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Stochastic controllability of a non-autonomous impulsive system with variable delays in control

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Abstract. In this paper, we study the sufficient conditions for the relative controllability of a non-autonomous stochastic impulsive differential equation with variable delays in control. The evolution family, Arzela-Ascoli theorem, Schauder's fixed point theorem, and stochastic analysis are used to achieve the main results. Two examples are provided in support of the main results.

1. Introduction

Many real-life phenomena are affected by the sudden change in their states at certain moments, such as heartbeats and blood flow in the human body. These phenomena are discussed in the form of impulses whose duration is negligible compared to the whole process. Impulsive differential equations are used to model such processes. The study of impulsive systems is crucial to analyzing more realistic mathematical models. It has a broad area of applications, including drug diffusion in the human body, control theory, frequency-modulated systems, population dynamics, metallurgy, theoretical physics, industrial robotics, engineering, etc. We refer to the papers [19, 27] and the references cited therein for more details on the basic theory of impulsive differential equations.

A stochastic differential equation is one in which one or more of the terms are random variables. Stochastic differential equations are used to describe various phenomena, notably unstable stock prices and physical devices with thermal fluctuations, population dynamics, biology, weather prediction model, molecular dynamics, and the textile industry. A brief summary of stochastic theory can be seen in [8, 18].

The goal of the controllability theory is the ability to control a specific system to the desired state by providing appropriate input functions during a finite time interval. Controllability theory is a fundamental concept in the dynamical control systems and plays a significant role in investigating and analyzing various dynamical control processes. Many research papers were presented about controllability theory where control function depends on stochastic parameters; see for instance [1, 21, 25]. There are numerous works on controllability results for autonomous [4, 28] and non-autonomous stochastic systems [1, 9, 10, 31].

Numerous dynamic systems have variable states subject to abrupt changes, which may result from stochastic phenomena such as sudden environmental changes. Stochastic disturbances are generally regarded as an essential component of systems as they can be used to represent effects caused by external

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perturbations. Due to its extensive need, the controllability of stochastic systems has received much attention. On the other hand, many systems derived from realistic models rely on histories or impulsive phenomena in addition to white noise. Thus, research on impulsive stochastic differential systems with delay is necessary. Due to their wide applicability in various domains, including automatic control, economics, physics, and industry, impulsive stochastic systems have been the subject of several works during the last few decades. One of the most crucial characteristic of dynamical systems is controllability, which is essential for solving various control issues, such as the stability of the system, with the appropriate controls. Using the Razumikhin technique and stochastic processes theory, Hu et al. [33] studied the exponential stability of impulsive stochastic delay differential systems. Using the Lyapunov approach and stochastic analysis theory, Hu and Zhu [34] established the stability criteria for impulsive stochastic functional differential systems with distributed-delay dependent impulsive effects. For more details on the impulsive stochastic system, the reader may refer to the papers [35, 36].

Time delay systems play an important role in many mathematical models. The time delays are present in either control or state or both. Several biological and physical systems involve time delays, such as population growth, epidemiology, immunology, etc. In these models, the time delays can be related to the duration of specific hidden processes. Different time delays have been taken into lots of studies, such as constant delays [6], time-variant delays [39], state-dependent delays [37], multi-proportional delays [17], and so on. The delay differential equations are widely used to analyze and predict many results in various areas of sciences, such as the time taken by a chemical reaction and its reactants in chemical kinetics and the production of new viruses in a biological system.

Several real-world systems (such as sudden stock price fluctuations brought on by war, epidemics, or market crashes, etc.) are subject to stochastic perturbations with impulsive effects. For these models, the path continuity supposition does not seem plausible. Consequently, when modeling such systems, we should take impulsive effects in stochastic processes into account. Due to the numerous applications of second-order abstract differential equations in physics and engineering, they have received much more attention. Second-order differential equations, for instance, can be used to analyze the system of dynamical buckling of a hinged extensible beam [26, 32]. The integrated process in continuous time, which can be made stationary, is best described by second-order stochastic differential equations. For instance, engineers can make use of second-order stochastic differential equations to describe mechanical vibrations or the charge on a capacitor. In recent years, numerous findings about the controllability of second-order stochastic control systems have been obtained [20, 24, 28, 30, 38].

In 2006, Balasubramaniam and Ntouyas [21] provided the controllability result for partial stochastic functional differential inclusions with infinite delay. In 2017, Haloi [23] gave sufficiency conditions for controllability of non-autonomous differential equations with a nonlocal finite delay with deviating arguments. In 2022, Afreen et al. [1] studied a semilinear stochastic system with constant delays in control.

More specifically, in 2012, Balachandran et al. [16] considered the following fractional delay differential equation

$$
\begin{cases}
{}^{C}\mathcal{D}^{q}\vartheta(t) = \mathcal{A}\vartheta(t) + \sum\limits_{i=0}^{M} \mathcal{B}_i v\left(h_i(t)\right) + f\left(t, \vartheta(t), v(t)\right), \quad t \in [0, T], \\
\vartheta(0) = \vartheta_0,
\end{cases}
$$

where $0 < q < 1$ and discussed the relative controllability of the considered problem.

In 2019, Haq and Sukavanam [3] obtained sufficient conditions for the controllability of the following semilinear delay system

$$
\begin{cases}
\vartheta''(t) = A\vartheta(t) + \mathcal{B}_1 v(t) + \mathcal{B}_2 v(t-b) + F\left(t, \vartheta_{a(t)}, v(t) + v(t-b)\right), \quad t \in (0, \beta], \\
\vartheta'(0) = \vartheta_1, \\
g(\vartheta) = \varphi, \quad v(t) = 0, \quad t \in [-b, 0].
\end{cases}
$$

Motivated by the mentioned works, our aim is to study the relative controllability of the following second-order non-autonomous stochastic impulsive system with variable delays in control:

$$\begin{cases} \vartheta''(t) = \mathrm{A}(t)\vartheta(t) + \sum\limits_{i=0}^{N} \mathrm{B}_i \upsilon(\alpha_i(t)) + f\left(t, \vartheta(t), \sum\limits_{i=0}^{N} \upsilon(\alpha_i(t))\right) + \rho\left(t, \vartheta(t), \sum\limits_{i=0}^{N} \upsilon(\alpha_i(t))\right) \dfrac{dw(t)}{dt}, \ t \in [0,\ell], \ t \neq t_q, \\ \vartheta(t_q^+) - \vartheta(t_q^-) = \mathrm{I}_q\big(\vartheta(t_q)\big), \quad q = 1,2,\ldots,r, \\ \vartheta'(t_q^+) - \vartheta'(t_q^-) = \mathrm{J}_q\big(\vartheta(t_q)\big), \quad q = 1,2,\ldots,r, \\ \vartheta(0) = \vartheta_0, \quad \vartheta'(0) = \vartheta_1, \end{cases} \tag{1}$$

where $0 = t_0 < t_1 < t_2 < \cdots < t_r < t_{r+1} = \ell$. Let X, Y be separable Hilbert spaces and the notation $\|\cdot\|$ denotes the norm of the spaces X, Y, and $L(Y, X)$. $\mathrm{A}(t)$ is a densely defined closed linear operator on X [5] and $\vartheta(\cdot) \in X$. The control function υ takes value in a separable Hilbert space Z and the linear operators $B_i \in L(Z, X)$, $i = 0, 1, \ldots, N$ are bounded. The time-dependent delays are given by the function α_i, $i = 0, 1, 2, \ldots, N$. f and ρ are nonlinear functions that satisfy some suitable conditions. I_q, J_q, $q = 1, 2, \ldots, r$ are impulsive functions. $\vartheta(t_q^+) - \vartheta(t_q^-)$ and $\vartheta'(t_q^+) - \vartheta'(t_q^-)$ denote the jumps at $t = t_q$ in the states ϑ and ϑ', respectively. $\vartheta(t_q^+), \vartheta(t_q^-)$ and $\vartheta'(t_q^+), \vartheta'(t_q^-)$, denote the right and left limit values of ϑ and ϑ' at $t = t_q$, respectively. Let w be a Y-valued Wiener process with a finite trace nuclear covariance operator $Q \geq 0$ on a complete probability space $(\Omega, \mathcal{F}, P)$, where $\mathcal{F}_t \subset \mathcal{F}$, $t \in [0, \ell]$ is a normal filtration. $\mathcal{F}_t$ is a right continuous increasing family and $\mathcal{F}_0$ contains all P-null sets. Let $L_0^2 = L_2(Q^{\frac{1}{2}} Y, X)$ denotes the space of all Hilbert-Schmidt operators from $Q^{\frac{1}{2}} Y$ to X, which is a separable Hilbert space with norm $\|\vartheta\|_{L_0^2} = \ tr[\vartheta Q \vartheta^*] < \infty$. The initial values ϑ_0 and ϑ_1 are $\mathcal{F}_0$-measurable, X-valued random variables. Let $L^p(\mathcal{F}, X)$ denotes the Banach space of all p-integrable, $\mathcal{F}$-measurable random variables and E is the expectation given by $E(\vartheta) = \int_\Omega \vartheta(\omega) dP$. The control function $\upsilon(\cdot) \in L^p_{\mathcal{F}}([0, \ell], Z)$, where $L^p_{\mathcal{F}}([0, \ell], Z)$ is a Banach space with

$$\|\upsilon\|_{L^p_{\mathcal{F}}} = \left(E \int\limits_0^\ell \|\upsilon(t)\|_Z^p \, dt \right)^{\frac{1}{p}},$$

satisfying $E \int\limits_0^\ell \|\upsilon(t)\|_Z^p \, dt < \infty$.

