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Conformal vector fields and geometric solitons on the tangent bundle with the ciconia metric

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Abstract. Let (X_{2k}, F, g) be an almost anti-paraHermitian manifold with an almost paracomplex structure F and a Riemannian metric g and let TX be its tangent bundle with the ciconia metric $\tilde{g}$. The purpose of this paper is divided into two folds. The first one is to examine the curvature properties of the tangent bundle TX with the ciconia metric $\tilde{g}$. The second one is to study conformal vector fields and almost Ricci and Yamabe solitons on the tangent bundle TX according to the ciconia metric $\tilde{g}$.

1. Introduction

Let us start with a $2k-$dimensional Riemannian manifold X with a Riemannian metric g. In mathematical terms, a paracomplex manifold on a Riemannian manifold is an almost product manifold (X_{2k}, F), $F^2 = id$, such that the two eigenbundles T^+X and T^-X linked to the two eigenvalues $+1$ and -1 of F are of the same rank. The fact that the Nijenhuis tensor specified by

$$N_F(A,B) = [FA, FB] - F[FA, B] - F[A, FB] + [A, B]$$

is zero means that an almost para complex structure is integrable. If an almost paracomplex structure is integrable, then it becomes a paracomplex structure. In the context of X_{2k}, an anti-paraHermitian metric is a Riemannian metric satisfying the following expression

$$g(FA, FB) = g(A, B)$$

or equivalent to this equation,

$$g(FA, B) = g(A, FB) \quad \text{(purity condition)}$$

for any vector fields A, B. The manifold X_{2k} equipped with an almost paracomplex structure and an anti-paraHermitian metric g is called an almost anti-paraHermitian manifold. It is also called anti-paraKähler if

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the paracomplex structure F is parallel with regard to the Levi-Civita connection ∇^g ($\nabla^g F = 0$). Recall that the condition $\nabla^g F = 0$ is equivalent to the paraholomorphicity of the Riemannian metric g, that is, $\check{T}_F g = 0$, where $\check{T}_F$ is the Tachibana operator applied to the Riemannian metric g [6, 8].

The paper [3] that we will talk about a little later is the most important motivation paper for this paper. The paper brings to light a new class of invariant metrics on the tangent bundle of any given almost Hermitian manifold. In here, the metric is called as a ciconia metric by author. Note that this metric generalises both the Sasaki and the Yano metrics with weights. Each ciconia metric is itself almost Hermitian. The present study focuses on the case of Riemannian surfaces, for which the equations of ciconia metrics are obtained. The next development concerns Kählerian and pseudo-Kählerian ciconia metrics, which yield new examples of Kählerian Ricci-flat manifolds in four real dimensions. Inspired by this paper, we construct a new metric on the tangent bundle over any anti-paraKähler manifold and study its geometry. We use the twin metric when defining this metric and call this metric as a ciconia metric. Because, the metric we are constructing is in the form of the ciconia metric given in [3].

1.1. *The ciconia metric on tangent bundle*

Let X_n be an n-dimensional Riemannian manifold having a Riemannian metric g and TX its tangent bundle. This article uses C^∞-category to explain everything, connected manifolds with dimension of $n > 1$ are provided in this study. Let $\pi : TX \to X_n$ be the natural projection. A system of local coordinates $\left(U, x^i\right)$ in X_n induces on TX a system of local coordinates $\left(\pi^{-1}(U), x^i, x^{\bar{i}} = u^i\right)$, $\bar{i} = n + i = n + 1, ..., 2n$. Here, $\left(u^i\right)$ are the cartesian coordinates in each tangent space $T_P X$ of $\forall p \in U$. Also, p is an arbitrary point on U.

Let the Levi-Civita connection of the Riemannian metric g be represented by ∇. In the horizontal distribution determined by ∇ and the vertical distribution defined by $\ker \pi_*$, the local frame is given as respectively

$$\eta_i = \frac{\partial}{\partial x^i} - u^s \Gamma_{is}^h \frac{\partial}{\partial u^h};\ i = 1, ..., n,$$

and

$$\eta_{\bar{i}} = \frac{\partial}{\partial u^i};\ \bar{i} = n + 1, ..., 2n,$$

where Γ_{is}^h denote the Christoffel's symbols of g. The local frame $\left\{\eta_\beta\right\} = \left(\eta_i, \eta_{\bar{i}}\right)$ is called the adapted frame. Let $A = A^i \frac{\partial}{\partial x^i} \in \chi(X_n)$ be a vector field. The horizontal and vertical lifts of A are obtained, with respect to the adapted frame, as follows: [7]

$$\begin{aligned} {}^H A &= A^i \eta_i \in \chi\left(\pi^{-1}(U)\right) \\ {}^V A &= A^i \eta_{\bar{i}} \in \chi\left(\pi^{-1}(U)\right). \end{aligned}$$

In TX, the local 1−form system $\left(dx^i, \delta u^i\right)$ is the dual frame of the adapted frame $\left\{\eta_\beta\right\}$, where

$$\delta u^i = {}^H\left(dx^i\right) = du^i + u^s \Gamma_{hs}^i dx^h.$$

Definition 1.1. *Let (X_{2k}, F, g) be an almost anti-paraHermitian manifold with an almost paracomplex structure F and a Riemannian metric g and let TX be its tangent bundle. The ciconia metric $\tilde{g}$ on the tangent bundle TX is defined as follows:*

$$\begin{aligned} i)\ \tilde{g}\left({}^V A, {}^V B\right) &= {}^V(bg(A, B)), \\ ii)\ \tilde{g}\left({}^V A, {}^H B\right) &= {}^V(G(A, B)) = {}^V(g(FA, B)), \\ iii)\ \tilde{g}\left({}^H A, {}^H B\right) &= {}^V(ag(A, B)) \end{aligned} \tag{1}$$

for all vector fields A, B on X_{2k}, where $G(A, B) = g(FA, B)$ is the twin metric and a, b are positive constants.

