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Anti-invariant and Clairaut anti-invariant pseudo-Riemannian submersions in para-Kenmotsu geometry

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Abstract. In this paper, we describe anti-invariant and Clairaut anti-invariant pseudo-Riemannian submersions (AIPR and CAIPR submersions, respectively, briefly) from para-Kenmotsu manifolds onto Riemannian manifolds. We introduce new Clairaut circumstances for anti-invariant submersions whose total space is para-Kenmotsu manifold. Also, we offer a obvious example of CAIPR submersion.

1. Introduction

In the the theory of the time-like geodesics upon a surface of revolution, the fundamental Clairaut's theorem states that for all time-like geodesic ϖ on a surface S the product $\Theta \cosh \beta$ is constant along time-like geodesic ϖ where Θ is the distance from a point on the surface to the axis of rotation and β is the angle between ϖ and the meridian curve through ϖ. This property was implemented to the pseudo-Riemannian submersions ([21]) by Allison ([2]). In ([25]), Şahin investigated Clairaut Riemannian map by using a geodesic curve on the total space and obtained necessary and sufficient conditions for Riemannian map to be Clairaut Riemannian map. Compared with the giant literature on Riemannian submersions, it seems that there are necessary new studies in anti-invariant Riemannian submersions; an interesting paper connecting these fields is ([26]).

Given a $C^\infty-$submersion ψ from a pseudo-Riemannian manifold $(\mathcal{B}, g_{\mathcal{B}})$ onto a pseudo-Riemannian manifold $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$, according to the conditions on the map $\psi : (\mathcal{B}, g_{\mathcal{B}}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$, we have the following: A pseudo-Riemannian submersion ([3],[9],[21],[23]), an almost Hermitian submersion ([24],[29]), a para-contact submersion ([10]), an anti-invariant submersion ([11],[13],[1],[8],[28],[14],[15]), a Clairaut submersion ([5],[20], [27], [12]), etc. As we know, Riemannian submersions were severally introduced by B. O'Neill ([21]) and A. Gray ([16]) in 1960s. In particular, by using the concept of almost Hermitian submersions, B. Watson ([30]) gave some differential geometric properties among fibers, base manifolds, and total manifolds. Actually Riemannian submersions have their implementations in the Yang-Mills theory ([7]), Kaluza-Klein theory ([6],[17]), supergravity and superstring theories ([18]).

Motivated by the above studies, we presented CAIPR submersions from para-Kenmotsu manifolds onto Riemannian manifolds. We organized our work in four sections. In section 2, we gather basic concepts

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and definitions needed in the following parts. In section 3, we examined AIPR submersions in para-Kenmotsu geometry that satisfies certain conditions. In section 4, we examined CAIPR submersion of these submersions which satisfy a new condition. Also, we give a obvious example of CAIPR submersion.

2. Preliminaries

2.1. Para-Kenmotsu manifolds

[31] Let $(\mathcal{B}, g_{\mathcal{B}})$ be an almost para-contact manifold of dimensionl (2m+1) with structure tensors (φ, ξ, η) where φ is a tensor field of type (1,1), ξ is a vector field, η is a 1-form. Then these tensors satisfy

$$\varphi\xi = 0, \ \ \eta o\varphi = 0, \ \ \eta(\xi) = 1 \ \ \ \varphi^2 = I - \eta \otimes \xi \tag{1}$$

If we take the $\mathcal{B}$ together with the pseudo-Riemannian metric $g_{\mathcal{B}}$ such that

$$g_{\mathcal{B}}(\varphi Y, \varphi \mathcal{Z}) = -g_{\mathcal{B}}(Y, \mathcal{Z}) + \eta(Y)\eta(\mathcal{Z}), \ \ Y, \mathcal{Z} \in \chi(\mathcal{B}) \tag{2}$$