Let U be a non-empty closed, bounded and convex subset of Z. The admissible control set is given by

$$U_{ad} = \left\{ \upsilon \in L^p_{\mathcal{F}}([0, \ell], Z) \mid \upsilon(t) \in U \text{ a.e. } t \in [0, \ell] \right\}.$$

Let $PC([0,\ell], X) = \left\{ \vartheta : [0,\ell] \to X : \vartheta \ \text{ is continuous at } t \neq t_q, \ \vartheta(t_q^+), \ \vartheta(t_q^-) \ \text{both exist and } \vartheta(t_q) = \vartheta(t_q^-) \right\}$. Similarly, $PC'([0,\ell], X) = \left\{ \vartheta : [0,\ell] \to X : \vartheta' \text{ is continuous at } t \neq t_q, \ \vartheta'(t_q^+), \ \vartheta'(t_q^-) \ \text{both exist and } \vartheta'(t_q) = \vartheta'(t_q^-) \right\}$. The space $PC := PC([0, \ell], X)$ is a Banach space with supremum norm defined by

$$\|\vartheta\|_{PC} = \left(\sup_{t \in [0,\ell]} E \|\vartheta(t)\|^p \right)^{\frac{1}{p}}.$$

This paper is divided into six sections. Sections 1 and 2 contain the introduction, assumptions, and basic lemmas. In Section 3, sufficient condition for the controllability of the corresponding linear system with time-dependent delays is discussed. Sections 4 and 5 are concerned with the main results. In Section 6, examples are given in support of the main results.

2. Preliminaries and Assumptions

Let $\psi : [0, \ell] \times [0, \ell] \to L(X)$ be the evolution operator [11], where *L(X)* denotes the Banach space of set of all bounded linear operators defined on X. Define

$$\xi(t, s) = -\frac{\partial \psi(t, s)}{\partial s}.$$

Lemma 2.1. *[13] For any $\vartheta_\ell \in L^p(\mathcal{F}, X)$ and $p \geq 2$, there exists a function $\varrho \in L^p_{\mathcal{F}}\left([0,\ell], L_0^2\right)$ such that*

$$\vartheta_\ell = E\vartheta_\ell + \int_0^\ell \varrho(s)dw(s).$$

Lemma 2.2. *[29] Let $\varrho : [0,\ell] \times \Omega \to L_0^2$ be strongly measurable function such that $\int_0^\ell E\left\|\varrho(s)\right\|^p_{L_0^2} ds < \infty$. Then*

$$E\left\|\int_0^t \varrho(s)dw(s)\right\|^p \leq L_\varrho \int_0^t E\left\|\varrho(s)\right\|^p ds,$$

for every $t \in [0,\ell]$, $p \geq 2$, where L_ϱ is the constant involving p and ℓ.

We assume the following assumptions which satisfy some necessary conditions.

(H1) $\left\|\psi(t,s)\right\| \leq l_1$, $\|\xi(t,s)\| \leq l_2$, $K = \sup\{\|\mathrm{B}_i\|, i = 0,1,2,\ldots,N\}$, where l_1, l_2 and K are positive constants.

(H2) The nonlinear function $f : [0,\ell] \times X \times Z \to X$ is continuous and there are real constants μ_1, μ_2 such that

$$\left\|f\left(t, \vartheta(t), \sum_{i=0}^N \upsilon(\alpha_i(t))\right) - f\left(t, \tilde{\vartheta}(t), \sum_{i=0}^N \tilde{\upsilon}(\alpha_i(t))\right)\right\|^p \leq \mu_1 \left\|\vartheta - \tilde{\vartheta}\right\|^p + \mu_2 \left\|\upsilon - \tilde{\upsilon}\right\|^p,$$

where $\left\|\upsilon - \tilde{\upsilon}\right\|^p = \sum_{i=0}^N \left\|\upsilon(\alpha_i(t)) - \tilde{\upsilon}(\alpha_i(t))\right\|^p$.

(H3) The nonlinear function $\rho : [0,\ell] \times X \times Z \to L_0^2$ is continuous and there are real constants μ_3, μ_4 such that

$$\left\|\rho\left(t, \vartheta(t), \sum_{i=0}^N \upsilon(\alpha_i(t))\right) - \rho\left(t, \tilde{\vartheta}(t), \sum_{i=0}^N \tilde{\upsilon}(\alpha_i(t))\right)\right\|^p_{L_0^2} \leq \mu_3 \left\|\vartheta - \tilde{\vartheta}\right\|^p + \mu_4 \left\|\upsilon - \tilde{\upsilon}\right\|^p,$$

where $\left\|\upsilon - \tilde{\upsilon}\right\|^p = \sum_{i=0}^N \left\|\upsilon(\alpha_i(t)) - \tilde{\upsilon}(\alpha_i(t))\right\|^p$.

(H4) The maps $\mathrm{I}_q, \mathrm{J}_q : X \to X$, $q = 1,2,\ldots,r$ are continuous and there exist positive constants $\nu_q, \tilde{\nu}_q, \varsigma_q, \tilde{\varsigma}_q$ such that

$$\begin{aligned} \left\|\mathrm{I}_q(\vartheta) - \mathrm{I}_q(\tilde{\vartheta})\right\|^p \leq \nu_q^p \|\vartheta - \tilde{\vartheta}\|^p, \quad \left\|\mathrm{I}_q(\vartheta)\right\|^p &\leq \tilde{\nu}_q^p, \\ \left\|\mathrm{J}_q(\vartheta) - \mathrm{J}_q(\tilde{\vartheta})\right\|^p \leq \varsigma_q^p \|\vartheta - \tilde{\vartheta}\|^p, \quad \left\|\mathrm{J}_q(\vartheta)\right\|^p &\leq \tilde{\varsigma}_q^p, \end{aligned}$$

for all ϑ, $\tilde{\vartheta} \in X$.

Remark 2.3. *In assumption (H1), the operators ψ, ξ and B_i are bounded by some positive constants. Assumption (H2) and (H3) mean that the nonlinear functions f and ρ satisfy Lipschitz like condition. Assumption (H4) means that impulsive functions I_q, J_q are bounded and also satisfy Lipschitz condition. We will use these assumptions to prove the main results.*

3. Linear system

Consider the non-autonomous linear impulsive system with time-dependent delays in control

$$\begin{cases} \vartheta''(t) = \mathrm{A}(t)\vartheta(t) + \sum\limits_{i=0}^{N} \mathrm{B}_i v\left(\alpha_i(t)\right), \quad t \in [0,\ell],\ t \neq t_q, \\ \vartheta(t_q^+) - \vartheta(t_q^-) = \mathrm{I}_q\Big(\vartheta(t_q)\Big), \quad q = 1,2,3,\ldots,r, \\ \vartheta'(t_q^+) - \vartheta'(t_q^-) = \mathrm{J}_q\Big(\vartheta(t_q)\Big), \quad q = 1,2,3,\ldots,r, \\ \vartheta(0) = \vartheta_0, \quad \vartheta'(0) = \vartheta_1. \end{cases} \tag{2}$$

Consider the following settings:

- The functions $\alpha_i : [0,\ell] \to \mathbb{R}$, $i = 0,1,2,\ldots,N$ are strictly increasing, absolutely continuous and satisfying $\alpha_i(t) \le t$.

  Also, the functions α_i can be represented as

  $$\alpha_i(t) = t - \tau_i(t), \quad i = 0,1,2,\ldots,N, \quad t \in [0,\ell],$$

  where $\tau_i(t)$ are time dependent delays.

- Define the time lead functions $\beta_i : [\alpha_i(0), \alpha_i(\ell)] \to [0,\ell]$ as

  $$\beta_i\left(\alpha_i(t)\right) = t, \quad i = 0,1,2,\ldots,N, \quad t \in [0,\ell].$$

  Also, assume that $\alpha_0(t) = t$ and for $t = \ell$, we have

  $$\alpha_N(\ell) \le \alpha_{N-1}(\ell) \le \cdots \le \alpha_{n+1}(\ell) \le 0 = \alpha_n(\ell) < \alpha_{n-1}(\ell) = \cdots = \alpha_1(\ell) = \alpha_0(\ell) = \ell. \tag{3}$$

- For a given $h > 0$, consider the control function $v : [-h,\ell] \to U_{ad}$ and $t \in [0,\ell]$, we take v_t to represent the map on $[-h,0]$ such that $v_t(s) = v(t+s)$ for $-h \le s < 0$.

Definition 3.1. *The complete state of the system (2) at time t is given by the set* $y(t) = \{\vartheta(t), v_t\}$.

Definition 3.2. *[14] The control system (1) is called relatively controllable on a given time interval* $[0,\ell]$ *if for all complete initial state* $y(0) = \{\vartheta(0), v_0\}$ *and each final instantaneous state* $\vartheta_\ell \in PC$*, there exists a control* $v(t) \in U_{ad}$ *such that* $\vartheta(\ell) = \vartheta_\ell$.

To study the relative controllability of (1), we will introduce the following equivalent conditions.

Lemma 3.3. *[15] The following conditions are equivalent:*

(1) The corresponding linear deterministic system with respect to (1) is relatively controllable on $[0,\ell]$.

(2) The controllability Grammian operator W defined in (8) is nonsingular.