From the manifold (X, g) to its tangent bundle TX, various Riemannian or pseudo-Riemannian metrics have been defined using the natural lifts of the Riemannian metric g. When used in this manner, these metrics are referred to as g-natural metrics. In [1], authors have obtained the family of all Riemannian g-natural metrics which depends on six arbitrary functions of the norm of a vector $u \in TX$. The Ciconia metric is also obtained by the lifts of the Riemannian metric and the twin metric in the base manifold. In the perspective, the ciconia metric is a natural metric. But the ciconia metric is never a subclass of $g-$natural metrics given in [1]. Thus, we present a new class for Riemannian geometries. In the adapted frame $\left\{\eta_\beta\right\}$, the ciconia metric and its inverse are as follows:

$$\left(\tilde{g}_{\gamma\beta}\right) = \begin{pmatrix} ag_{ji} & G_{ji} \\ G_{ji} & bg_{ji} \end{pmatrix}$$

and

$$\left(\tilde{g}^{\gamma\varepsilon}\right) = \begin{pmatrix} \frac{b}{\alpha} g^{jk} & \frac{-1}{\alpha} G^{jk} \\ \frac{-1}{\alpha} G^{jk} & \frac{a}{\alpha} g^{jk} \end{pmatrix},$$

where $\alpha = a.b - 1 \neq 0$, the twin metric $G(A, B) = g(FA, B)$ is locally expressed as $G_{ji} = g_{jh} F_i^h$.

Let us start a linear connection ∇ on a Riemannian manifold (X_n, g). The linear connection ∇ is called a metric connection if $\nabla g = 0$. Otherwise, it is called a non-metric connection. As it is known, if the metric connection ∇ is a torsion-free connection, then it becomes the Levi-Civita connection of g. We will search whether the horizontal lift ${}^H\nabla$ of the Levi-Civita connection of g is a metric connection of the ciconia metric $\tilde{g}$ or not. We know that the horizontal lift connection ${}^H\nabla$ is a linear connection with torsion on TX even if the connection ∇ on the base manifold is a torsion-free connection [7]. The horizontal lift of any torsion free connection ∇ to the tangent bundle TX is defined by the following relations:

$$\begin{cases} {}^H\nabla_{{}^VA} {}^VB = 0, & {}^H\nabla_{{}^VA} {}^HB = 0, \\ {}^H\nabla_{{}^HA} {}^VB = {}^V(\nabla_A B), & {}^H\nabla_{{}^HA} {}^HB = {}^H(\nabla_A B) \end{cases} \tag{2}$$

for all vector fields A, B on (X_n, g).

Theorem 1.2. *Let (X_{2k}, F, g) be an anti-paraKähler manifold and TX be its tangent bundle with the ciconia metric $\tilde{g}$. The horizontal lift connection ${}^H\nabla$ of the Levi-Civita connection of g becomes a metric connection with torsion of the ciconia metric $\tilde{g}$.*

Proof. It is known that

$$(\nabla_A g)(B, C) = Ag(B, C) - g(\nabla_A B, C) - g(B, \nabla_A C).$$

Using the above formula and (2) we will show that the horizontal lift connection ${}^H\nabla$ of the Levi-Civita connection of g is a metric connection of the ciconia metric $\tilde{g}$. For this we have to show that ${}^H\nabla\tilde{g} = 0$. The following equations are obtained

$$\begin{aligned} (i).\ \left({}^H\nabla_{{}^HA}\tilde{g}\right)\left({}^HB, {}^HC\right) &= {}^HA\tilde{g}(({}^HB, {}^HC) - \tilde{g}({}^H\nabla_{{}^HA} {}^HB, {}^HC) - \tilde{g}({}^HB, {}^H\nabla_{{}^HA} {}^HC) \\ &= {}^HA({}^V(ag(B, C))) - \tilde{g}({}^H(\nabla_A B), {}^HC) - \tilde{g}({}^HB, {}^H(\nabla_A C)) \\ &= {}^V(Aag(B, C)) - {}^V(ag(\nabla_A B, C)) - {}^V(ag(B, \nabla_A C)) \\ &= a^V[Ag(B, C) - g(\nabla_A B, C) - g(B, \nabla_A C)] \\ &= a^V[(\nabla_A g)(B, C)] = 0, \end{aligned}$$

$$
\begin{aligned}
(ii).\ ({}^H\nabla_{{}^HA}\tilde{g})\left({}^VB,{}^HC\right) &= {}^HA\tilde{g}({}^VB,{}^HC)-\tilde{g}({}^H\nabla_{{}^HA}\,{}^VB,{}^HC)-\tilde{g}({}^VB,{}^H\nabla_{{}^HA}\,{}^HC)\\
&= {}^HA\,{}^V(g(FB,C))-\tilde{g}({}^V(\nabla_AB),{}^HC)-\tilde{g}({}^VB,{}^H(\nabla_AC))\\
&= {}^HA\,{}^V(g(FB,C))-{}^V(g(F(\nabla_AB),C))-{}^V(g(B,F\nabla_AC)\\
&= {}^V(Ag(FB,C))-{}^V(g(F(\nabla_AB),C))-{}^V(g(B,F\nabla_AC))\\
&= {}^V[Ag(FB,C)-g(F(\nabla_AB),C)-g(B,F\nabla_AC)]\\
&= {}^V[(\nabla_Ag)(FB,C)]=0,
\end{aligned}
$$

$$
\begin{aligned}
(iii).\ ({}^H\nabla_{{}^HA}\tilde{g})({}^VB,{}^VC) &= {}^HA\tilde{g}({}^VB,{}^VC)-\tilde{g}({}^H\nabla_{{}^HA}\,{}^VB,{}^VC)-\tilde{g}({}^VB,{}^H\nabla_{{}^HA}\,{}^VC)\\
&= {}^HA\,{}^V(bg(B,C))-\tilde{g}({}^V(\nabla_AB),{}^VC)-\tilde{g}({}^VB,{}^V(\nabla_AC))\\
&= {}^V[Abg(B,C)-bg(\nabla_AB,C)-bg(B,\nabla_AC)]\\
&= {}^V[b(\nabla_Ag)(B,C)]=0,
\end{aligned}
$$

