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Characterizations of zero-divisor graphs of certain rings

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Abstract. The aim of this paper is to study zero-divisor graphs of some polarity rings, and certain special rings whose zero-divisor graphs are of tournament. Especially, zero-divisor graphs of polar rings, J-polar rings and nil-polar rings are connected. In addition, a ring whose zero-divisor graph is a tournament, must be quasinormal, but the converse is not true.

1. Introduction

Let R be an associate ring with unit 1. As usual, denote by $U(R)$, $E(R)$ and $N(R)$ the set of all invertible elements of R, the set of all idempotents of R and the set of all nilpotent elements of R, respectively. In 1988, Beck [2] introduced the coloring properties of a graph, whose vertices are all the elements of the ring and two vertices are adjacent if their product is 0. In 1999, Anderson and Livingston [1] simplified this definition by zero-divisor graph, and proved that the zero-divisor graphs of commutative rings are always connected with the diameter at most three. In 2012, Dolžan and Oblak [11] proved that the zero-divisor graphs of semirings are always connected and have diameters at most 3.

In 2002, Koliha and Patrício [16] defined a set $comm(a) = \{y \in R | ay = ya\}$, the commutant of a in ring R, and introduced the notion of quasipolar elements of rings. In 2012, Ying and Chen [26] showed that every strongly π-regular ring is quasipolar, and if a ring R is quasipolar, then so is eRe, for any $e \in E(R)$. Furthermore, J-quasipolar rings and nil-quasipolar rings were studied in [6, 12], and every J-quasipolar ring was quasipolar. In 2015, Calci, Halicioglu and Harmanci [7] extended the results of J-quasipolar rings to weakly J-quasipolar rings, and proved that if a ring R is weakly J-quasipolar (or J-quasipolar), then it must be directly finite. In 2017, Pekacar Calci, Halicioglu and Harmanci [23] introduced δ-quasipolar rings, and proved that every abelian δ-quasipolar ring is strongly regular, and established the relation between δ-quasipolar ring and directly finite ring.

Motivated by these classes of quasipolarity versions of rings, we introduce polar rings, J-polar rings and nil-polar rings. A ring R is called a polar (J-polar) ring, if for each $a \in R$, there is an idempotent $p \in comm(a)$

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such that $a+p \in U(R)$ ($a+p \in J(R)$) and $ap \in N(R)$ ($a(1-p) \in N(R)$), where $J(R)$ is the Jacobson radical of R. A ring R is called a nil-polar ring, if for each $a \in R$, there is an idempotent $p \in comm(a)$ such that $a+p \in N(R)$.

There are several useful rings with special characteristics as follows. An element a in a ring R is said to be π-regular if there is an integer $n \geq 1$ and $b \in R$ satisfying $a^n = a^n b a^n$. A ring R is said to be π-regular if for any $a \in R$ is π-regular, and is called a left C_2 ring, if for any $a \in R$, $Ra \cong R$ as left R-module implies $Ra = Re$, for some $e \in E(R)$. A ring R is said to be semiprime if $a \in R$ and $aRa = 0$ imply $a = 0$. The centralizer of semiprime ring is studied in [28].

In this paper, we study zero-divisor graphs of some polarity rings, and certain special rings whose zero-divisor graphs are of tournament. In section 3, we first prove that zero-divisor graphs of polar rings, J-polar rings and nil-polar rings are connected. Motivated by the relation between quasipolar ring and directly finite ring (or strongly π-regular ring) [23, 26], we present that polar rings, J-polar rings and nil-polar rings are directly finite, but the converse is not true. Moreover, we prove that a π-regular ring (or left C_2 ring) is directly finite if and only if its zero-divisor graph is connected. In section 4, we show that a ring whose zero-divisor graph is a tournament, must be quasinormal, but the converse is not true. Furthermore, we prove that a semiprime ring must be reduced, under the condition mentioned above.

2. Preliminaries

Let $G = \{V, E\}$ be a graph. G is said to be complete if there is an edge between every pair of the vertices, that is, any two vertices are adjacent. A graph G is said to be connected if there is at least one path between any two vertices in G. A directed graph G is called a tournament if for every two vertices x and y in G, either $x \to y$ or $y \to x$ is an edge of G. The distance $d(x, y)$ in G of two vertices x and y is the length of a short $x - y$ path in G, if no such path exists, we write $d(x, y) = \infty$. The greatest distance between any two vertices in G is the diameter of G, denoted by $diam(G)$.