The solution of the system (2) can be written as

$$\begin{aligned} \vartheta(t) &= \xi(t,0)\vartheta_0 + \psi(t,0)\vartheta_1 + \int_0^t \psi(t,s)\sum_{i=0}^{N} \mathrm{B}_i v\left(\alpha_i(s)\right) ds \quad + \sum_{0<t_q<t} \xi(t,t_q)\mathrm{I}_q\Big(\vartheta(t_q)\Big) \\ &\quad + \sum_{0<t_q<t} \psi(t,t_q)\mathrm{J}_q\left(\vartheta(t_q)\right). \end{aligned} \tag{4}$$

Using the time lead function β_i in (4), the solution becomes

$$\begin{aligned}\vartheta(t) &= \xi(t,0)\vartheta_0 + \psi(t,0)\vartheta_1 + \sum_{i=0}^{N}\int_{\alpha_i(0)}^{\alpha_i(t)} \psi\left(t,\beta_i(s)\right) \mathrm{B}_i\beta_i^{(1)}(s)v(s)ds + \sum_{0<t_q<t}\xi(t,t_q)\mathrm{I}_q\Big(\vartheta(t_q)\Big) \\ &\quad + \sum_{0<t_q<t}\psi(t,t_q)\mathrm{J}_q\Big(\vartheta(t_q)\Big). \end{aligned} \tag{5}$$

Substituting $t=\ell$ and using (3) in (5), we have

$$\begin{aligned}\vartheta(\ell) &= \xi(\ell,0)\vartheta_0 + \psi(\ell,0)\vartheta_1 + \sum_{i=0}^{n}\int_{\alpha_i(0)}^{0} \psi\left(\ell,\beta_i(s)\right) \mathrm{B}_i\beta_i^{(1)}(s)v_0(s)ds + \sum_{i=0}^{n}\int_{0}^{\ell} \psi\left(\ell,\beta_i(s)\right) \mathrm{B}_i\beta_i^{(1)}(s)v(s)ds \\ &\quad + \sum_{i=n+1}^{N}\int_{\alpha_i(0)}^{\alpha_i(\ell)} \psi\left(\ell,\beta_i(s)\right) \mathrm{B}_i\beta_i^{(1)}(s)v_0(s)ds + \sum_{0<t_q<\ell}\xi(\ell,t_q)\mathrm{I}_q\Big(\vartheta(t_q)\Big) + \sum_{0<t_q<\ell}\psi(\ell,t_q)\mathrm{J}_q\Big(\vartheta(t_q)\Big). \end{aligned} \tag{6}$$

Define the following notations

$$G(\ell) = \sum_{i=0}^{n}\int_{\alpha_i(0)}^{0} \psi\left(\ell,\beta_i(s)\right) \mathrm{B}_i\beta_i^{(1)}(s)v_0(s)ds + \sum_{i=n+1}^{N}\int_{\alpha_i(0)}^{\alpha_i(\ell)} \psi\left(\ell,\beta_i(s)\right) \mathrm{B}_i\beta_i^{(1)}(s)v_0(s)ds, \tag{7}$$

and the controllability Grammian operator

$$W(0,\ell) = \sum_{i=0}^{n}\int_{0}^{\ell} \psi\left(\ell,\beta_i(s)\right) \mathrm{B}_i\beta_i^{(1)}(s)\left\{\psi\left(\ell,\beta_i(s)\right) \mathrm{B}_i\beta_i^{(1)}(s)\right\}^* ds, \tag{8}$$

where $*$ denotes the adjoint.

Theorem 3.4. *The linear system (2) is relatively controllable on* $[0,\ell]$, *if the controllability Grammian operator (8) is nonsingular.*

Proof. Since W is nonsingular. Therefore, its inverse exists. Define $v(t)$ as

$$\begin{aligned} v(t) &= \left\{\mathrm{B}_i^*\psi^*\left(\ell,\beta_i(t)\right)\beta_i^{(1)}(t)\right\}W^{-1}\Big[\vartheta_\ell - \xi(\ell,0)\vartheta_0 - \psi(\ell,0)\vartheta_1 - G(\ell) - \sum_{0<t_q<\ell}\xi(\ell,t_q)\mathrm{I}_q\Big(\vartheta(t_q)\Big) \\ &\quad - \sum_{0<t_q<\ell}\psi(\ell,t_q)\mathrm{J}_q\Big(\vartheta(t_q)\Big)\Big]. \end{aligned}$$

Substituting the value of $v(t)$ in (6) and using (7), we have

$$\begin{aligned}\vartheta(\ell) &= \xi(\ell,0)\vartheta_0 + \psi(\ell,0)\vartheta_1 + \sum_{i=0}^{n}\int_{0}^{\ell} \psi\left(\ell,\beta_i(s)\right) \mathrm{B}_i\beta_i^{(1)}(s)\left\{\mathrm{B}_i^*\psi^*\left(\ell,\beta_i(s)\right)\beta_i^{(1)}(s)\right\}\times \\ &\quad W^{-1}\left[\vartheta_\ell - \xi(\ell,0)\vartheta_0 - \psi(\ell,0)\vartheta_1 - G(\ell) - \sum_{0<t_q<\ell}\xi(\ell,t_q)\mathrm{I}_q\Big(\vartheta(t_q)\Big) - \sum_{0<t_q<\ell}\psi(\ell,t_q)\mathrm{J}_q\Big(\vartheta(t_q)\Big)\right]ds \\ &\quad + G(\ell) + \sum_{0<t_q<\ell}\xi(\ell,t_q)\mathrm{I}_q\Big(\vartheta(t_q)\Big) + \sum_{0<t_q<\ell}\psi(\ell,t_q)\mathrm{J}_q\Big(\vartheta(t_q)\Big) \\ &= \vartheta_\ell. \end{aligned}$$

Thus, $v(t)$ steers the initial state to the desired state ϑ_ℓ at $t=\ell$. Hence, the system (2) is relatively controllable. □

4. Nonlinear system

Consider the Banach space $M = PC([0,\ell],X) \times C([0,\ell],U_{ad})$ with norm defined by

$\left\|(z,\varphi)\right\|^p = \|z\|_{PC}^p + \left\|\varphi\right\|^p.$

For every $(z,\varphi) \in M$, consider the linear system

$$\begin{cases} \vartheta''(t) = \mathrm{A}(t)\vartheta(t) + \sum\limits_{i=0}^{N} \mathrm{B}_i v\left(\alpha_i(t)\right) + f\left(t, z(t), \sum\limits_{i=0}^{N} \varphi\left(\alpha_i(t)\right)\right) \\ \qquad\qquad +\rho\left(t, z(t), \sum\limits_{i=0}^{N} \varphi\left(\alpha_i(t)\right)\right)\dfrac{dw(t)}{dt}, \quad t \in [0,\ell],\ t \neq t_q, \\ \vartheta(t_q^+) - \vartheta(t_q^-) = \mathrm{I}_q(\vartheta(t_q)), \quad q = 1,2,\ldots,r, \\ \vartheta'(t_q^+) - \vartheta'(t_q^-) = \mathrm{J}_q(\vartheta(t_q)), \quad q = 1,2,\ldots,r, \\ \vartheta(0) = \vartheta_0, \quad \vartheta'(0) = \vartheta_1. \end{cases} \tag{9}$$

The solution of the system (9) can be written as [2]

$$\begin{aligned} \vartheta(t) &= \xi(t,0)\vartheta_0 + \psi(t,0)\vartheta_1 + \int_0^t \psi(t,s)\sum_{i=0}^{N} \mathrm{B}_i v\left(\alpha_i(s)\right) ds + \int_0^t \psi(t,s) f\left(s, z(s), \sum_{i=0}^{N} \varphi\left(\alpha_i(s)\right)\right) ds \\ &\quad + \int_0^t \psi(t,s)\rho\left(s, z(s), \sum_{i=0}^{N} \varphi\left(\alpha_i(s)\right)\right) dw(s) + \sum_{0<t_q<t} \xi(t,t_q)\mathrm{I}_q\Big(\vartheta(t_q)\Big) + \sum_{0<t_q<t} \psi(t,t_q)\mathrm{J}_q\Big(\vartheta(t_q)\Big). \end{aligned} \tag{10}$$

Using the time-lead functions β_i in (10), we have

$$\begin{aligned} \vartheta(t) &= \xi(t,0)\vartheta_0 + \psi(t,0)\vartheta_1 + \sum_{i=0}^{N} \int_{\alpha_i(0)}^{\alpha_i(t)} \psi\left(t, \beta_i(s)\right) \mathrm{B}_i \beta_i^{(1)}(s) v(s) ds + \int_0^t \psi(t,s) f\left(s, z(s), \sum_{i=0}^{N} \varphi\left(\alpha_i(s)\right)\right) ds \\ &\quad + \int_0^t \psi(t,s)\rho\left(s, z(s), \sum_{i=0}^{N} \varphi\left(\alpha_i(s)\right)\right) dw(s) + \sum_{0<t_q<t} \xi(t,t_q)\mathrm{I}_q\Big(\vartheta(t_q)\Big) + \sum_{0<t_q<t} \psi(t,t_q)\mathrm{J}_q\Big(\vartheta(t_q)\Big). \end{aligned} \tag{11}$$