$$
\begin{aligned}
(iv).\ ({}^H\nabla_{{}^VA}\tilde{g})({}^HB,{}^HC) &= {}^VA\tilde{g}({}^HB,{}^HC)-\tilde{g}({}^H\nabla_{{}^VA}\,{}^HB,{}^HC)-\tilde{g}({}^HB,{}^H\nabla_{{}^VA}\,{}^HC)\\
&= {}^VA\tilde{g}({}^HB,{}^HC)={}^VA\,{}^V(ag(B,C))=0,
\end{aligned}
$$

$$
\begin{aligned}
(v).\ ({}^H\nabla_{{}^VA}\tilde{g})({}^VB,{}^HC) &= {}^VA\tilde{g}({}^VB,{}^HC)-\tilde{g}({}^H\nabla_{{}^VA}\,{}^VB,{}^HC)-\tilde{g}({}^VB,{}^H\nabla_{{}^VA}\,{}^HC)\\
&= {}^VA\tilde{g}({}^VB,{}^HC)={}^VA\,{}^V(g(FB,C))=0,
\end{aligned}
$$

$$
\begin{aligned}
(vi).\ ({}^H\nabla_{{}^VA}\tilde{g})({}^VB,{}^VC) &= {}^VA\tilde{g}({}^VB,{}^VC)-\tilde{g}({}^H\nabla_{{}^VA}\,{}^VB,{}^VC)-\tilde{g}({}^VB,{}^H\nabla_{{}^VA}\,{}^VC)\\
&= {}^VA\,{}^V(bg(B,C))=0.
\end{aligned}
$$

Hence, ${}^H\nabla\tilde{g}=0$. The horizontal lift connection ${}^H\nabla$ of the Levi-Civita connection ∇ of g becomes a metric connection with torsion of the ciconia metric $\tilde{g}$. □

Remark 1.3. *Let (X_{2k},F,g) be an anti-paraKähler manifold and TX be its tangent bundle with the ciconia metric $\tilde{g}$. The horizontal lift connection ${}^H\nabla$ of the Levi-Civita connection of g coincides with the Levi-Civita connection of the ciconia metric if and only if (X_{2k},F,g) is a flat manifold.*

Now, we will calculate the Levi-Civita connection $\tilde{\nabla}$ of the ciconia metric $\tilde{g}$. The coefficients of the Levi-Civita connection can be found with

$$
\tilde{\Gamma}^{\alpha}_{\gamma\beta}=\frac{1}{2}\tilde{g}^{\alpha\varepsilon}(\eta_{\gamma}\tilde{g}_{\varepsilon\beta}+\eta_{\beta}\tilde{g}_{\gamma\varepsilon}-\eta_{\varepsilon}\tilde{g}_{\gamma\beta})+\frac{1}{2}(\Omega_{\gamma\beta}{}^{\alpha}+\Omega^{\alpha}_{\gamma\beta}+\Omega^{\alpha}_{\beta\gamma}),
$$

where

$$
\begin{cases}
\Omega^{\alpha}_{\gamma\beta}=\tilde{g}^{\alpha\varepsilon}\tilde{g}_{\delta\beta}\Omega_{\varepsilon\gamma}{}^{\delta},\\
\Omega_{ji}{}^{\bar{h}}=-\Omega_{ij}{}^{\bar{h}}=-\mathcal{R}_{jis}{}^{h}y^{s}\ ,\\
\Omega_{\bar{j}i}{}^{\bar{h}}=-\Omega_{\bar{i}j}{}^{\bar{h}}=\Gamma^{h}_{ji}
\end{cases}
$$

and it will be used as $\gamma=j;\bar{j}\quad\beta=i;\bar{i}\quad\alpha=h;\bar{h}\quad\varepsilon=k;\bar{k}\quad\delta=m;\overline{m}$.

For the Levi-Civita connection $\tilde{\nabla}$ of the ciconia metric $\tilde{g}$, we give the following proposition.

Proposition 1.4. *Let (X_{2k}, F, g) be an anti-paraKähler manifold and $(TX, \tilde{g})$ be its tangent bundle with the ciconia metric. The Levi-Civita connection $\tilde{\nabla}$ of the ciconia metric $\tilde{g}$ on TX is locally given by:*

$$\begin{aligned}
\tilde{\nabla}_{\eta_j}\eta_i &= \left(\Gamma^h_{ji} - \frac{b}{2\alpha}u^s F^t_j \left(2\mathcal{R}_{tsi}{}^h - \mathcal{R}_{tis}{}^h\right)\right)\eta_h + \left(\frac{1}{2\alpha}u^s\mathcal{R}_{jsi}{}^h + \frac{1}{2\alpha}u^s\mathcal{R}_{isj}{}^h - \frac{1}{2}u^s\mathcal{R}_{jis}{}^h\right)\eta_{\bar{h}},\\
\tilde{\nabla}_{\eta_j}\eta_{\bar{i}} &= \left(\frac{b^2}{2\alpha}u^s\mathcal{R}_{sij}{}^h\right)\eta_h + \left(\Gamma^h_{ji} + \frac{b}{2\alpha}u^s F^t_j\mathcal{R}_{ist}{}^h\right)\eta_{\bar{h}},\\
\tilde{\nabla}_{\eta_{\bar{j}}}\eta_i &= \left(\frac{b^2}{2\alpha}u^s\mathcal{R}_{sji}{}^h\right)\eta_h + \left(\frac{b}{2\alpha}u^s F^t_j\mathcal{R}_{tsi}{}^h\right)\eta_{\bar{h}},\\
\tilde{\nabla}_{\eta_{\bar{j}}}\eta_{\bar{i}} &= 0,
\end{aligned}$$

where $\mathcal{R}$ is the Riemannian curvature tensor of g.

1.2. Some curvature properties of the ciconia metric on tangent bundle

The Riemannian curvature tensors are found with

$$\widetilde{\mathcal{R}}_{\delta\gamma\beta}{}^{\alpha} = \eta_\delta\tilde{\Gamma}^\alpha_{\gamma\beta} - \eta_\gamma\tilde{\Gamma}^\alpha_{\delta\beta} + \tilde{\Gamma}^\alpha_{\delta\varepsilon}\tilde{\Gamma}^\varepsilon_{\gamma\beta} - \tilde{\Gamma}^\alpha_{\gamma\varepsilon}\tilde{\Gamma}^\varepsilon_{\delta\beta} - \Omega_{\delta\gamma}{}^{\varepsilon}\tilde{\Gamma}^\alpha_{\varepsilon\beta}.$$

Here it will be used as $\gamma = i;\bar{i}$ $\beta = j;\bar{j}$ $\alpha = k;\bar{k}$ $\varepsilon = h;\bar{h}$ $\delta = m;\bar{m}$. In the following proposition we give some components of the Riemannian curvature tensor of the ciconia metric which will use the Ricci tensor.