$(\varphi, \xi, \eta, g_{\mathcal{B}})$ is an almost para-contact metric structure and we can say that $(\mathcal{B}^{2m+1}, \varphi, \xi, \eta, g_{\mathcal{B}})$ is an almost para-contact metric manifold. From (1) and (2) can be deduced the following conclusion

$$g_{\mathcal{B}}(Y, \varphi\mathcal{Z}) = -g_{\mathcal{B}}(\varphi Y, \mathcal{Z}), \ \ \eta(U) = g_{\mathcal{B}}(U, \xi), \ \ \ U \in \chi(\mathcal{B}) \tag{3}$$

Furthermore, the fundamental 2−form Φ is defined $\Phi(Y, \mathcal{Z}) = g_{\mathcal{B}}(Y, \varphi\mathcal{Z})$ for any $Y, \mathcal{Z} \in T\mathcal{B}$.

$(\mathcal{B}, \varphi, \xi, \eta, g_{\mathcal{B}})$ is known to be a para-Kenmotsu manifold ([31]) if and only if

$$(\nabla_Y \varphi)\mathcal{Z} = g_{\mathcal{B}}(Y, \varphi\mathcal{Z})\xi + \eta(\mathcal{Z})\varphi Y, \ \ \ Y, \mathcal{Z} \in \chi(\mathcal{B}) \tag{4}$$

From (4) can be deduced the following conclusion

$$\nabla_Y \xi = -Y + \eta(Y)\xi \tag{5}$$

2.2. Pseudo-Riemannian submersions

Let $(\mathcal{B}, g_{\mathcal{B}})$ and $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be two pseudo-Riemannian manifolds. Being a pseudo-Riemannian submersion $\psi : \mathcal{B} \to \tilde{\mathcal{B}}$ provides the following three properties;
(i) $\psi_{*|p}$ is onto for all $p \in \mathcal{B}$,
(ii) the fibres $\psi^{-1}(q)$, $q \in \tilde{\mathcal{B}}$, are $r-$ dimensional pseudo-Riemannian submanifolds of $\mathcal{B}$, where $r = dim(\mathcal{B}) - dim(\tilde{\mathcal{B}})$.
(iii) ψ_* preserves scalar products of vectors normal to fibres.
The vectors tangent to the fibres are called vertical and those normal to the fibres are called horizontal. A vector field U on $\mathcal{B}$ is called basic if U is horizontal and ψ- related to a vector field U_* on $\tilde{\mathcal{B}}$, i.e., $\psi_* U_p = U_{*\psi_p}$ for all $p \in \mathcal{B}$. We indicate by $\mathcal{V}$ the vertical distribution, by $\mathcal{H}$ the horizontal distribution and by v and h the vertical and horizontal projection. We know that $(\mathcal{B}, g_{\mathcal{B}})$ is called total manifold and $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ is called base manifold of the submersion $\psi : (\mathcal{B}, g_{\mathcal{B}}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$.

Lemma 2.1. *([9], [21]) If $\psi : \mathcal{B} \to \tilde{\mathcal{B}}$ is a pseudo-Riemannian submersion and Y_1, Y_2 basic vector fields on $\mathcal{B}$, $\psi-$related to Y_{1*} and Y_{2*} on $\tilde{\mathcal{B}}$ then we have the following properties*

1. *$h[Y_1, Y_2]$ is the basic vector field associated to $\psi_* h[Y_{1*}, Y_{2*}] \circ \psi$;*
2. *$h(\nabla^1_{Y_1} Y_2)$ is the basic vector field $\psi-$related to $(\nabla^2_{Y_{1*}} Y_{2*})$, where ∇^1 and ∇^2 are the Levi-Civita connection on $\mathcal{B}$ and $\tilde{\mathcal{B}}$.*
3. *$[F, Y_3] \in \Gamma(ker\psi_*)$ for basic vector field F and $Y_3 \in \Gamma(ker\psi_*)$.*