Substituting $t = \ell$ and using (3) in (11), we have

$$\begin{aligned} \vartheta(\ell) &= \xi(\ell,0)\vartheta_0 + \psi(\ell,0)\vartheta_1 + \sum_{i=0}^{n} \int_{\alpha_i(0)}^{0} \psi\left(\ell, \beta_i(s)\right) \mathrm{B}_i \beta_i^{(1)}(s) v_0(s) ds + \sum_{i=0}^{n} \int_0^{\ell} \psi\left(\ell, \beta_i(s)\right) \mathrm{B}_i \beta_i^{(1)}(s) v(s) ds \\ &\quad + \sum_{i=n+1}^{N} \int_{\alpha_i(0)}^{\alpha_i(\ell)} \psi\left(\ell, \beta_i(s)\right) \mathrm{B}_i \beta_i^{(1)}(s) v_0(s) ds + \int_0^{\ell} \psi(\ell,s) f\left(s, z(s), \sum_{i=0}^{N} \varphi\left(\alpha_i(s)\right)\right) ds \\ &\quad + \int_0^{\ell} \psi(\ell,s)\rho\left(s, z(s), \sum_{i=0}^{N} \varphi\left(\alpha_i(s)\right)\right) dw(s) + \sum_{0<t_q<\ell} \xi(\ell,t_q)\mathrm{I}_q\Big(\vartheta(t_q)\Big) + \sum_{0<t_q<\ell} \psi(\ell,t_q)\mathrm{J}_q\Big(\vartheta(t_q)\Big). \end{aligned} \tag{12}$$

Using (7), we define

$$\begin{aligned} \phi\Big(y(0), \vartheta_\ell; z, \varphi\Big) &= E\vartheta_\ell - \xi(\ell,0)\vartheta_0 - \psi(\ell,0)\vartheta_1 + \int_0^{\ell} \varrho(s) dw(s) - G(\ell) - \int_0^{\ell} \psi(\ell,s) f\left(s, z(s), \sum_{i=0}^{N} \varphi\left(\alpha_i(s)\right)\right) ds \\ &\quad - \int_0^{\ell} \psi(\ell,s)\rho\left(s, z(s), \sum_{i=0}^{N} \varphi\left(\alpha_i(s)\right)\right) dw(s) - \sum_{0<t_q<\ell} \xi(\ell,t_q)\mathrm{I}_q\Big(\vartheta(t_q)\Big) \\ &\quad - \sum_{0<t_q<\ell} \psi(\ell,t_q)\mathrm{J}_q\Big(\vartheta(t_q)\Big). \end{aligned} \tag{13}$$

Define the controllability Grammian operator

$$W(0,\ell)\{\cdot\} = \sum_{i=0}^{n}\int_0^{\ell}\psi(\ell,\beta_i(s))\,\mathrm{B}_i\beta_i^{(1)}(s)\left\{\psi(\ell,\beta_i(s))\,\mathrm{B}_i\beta_i^{(1)}(s)\right\}^* E\{\cdot|\mathcal{F}_t\}ds, \tag{14}$$

and control function

$$v(t) = \left\{\mathrm{B}_i^*\psi^*(\ell,\beta_i(t))\,\beta_i^{(1)}(t)\right\}W^{-1}\phi\left(y(0),\vartheta_\ell;z,\varphi\right), \tag{15}$$

where $i=0,1,2,\dots,n$. Here $y(0)$ and ϑ_ℓ are taken arbitrarily. Substituting the value of $v(t)$ in (12) and using (13), $v(t)$ leads the initial complete state $y(0)$ to final state ϑ_ℓ at $t=\ell$ for every $(z,\varphi)\in M$. Also, using (13) and (14), (12) can be written as

$$\begin{aligned}\vartheta(t) = {} & \xi(t,0)\vartheta_0+\psi(t,0)\vartheta_1+\sum_{i=0}^{n}\int_{\alpha_i(0)}^{0}\psi(t,\beta_i(s))\,\mathrm{B}_i\beta_i^{(1)}(s)v_0(s)ds+\sum_{i=0}^{n}\int_0^{t}\psi(t,\beta_i(s))\,\mathrm{B}_i\beta_i^{(1)}(s)\times\\ & \left\{\mathrm{B}_i^*\psi^*(\ell,\beta_i(s))\,\beta_i^{(1)}(s)\right\}W^{-1}\phi\left(y(0),\vartheta_\ell;z,\varphi\right)ds+\sum_{i=n+1}^{N}\int_{\alpha_i(0)}^{\alpha_i(t)}\psi(t,\beta_i(s))\,\mathrm{B}_i\beta_i^{(1)}(s)v_0(s)ds\\ & +\int_0^t\psi(t,s)f\left(s,z(s),\sum_{i=0}^{N}\varphi(\alpha_i(s))\right)ds+\int_0^t\psi(t,s)\rho\left(s,z(s),\sum_{i=0}^{N}\varphi(\alpha_i(s))\right)dw(s)+\sum_{0<t_q<t}\xi(t,t_q)\mathrm{I}_q\big(\vartheta(t_q)\big)\\ & +\sum_{0<t_q<t}\psi(t,t_q)\mathrm{J}_q\big(\vartheta(t_q)\big).\end{aligned} \tag{16}$$

Theorem 4.1. *Assume that for* $(\vartheta,\tilde{v})\in M$, *where* $\tilde{v}=\sum_{i=0}^{N}v(\alpha_i(t))$,

$$\lim_{\|(\vartheta,\tilde{v})\|^p\to\infty}\frac{\left\|f(t,\vartheta,\tilde{v})\right\|^p+\left\|\rho(t,\vartheta,\tilde{v})\right\|^p_{L_0^2}}{\|(\vartheta,\tilde{v})\|^p}=0,$$

uniformly on $[0,\ell]$ *and (H1), (H4) hold. Then, the nonlinear system (9) is globally relatively controllable on* $[0,\ell]$, *if the linear system (2) is globally relatively controllable.*

Proof. Let the map $\mathcal{G}:H\subseteq M\to H$ is given by

$$\mathcal{G}(z,\tilde{\varphi})=(\vartheta,\tilde{v}),$$

where $\tilde{\varphi}=\sum_{i=0}^{N}\varphi(\alpha_i(t))$,

$$\begin{aligned}v(t) = {} & \left\{\mathrm{B}_i^*\psi^*(\ell,\beta_i(t))\,\beta_i^{(1)}(t)\right\}W^{-1}\Bigg[E\vartheta_\ell-\xi(\ell,0)\vartheta_0-\psi(\ell,0)\vartheta_1+\int_0^{\ell}\varrho(s)dw(s)-\sum_{i=0}^{n}\int_{\alpha_i(0)}^{0}\psi(\ell,\beta_i(s))\,\mathrm{B}_i\beta_i^{(1)}(s)\times\\ & v_0(s)ds-\sum_{i=n+1}^{N}\int_{\alpha_i(0)}^{\alpha_i(\ell)}\psi(\ell,\beta_i(s))\,\mathrm{B}_i\beta_i^{(1)}(s)v_0(s)ds-\int_0^{\ell}\psi(\ell,s)f\left(s,z(s),\sum_{i=0}^{N}\varphi(\alpha_i(s))\right)ds\\ & -\int_0^{\ell}\psi(\ell,s)\rho\left(s,z(s),\sum_{i=0}^{N}\varphi(\alpha_i(s))\right)dw(s)-\sum_{0<t_q<\ell}\xi(\ell,t_q)\mathrm{I}_q\big(\vartheta(t_q)\big)-\sum_{0<t_q<\ell}\psi(\ell,t_q)\mathrm{J}_q\big(\vartheta(t_q)\big)\Bigg],\end{aligned}$$

and

$$\begin{aligned}\vartheta(t) &= \xi(t,0)\vartheta_0 + \psi(t,0)\vartheta_1 + \sum_{i=0}^{n}\int_{\alpha_i(0)}^{0}\psi\left(t,\beta_i(s)\right)\mathrm{B}_i\beta_i^{(1)}(s)v_0(s)ds + \sum_{i=0}^{n}\int_0^t \psi\left(t,\beta_i(s)\right)\mathrm{B}_i\beta_i^{(1)}(s)\times \\ &\quad \left\{\mathrm{B}_i^*\psi^*\left(\ell,\beta_i(s)\right)\beta_i^{(1)}(s)\right\}W^{-1}\phi\left(y(0),\vartheta_\ell : z,\varphi\right)ds + \sum_{i=n+1}^{N}\int_{\alpha_i(0)}^{\alpha_i(t)}\psi\left(t,\beta_i(s)\right)\mathrm{B}_i\beta_i^{(1)}(s)v_0(s)ds \\ &\quad + \int_0^t \psi(t,s)f\left(s,z(s),\sum_{i=0}^{N}\varphi\left(\alpha_i(s)\right)\right)ds + \int_0^t \psi(t,s)\rho\left(s,z(s),\sum_{i=0}^{N}\varphi\left(\alpha_i(s)\right)\right)dw(s) + \sum_{0<t_q<t}\xi(t,t_q)\mathrm{I}_q\Big(\vartheta(t_q)\Big) \\ &\quad + \sum_{0<t_q<t}\psi(t,t_q)\mathrm{J}_q\Big(\vartheta(t_q)\Big). \qquad (17)\end{aligned}$$