Proposition 1.5. *Let (X_{2k}, F, g) be an anti-paraKähler manifold and $(TX, \tilde{g})$ be its tangent bundle with the ciconia metric. Then the corresponding Riemannian curvature tensor $\widetilde{\mathcal{R}}$ is locally given by:*

$$\begin{aligned}
\widetilde{\mathcal{R}}_{mij}{}^k &= \mathcal{R}_{mij}{}^k + \frac{b^2}{4\alpha^2}u^s u^p\left(\mathcal{R}_{shm}{}^k\mathcal{R}_{ipj}{}^h + \mathcal{R}_{shm}{}^k\mathcal{R}_{jpi}{}^h - \mathcal{R}_{shi}{}^k\mathcal{R}_{mpj}{}^h - \mathcal{R}_{shi}{}^k\mathcal{R}_{jpm}{}^h\right)\\
&\quad -\frac{b^2}{4\alpha}u^s u^p\left(\mathcal{R}_{shm}{}^k\mathcal{R}_{ijp}{}^h - \mathcal{R}_{shi}{}^k\mathcal{R}_{mjp}{}^h - 2\mathcal{R}_{phj}{}^k\mathcal{R}_{mis}{}^h\right)\\
&\quad +\frac{b}{2\alpha}u^s F^t_m\left(\nabla_s\mathcal{R}_{tij}{}^k + \nabla_i\mathcal{R}_{jst}{}^k + \nabla_t\mathcal{R}_{jsi}{}^k\right),
\end{aligned}$$

$$\begin{aligned}
\widetilde{\mathcal{R}}_{m\bar{i}j}{}^k &= \frac{b^3}{4\alpha^2}u^s u^p\left(F^t_m\mathcal{R}_{sih}{}^k\left(2\mathcal{R}_{tpj}{}^h - \mathcal{R}_{tjp}{}^h\right) - F^t_m\mathcal{R}_{sij}{}^h\left(2\mathcal{R}_{tph}{}^k - \mathcal{R}_{thp}{}^k\right) + \mathcal{R}_{shm}{}^k\mathcal{R}_{tpj}{}^h\right)\\
&\quad -\frac{b}{2\alpha}F^t_m\ \left(2\mathcal{R}_{tij}{}^k - \mathcal{R}_{tji}{}^k\right) + \frac{b^2}{2\alpha}u^s\nabla_m\mathcal{R}_{sij}{}^k,
\end{aligned}$$

$$\begin{aligned}
\widetilde{\mathcal{R}}_{\overline{m}ij}{}^{\bar{k}} &= \frac{1}{2}\mathcal{R}_{jim}{}^k + \frac{1}{2\alpha}\left(\mathcal{R}_{imj}{}^k + \mathcal{R}_{jmi}{}^k\right) + \frac{b}{2\alpha}u^s F^h_m\nabla_i\mathcal{R}_{shj}{}^k\\
&\quad +\frac{b^2}{4\alpha^2}u^s u^p\left(\mathcal{R}_{pmj}{}^h\left(2\mathcal{R}_{his}{}^k + \mathcal{R}_{hsi}{}^k\right) + \mathcal{R}_{mph}{}^k\left(2\mathcal{R}_{isj}{}^h - \mathcal{R}_{ijs}{}^h\right)\right),\\
\widetilde{\mathcal{R}}_{m i\bar{j}}{}^k &= \frac{b^2}{2\alpha}\mathcal{R}_{jim}{}^k - \frac{b^4}{4\alpha^2}u^s u^p\mathcal{R}_{sih}{}^k\mathcal{R}_{pjm}{}^h,\\
\widetilde{\mathcal{R}}_{\overline{mi}j}{}^{\bar{k}} &= \frac{b}{\alpha}F^k_l\mathcal{R}_{mij}{}^l + \frac{b^3}{4\alpha^2}u^s u^p F^k_l\left(\mathcal{R}_{smh}{}^l\mathcal{R}_{pij}{}^h - \mathcal{R}_{sih}{}^l\mathcal{R}_{pmj}{}^h\right),\\
\mathcal{R}_{\overline{mi}\bar{j}}{}^{\bar{k}} &= 0.
\end{aligned}$$

Next, we consider the Ricci and scalar curvature tensors. With the help of Proposition 1.5 and standard calculations give the following results.

Proposition 1.6. *Let (X_{2k}, F, g) be an anti-paraKähler manifold and $(TX, \tilde{g})$ be its tangent bundle with the ciconia metric. Then the corresponding Ricci curvature tensor is given locally by:*

$$\begin{aligned}
\widetilde{\mathcal{R}}_{ij} &= \frac{\alpha-1}{\alpha}\mathcal{R}_{ij} + \frac{b}{2\alpha}u^s\left(F_i^h\nabla_s\mathcal{R}_{hj} - F_s^m\nabla_i\mathcal{R}_{mj} + F_m^h\nabla_h\mathcal{R}_{jsi}{}^m\right)\\
&+\frac{b^2}{4\alpha}u^su^p\left(\mathcal{R}_{shi}{}^m\,\mathcal{R}_{mjp}^{\;\;\;h} + 2\mathcal{R}_{phj}{}^m\mathcal{R}_{mis}^{\;\;\;h}\right)\\
&+\frac{b^2}{4\alpha^2}u^su^p\left(2\mathcal{R}_{pmj}^{\;\;\;h}\mathcal{R}_{his}{}^m - \mathcal{R}_{shi}{}^m\,\mathcal{R}_{jpm}^{\;\;\;h} + \mathcal{R}_{sh}\left(2\mathcal{R}_{ipj}{}^h - \mathcal{R}_{ijp}^{\;\;\;h}\right)\right),
\end{aligned}$$