Let R be a ring. An element $0 \neq a \in R$ is called a left (right) zero-divisor if there exists $0 \neq x \in R$ such that $ax = 0$ ($xa = 0$). Denote by $Z_L(R)$ ($Z_R(R)$) the set of all left (right) zero-divisors of R. The zero-divisor graph of a ring R, denoted by $\Gamma(R)$, is a directed graph with the vertex set $Z(R)$ in which for any two vertices x and y, $x \to y$ is an edge if and only if $x \neq y$ and $xy = 0$.

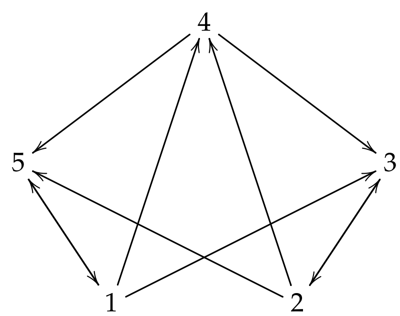
3. Zero-divisor graph and polarity rings

In this section, we work in an associative ring with unit 1 unless otherwise stated. We discuss the relation between polarity rings (or directly finite rings) and their zero-divisor graphs. It is well known that a zero-divisor graph which is connected, is not complete in general as follows.

Example 3.1. Let $R = T_2(\mathbb{Z}_2) = \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \middle| x, y, z \in \mathbb{Z}_2 \right\}$. It is easy to check that

$$Z(R) = \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\}.$$

We denote the five elements of $Z(R)$ by 1, 2, 3, 4 and 5, respectively. Then the corresponding zero-divisor graph $\Gamma(R)$ is



which is connected, but is not complete.

We first consider when the zero-divisor graph $\Gamma(R)$ of a ring R is connected. There is a relation between connected zero-divisor graph $\Gamma(R)$ and left (right) zero-divisor of R.

Lemma 3.2. [24, Theorem 2.3] Let R be a ring. Then the zero-divisor graph $\Gamma(R)$ is connected if and only if $Z_L(R) = Z_R(R)$.

Here, there is an example of noncommutative polar ring as follows.

Example 3.3. $R = M_2(\mathbb{Z}_2)$ is a ring with addition and multiplication of matrices.
$R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \right.$
$\left. \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}.$
Since $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, R is noncommutative. Moreover,
if $a = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, then $p = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. If $a = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then $p = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.
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If $a = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, then $p = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. If $a = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, then $p = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
Therefore, R is a polar ring. That is, R is a noncommutative polar ring.

From Lemma 3.2, in order to discuss when the zero-divisor graph $\Gamma(R)$ of a ring R is connected, it only remains to verify that $Z_L(R) = Z_R(R)$. Next, we will prove that $Z_L(R) = Z_R(R)$ in polar ring R (or J-polar ring, or nil-polar ring).

Proposition 3.4. *If R is a polar ring, then $Z_L(R) = Z_R(R)$.*

Proof. Assume that $a \notin Z_R(R)$. Then there exists $p^2 = p \in comm(a)$ such that $a + p \in U(R)$ and $ap \in N(R)$. Moreover, there is an integer $n \geq 1$ satisfying $(ap)^n = 0$, which implies $(pa^{n-1})a = 0$. Since $a \notin Z_R(R)$, we get $pa^{n-1} = 0$. Repeating the above process, we have $p = 0$, which gives $a \in U(R)$. That is, $a \notin Z_L(R)$. Hence $Z_L(R) \subseteq Z_R(R)$. In the same manner, we can see that $Z_R(R) \subseteq Z_L(R)$. □

Motivated by the proof of Proposition 3.4, we have the following two propositions.