For simplicity, take

$$\begin{aligned}& b_i = \sup\left\|\beta_i^{(1)}(s)\right\|,\ i=0,1,2,\dots,N; \quad \tilde{a} = \sup\|v_0(s)\|; \quad c' = 10^{p-1}K^p l_1^p b_i^p\left\|W^{-1}\right\|^p; \lambda = \int_0^\ell E\left\|\varrho(s)\right\|^p ds; \\ & L_i = \int_{\alpha_i(0)}^{0}\left\|\psi\left(\ell,\beta_i(s)\right)\right\|ds; \quad M_i = \int_{\alpha_i(0)}^{\alpha_i(\ell)}\left\|\psi\left(\ell,\beta_i(s)\right)\right\|ds; \quad N_i = \int_0^\ell\left\|\psi\left(\ell,\beta_i(s)\right)\right\|ds; \\ & \tilde{f} = \sup E\left\|f\left(s,z(s),\sum_{i=0}^{N}\varphi\left(\alpha_i(s)\right)\right)\right\|; \quad \tilde{\rho} = \sup E\left\|\rho\left(s,z(s),\sum_{i=0}^{N}\varphi\left(\alpha_i(s)\right)\right)\right\|; \quad \tilde{b} = \max\left\{c' l_1^p \ell^{1+p/q}, c' l_1^p \ell L_\rho\right\}; \\ & c = c'\Bigg[E\left\|E\vartheta_\ell\right\|^p + l_2^p E\left\|\vartheta_0\right\|^p + l_1^p E\left\|\vartheta_1\right\|^p + L_\varrho\lambda + (-\alpha_i(0))^{p/q}\tilde{a}^p K^p\sum_{i=0}^{n}L_i^p b_i^p + (\alpha_i(\ell)-\alpha_i(0))^{p/q}\tilde{a}^p K^p\sum_{i=n+1}^{N}M_i^p b_i^p \\ & + \sum_{k=1}^{r}\left\{l_2^p\tilde{v}_k^p + l_1^p\tilde{\varsigma}_k^p\right\}\Bigg]; \quad \gamma_1 = 9^{p-1}\ell^{p/q}\left[K^p\sum_{i=0}^{n}N_i^p b_i^p\tilde{b} + l_1^p\ell\right]; \quad \gamma_2 = 9^{p-1}\left[\ell^{p/q}K^p\sum_{i=0}^{n}N_i^p b_i^p\tilde{b} + L_\rho l_1^p\ell\right]; \\ & d = 9^{p-1}\Bigg[l_2^p E\|\vartheta_0\|^p + l_1^p E\|\vartheta_1\|^p + (-\alpha_i(0))^{p/q}\tilde{a}^p K^p\sum_{i=0}^{n}L_i^p b_i^p + \ell^{p/q}K^p\sum_{i=0}^{n}N_i^p b_i^p c + (\alpha_i(\ell)-\alpha_i(0))^{p/q}\tilde{a}^p K^p\times \\ & \sum_{i=n+1}^{N}M_i^p b_i^p + \sum_{q=1}^{r}\{l_2^p\tilde{v}_q^p + l_1^p\tilde{\varsigma}_q^p\}\Bigg]; \quad \gamma = \max\{\gamma_1,\gamma_2\}; \quad \delta_1 = \max\{c,d\}; \quad \delta_2 = \max\left\{\tilde{b},\gamma\right\}.\end{aligned}$$

Using Lemma 2.2 and Hölder's inequality, we have,

$$\begin{aligned}E\|v(t)\|^p &\le 10^{p-1}E\left\|\left\{\mathrm{B}_i^*\psi^*\left(\ell,\beta_i(t)\right)\beta_i^{(1)}(t)\right\}W^{-1}\right\|^p\Bigg[\|E\vartheta_\ell\|^p + E\|\xi(\ell,0)\vartheta_0\|^p + E\left\|\psi(\ell,0)\vartheta_1\right\|^p + L_\varrho\int_0^\ell E\left\|\varrho(s)\right\|^p ds \\ &\quad + \sum_{i=0}^{n}E\left\|\int_{\alpha_i(0)}^{0}\psi\left(\ell,\beta_i(s)\right)\mathrm{B}_i\beta_i^{(1)}(s)v_0(s)ds\right\|^p + \sum_{i=n+1}^{N}E\left\|\int_{\alpha_i(0)}^{\alpha_i(\ell)}\psi\left(\ell,\beta_i(s)\right)\mathrm{B}_i\beta_i^{(1)}(s)v_0(s)ds\right\|^p \\ &\quad + E\left\|\int_0^\ell\psi(\ell,s)f\left(s,z(s),\sum_{i=0}^{N}\varphi\left(\alpha_i(s)\right)\right)ds\right\|^p + E\left\|\int_0^\ell\psi(\ell,s)\rho\left(s,z(s),\sum_{i=0}^{N}\varphi\left(\alpha_i(s)\right)\right)dw(s)\right\|^p \\ &\quad + E\left\|\sum_{0<t_q<\ell}\xi(\ell,t_q)\mathrm{I}_q\Big(\vartheta(t_q)\Big)\right\|^p + E\left\|\sum_{0<t_q<\ell}\psi(\ell,t_q)\mathrm{J}_q\Big(\vartheta(t_q)\Big)\right\|^p\Bigg].\end{aligned}$$

Using the assumptions (H1), (H4), we get

$$\begin{aligned} E\|v(t)\|^p &\le c'\Bigg[\|E\vartheta_\ell\|^p + l_2^p E\|\vartheta_0\|^p + l_1^p E\|\vartheta_1\|^p + L_\varrho\lambda + (-\alpha_i(0))^{p/q}\tilde{a}^p K^p \sum_{i=0}^{n} L_i^p b_i^p + (\alpha_i(\ell)-\alpha_i(0))^{p/q}\tilde{a}^p K^p \times \\ &\quad \sum_{i=n+1}^{N} M_i^p b_i^p + l_1^p \tilde{f}^p \ell^{1+p/q} + L_\rho l_1^p \tilde{\rho}^p \ell + \sum_{q=1}^{r}\left\{l_2^p \tilde{v}_q^p + l_1^p \tilde{\varsigma}_q^p\right\}\Bigg] \\ &\le c + \tilde{b}\left(\tilde{f}^p + \tilde{\rho}^p\right) \\ &\le \delta_1 + \delta_2\left(\tilde{f}^p + \tilde{\rho}^p\right) \\ \Rightarrow E\|\tilde{v}(t)\|^p &\le (N+1)\left[\delta_1 + \delta_2\left(\tilde{f}^p + \tilde{\rho}^p\right)\right], \end{aligned}$$

and

$$\begin{aligned} E\|\vartheta(t)\|^p &\le 9^{p-1}\Bigg[l_2^p E\|\vartheta_0\|^p + l_1^p E\|\vartheta_1\|^p + (-\alpha_i(0))^{p/q}\tilde{a}^p K^p \sum_{i=0}^{n} L_i^p b_i^p + \ell^{p/q} K^p \sum_{i=0}^{n} N_i^p b_i^p \left\{c + \tilde{b}\left(\tilde{f}^p + \tilde{\rho}^p\right)\right\} \\ &\quad + (\alpha_i(\ell)-\alpha_i(0))^{p/q}\tilde{a}^p K^p \sum_{i=n+1}^{N} M_i^p b_i^p + \ell_1^p \ell^{1+p/q}\tilde{f}^p + L_\rho l_1^p \ell\tilde{\rho}^p + \sum_{q=1}^{r}\left\{l_2^p \tilde{v}_q^p + l_1^p \tilde{\varsigma}_q^p\right\}\Bigg] \\ &\le d + \gamma\left(\tilde{f}^p + \tilde{\rho}^p\right) \\ &\le \delta_1 + \delta_2\left(\tilde{f}^p + \tilde{\rho}^p\right). \end{aligned}$$

Since the function f and ρ satisfy [12, Proposition 1]. Therefore, for each pair of constants δ_1 and δ_2, there exists $\varepsilon > 0$ such that, if $E\|z(t)\|^p \le \frac{\varepsilon}{2}$ and $E\left\|\tilde{\varphi}(t)\right\|^p \le \frac{\varepsilon}{2}$, i.e., $\left\|(z,\tilde{\varphi})\right\|^p \le \varepsilon$, then $\delta_1 + \delta_2\left(\tilde{f}^p + \tilde{\rho}^p\right) \le \varepsilon$. Therefore, $E\|\tilde{v}(t)\|^p \le (N+1)\varepsilon$, and $E\|\vartheta(t)\|^p \le \varepsilon$. Thus, we have proved that, if $H(\varepsilon') = \left\{(z,\tilde{\varphi}) \in H : \|z\|^p \le \frac{\varepsilon}{2} \text{ and } \left\|\tilde{\varphi}\right\|^p \le \frac{\varepsilon}{2}\right\}$, where $\varepsilon' = \max\{(N+2)\varepsilon, \varepsilon\}$, then $\mathcal{G}$ maps $H(\varepsilon')$ into itself. Since f, ρ are continuous, therefore $\mathcal{G}$ is continuous. The complete continuity of $\mathcal{G}$ is followed by Arzela-Ascoli theorem. As $H(\varepsilon')$ is closed, bounded and convex set, therefore by Schauder's fixed point theorem, $\mathcal{G}$ has a fixed point $(z,\tilde{\varphi}) \in H(\varepsilon')$, i.e., $\mathcal{G}(z,\tilde{\varphi}) = (z,\tilde{\varphi}) \equiv (\vartheta,\tilde{v})$. Hence, we have

$$\begin{aligned} \vartheta(t) &= \xi(t,0)\vartheta_0 + \psi(t,0)\vartheta_1 + \sum_{i=0}^{n}\int_{\alpha_i(0)}^{0}\psi(t,\beta_i(s))\,\mathrm{B}_i\beta_i^{(1)}(s)v_0(s)ds + \sum_{i=0}^{n}\int_0^t \psi(t,\beta_i(s))\,\mathrm{B}_i\beta_i^{(1)}(s)v(s)ds \\ &\quad + \sum_{i=n+1}^{N}\int_{\alpha_i(0)}^{\alpha_i(t)}\psi(t,\beta_i(s))\,\mathrm{B}_i\beta_i^{(1)}(s)v_0(s)ds + \int_0^t \psi(t,s) f\left(s, z(s), \sum_{i=0}^{N}\varphi(\alpha_i(s))\right)ds \\ &\quad + \int_0^t \psi(t,s)\rho\left(s, z(s), \sum_{i=0}^{N}\varphi(\alpha_i(s))\right)dw(s) + \sum_{0<t_q<t}\xi(t,t_q)\mathrm{I}_q\big(\vartheta(t_q)\big) + \sum_{0<t_q<t}\psi(t,t_q)\mathrm{J}_q\big(\vartheta(t_q)\big). \end{aligned} \tag{18}$$