Now, let's denote O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$([21]):

$$\mathcal{T}_U \mathcal{W} = h\nabla_{vU} v\mathcal{W} + v\nabla_{vU} h\mathcal{W} \tag{6}$$

and

$$\mathcal{A}_U \mathcal{W} = v\nabla_{hU} h\mathcal{W} + h\nabla_{hU} v\mathcal{W} \tag{7}$$

for every $U, \mathcal{W} \in \chi(\mathcal{B})$, on $\mathcal{B}$ where ∇ is the Levi-Civita connection of $g_{\mathcal{B}}$.
Further, a pseudo-Riemannian submersion $\psi : \mathcal{B} \to \tilde{\mathcal{B}}$ has totally geodesic fibers if and only if $\mathcal{T} \equiv 0$. Also, if $\mathcal{A}$ vanishes then the horizontal distribution is integrable(see [3],[9]). Using (6) and (7), we get

$$\nabla_U W = \mathcal{T}_U W + \hat{\nabla}_U W; \tag{8}$$

$$\nabla_U \zeta = \mathcal{T}_U \zeta + h\nabla_U \zeta; \tag{9}$$

$$\nabla_\zeta U = \mathcal{A}_\zeta U + v\nabla_\zeta U; \tag{10}$$

$$\nabla_\zeta \eta = \mathcal{A}_\zeta \eta + h\nabla_\zeta \eta, \tag{11}$$

for any $\zeta, \eta \in \Gamma((ker\psi_*)^\perp)$, $U, W \in \Gamma(ker\psi_*)$. Also, if ζ is basic then $h\nabla_U \zeta = h\nabla_\zeta U = \mathcal{A}_\zeta U$.
We can easily see that $\mathcal{T}$ is symmetric on the vertical distribution and $\mathcal{A}$ is alternating on the horizontal distribution such that

$$\mathcal{T}_{\mathcal{W}} U = \mathcal{T}_U \mathcal{W}, \quad \mathcal{W}, U \in \Gamma(ker\psi_*); \tag{12}$$

$$\mathcal{A}_Y V = -\mathcal{A}_V Y = \frac{1}{2} v[Y, V], \quad Y, V \in \Gamma((ker\psi_*)^\perp). \tag{13}$$

Assume that $\mathcal{W}_1, ..., \mathcal{W}_{m-n}$ be an orthonormal frame of $\Gamma(ker\psi_*)$. Further, H which is the horizontal vector field is known as the mean curvature vector field of the fiber such that $H = \frac{1}{m-n} \sum_{j=1}^{m-n} \mathcal{T}_{\mathcal{W}_j} \mathcal{W}_j$. A pseudo-Riemannian submersion is said to be minimal if and only if $H = 0$. Also, a pseudo-Riemannian submersion is known as pseudo-Riemannian submersion with totally umbilical fibers if

$$\mathcal{T}_V \mathcal{W} = g_{\mathcal{B}}(V, \mathcal{W}) H \tag{14}$$

for $V, \mathcal{W} \in \Gamma(ker\psi_*)$.

It is easily seen that for any $W \in \Gamma(T\mathcal{B})$, $\mathcal{T}_W$ and $\mathcal{A}_W$ are skew-symmetric operators on $\Gamma(T\mathcal{B})$, such that

$$g_{\mathcal{B}}(\mathcal{T}_{\mathcal{W}} U, \mathcal{X}) = -g_{\mathcal{B}}(\mathcal{T}_{\mathcal{W}} \mathcal{X}, U) \tag{15}$$

$$g_{\mathcal{B}}(\mathcal{A}_{\mathcal{W}} U, \mathcal{X}) = -g_{\mathcal{B}}(\mathcal{A}_{\mathcal{W}} \mathcal{X}, U) \tag{16}$$

$\psi : \mathcal{B} \to \tilde{\mathcal{B}}$ is a differentiable map. Then, $(\mathcal{B}, g_{\mathcal{B}})$ and $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be pseudo-Riemannian manifolds. Then, the second fundamental form of ψ is described by

$$(\nabla\psi_*)(\zeta, V) = \nabla_\zeta^\psi \psi_* V - \psi_*(\nabla_\zeta V) \tag{17}$$

for $\zeta, V \in \Gamma(\mathcal{B})$. When $trace(\nabla\psi_*) = 0$, we can say that ψ is harmonic and ψ is a totally geodesic map when $(\nabla\psi_*)(\zeta, V) = 0$ for $\zeta, V \in \Gamma(T\mathcal{B})$ ([19]). Recall that ∇^ψ is the pullback connection.