Proposition 3.5. *If R is a J-polar ring, then $Z_L(R) = Z_R(R)$.*

Proof. Assume that $a \notin Z_R(R)$. Then there exists $p^2 = p \in comm(a)$ such that $a + p \in J(R)$ and $a(1 - p) \in N(R)$. From the proof of Proposition 3.4, we obtain $p = 1$. That is, $a + 1 \in J(R)$, which implies $a \in U(R)$. It means that $a \notin Z_L(R)$. Therefore, $Z_L(R) \subseteq Z_R(R)$. Similarly, $Z_R(R) \subseteq Z_L(R)$. □

Proposition 3.6. *If R is a nil-polar ring, then $Z_L(R) = Z_R(R)$.*

Proof. Assume that $a \notin Z_R(R)$. Then there exists $p^2 = p \in comm(a)$ such that $a+p \in N(R)$. Set $a+p = m \in N(R)$. Then $mp = pm$, because $p \in comm(a)$. On the other hand, since $m \in N(R)$, there exists an integer $n \geq 1$ satisfying $m^n = 0$. Thus, $[(1-p)m]^n = (1-p)m^n = 0$, which leads to $[(1-p)a]^n = 0$. As in the proof of Proposition 3.4, we infer that $p = 1$. It follows that $a = (a+1) - 1 = (a+p) - 1 = m - 1 \in U(R)$. That is, $a \notin Z_L(R)$. Consequently, $Z_L(R) \subseteq Z_R(R)$. In the same way, we obtain that $Z_R(R) \subseteq Z_L(R)$. □

Therefore, zero-divisor graphs of polar ring, J-polar ring, and nil-polar ring are connected from Lemma 3.2 and Proposition 3.4-3.6. Here we introduce a set $SN(R) = \{x \in R | x^n \neq 0, \textit{for all } n \geq 1\}$. Then, we can replace the condition $Z_L(R) = Z_R(R)$ by $Z(R) \cap SN(R) \subseteq Z_L(R) \cap Z_R(R)$ in Lemma 3.2, and the result still holds.

Theorem 3.7. *Let R be a ring. Then the zero-divisor graph $\Gamma(R)$ is connected if and only if $Z(R) \cap SN(R) \subseteq Z_L(R) \cap Z_R(R)$.*

Proof. "$\Leftarrow$" Take $a, b \in Z(R)$ and $a \neq b$.

(1) $ab = 0$. In this case, $a \to b$ is a path from a to b.

(2) $ab \neq 0$.

1) $a^n = 0$ and $a^{n-1} \neq 0$.

If there is an integer $k \geq 2$ such that $a^{n-1}b^{k-1} \neq 0$ and $a^{n-1}b^k = 0$, then there is a path $a \to a^{n-1}b^{k-1} \to b$.

If $a^{n-1}b^k \neq 0$, for all $k \geq 1$, then $b \in Z(R) \cap SN(R) \subseteq Z_L(R) \cap Z_R(R)$. Thus, there exists $0 \neq x \in R$ such that $xb = 0$.

If $a^{n-1}x = 0$, then there is a path $a \to a^{n-1} \to x \to b$.

If $a^{n-1}x \neq 0$, then there is also a path $a \to a^{n-1}x \to b$.

2) $b^m = 0$ and $b^{m-1} \neq 0$.

If there is an integer $k \geq 2$ such that $a^{k-1}b^{m-1} \neq 0$ and $a^k b^{m-1} = 0$, then there is a path $a \to a^{k-1}b^{m-1} \to b$.

If $a^k b^{m-1} \neq 0$, for all $k \geq 1$, then $a \in Z(R) \cap SN(R) \subseteq Z_L(R) \cap Z_R(R)$. Thus, there exists $0 \neq y \in R$ satisfying $ay = 0$.

If $yb^{m-1} = 0$, then there is a path $a \to y \to b^{m-1} \to b$.

If $yb^{m-1} \neq 0$, then there is also a path $a \to yb^{m-1} \to b$.

3) $a, b \in SN(R)$.

In this case, $a, b \in Z(R) \cap SN(R) \subseteq Z_L(R) \cap Z_R(R)$. Thus there exist $0 \neq x, y \in R$ such that $ax = 0 = yb$.

If $xy = 0$, then there is a path $a \to x \to y \to b$.

If $xy \neq 0$, then there is also a path $a \to xy \to b$.

Summarizing, there is always a path form a to b, and its distance $d(a, b)$ is no more than 3, which imply that the zero-divisor graph $\Gamma(R)$ is connected and $diam(\Gamma(R)) \leq 3$.

"$\Rightarrow$" Take $x \in Z(R) \cap SN(R)$. Then $x \in Z(R)$. Since $\Gamma(R)$ is connected, there exists $0 \neq y \in R$ such that $xy = 0$ or $yx = 0$.