Thus, $\vartheta(t)$ is the solution of the nonlinear system with the control $v(t)$ satisfying $\vartheta(\ell) = \vartheta_\ell$ which steers the stochastic system (9) from its initial state $y(0)$ to desired state ϑ_ℓ on $[0,\ell]$. Hence the result follows. □

5. Optimal Controllability

Consider the Lagrange problem to obtain an optimal state-control pair $(\vartheta^0, v^0) \in PC \times U_{ad}$ satisfying [22]

$$\mathcal{I}\left(\vartheta^0, v^0\right) \le \mathcal{I}(\vartheta, v), \text{ for all } (\vartheta, v) \in PC \times U_{ad},$$

where ϑ denotes the mild solution of the stochastic system (1) corresponding to the control $v \in U_{ad}$ and

$$\mathcal{I}(\vartheta, v) = E\left\{\int_0^\ell \tilde{\mathcal{K}}\left(t, \vartheta(t), \sum_{i=0}^{N} v\left(\alpha_i(t)\right)\right) dt\right\}. \tag{19}$$

In order to discuss (19), we assume the following conditions

(H5) The Borel measurable function $\tilde{\mathcal{K}} : [0, \ell] \times X \times Z \to \mathbb{R} \cup \{\infty\}$ satisfies

(a) For almost all $t \in [0, \ell]$, $\tilde{\mathcal{K}}(t, \vartheta, \cdot)$ is convex on Z for each $\vartheta \in X$.

(b) For almost all $t \in [0, \ell]$, $\tilde{\mathcal{K}}(t, \cdot, \cdot)$ is sequentially lower semicontinuous on $X \times Z$.

(c) There exist constants $e_1 \geq 0,\ e_2 > 0$ and Φ is a non-negative function in $\mathcal{L}^1([0, \ell], \mathbb{R})$ such that

$$\tilde{\mathcal{K}}\left(t, \vartheta(t), \sum_{i=0}^{N} v\left(\alpha_i(t)\right)\right) \geq \Phi(t) + e_1 \|\vartheta\|^p + e_2 \|v\|^p,$$

where $\|v\|^p = \sum_{i=0}^{N} \|v(\alpha_i(t))\|^p$.

Theorem 5.1. *Let the assumptions (H1)-(H5) hold. If all the hypotheses of Theorem 4.1 are satisfied, then there exists an optimal state control pair of (1) provided that*

$$5^{p-1}\left[l_1^p (N+1) K^p \ell^{1+p/q} + l_1^p \ell^{1+p/q} \mu_1 + L_\rho l_1^p \ell \mu_3 + l_2^p \sum_{q=1}^{r} \nu_q^p + l_1^p \sum_{q=1}^{r} \varsigma_q^p\right] < 1.$$

Proof. It is enough to show that there exists at least one optimal pair $\left(\vartheta^0, v^0\right) \in PC \times U_{ad}$ which minimize $\mathcal{I}(\vartheta, v)$.
If $\inf\{\mathcal{I}(\vartheta, v) : (\vartheta, v) \in PC \times U_{ad}\} = \infty$, then result holds trivially. If $\inf\{\mathcal{I}(\vartheta, v) : (\vartheta, v) \in PC \times U_{ad}\} = \epsilon_0 < \infty$, then there exists a minimizing sequence $\{(\vartheta^n, v^n)\}$ such that $\mathcal{I}(\vartheta^n, v^n) \to \epsilon_0$ as $n \to \infty$. Since U_{ad} is a convex and closed, therefore, sequence $\{v^n\}$ has a weakly convergent subsequence $\{v^m\}$ such that $v^m \to v^0 \in U_{ad}$ by Marzur lemma. Using Theorem 4.1, for each $v^m \in U_{ad}$, there exists a mild solution ϑ^m of (1),

$$\begin{aligned}\vartheta^m(t) &= \xi(t,0)\vartheta_0 + \psi(t,0)\vartheta_1 + \int_0^t \psi(t,s) \sum_{i=0}^{N} \mathrm{B}_i v^m\left(\alpha_i(s)\right) ds + \int_0^t \psi(t,s) f\left(s, \vartheta^m(s), \sum_{i=0}^{N} v^m\left(\alpha_i(s)\right)\right) ds \\ &\quad + \int_0^t \psi(t,s)\rho\left(s, \vartheta^m(s), \sum_{i=0}^{N} v^m\left(\alpha_i(s)\right)\right) dw(s) + \sum_{0<t_q<t} \xi(t,t_q)\mathrm{I}_q\left(\vartheta^m(t_q)\right) + \sum_{0<t_q<t} \psi(t,t_q)\mathrm{J}_q\left(\vartheta^m(t_q)\right).\end{aligned}$$

Similarly, corresponding to v^0, we have

$$\begin{aligned}\vartheta^0(t) &= \xi(t,0)\vartheta_0 + \psi(t,0)\vartheta_1 + \int_0^t \psi(t,s) \sum_{i=0}^{N} \mathrm{B}_i v^0\left(\alpha_i(s)\right) ds + \int_0^t \psi(t,s) f\left(s, \vartheta^0(s), \sum_{i=0}^{N} v^0\left(\alpha_i(s)\right)\right) ds \\ &\quad + \int_0^t \psi(t,s)\rho\left(s, \vartheta^0(s), \sum_{i=0}^{N} v^0\left(\alpha_i(s)\right)\right) dw(s) + \sum_{0<t_q<t} \xi(t,t_q)\mathrm{I}_q\left(\vartheta^0(t_q)\right) + \sum_{0<t_q<t} \psi(t,t_q)\mathrm{J}_q\left(\vartheta^0(t_q)\right).\end{aligned}$$

Using Lemma 2.2 and Hölder's inequality with the assumptions, we have

$$\begin{aligned}
E\left\|\vartheta^m(t)-\vartheta^0(t)\right\|^p &\leq 5^{p-1}E\left\|\int_0^t \psi(t,s)\sum_{i=0}^{N}\left\{\mathrm{B}_i\vartheta^m(s)-\mathrm{B}_i\vartheta^0(s)\right\}ds\right\|^p \\
&+5^{p-1}E\left\|\int_0^t \psi(t,s)\left\{f\left(s,\vartheta^m(s),\sum_{i=0}^{N}\upsilon^m(\alpha_i(s))\right)-f\left(s,\vartheta^0(s),\sum_{i=0}^{N}\upsilon^0(\alpha_i(s))\right)\right\}ds\right\|^p \\
&+5^{p-1}E\left\|\int_0^t \psi(t,s)\left\{\rho\left(s,\vartheta^m(s),\sum_{i=0}^{N}\upsilon^m(\alpha_i(s))\right)-\rho\left(s,\vartheta^0(s),\sum_{i=0}^{N}\upsilon^0(\alpha_i(s))\right)\right\}dw(s)\right\|^p \\
&+5^{p-1}E\left\|\sum_{0<t_q<t}\xi(t,t_q)\mathrm{I}_q\big(\vartheta^m(t_q)\big)-\sum_{0<t_q<t}\xi(t,t_q)\mathrm{I}_q\big(\vartheta^0(t_q)\big)\right\|^p \\
&+5^{p-1}E\left\|\sum_{0<t_q<t}\psi(t,t_q)\mathrm{J}_q\big(\vartheta^m(t_q)\big)-\sum_{0<t_q<t}\psi(t,t_q)\mathrm{J}_q\big(\vartheta^0(t_q)\big)\right\|^p \\
&\leq 5^{p-1}l_1^p(N+1)K^p\ell^{1+p/q}\left\|\vartheta^m-\vartheta^0\right\|_{PC}^p+5^{p-1}l_1^p\ell^{1+p/q}\left\{\mu_1\left\|\vartheta^m-\vartheta^0\right\|_{PC}^p+\mu_2\left\|\upsilon^m-\upsilon^0\right\|_{L_{\mathcal{F}}^p}^p\right\} \\
&\quad 5^{p-1}L_\rho l_1^p\ell\left\{\mu_3\left\|\vartheta^m-\vartheta^0\right\|_{PC}^p+\mu_4\left\|\upsilon^m-\upsilon^0\right\|_{L_{\mathcal{F}}^p}^p\right\}+5^{p-1}l_2^p\sum_{q=1}^{r}\nu_q^p\left\|\vartheta^m-\vartheta^0\right\|_{PC}^p \\
&+5^{p-1}l_1^p\sum_{q=1}^{r}\varsigma_q^p\left\|\vartheta^m-\vartheta^0\right\|_{PC}^p \\
&\leq 5^{p-1}\left[l_1^p(N+1)K^p\ell^{1+p/q}+l_1^p\ell^{1+p/q}\mu_1+L_\rho l_1^p\ell\mu_3+l_2^p\sum_{q=1}^{r}\nu_q^p+l_1^p\sum_{q=1}^{r}\varsigma_q^p\right]\left\|\vartheta^m-\vartheta^0\right\|_{PC}^p \\
&+5^{p-1}\left[l_1^p\ell^{1+p/q}\mu_2+L_\rho l_1^p\ell\mu_4\right]\left\|\upsilon^m-\upsilon^0\right\|_{L_{\mathcal{F}}^p}^p.
\end{aligned}$$

Since $5^{p-1}\left[l_1^p(N+1)K^p\ell^{1+p/q}+l_1^p\ell^{1+p/q}\mu_1+L_\rho l_1^p\ell\mu_3+l_2^p\sum_{q=1}^{r}\nu_q^p+l_1^p\sum_{q=1}^{r}\varsigma_q^p\right]<1$, and $\left\|\upsilon^m-\upsilon^0\right\|_{L_{\mathcal{F}}^p}^p\to 0$, we conclude that $\vartheta^m\to\vartheta^0$.