3. Anti-invariant pseudo-Riemannian submersions

In this section, while we investigate AIPR submersions from a para-Kenmotsu manifold onto a Riemannian manifold suppose that the characteristic vector field ξ is a horizontal vector field.

Definition 3.1. *Suppose that $\psi : (\mathcal{B}, \varphi, \xi, \eta, g_{\mathcal{B}}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a pseudo-Riemannian submersion. Let assume that the total manifold as a para-Kenmotsu manifold and base manifold as a Riemannian manifold. Then, there exists a pseudo-Riemannian submersion ψ such that $ker\psi_*$ is anti-invariant with respect to φ, i.e., $\varphi(ker\psi_*) \subseteq (ker\psi_*)^{\perp}$. So, we can say ψ is an anti-invariant pseudo-Riemannian submersion.*

Remark 3.2. In references ([2]) and ([4]), authors described the pseudo-Riemannian submersions from a pseudo-Riemannian manifold onto a Riemannian manifold. Therefore, we have defined above AIPR submersions from a para-Kenmotsu manifold onto a Riemannian manifold.

Suppose that $\psi : (\mathcal{B}, \varphi, \xi, \eta, g_{\mathcal{B}}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be an AIPR submersions with total manifold as a para-Kenmotsu manifold and base manifold as a Riemannian manifold. Using Definition 3.1, we obtain $\varphi(ker\psi_*) \cap (ker\psi_*)^{\perp} \neq 0$. Now, we can indicate the complementary orthogonal distribution to $\varphi(ker\psi_*)$ in $(ker\psi_*)^{\perp}$ by μ. Such that

$$(ker\psi_*)^{\perp} = \varphi(ker\psi_*) \oplus \mu \tag{18}$$

We can say that for $\wp \in \mathcal{B}$, $(ker\psi_*)$ is a time-like subspace and $(ker\psi_*)^{\perp}$is a space-like subspace of $T_{\wp}\mathcal{B}$. Then, for any space-like vector field $U \in \Gamma((ker\psi_*)^{\perp})$, we obtain

$$\varphi U = EU + FU \tag{19}$$

where $EU \in \Gamma(ker\psi_*)$ and $FU \in \Gamma(\mu)$. We know that ψ a pseudo-Riemannian submersion and in addition to $\psi_*((ker\psi_*)^{\perp})) = T\tilde{\mathcal{B}}$. Then, using (19) we get $g_{\tilde{\mathcal{B}}}(\psi_*\varphi V, \psi_* FU) = 0$, for all space-like vector field $U \in \Gamma((ker\psi_*)^{\perp})$ and time-like vector field $V \in \Gamma((ker\psi_*))$ such that

$$T\tilde{\mathcal{B}} = \psi_*(\varphi(ker\psi_*)) \oplus \psi_*(\mu) \tag{20}$$

The concept of Lagrangian submersion is considered a special case from the concept of anti-invariant submersion. Now, let us remember the definition of Lagrangian submersion.

Definition 3.3. Suppose that $\psi : (\mathcal{B}, \varphi, \xi, \eta, g_{\mathcal{B}}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be an AIPR submersions with total manifold as a para-Kenmotsu manifold and base manifold as a Riemannian manifold. We can say that $\mu = \{0\}$ or $\mu = span\{\xi\}$, i.e. $\varphi(ker\psi_*) = (ker\psi_*)^{\perp}$ or $\varphi(ker\psi_*) \oplus < \xi >= (ker\psi_*)^{\perp}$, in the same order, ψ is known the Lagrangian submersion.