If $xy = 0$, then $x \in Z_L(R)$. Since $\Gamma(R)$ is connected, there is a path $P = (V, E)$ from y to x, where $V = \{y = z_1, z_2, \cdots, z_r, z_{r+1} = x\}$ and $E = \{z_1z_2, z_2z_3, \cdots, z_rz_{r+1}\}$. That is, there exist distinct inner vertices $y = z_1, z_2, \cdots, z_{r+1} = x$ of P such that $z_1 \to z_2, z_2 \to z_3, \cdots, z_r \to z_{r+1}$ in $\Gamma(R)$. Thus $z_r x = 0$ which yields $x \in Z_R(R)$. Hence $x \in Z_L(R) \cap Z_R(R)$. Similarly, if $yx = 0$, then $x \in Z_L(R) \cap Z_R(R)$. □

Recall that a ring R is a directly finite ring if $ab = 1$ implies $ba = 1$, where $a, b \in R$. Based on the relation between polar ring and its left (right) zero-divisor, we will discuss the relation between directly finite ring and its left (right) zero-divisor.

Lemma 3.8. *Let R be a ring. If $Z_L(R) \cap SN(R) \subseteq Z_R(R)$, then R is a directly finite ring.*

Proof. Assume that $ab = 1$, where $a, b \in R$. Then $a(1 - ba) = 0$. If $1 - ba \neq 0$, then $a \in Z_L(R) \cap SN(R) \subseteq Z_R(R)$. In fact, if $a \notin SN(R)$, then there is an integer n such that $a^n = 0$. Moreover, $a^{n-1} = a^n b = 0$. Repeating the above step, we have $a = a^2 b = 0$, which is a contradiction. Thus, there exists $0 \neq c \in R$ such that $ca = 0$. It follows that $c = c1 = cab = 0$, which is a contradiction. Hence $ba = 1$. □

However, the converse of Lemma 3.8 is not true from the following example.

Example 3.9. Let $R = \left\{ \begin{pmatrix} a & a & b \\ 0 & 0 & c \\ 0 & 0 & d \end{pmatrix} \middle| a, b, c, d \in \mathbb{R} \right\}$. It is easy to verify that R is a directly finite ring. Take $0 \neq A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, and $0 \neq B = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix} \in R$. Then $AB = 0$, and $A^n = \begin{pmatrix} 1 & 1 & n-1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \neq 0$, for all $n \geq 1$, which imply $A \in Z_L(R) \cap SN(R)$. Assume that $A \in Z_R(R)$. Then there exists $0 \neq C \in R$ such that $CA = 0$. Write $C = \begin{pmatrix} x & x & y \\ 0 & 0 & z \\ 0 & 0 & r \end{pmatrix} \in R$. Then we have $CA = \begin{pmatrix} x & x & x+y \\ 0 & 0 & z \\ 0 & 0 & r \end{pmatrix} = 0$. This gives $x = y = z = r = 0$, that is, $C = 0$, which is a contradiction. Hence $A \notin Z_R(R)$.

Therefore, polar ring, J-polar ring, and nil-polar ring are directly finite, but the converse is not true. In what follows, we consider the relation between zero-divisor graph $\Gamma(R)$ of a ring R is connected and it is a directly finite ring.

Proposition 3.10. *Let R be a left C_2 ring. Then $\Gamma(R)$ is connected if and only if R is a directly finite ring.*

Proof. Assume that R is a directly finite ring. Fix $0 \neq a \in R \setminus Z_R(R)$. We first define a mapping $\sigma : R \to R$, $r \mapsto ra$. It is easy to check that the mapping σ is a left R-homomorphism and $Ra = Im\sigma \cong R$. Thus, there exists $e \in E(R)$ such that $Ra = Re$. Write $e = ba$. Then we have $a = aba$, which implies $ab = 1$. Moreover, $ba = 1$. If $a \in Z_L(R)$, then there is $0 \neq x \in R$ such that $ax = 0$, that is, $x = bax = 0$, which is a contradiction. Hence $a \notin Z_L(R)$. It means that $Z_L(R) \subseteq Z_R(R)$. Similarly, we obtain $Z_R(R) \subseteq Z_L(R)$. From Lemma 3.2, $\Gamma(R)$ is connected. The converse is obvious by Lemma 3.2 and 3.8. □

Recall that an element a in a ring R is said to be regular if there exists $b \in R$ such that $aba = a$, and is said to be strongly π-regular if there is an integer $n \geq 1$ and $b, c \in R$ satisfying $a^n = a^{n+1}b$ and $a^n = ca^{n+1}$. It is obvious that if $a \in R$ is regular or strongly π-regular, then a must be π-regular. Denote by R^{reg} the set of all regular elements of R. A ring is said to be regular (strongly π-regular) if every element in ring is regular (strongly π-regular). It is not a necessary condition that a ring R is a left C_2 ring in Proposition 3.10. Next, we consider π-regular ring, and the conclusion still holds.