$$\begin{aligned}
\widetilde{\mathcal{R}}_{\bar{i}j} &= \frac{b}{2\alpha}F_j^m\mathcal{R}_{mi} + \frac{b^2}{2\alpha}u^s\left(\nabla_s\,\mathcal{R}_{ij} - \nabla_i\mathcal{R}_{sj}\right)\\
&+\frac{b^3}{4\alpha^2}u^su^pF_h^m\left(2\mathcal{R}_{sm}\mathcal{R}_{ipj}{}^h + 3\,\mathcal{R}_{sil}{}^h\mathcal{R}_{pmj}^{\;\;\;l} - \mathcal{R}_{sil}{}^h\mathcal{R}_{mjp}^{\;\;\;l}\right),\\
\widetilde{\mathcal{R}}_{i\bar{j}} &= \frac{b}{2\alpha}F_i^m\mathcal{R}_{mj} + \frac{b^2}{2\alpha}u^s\left(\nabla_s\,\mathcal{R}_{ji} - \nabla_j\mathcal{R}_{si}\right)\\
&+\frac{b^3}{4\alpha^2}u^su^pF_h^m\left(2\mathcal{R}_{sm}\mathcal{R}_{jpi}{}^h + 3\,\mathcal{R}_{sjl}{}^h\mathcal{R}_{pmi}^{\;\;\;l} - \mathcal{R}_{sjl}{}^h\mathcal{R}_{mip}^{\;\;\;l}\right),\\
\widetilde{\mathcal{R}}_{\bar{i}\bar{j}} &= \frac{b^4}{4\alpha^2}u^su^p\mathcal{R}_{ish}{}^m\mathcal{R}_{pjm}^{\;\;\;h}.
\end{aligned}$$

Proposition 1.7. *Let (X_{2k}, F, g) be an anti-paraKähler manifold and $(TX, \tilde{g})$ be its tangent bundle with the ciconia metric. Then the corresponding scalar curvature tensor $\tilde{r}$ is locally given by:*

$$\begin{aligned}
\tilde{r} &= \frac{b\,(\alpha-2)}{\alpha^2}r + \frac{b^2(ab^2-2\alpha)}{4\alpha^3}u^su^p\mathcal{R}_{phjn}\mathcal{R}_s{}^{hjn}\\
&+\frac{b^3}{4\alpha^3}u^su^p\left(\mathcal{R}_{phjn}\mathcal{R}_s{}^{jhn} + \mathcal{R}_{phjn}\mathcal{R}_s{}^{nhj}\right),
\end{aligned}$$

where r is the scalar curvature of g and $\mathcal{R}_s{}^{nhj} = g^{mn}g^{ht}g^{ij}\mathcal{R}_{smti}$.

2. Main results

2.1. Conformal vector field with respect to the ciconia metric

Let $L_{\widetilde{A}}$ be the Lie derivation with respect to the vector field $\widetilde{A}$. A vector field $\widetilde{A}$ with components $(v^h, v^{\bar{h}})$ is fibre-preserving if and only if v^h depend only on the variables (x^h). We shall first state following lemma which are needed later on.

Lemma 2.1. *Let (X_{2k}, F, g) be an anti-paraKähler manifold and TX its tangent bundle with the ciconia metric $\widetilde{g}$. The Lie derivative of the ciconia metric $\tilde{g}$ with respect to the fibre-preserving vector field $\widetilde{A}$ is given as follows*

$$\begin{aligned}
L_{\widetilde{A}}\tilde{g} &= \left[aL_Vg_{ij} + 2G_{ih}\left(u^sv^m\mathcal{R}_{mjs}^h + \Gamma_{mj}^hv^{\overline{m}} + \eta_jv^{\bar{h}}\right)\right]dx^idx^j\\
&+2\left[L_VG_{ij} - G_{ih}\nabla_jv^h - G_{ih}\left(\eta_{\bar{j}}v^{\bar{h}}\right)\right.\\
&\left.+bg_{hj}\left(u^sv^m\mathcal{R}_{mis}^{\;\;\;h} + \Gamma_{mi}^hv^{\overline{m}} + \eta_iv^{\bar{h}}\right)\right]dx^i\delta u^j\\
&+2b\left[g_{hj}\left(\eta_{\bar{i}}v^{\bar{h}}\right)\right]\delta u^i\delta u^j,
\end{aligned}$$

where L_Vg_{ij} and L_VG_{ij} denote the components of the Lie derivative L_Vg and L_VG.

Let $\widetilde{A}$ be an infinitesimal fibre-preserving conformal transformation on TX such that $L_{\widetilde{A}}\tilde{g} = 2\Omega\tilde{g}$. From Lemma 2.1, we have

$$\begin{aligned} L_V G_{ij} - 2\Omega G_{ij} &= G_{ih}\nabla_j v^h + G_{ih}\left(\eta_{\bar{j}} v^{\bar{h}}\right) \\ &\quad - b g_{hj}\left(u^s v^m \mathcal{R}_{mis}{}^h + \Gamma^h_{mi} v^{\overline{m}} + \eta_i v^{\bar{h}}\right), \end{aligned} \quad (3)$$

$$L_V g_{ij} - 2\Omega g_{ij} = -\frac{2}{a} G_{ih}\left(u^s v^m \mathcal{R}_{mjs}{}^h + \Gamma^h_{mj} v^{\overline{m}} + \eta_j v^{\bar{h}}\right), \quad (4)$$

$$\Omega g_{ij} = g_{hj}\left(\eta_{\bar{i}} v^{\bar{h}}\right). \quad (5)$$

Proposition 2.2. *Let (X_{2k}, F, g) be an anti-paraKähler manifold and TX its tangent bundle with the ciconia metric $\widetilde{g}$. The scalar function Ω on TX, depends only on the variables $\left(x^h\right)$ with respect to the induced coordinates $\left(x^h, u^h\right)$.*

Proof. Applying $\eta_{\bar{k}}$ to both sides of the equation (5), we have

$$g_{hj}\left(\eta_{\bar{k}}\eta_{\bar{i}} v^{\bar{h}}\right) = \left(\eta_{\bar{k}}\Omega\right) g_{ij}. \quad (6)$$