Lemma 3.4. Suppose that $\psi : (\mathcal{B}, \varphi, \xi, \eta, g_{\mathcal{B}}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be an AIPR submersions with total manifold as a para-Kenmotsu manifold and base manifold as a Riemannian manifold. Then we obtain

$$A_{\mathcal{W}}\xi = 0 \tag{21}$$

$$T_{\mathcal{Z}}\xi = -\mathcal{Z} \tag{22}$$

$$g_{\mathcal{B}}(F\mathcal{W}, \varphi\mathcal{Z}) = 0 \tag{23}$$

$$g_{\mathcal{B}}(\nabla_{\mathcal{W}} F\mathcal{Y}, \varphi\mathcal{Z}) = -g_{\mathcal{B}}(F\mathcal{Y}, \varphi A_{\mathcal{W}}\mathcal{Z}) \tag{24}$$

for space-like vector fields $\mathcal{W}, \mathcal{Y} \in \Gamma((ker\psi_*)^{\perp})$ and time-like vector fields $\mathcal{Z} \in \Gamma(ker\psi_*)$.

Proof. From (15) and (5), we obtain (21). Further, from (9) and (5), we obtain (22). Then from (2) and (19) we obtain

$$\begin{aligned} g_{\mathcal{B}}(F\mathcal{W}, \varphi\mathcal{Z}) &= g_{\mathcal{B}}(\varphi\mathcal{W} - E\mathcal{W}, \varphi\mathcal{Z}) \\ &= -g_{\mathcal{B}}(\mathcal{W}, \mathcal{Z}) + \eta(\mathcal{W})\eta(\mathcal{Z}) + g_{\mathcal{B}}(\varphi E\mathcal{W}, \mathcal{Z}) \end{aligned}$$

for space-like vector fields $\mathcal{W} \in \Gamma((ker\psi_*)^\perp)$ and time-like vector fields $\mathcal{Z} \in \Gamma(ker\psi_*)$. Since $\varphi E\mathcal{W} \in \Gamma((ker\psi_*)^\perp)$ and $\xi \in \Gamma((ker\psi_*)^\perp)$ we arrive at (23). After from (10) and (4), we obtain

$$g_{\mathcal{B}}(\nabla_{\mathcal{W}} F\mathcal{Y}, \varphi\mathcal{Z}) = -g_{\mathcal{B}}(F\mathcal{Y}, \varphi A_{\mathcal{W}}\mathcal{Z}) - g_{\mathcal{B}}(F\mathcal{Y}, \varphi v\nabla_{\mathcal{W}}\mathcal{Z})$$

Since $\varphi v\nabla_{\mathcal{W}}\mathcal{Z} \in \Gamma((ker\psi_*)^\perp)$ we arrive at (24).

We accept that ψ is Lagrangian submersion for space-like vector fields $\mathcal{W} \in \Gamma((ker\psi_*)^\perp)$, we have

$$E\mathcal{W} = \varphi\mathcal{W}, \quad F\mathcal{W} = 0. \tag{25}$$

Lemma 3.5. Suppose that $\psi : (\mathcal{B}, \varphi, \xi, \eta, g_{\mathcal{B}}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be an AIPR submersion with $\varphi(ker\psi_*) \oplus <\xi>= (ker\psi_*)^\perp$. If the total manifold is a para-Kenmotsu manifold, then we obtain

$$T_{\mathcal{Z}}\varphi\zeta = \varphi T_{\mathcal{Z}}\zeta \tag{26}$$

$$A_{\mathcal{W}}\varphi\mathcal{Z} = \varphi A_{\mathcal{W}}\mathcal{Z} \tag{27}$$

for space-like vector field $\mathcal{W} \in \Gamma((ker\psi_*)^\perp)$ and time-like vector fields $\mathcal{Z}, \zeta \in \Gamma(ker\psi_*)$.

We now investigate the integrability of the distribution $(ker\psi_*)^\perp$ and we research the geometry of leaves of $(ker\psi_*)$ and $(ker\psi_*)^\perp$.