Proposition 3.11. *Let R be a π-regular ring. Then $\Gamma(R)$ is connected if and only if R is a directly finite ring.*

Proof. Suppose that R is a directly finite ring. Fix $0 \neq a \in R \setminus Z_R(R)$. Since a is π-regular, there is an integer $n \geq 1$ and an element $b \in R$ such that $a^n = a^n ba^n$, which gives $(1-a^n b)a^n = 0$. Thus, $a^n b = 1$, because $a \notin Z_R(R)$. Since R is a directly finite ring, we have $ba^n = 1$, which leads to $a \notin Z_L(R)$ by the proof of Proposition 3.10. That is, $Z_L(R) \subseteq Z_R(R)$. Similarly, we get $Z_R(R) \subseteq Z_L(R)$. □

As all we know, if a ring R is regular, then it is π-regular. From Proposition 3.11, we have the following corollary.

Corollary 3.12. *Let R be a regular ring. Then $\Gamma(R)$ is connected if and only if R is a directly finite ring.*

Moreover, if a ring R is strongly π-regular, then it must be a directly finite ring by the following corollary. From Proposition 3.11, the zero-divisor graph $\Gamma(R)$ of R is connected.

Corollary 3.13. *If R is a strongly π-regular ring, then $\Gamma(R)$ is connected.*

Proof. Assume that $ab = 1$, where $a, b \in R$. There is an integer $n \geq 1$ and an element $c \in R$ such that $b^n = b^{n+1}c$. It is easy to see that $1 = a^n b^n = a^n b^{n+1} c = bc$, which implies $a = a1 = abc = 1c = c$. That is, $ba = bc = 1$. Consequently, R is a directly finite ring. From Proposition 3.11, we obtain that $\Gamma(R)$ is connected. □

4. Tournament and some special rings

Motivated by Section 3, this section is devoted to the study of some rings, whose zero-divisor graph are of tournament. Recall that a ring R is quasinormal, if $eR(1-e)Re = 0$ for each $e \in E(R)$ [25]. The following proposition describes that thus a ring must be quasinormal, under the condition stated above.

Proposition 4.1. *Let R be a ring. If $\Gamma(R)$ is a tournament, then R is a quasinormal ring.*

Proof. Assume that $e \in E(R)$. If $eR(1-e)Re \neq 0$, then there exist $x, y \in R$ satisfying $ex(1-e)ye \neq 0$. Thus, $1-e \neq 0$. It follows that there is a path $1-e \to ex(1-e)ye \to 1-e$, which proves that $1-e = ex(1-e)ye$. It is easy to see that $ex(1-e)ye = 0$, which is a contradiction. Therefore, $eR(1-e)Re = 0$. □

The converse of Proposition 4.1 is not true from the following example.

Example 4.2. Let $R = T_2(\mathbb{Z}_2) = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| a, b, c \in \mathbb{Z}_2 \right\}$. It is easy to check that

$$E(R) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

By direct computation, we infer that R is a quasinormal ring. Furthermore, there is a path $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Therefore, $\Gamma(R)$ is not a tournament.

Then, we will discuss regular elements in a ring with the condition stated above. Recall that the group inverse of a in a ring R is the element $a^\# \in R$ satisfying $aa^\#a = a$, $a^\#aa^\# = a^\#$, $aa^\# = a^\#a$. Note that if $a^\#$ exists, then it is unique [3]. We denote the set of all group invertible elements of R by $R^\#$. An element $a \in R$ is group invertible if and only if $a \in a^2R \cap Ra^2$ [8, 22].