Interchanging i by k in the equation (6) we get

$$\begin{aligned} \eta_{\bar{k}}\left(\eta_{\bar{i}} v^{\bar{h}}\right) g_{hj} &= \left(\eta_{\bar{k}}\Omega\right) g_{ij} \\ \eta_{\bar{i}}\left(\eta_{\bar{k}} v^{\bar{h}}\right) g_{hj} &= \left(\eta_{\bar{i}}\Omega\right) g_{kj} \end{aligned}$$

and

$$\left(\eta_{\bar{k}}\Omega\right) g_{ij} = \left(\eta_{\bar{i}}\Omega\right) g_{kj}.$$

Contracting with g^{ij} in the above equation we get

$$2k\left(\eta_{\bar{k}}\Omega\right) = \left(\eta_{\bar{k}}\Omega\right) \Rightarrow \left(\eta_{\bar{k}}\Omega\right) = 0. \quad (7)$$

This shows that the scalar function Ω on TX depends only on the variables $\left(x^h\right)$ with respect to the induced coordinates $\left(x^h, u^h\right)$, thus we can think Ω as a function on X_{2k}. Substituting (7) into (6) we have

$$\eta_{\bar{k}}\left(\eta_{\bar{i}} v^{\bar{h}}\right) = 0 \Rightarrow v^{\bar{h}} = u^s \phi^h_s + B^h, \quad (8)$$

where ϕ^h_s and B^h are certain functions which depend only on the variables $\left(x^h\right)$. $\square$

Theorem 2.3. *Let (X_{2k}, F, g) be an anti-paraKähler manifold and TX its tangent bundle with the ciconia metric $\widetilde{g}$. The fibre-preserving vector field $\widetilde{A}$ is an infinitesimal fibre-preserving conformal vector field on $(TX, \widetilde{g})$ if and only if the vector field $\widetilde{A}$ is defined by*

$$\begin{aligned} \widetilde{A} &= v^h E_h + (u^s \phi^h_s + B^h) E_{\bar{h}} \\ &= {}^H V + {}^V B + \gamma\phi \end{aligned}$$

and the following conditions are satisfied

i) $\Omega = \frac{1}{2k}\left(\phi^i_i\right)$,
ii) $L_V g_{ij} = 2\Omega g_{ij} - \frac{2}{a} G_{ih}\nabla_j B^h$,
iii) $v^m \mathcal{R}_{mjs}{}^h + \nabla_j \phi^h_s = 0$,
iv) $L_V G_{ij} = 2\Omega G_{ij} + G_{ih}\nabla_j v^h + G_{ih}\phi^h_j - b g_{hj}\nabla_i B^h$,

where $\nabla_i B^h$ and $\nabla_j \phi_s^h$ denote the components of the covariant derivative of the vector field $B = \left(B^h\right)$ and the $(1,1)$ −tensor field $\phi = \left(\phi_j^h\right)$ on X_{2k}.

Proof. Substituting the equation (8) into the equation (5), we have

$$g_{hj}\phi_i^h = \Omega g_{ij}.$$

Contracting with g^{ij} in the above equation, we get

$$\Omega = \frac{1}{2k}\left(\phi_i^i\right).$$

From the equations (8) and (4), we get the following results

$$aL_V g_{ij} + 2G_{ih}\nabla_j B^m = 2a\Omega g_{ij},$$

$$v^m R_{mjs}^h + \nabla_j \phi_s^m = 0.$$

Similarly, substituting the equation (8) into the equation (3) we have

$$L_V G_{ij} - 2\Omega G_{ij} - G_{ih}\nabla_j v^h - G_{ih}\phi_j^h + bg_{hj}\nabla_i B^h = 0.$$

Conversely, it is easy to see that $\widetilde{A} = {}^H V + {}^V B + \gamma\phi$ is an infinitesimal fibre-preserving conformal vector field on $(TX, \tilde{g})$ under which of the conditions $(i) - (iv)$. □

2.2. Almost Ricci soliton on the tangent bundle according to the ciconia metric

Let X_{2k} ($k \geq 1$) be a smooth manifold. A Ricci soliton on X_{2k} is a triple (g, V, λ), where g is a pseudo-Riemannian metric on X_{2k}, Ric is the associated Ricci tensor, V is a vector field and λ is a real constant, satisfying the equation

$$Ric + \frac{1}{2}L_V g = \lambda g,$$

where L_V is the Lie derivative with respect to V. The Ricci soliton is said to be either shrinking, steady, or expanding, according as λ is positive, zero, or negative, respectively. The importance of Ricci solitons comes from the fact that they correspond to self-similar solutions of the Ricci flow [4] and at the same time they are natural generalizations of Einstein metrics. Recently, Pigola et al. [5] introduced the notion of almost Ricci soliton. Here, the soliton constant λ in Ricci soliton is a smooth function. An almost Ricci soliton is said to be expanding, steady or shrinking according as $\lambda < 0, \lambda = 0$ or $\lambda > 0$, respectively.

An almost Ricci soliton on the tangent bundle TX with the ciconia metric $\widetilde{g}$ over an anti-paraKähler manifold (X_{2k}, F, g) ($k \geq 1$) is defined by

$$\widetilde{Ric} + \frac{1}{2}L_{\widetilde{A}}\widetilde{g} = \widetilde{\lambda}\widetilde{g}, \tag{9}$$

where $\widetilde{Ric}$ is the Ricci tensor of $\widetilde{g}$, $\widetilde{A}$ is a vector field on TX and $\widetilde{\lambda}$ is a smooth function on TX.