Theorem 3.6. *Suppose that $\psi : (\mathcal{B}, \varphi, \xi, \eta, g_{\mathcal{B}}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be an AIPR submersions with total manifold as a para-Kenmotsu manifold and base manifold as a Riemannian manifold. Then we obtain the following assertion;*

(i) $(ker\psi_*)^\perp$ *is integrable.*

(ii) $g_{\mathcal{B}}((\nabla\psi_*)(V, E\mathcal{W}, \psi_*\varphi\mathcal{Z}) = (\nabla\psi_*)(\mathcal{W}, EV), \psi_*\varphi\mathcal{Z})$
$\quad - g_{\mathcal{B}}(F\mathcal{W}, \varphi A_V\mathcal{Z}) + g_{\mathcal{B}}(FV, \varphi A_{\mathcal{W}}\mathcal{Z})$

(iii) $g_{\mathcal{B}}(A_{\mathcal{W}}EV - A_V E\mathcal{W}, \varphi\mathcal{Z}) = -g_{\mathcal{B}}(F\mathcal{W}, \varphi A_V\mathcal{Z}) + g_{\mathcal{B}}(FV, \varphi A_{\mathcal{W}}\mathcal{Z})$

for space-like vector fields $V, \mathcal{W} \in \Gamma((ker\psi_*)^\perp)$ and time-like vector fields $\mathcal{Z} \in \Gamma(ker\psi_*)$.

Proof. For space-like vector fields $V, \mathcal{W} \in \Gamma((ker\psi_*)^\perp)$ and time-like vector fields $\mathcal{Z} \in \Gamma(ker\psi_*)$. From (2), (5) and (19) we obtain

$$\begin{aligned} g_{\mathcal{B}}([V, \mathcal{W}], \mathcal{Z}) &= -g_{\mathcal{B}}(\nabla_V\varphi\mathcal{W}, \varphi\mathcal{Z}) + g_{\mathcal{B}}(\nabla_{\mathcal{W}}\varphi V, \varphi\mathcal{Z}) \\ &= -g_{\mathcal{B}}(\nabla_V E\mathcal{W}, \varphi\mathcal{Z}) - g_{\mathcal{B}}(\nabla_V F\mathcal{W}, \varphi\mathcal{Z}) \\ &+ g_{\mathcal{B}}(\nabla_{\mathcal{W}} EV, \varphi\mathcal{Z}) + g_{\mathcal{B}}(\nabla_{\mathcal{W}} FV, \varphi\mathcal{Z}). \end{aligned}$$

Using (10), (24) and we know that ψ is a pseudo-Riemannian submersion, we get

$$\begin{aligned} g_{\mathcal{B}}([V, \mathcal{W}], \mathcal{Z}) &= -g_{\mathcal{B}}(\psi_*\nabla_V E\mathcal{W}, \psi_*\varphi\mathcal{Z}) + g_{\mathcal{B}}(F\mathcal{W}, \varphi A_V\mathcal{Z}) \\ &+ g_{\mathcal{B}}(\psi_*\nabla_{\mathcal{W}} EV, \psi_*\varphi\mathcal{Z}) - g_{\mathcal{B}}(FV, \varphi A_{\mathcal{W}}\mathcal{Z}) \end{aligned}$$

Then, using (16) we obtain;

$$\begin{aligned} g_{\mathcal{B}}([V, \mathcal{W}], \mathcal{Z}) &= g_{\mathcal{B}}((\nabla\psi_*)(V, E\mathcal{W}) - (\nabla\psi_*)(\mathcal{W}, EV), \psi_*\varphi\mathcal{Z}) - g_{\mathcal{B}}(FV, \varphi A_{\mathcal{W}}\mathcal{Z}) \\ &+ g_{\mathcal{B}}(F\mathcal{W}, \varphi A_V\mathcal{Z}) \end{aligned}$$

which proves $(i) \iff (ii)$. Similarly, using (16) we obtain;

$$(\nabla\psi_*)(V, E\mathcal{W}) - (\nabla\psi_*)(\mathcal{W}, EV) = -\psi_*(\nabla_V E\mathcal{W} - \nabla_{\mathcal{W}} EV)$$

Thus, from (10) easily obtained

$$(\nabla\psi_*)(V, E\mathcal{W}) - (\nabla\psi_*)(\mathcal{W}, EV) = -\psi_*(A_V E\mathcal{W} - A_{\mathcal{W}} EV).$$

Since $A_V E\mathcal{W} - A_{\mathcal{W}} EV \in \Gamma((ker\psi_*)^\perp)$, we arrive at (*ii*) $\iff$ (*iii*).

