), \quad k\in\mathbb{Z}, \tag{5.7b}$$

with $y(0) = -10$. *Eq. (5.7) is a special case of the IDEPCA (1.2) with* $a = -\frac{5}{1+e^2}$, $b(t) = \frac{1+\sqrt{5}}{2}\cos(t)$, $f(t) = \frac{100\cdot\sin(t)}{(t+1)^{1.8}}$, $d_k = (-1)^k \cdot \frac{8+\sqrt{2}}{5}$, $k \in \mathbb{Z}$, $p = 2\pi$ *and* $l = \pi$. *It is easy to see that* $\lambda(2k\pi - \pi, 2k\pi) \neq 0$, $k \in \mathbb{Z}$.
We calculate

$$\begin{aligned}
&\left|(1+d_j)\frac{\lambda(jp-l,(j-1)p)}{\lambda(jp-l,jp)}\right| \\
&\leq \left|\frac{13+\sqrt{2}}{5}\right| \cdot \sup_{j\in\mathbb{Z}} \left|\frac{e^{-\frac{5\pi}{1+e^2}} + \int_{2(j-1)\pi}^{2\pi j-\pi} e^{-\frac{5}{1+e^2}(2\pi j-\pi-s)}\frac{1+\sqrt{5}}{2}\cos(s)ds}{e^{\frac{5\pi}{1+e^2}} + \int_{2\pi j}^{2\pi j-\pi} e^{-\frac{5}{1+e^2}(2\pi j-\pi-s)}\frac{1+\sqrt{5}}{2}\cos(s)ds}\right| \\
&\approx 2.88284 \cdot 0.15432 \approx 0.44487 < 1.
\end{aligned}$$

Then the hypotheses (5.4) *and* $\lim_{t\to\infty} \frac{100\cdot\sin(t)}{(t+1)^{1.8}} = 0$ *are satisfied, by Theorem 5.5 ii), we conclude that the solution of the IDEPCA (5.7) is globally asymptotically stable. The numerical simulation is given in Figure 2.*

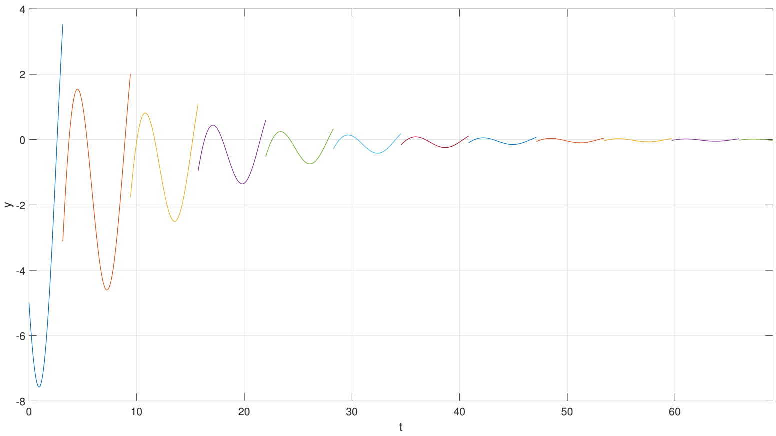

Figure 2. *The global asymptotic stability of the oscillatory solution for the IDEPCA* (5.7).

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