Theorem 2.4. *Let (X_{2k}, F, g) ($k \geq 1$) be an anti-paraKä hler manifold and TX its tangent bundle with the ciconia metric $\widetilde{g}$. The $\left(TX, \widetilde{g}, \widetilde{A}, {}^V\lambda\right)$ is an almost Ricci soliton if and only if the base manifold X_{2k} is flat and the following conditions are satisfied:*

i) $\lambda = \frac{1}{2k}\phi_j^j$,
ii) $\frac{5}{3}\nabla_j v^j + \frac{2}{a}F_h^j\nabla_j B^h = 0$,
iii) $\phi_j^j = \frac{-2}{5\alpha}F_h^j\nabla_j B^h$,
iv) $\nabla_j\left(\nabla_i B^h\right) = 0$,

where $\alpha = ab-1$, ${}^{V}\lambda$ *is the vertical lift of a smooth function* λ *to* TX *defined by* ${}^{V}\lambda = \lambda \circ \pi$, $\widetilde{A} = v^h \eta_h + \left(u^s \phi_s^h + B^h\right) \eta_{\bar{h}}$ *is a fibre-preserving vector field on* TX, ϕ_s^h *and* B^h *are the components of a certain* $(1,1)-$*tensor field and a certain contravariant vector field on* X_{2k}, *respectively.*

Proof. We will show that the existence of the smooth fuction λ. If the expression of $L_{\widetilde{A}}\tilde{g}$ in Lemma 2.1 is used in the equation (9), we have

$$\begin{aligned} \lambda a g_{ij} = {} & \frac{1}{2} a L_V g_{ij} + G_{ih}\left(u^s v^m \mathcal{R}_{mjs}{}^{h} + \Gamma^h_{mj} v^{\bar{m}} + \eta_j v^{\bar{h}}\right) \\ & + \frac{2\alpha+1}{2\alpha} R_{ij} + \frac{b}{2\alpha} u^s \left(F_i^t \nabla_t R_{sj} - F_j^t \nabla_t R_{si}\right) \\ & + \frac{b^2}{4\alpha} u^s u^p \left(3\mathcal{R}_{phj}{}^{m} \mathcal{R}_{sim}{}^{h} - 3\mathcal{R}_{phj}{}^{m} \mathcal{R}_{smi}{}^{h} + \mathcal{R}_{shi}{}^{m} \mathcal{R}_{pmj}{}^{h} - \mathcal{R}_{pjm}{}^{h} \mathcal{R}_{shi}{}^{m}\right) \\ & + \frac{b^2}{4\alpha^2} u^s u^p \left(\mathcal{R}_{sih}{}^{m} \mathcal{R}_{pjm}{}^{h} + \mathcal{R}_{shi}{}^{m} \mathcal{R}_{pmj}{}^{h} - \mathcal{R}_{shi}{}^{m} \mathcal{R}_{pjm}{}^{h} - 2\mathcal{R}_{pmj}{}^{h} \mathcal{R}_{sih}{}^{m}\right), \end{aligned} \tag{10}$$

$$\begin{aligned} \lambda G_{ij} = {} & L_V G_{ij} - G_{ih} \nabla_j v^h - G_{ih}\left(\eta_{\bar{j}} v^{\bar{h}}\right) \\ & + \frac{b^2}{2\alpha} u^s \left(\nabla_s \mathcal{R}_{ij} - \nabla_j \mathcal{R}_{si}\right) + \frac{b}{2\alpha} F_i^m \mathcal{R}_{mj} + \frac{b^3}{4\alpha^2} u^s u^t F_l^p \mathcal{R}_{sih}{}^{l} \mathcal{R}_{jpt}{}^{h} \\ & + b g_{hj}\left(u^s v^m \mathcal{R}_{mis}{}^{h} + v^{\overline{m}} \Gamma^h_{mi} + \eta_i v^{\bar{h}}\right) \end{aligned} \tag{11}$$

and

$$\lambda g_{ij} = \frac{b^3}{4\alpha^2} u^s u^p \mathcal{R}_{ish}{}^{m} \mathcal{R}_{pjm}{}^{h} + g_{hj}\left(\eta_{\bar{i}} v^{\bar{h}}\right). \tag{12}$$

Substituting $v^{\bar{h}} = u^s \phi_s^h + B^h$ into the equation (12), we get

$$\lambda g_{ij} = \frac{b^3}{4\alpha^2} u^s u^p \mathcal{R}_{ish}{}^{m} \mathcal{R}_{pjm}{}^{h} + g_{hj} \phi_i^h.$$

Contraction with g^{ij} into the last equation gives

$$\lambda = \frac{1}{2k} \frac{b^3}{4\alpha^2} u^s u^p g^{ij} \mathcal{R}_{ish}{}^{m} \mathcal{R}_{pjm}{}^{h} + \frac{1}{2k} \phi_j^j,$$

from which we get

$$\lambda = \frac{1}{2k} \phi_j^j \tag{13}$$

and

$$-g^{ij} \mathcal{R}_{sih}{}^{m} \mathcal{R}_{pjm}{}^{h} = 0. \tag{14}$$

From (14), we can write

$$\begin{aligned} g^{ij} g^{hl} g^{nm} \mathcal{R}_{sihn} \mathcal{R}_{pjlm} &= 0 \\ \|\mathcal{R}\| &= 0 \\ \mathcal{R} &= 0. \end{aligned}$$

So the base manifold is flat.

Because of the base manifold is flat, the equations (10), (11) and (12) turn into the following equations

$$\lambda a g_{ij} = \frac{1}{2} a L_V g_{ij} + G_{ih}\left(u^s v^m \mathcal{R}_{mjs}{}^{h} + \Gamma^h_{mj} v^{\bar{m}} + \eta_j v^{\bar{h}}\right), \tag{15}$$

$$\lambda G_{ij} = L_V G_{ij} - G_{ih}\nabla_j v^h - G_{ih}\left(\eta_{\bar{j}} v^{\bar{h}}\right), \tag{16}$$

$$\lambda g_{ij} = g_{hj}\left(\eta_{\bar{i}} v^{\bar{h}}\right).$$

Substituting $v^{\bar{h}} = u^s \phi_s^h + B^h$ into the equation (15) we obtain

$$\frac{a}{2} L_V g_{ij} + G_{ih}\nabla_j B^h = \lambda a g_{ij} \tag{17}$$

and

$$\nabla_j \phi_s^h = 0. \tag{18}$$

Contracting with g^{ij} in the equation (17), we have

$$\lambda = \frac{1}{ka}\left(a\nabla_j v^j + F_h^j \nabla_j B^h\right).$$

If we use (13) into the last equation, we have

$$\phi_j^j = 2\nabla_j v^j + \frac{2}{a} F_h^j \nabla_j B^h. \tag{19}$$

Substitution $v^{\bar{h}} = u^s \phi_s^h + B^h$ into the equation (16) gives that