Applying Balder's theorem (see [7, Theorem 2.1]), we obtain

$$\begin{aligned}
\epsilon_0 &= \lim_{m\to\infty}E\left\{\int_0^\ell \tilde{\mathcal{K}}\left(t,\vartheta^m(t),\sum_{i=0}^{N}\upsilon^m(\alpha_i(t))\right)dt\right\} \\
&\geq E\left\{\int_0^\ell \tilde{\mathcal{K}}\left(t,\vartheta^0(t),\sum_{i=0}^{N}\upsilon^0(\alpha_i(t))\right)dt\right\} \\
&= \mathcal{I}\left(\vartheta^0,\upsilon^0\right)\geq\epsilon_0.
\end{aligned}$$

This completes the proof. □

Remark 5.2. *The trivial case for $N=0$, f and ρ are independent of υ has been considered in the paper [30].*

Remark 5.3. *For $N=1$, if we define $\sum_{i=0}^{1}\mathrm{B}_i\upsilon(\alpha_i(t))=\mathrm{B}_1\upsilon(t)+\mathrm{B}_2\upsilon(t-b)$, $f(t,\vartheta(t),\sum_{i=0}^{1}\upsilon(\alpha_i(t)))=F(t,\vartheta_{a(t)},\upsilon(t)+\upsilon(t-b))$, $\rho=0$, and taking the autonomous system without impulse effect, has been considered in the paper [3].*

Remark 5.4. *If we define* $\sum_{i=0}^{N} \mathrm{B}_i \upsilon(\alpha_i(t)) = \sum_{i=0}^{N} \mathrm{B}_i \upsilon(t-\nu_i)$, $f\left(t, \vartheta(t), \sum_{i=0}^{N} \upsilon(\alpha_i(t))\right) = g(t, \vartheta(t), \upsilon(t), \upsilon(t-\nu_1), \cdots, \upsilon(t-\nu_N))$, $\rho\left(t, \vartheta(t), \sum_{i=0}^{N} \upsilon(\alpha_i(t))\right) = \rho(t, \vartheta(t), \upsilon(t), \upsilon(t-\nu_1), \cdots, \upsilon(t-\nu_N))$, *and taking the non-autonomous system without impulse effect, has been discussed in the paper [1].*

6. Applications

In this section, we consider the following examples to illustrate the applicability of the result.

Example 6.1. *Consider the following stochastic impulsive system:*

$$\begin{cases} \vartheta''(t) = \mathrm{A}(t)\vartheta(t) + \mathrm{B}_0\upsilon\big(\alpha_1(t)\big) + \mathrm{B}_1\upsilon\big(\alpha_1(t)\big) + \mathrm{B}_2\upsilon\big(\alpha_2(t)\big) + f\left(t, \vartheta(t), \sum_{i=0}^{2} \upsilon(\alpha_i(t))\right) \\ \qquad\qquad + \rho\left(t, \vartheta(t), \sum_{i=0}^{2} \upsilon(\alpha_i(t))\right) \dfrac{dw(t)}{dt}, \ t \in [0,3], \ t \neq t_q, \\ \vartheta(t_q^+) - \vartheta(t_q^-) = \mathrm{I}_q\big(\vartheta(t_q)\big), \quad q = 1, 2, \dots, 10, \\ \vartheta'(t_q^+) - \vartheta'(t_q^-) = \mathrm{J}_q\big(\vartheta(t_q)\big), \quad q = 1, 2, \dots, 10, \\ \vartheta(0) = \vartheta_0, \quad \vartheta'(0) = \vartheta_1, \end{cases} \tag{20}$$

where

$$\vartheta(t) = \begin{bmatrix} \vartheta_1(t) \\ \vartheta_2(t) \end{bmatrix}, \ \upsilon(t) = \begin{bmatrix} \upsilon_1(t) \\ \upsilon_2(t) \end{bmatrix}, \ w(t) = \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}, \ A(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ B_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ B_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \ f\left(t, \vartheta(t), \sum_{i=0}^{2} \upsilon(\alpha_i(t))\right) = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \ \rho\left(t, \vartheta(t), \sum_{i=0}^{2} \upsilon(\alpha_i(t))\right) = \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix},$$

$$\mathrm{I}_q(\vartheta(t_q)) = \begin{bmatrix} \mathrm{I}_{q1} & 0 \\ 0 & \mathrm{I}_{q2} \end{bmatrix}, \ \mathrm{J}_q(\vartheta(t_q)) = \begin{bmatrix} \mathrm{J}_{q1} & 0 \\ 0 & \mathrm{J}_{q2} \end{bmatrix},$$

where

$$f_1 = \frac{\upsilon_1(t) + \upsilon_1\left(\frac{2t}{3}\right) + \upsilon_1\left(\frac{t}{3}\right)}{1 + \vartheta_1^2(t) + \vartheta_2^2(t)}, \ f_2 = \frac{1}{1 + \vartheta_1^2(t) + \vartheta_2^2(t)}, \ \rho_1 = \frac{\vartheta_1(t)\cos(\vartheta_2(t))}{1 + \vartheta_1^2(t) + \vartheta_2^2(t)}, \ \rho_2 = \frac{\vartheta_2(t)\cos(\vartheta_1(t))}{1 + \vartheta_1^2(t) + \vartheta_2^2(t)},$$

$$\mathrm{I}_{q1} = \vartheta_1(t_q), \ \mathrm{I}_{q2} = \frac{\vartheta_2(t_q)}{2}, \ \mathrm{J}_{q1} = \vartheta_1(t_q), \ \mathrm{J}_{q2} = \sin(\vartheta_2(t_q)).$$

The space $X = Y = Z = \mathbb{R}^2$, with the Euclidean norm $\|\cdot\|$. If P is a matrix, its trace norm is $\|\cdot\| := \sqrt{tr(P^T P)}$, P^T is the transpose of P. $w(t)$ represents a two dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$, with the norm $\|\phi\| = \sup_{0\le t\le 3} E\left(\phi(t)\right) < \infty$, where ϕ is bounded, $\mathcal{F}_0$ measurable, $C\left([0,3], \mathbb{R}^2\right)$-valued random variables.
Moreover, for $t \in [0,3]$

$$\alpha_0(t) = t, \ \alpha_1(t) = \frac{2t}{3}, \ \alpha_2(t) = \frac{t}{3}.$$

Consider the following time lead functions:

$$\beta_0(t) = t, \ \beta_1(t) = \frac{3t}{2}, \ \beta_2(t) = 3t.$$

For $t = 3$, we have

$$\alpha_2(3) < \alpha_1(3) < \alpha_0(3) = 3, \quad t \in [0,3].$$

Here, we take

$$\psi(t,s) = \begin{bmatrix} \sin(t-s) & 0 \\ 0 & \sin(t-s) \end{bmatrix}, \ \xi(t,s) = \begin{bmatrix} \cos(t-s) & 0 \\ 0 & \sin(t-s) \end{bmatrix}.$$

We calculate the controllability Grammian matrix defined in (8) as

$$\begin{aligned} W(0,3) &= \sum_{i=0}^{2} \int_0^3 \psi\left(3, \beta_i(s)\right) \mathrm{B}_i \beta_i^{(1)}(s) \left\{\psi\left(3, \beta_i(s)\right) \mathrm{B}_i \beta_i^{(1)}(s)\right\}^* ds \\ &= W_0(0,3) + W_1(0,3) + W_2(0,3). \end{aligned} \tag{21}$$