Proposition 4.3. *Let R be a ring. If $\Gamma(R)$ is a tournament, then for any regular element $a \in R$ is either $a \in U(R)$ or $a^2 = 0$.*

Proof. Since a is regular, there is an element $b \in R$ such that $a = aba$. Write $e = ba$. Then $e^2 = e$ and $a = ae$. It means that $eb(1-e)a = eb(1-e)ae \in eR(1-e)Re$. From Proposition 4.1, we get $eb(1-e)a = 0$, which implies $e = ebea = eb^2a^2$. It follows that $a = ae = ab^2a^2 \in Ra^2$. We now apply this argument again, with $e = ba$ replaced by $g = ab$, to obtain $a = a^2b^2a \in a^2R$. So $a \in R^\#$, that is, $a(1 - a^\#a) = 0 = (1 - a^\#a)a$. If $1 - a^\#a \neq 0$, then there is a path $a \to 1 - a^\#a \to a$, which yields $a = 1 - a^\#a$. Hence $a^2 = (1 - a^\#a)a = 0$. If $1 - a^\#a = 0$, then $a \in U(R)$. □

Corollary 4.4. *Let R be a ring. If $\Gamma(R)$ is a tournament, then $R^\# = U(R) \bigcup \{0\}$.*

Proof. Suppose that $a \in R^\#$. Then a is a regular element. If $a \notin U(R)$, then $a^2 = 0$ by Proposition 4.3, which infers that $a = a^\#a^2 = 0$. □

Let R be a $*$-ring. The Moore-Penrose inverse (or MP-inverse) [21] of $a \in R$ is the element $a^\dagger \in R$ satisfying $aa^\dagger a = a$, $a^\dagger aa^\dagger = a^\dagger$, $(aa^\dagger)^* = aa^\dagger$, $(a^\dagger a)^* = a^\dagger a$. There is at most one $a^\dagger$ satisfying the above equations [13, 14, 17]. Denote by $R^\dagger$ the set of all MP-invertible elements of R. An element $a \in R^\dagger$ satisfying $aa^\dagger = a^\dagger a$ is said to be EP. Denote by R^{EP} the set of all EP elements of R. Various characterizations of EP element in complex matrices, Hilbert spaces and rings with involution, are presented in [4, 5, 9, 10, 18–20, 27].

Corollary 4.5. *Let R be a $*$-ring. If $\Gamma(R)$ is a tournament, then $R^\# = R^\dagger$.*

Proof. Assume that $0 \neq a \in R^{\dagger}$. Then $aa^* \in R^{\#}$ and $aa^* \neq 0$. In fact, if $aa^* = 0$, then $a = aa^*(a^{\dagger})^* = 0$, which is a contradiction. From Proposition 4.1 and Corollary 4.4, $aa^* \in U(R)$ and R is a quasinormal ring. From [25, Theorem 2.4], R is a directly finite ring. Hence $a \in U(R) \subseteq R^{\#}$. Thus, $R^{\dagger} \subseteq R^{\#}$. On the other hand, it is clear that $U(R) \subseteq R^{\dagger}$. From Corollary 4.4, $R^{\#} \subseteq R^{\dagger}$. Therefore, $R^{\#} = R^{\dagger}$. □

From Corollary 4.4 and the proof of Corollary 4.5, we have the following corollary.

Corollary 4.6. *Let R be a $*$-ring. If $\Gamma(R)$ is a tournament, then $R^{\#} = R^{EP} = R^{reg}$.*

Recall that a ring R is called a *CN* ring if $N(R) \subseteq C(R)$, where $C(R)$ is the center of R, and is called a reduced ring if $N(R) = 0$. In [15], it is shown that a ring R is reduced if and only if the classical right quotient ring of R is reduced. Next, we will find out that in what conditions can a ring R, whose zero-divisor graph $\Gamma(R)$ is a tournament, be a reduced ring (or *CN* ring)? Thus, we first consider a ring, which is semiprime.