$$L_V G_{ij} - G_{ih}\nabla_j v^h - G_{ih}\phi_j^h = \lambda G_{ij}. \tag{20}$$

Contracting with G^{ij} the last equation and using (13), we find

$$\phi_j^j = \frac{1}{3}\nabla_j v^j. \tag{21}$$

From (19) and (21) we obtain

$$\frac{5}{3}\nabla_j v^j + \frac{2}{\alpha} F_h^j \nabla_j B^h = 0$$

and

$$\phi_j^j = \frac{-2}{5\alpha} F_h^j \nabla_j B^h,$$

also because of $\nabla_i \phi_s^h = 0$, the following result is obtained

$$\nabla_i\left(\nabla_j B^h\right) = 0.$$

Also, it is easy to see that the equation (18) is verified by means of (20).

Conversely by a routine calculation, the accuracy of the theorem can be easily checked under conditions the base manifold being flat and $(i) - (iv)$. □

2.3. *Almost Yamabe soliton on tangent bundle according to the ciconia metric*

On a complete Riemannian manifold (X, g), the Riemannian metric g admits a Yamabe soliton if the satisfies

$$\frac{1}{2} L_V g = (r - \lambda)\, g,$$

where λ is a scalar constant and V known as soliton vector field is a differentiable vector field and r is the scalar curvature of (X, g). If λ is a smooth function, then the Riemannian metric g admits an almost Yamabe soliton [2]. Almost Yamabe solitons are the natural generalization of Yamabe solitons. Moreover, we say that an almost Yamabe soliton is steady, expanding or shrinking if $\lambda = 0$, $\lambda < 0$ or $\lambda > 0$, respectively. It is obvious that Einstein manifolds are almost Yamabe solitons. On the tangent bundle TX with the ciconia metric $\widetilde{g}$, an almost Yamabe soliton satisfies the following equation:

$$\frac{1}{2}L_{\widetilde{A}}\widetilde{g} = \left(\widetilde{r} - \widetilde{\lambda}\right)\widetilde{g}, \tag{22}$$

where $\widetilde{A}$ and $\widetilde{\lambda}$ is a vector field and a smooth function on TX, respectively.

Theorem 2.5. *Let (X_{2k}, F, g) ($k \geq 1$) be an anti-paraKähler manifold and TX its tangent bundle with the ciconia metric $\widetilde{g}$. The $\left(TX, \widetilde{g}, \widetilde{A}, \widetilde{\lambda}\right)$ is an almost Yamabe soliton if and only if*

i) $\widetilde{A} = (v^h, v^{\overline{h}}) = (v^h, u^s\phi_s^h + B^h)$,
ii) $v^m\mathcal{R}_{mis}{}^{h} + \nabla_i\phi_s^h = 0$,
iii) $\widetilde{\lambda} = \tilde{r} - \frac{1}{2k}\left(\nabla_h v^h - \phi_h^h - bF_h^i\nabla_i B^h\right)$,

where $\widetilde{A} = v^h\eta_h + v^{\overline{h}}\eta_{\overline{h}}$ is a fibre-preserving vector field on TX and $\widetilde{\lambda}$ is a smooth function on TX.

Proof. We will show that the existence of the smooth function $\widetilde{\lambda}$ on TX. If the expression of $L_{\widetilde{A}}\widetilde{g}$ in Lemma 2.1 is used in the equation (22), we get

$$\frac{1}{2}\left[aL_V g_{ij} + 2G_{ih}\left(u^s v^m \mathcal{R}_{mjs}^h + \Gamma_{mj}^h v^{\overline{m}} + \eta_j v^{\overline{h}}\right)\right] = \left(\tilde{r} - \widetilde{\lambda}\right)ag_{ij}, \tag{23}$$

$$\begin{aligned}\left(\tilde{r} - \widetilde{\lambda}\right)G_{ij} &= L_V G_{ij} - G_{ih}\nabla_j v^h - G_{ih}\left(\eta_{\bar{j}} v^{\overline{h}}\right)\\ &\quad + bg_{hj}\left(u^s v^m \mathcal{R}_{mis}^h + \Gamma_{mi}^h v^{\overline{m}} + \eta_i v^{\overline{h}}\right),\end{aligned} \tag{24}$$

$$g_{hj}\left(\eta_{\bar{i}} v^{\overline{h}}\right) = g_{ij}. \tag{25}$$

Applying $\eta_{\overline{k}}$ to both sides of the equation (25), we get

$$\begin{aligned}\eta_{\overline{k}}\left(\eta_{\bar{i}} v^{\overline{h}}\right) &= 0\\ v^{\overline{h}} &= u^s\phi_s^h + B^h,\end{aligned} \tag{26}$$

where $B = \left(B^h\right)$ and $A = \left(\phi_h^s\right)$ are $(1,0)$ and $(1,1)-$ tensor fields on X_{2k}, respectively, and $\tilde{r}$ is the scalar curvature of the ciconia metric $\tilde{g}$. Substituting the equation (26) into the equation (24), we have

$$L_V G_{ij} - G_{ih}\nabla_j v^h - G_{ih}A_j^h + bg_{hj}\nabla_i B^h = \left(\tilde{r} - \widetilde{\lambda}\right)G_{ij} \tag{27}$$

and

$$v^m\mathcal{R}_{mis}{}^{h} + \nabla_i\phi_s^h = 0.$$

Contracting with G^{ij} in the equation (27), we have

$$\nabla_h v^h - \phi_h^h + bF_h^i\nabla_i B^h = \left(\tilde{r} - \widetilde{\lambda}\right)2k,$$

from which, we get

$$\widetilde{\lambda} = \tilde{r} - \frac{1}{2k}\left(\nabla_h v^h - \phi_h^h - bF_h^i\nabla_i B^h\right).$$

Conversely, by routine calculations, we can check the accuracy of theorem under the conditions (*i*), (*ii*) and (*iii*) of the theorem. □

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