Here,

$$\begin{aligned} W_0(0,3) &= \int_0^3 \psi\left(3, \beta_0(s)\right) \mathrm{B}_0 \beta_0^{(1)}(s) \left\{\psi\left(3, \beta_0(s)\right) \mathrm{B}_0 \beta_0^{(1)}(s)\right\}^* ds \\ &= \int_0^3 \left\{\begin{bmatrix} \sin(3-s) & 0 \\ 0 & \sin(3-s) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right\} \left\{\begin{bmatrix} \sin(3-s) & 0 \\ 0 & \sin(3-s) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right\}^* ds \\ &= \int_0^3 \begin{bmatrix} \sin^2(3-s) & 0 \\ 0 & 0 \end{bmatrix} ds \\ &= \begin{bmatrix} 1.4739 & 0 \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} W_1(0,3) &= \int_0^3 \psi\left(3, \beta_1(s)\right) \mathrm{B}_1 \beta_1^{(1)}(s) \left\{\psi\left(3, \beta_1(s)\right) \mathrm{B}_1 \beta_1^{(1)}(s)\right\}^* ds \\ &= \int_0^3 \left\{\frac{3}{2}\begin{bmatrix} \sin\left(3-\frac{3s}{2}\right) & 0 \\ 0 & \sin\left(3-\frac{3s}{2}\right) \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right\} \left\{\frac{3}{2}\begin{bmatrix} \sin\left(3-\frac{3s}{2}\right) & 0 \\ 0 & \sin\left(3-\frac{3s}{2}\right) \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right\}^* ds \\ &= \frac{9}{4}\int_0^3 \begin{bmatrix} 2\sin^2\left(3-\frac{3s}{2}\right) & 0 \\ 0 & 0 \end{bmatrix} ds \\ &= \begin{bmatrix} 6.6324 & 0 \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

and

$$\begin{aligned} W_2(0,3) &= \int_0^3 \psi\left(3, \beta_2(s)\right) \mathrm{B}_2 \beta_2^{(1)}(s) \left\{\psi\left(3, \beta_2(s)\right) \mathrm{B}_2 \beta_2^{(1)}(s)\right\}^* ds \\ &= \int_0^3 \left\{3\begin{bmatrix} \sin(3-3s) & 0 \\ 0 & \sin(3-3s) \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right\} \left\{3\begin{bmatrix} \sin(3-3s) & 0 \\ 0 & \sin(3-3s) \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right\}^* ds \\ &= 9\int_0^3 \begin{bmatrix} 2\sin^2(3-3s) & \sin^2(3-3s) \\ \sin^2(3-3s) & \sin^2(3-3s) \end{bmatrix} ds \\ &= \begin{bmatrix} 26.5314 & 13.2657 \\ 13.2657 & 13.2657 \end{bmatrix}. \end{aligned}$$

Substituting the values of $W_0(0,3)$, $W_1(0,3)$ and $W_2(0,3)$ in (21), we have

$$W(0,3) = \begin{bmatrix} 34.6377 & 13.2657 \\ 13.2657 & 13.2657 \end{bmatrix},$$

which is invertible. Therefore, the deterministic system corresponding to (20)

$$\begin{cases} \vartheta''(t) = \mathrm{A}(t)\vartheta(t) + \mathrm{B}_0 \upsilon\Big(\alpha_1(t)\Big) + \mathrm{B}_1 \upsilon\Big(\alpha_1(t)\Big) + \mathrm{B}_2 \upsilon\Big(\alpha_2(t)\Big), \quad t \in [0,3],\ t \neq t_q, \\ \vartheta(t_q^+) - \vartheta(t_q^-) = \mathrm{I}_q\Big(\vartheta(t_q)\Big), \quad q = 1,2,\dots,10, \\ \vartheta'(t_q^+) - \vartheta'(t_q^-) = \mathrm{J}_q\Big(\vartheta(t_q)\Big), \quad q = 1,2,\dots,10, \\ \vartheta(0) = \vartheta_0, \quad \vartheta'(0) = \vartheta_1, \end{cases} \tag{22}$$

is relatively controllable.

Since $\psi(t,s)$, $\xi(t,s)$, B_i, $i = 0,1,2$ are bounded. As a result, (H1) is satisfied. Also, the impulse functions I_q and J_q are bounded and satisfy Lipschitz conditions. Thus, (H4) is satisfied. Further, the nonlinear function f and ρ satisfy the hypotheses of Theorem 4.1. Hence, the system (20) is globally relatively controllable.

Example 6.2. *Consider the following system*

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}\varrho(t,y) = \dfrac{\partial^2}{\partial y^2}\varrho(t,y) + b(t)\dfrac{\partial}{\partial t}\varrho(t,y) + \mathrm{B}_0\upsilon(t,y) + \upsilon\left(\frac{t}{2},y\right) + \upsilon\left(\frac{t}{4},y\right) \\ \qquad\qquad + f\left(t,\varrho(t,y),\upsilon(t,y) + \upsilon\left(\frac{t}{2},y\right) + \upsilon\left(\frac{t}{4},y\right)\right) \\ \qquad\qquad + \rho\left(t,\varrho(t,y),\upsilon(t,y) + \upsilon\left(\frac{t}{2},y\right) + \upsilon\left(\frac{t}{4},y\right)\right)\dfrac{dw(t)}{dt}, \ \ y \in [0,\pi], \quad t \in [0,4], \quad t \neq t_q, \\ \varrho(t,0) = \varrho(t,\pi) = 0, \quad t \in [0,4], \\ \varrho(0,y) = y, \quad y \in [0,\pi], \\ \dfrac{\partial}{\partial t}\varrho(0,y) = 1 + y, \quad y \in [0,\pi], \\ \varrho\left(t_q^+,y\right) - \varrho\left(t_q^-,y\right) = \displaystyle\int_0^{t_q} \sin\left(t_q - s\right)\varrho(s,y)ds, \quad q = 1,2,\dots,5, \\ \dfrac{\partial}{\partial t}\varrho\left(t_q^+,y\right) - \dfrac{\partial}{\partial t}\varrho\left(t_q^-,y\right) = \displaystyle\int_0^{t_q} \cos\left(t_q - s\right)\varrho(s,y)ds, \quad q = 1,2,\dots,5, \end{cases} \tag{23}$$

where $b : [0,4] \to \mathbb{R}$ be such that $\sup_{t\in[0,4]} \|b(t)\| = b_0$ and $0 = t_0 < t_1 < \cdots < t_4 < t_5 = 4$. $\upsilon(t)$ is an admissible control function and w be a standard Brownian motion. f and ρ are continuous functions that satisfy the assumption of Theorem 4.1, respectively. Set $\varrho(t)y = \varrho(t,y)$ and $\upsilon(t)(y) = \upsilon(t,y)$ for almost every $t \in [0,\pi]$. We define $\mathcal{V}(t)$ on the space $W^1(\mathbb{T},\mathbb{C})$ given by $\mathcal{V}(t)\varrho(y) = b(t)\varrho'(t)$. Clearly the operator $\mathrm{A}(t) = \mathrm{A} + \mathcal{V}(t)$ is a closed linear operator which generates an evolution operator [29].

Let $X = L^2(\mathbb{T},\mathbb{C})$, the space of 2-integrable and 2π-periodic functions from $\mathbb{R}$ into the set of complex numbers $\mathbb{C}$ with the use of identification between functions on $\mathbb{T}$ and 2π-periodic functions on $\mathbb{R}$, where $\mathbb{T}$ represents the group defined as the quotient $\frac{\mathbb{R}}{2\pi\mathbb{Z}}$. Also $W^2(\mathbb{T},\mathbb{C})$ denotes the Sobolev space of 2π-periodic and 2-integrable functions $\vartheta : \mathbb{R} \to \mathbb{C}$ with $\vartheta'' \in W^2(\mathbb{T},\mathbb{C})$.

Define $\mathrm{A}(t) = \mathrm{A} + \mathcal{V}(t)$, where $\mathcal{V}(t) : D(\mathcal{V})(t) \subseteq X \to X$ is a closed linear operator with $D(\mathrm{A})(t) \subseteq D(\mathcal{V})(t)$ for all $t \in [0,4]$ and $\mathrm{A} : X \to X$ is defined by

$$\mathrm{A}z = z'' \ \text{ with the domain } \ D(\mathrm{A}) := W^2(\mathbb{T},\mathbb{C}).$$

The operator A has normalized eigenvectors $e_n(x) = \frac{1}{\sqrt{2\pi}}e^{inx}$ corresponding to the eigenvalues $-n^2$, where $n \in \mathbb{N}$. Moreover, $\{e_n : n \in \mathbb{Z}\}$ forms an orthogonal basis for X. Thus, we have

$$\mathrm{A}z = \sum_{n\in\mathbb{Z}} -n^2 \langle z, e_n\rangle e_n, \ z \in D(\mathrm{A}).$$

Clearly, A is the infinitesimal generator of cosine family $\xi(t)$ defined as

$$\xi(t)z = \sum_{n\in\mathbb{Z}} \cos nt \langle z, e_n\rangle e_n,$$

associated with the sine family

$$\psi(t)z = \sum_{n\in\mathbb{Z}, n\neq 0} \frac{\sin nt}{n} \langle z, e_n \rangle e_n + t \langle z, e_0 \rangle e_0.$$

Now, define $Z = \left\{ \upsilon : \upsilon = \sum_{n=2}^{\infty} \upsilon_n e_n(x) \middle| \sum_{n=2}^{\infty} \upsilon_n^2 < \infty \right\}$ with norm $\|\upsilon\| = \left(\sum_{n=2}^{\infty} \upsilon_n^2 \right)^{1/2}$.
Define the operator $B_0 : Z \to X$ by $B_0\upsilon = (B\upsilon)(t)$, where $B \in L(Z, X)$ be such that $B\upsilon(t) = 2\upsilon_2(t)e_1(x) + \sum_{n=2}^{\infty} \upsilon_n(t)e_n(x)$. Collecting these definitions, it is clear that the control system (23) can be represented in the abstract form (1). Furthermore, it is easy to check that all the assumptions of Theorem 4.1 are fulfilled. Therefore, the nonlinear system (23) is globally relatively controllable on $[0, 4]$.

7. Conclusions

This paper is concerned with the relative controllability of a non-autonomous stochastic impulsive differential equation with variable delays in control. We have given sufficient condition for the controllability of the corresponding linear system using the controllability Grammian operator. We have used Schauder's fixed point theorem, Arzela -Ascoli theorem, stochastic analysis and controllability of the linear system to achieve the sufficient conditions for relative controllability of the nonlinear system. The above work can be extended for the fractional-order non-autonomous system.

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