Theorem 4.7. *If R is a semiprime ring and $\Gamma(R)$ is a tournament, then R is a reduced ring.*

Proof. Suppose that the assertion of the theorem is false. Then there exists an element $0 \neq a \in R$ and an integer $n \geq 2$ such that $a^n = 0$ and $a^{n-1} \neq 0$. This means that there is a path $a \to a^{n-1} \to a$. Since $\Gamma(R)$ is a tournament, it follows that $a = a^{n-1}$, which implies $a^2 = 0$. Next, we only need to show that $aRa = 0$. If there is an element $x \in R$ satisfying $axa \neq 0$, then $axa = a$, because there is a path $a \to axa \to a$. That is, $a = axaxa$ and $xax \neq 0$. If $ax^2a \neq 0$, then there is a path $a \to ax^2a \to a$, which leads to $a = ax^2a$. Thus, $ax(1 - xax) = 0 = (1 - xax)xa$. We claim that $1 - xax \neq 0$. In fact, if $1 - xax = 0$, then $xax = 1$. It follows that $a = axax = ax$ and $1 = xax = xa$. Thus $a = xa^2 = 0$, which is a contradiction. Furthermore, there is a path $xax \to 1 - xax \to xax$, which yields $xax = 1 - xax$. That is, $xax^2 = x - xax^2$. It follows that $xa = xax^2a = (x - xax^2)a = xa - xa = 0$, that is, $a = axa = 0$, which is a contradiction. Thus, $ax^2a = 0$. It means that there is a path $ax \to xa \to ax$, which gives $ax = xa$. We thus get $a = axa = xa^2 = 0$, which is a contradiction. From the above discussions, we obtain $aRa = 0$. Since R is a semiprime ring, we have $a = 0$, which is also a contradiction. Therefore, R is a reduce ring. □

According to the above result, in what follows, we will discuss a ring R with the condition that there is an integer $n \geq 1$ such that $a^n \in C(R)$ for any $a \in SN(R)$.

Theorem 4.8. *Let R be a ring. If $\Gamma(R)$ is a tournament, and there is an integer $n \geq 1$ such that $a^n \in C(R)$ for any $a \in SN(R)$, then R is a CN ring.*

Proof. Assume that $a \in N(R)$. If $a = 0$, then $a \in C(R)$. If $a \neq 0$, then there exists an integer $n \geq 2$ such that $a^n = 0$ and $a^{n-1} \neq 0$. Assume that there is an element $x \in R$ satisfying $ax - xa \neq 0$.
If $a^{n-1}(ax - xa) \neq 0$, then there is a path $a^{n-1}(ax - xa) \to a^{n-1} \to a^{n-1}(ax - xa)$, which gives $a^{n-1}(ax - xa) = a^{n-1}$. It follows that $a^{n-1}(ax - xa)a = a^{n-1}a = 0$. There is also a path $a^{n-1}(ax - xa) \to a \to a^{n-1}(ax - xa)$, which yields $a = a^{n-1}(ax - xa) = a^{n-1}$. So $a^2 = 0$ and $a \neq 0$. Moreover, $a(ax - xa) = a^{n-1}(ax - xa) = a \neq 0$, that is, $a = -axa$. Set $e = -ax$. Then $e^2 = e \in SN(R)$. By the hypothesis, we have $e \in C(R)$. The result is $a = ea = ae = -a^2x = 0$, which is a contradiction. Therefore, $a^{n-1}(ax - xa) = 0$. By a similar argument, we can get $(ax - xa)a^{n-1} = 0$.
From the above discussions, there is a path $a^{n-1} \to ax - xa \to a^{n-1}$, which shows that $a^{n-1} = ax - xa$. It follows that $a(ax - xa) = aa^{n-1} = 0 = a^{n-1}a = (ax - xa)a$. There is also a path $a \to ax - xa \to a$, which leads to $a = ax - xa = a^{n-1}$. Hence $a^2 = 0$ and $axa = 0$. If $ax \neq 0$, then there is a path $ax \to a \to ax$, which implies $ax = a$ and $xa = 0$. This means $a = ax = ax^2 = \cdots = ax^n = \cdots$. Since $a \neq 0$, we have $x \in SN(R)$. By assumption, there exists an integer $n \geq 1$ such that $x^n \in C(R)$. This forces $a = ax^n = x^na = 0$, which is a contradiction. Consequently, $ax = 0$.
In conclusion, we can deduce that $a = ax - xa = -xa = (-x)^2a = \cdots = (-x)^ka = \cdots$. Since $a \neq 0$, there is an integer $k \geq 1$ satisfying $(-x)^k \in C(R)$. Thus $a = a(-x)^k = 0$, which is a contradiction. Hence, $ax - xa = 0$ for any $x \in R$, that is, $a \in C(R)$. The proof is completed. □

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