Corollary 3.7. *Suppose that total manifold as a para-Kenmotsu manifold and base manifold as a Riemannian manifold. Let* $\psi : (\mathcal{B}, \varphi, \xi, \eta, g_{\mathcal{B}}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ *be an AIPR submersions such that* $\varphi(ker\psi_*) \oplus \xi = (ker\psi_*)^\perp$. *Then we obtain the following assertion;*

(i) $(ker\psi_*)^\perp$ *is integrable.*
(ii) $(\nabla\psi_*)(V, \varphi\mathcal{W}) = (\nabla\psi_*)(\mathcal{W}, \varphi V)$

for space-like vector fields $V, \mathcal{W} \in \Gamma((ker\psi_*)^\perp)$.

Theorem 3.8. *Suppose that* $\psi : (\mathcal{B}, \varphi, \xi, \eta, g_{\mathcal{B}}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ *be an AIPR submersions with total manifold as a para-Kenmotsu manifold and base manifold as a Riemannian manifold. Then we obtain the following assertion;*

(i) $(ker\psi_*)^\perp$ is describes a totally geodesic foliation on $\mathcal{B}$,
(ii) $g_{\mathcal{B}}(A_V E\mathcal{W}, \varphi\mathcal{Z}) = g_{\mathcal{B}}(\varphi A_V \mathcal{Z}, F\mathcal{W})$
(iii) $g_{\mathcal{B}}((\nabla\psi_*)(V, E\mathcal{W}), \psi_*\varphi Z) = -g_{\mathcal{B}}(\varphi A_V \mathcal{Z}, F\mathcal{W})$

Proof. For space-like vector fields $V, \mathcal{W} \in \Gamma((ker\psi_*)^\perp)$ and time-like vector field $\mathcal{Z} \in \Gamma(ker\psi_*)$. From (2), (4), (10) and (19) we obtain

$$g_{\mathcal{B}}(\nabla_V \mathcal{W}, \mathcal{Z}) = -g_{\mathcal{B}}(A_V E\mathcal{W}, \varphi\mathcal{Z}) - g_{\mathcal{B}}(A_V F\mathcal{W}, \varphi\mathcal{Z}).$$

By using (24), we obtain

$$g_{\mathcal{B}}(\nabla_V \mathcal{W}, \mathcal{Z}) = -g_{\mathcal{B}}(A_V E\mathcal{W}, \varphi\mathcal{Z}) + g_{\mathcal{B}}(\varphi A_V \mathcal{Z}, F\mathcal{W}) \tag{28}$$

which shows (*i*) $\iff$ (*ii*).
We know that ψ is a pseudo-Riemannian submersion and using (10), (24) in (28) we get

$$g_{\mathcal{B}}(\varphi A_V \mathcal{Z}, F\mathcal{W}) = g_{\mathcal{B}}((-\nabla\psi_*)(V, E\mathcal{W}), \psi_*\varphi Z)$$

which shows that (*ii*) $\iff$ (*iii*).

Corollary 3.9. *Suppose that total manifold as a para-Kenmotsu manifold and base manifold as a Riemannian manifold. Let* $\psi : (\mathcal{B}, \varphi, \xi, \eta, g_{\mathcal{B}}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ *be an AIPR submersions such that* $\varphi(ker\psi_*) \oplus \xi = (ker\psi_*)^\perp$. *Then we obtain the following assertion;*

(i) $(ker\psi_*)^\perp$ *is describes a totally geodesic foliation on* $\mathcal{B}$,
(ii) $A_V \varphi\mathcal{W} = 0$
(iii) $(\nabla\psi_*)(V, \varphi\mathcal{W}) = 0$

for space-like vector fields $V, \mathcal{W} \in \Gamma((ker\psi_*)^\perp)$.

Theorem 3.10. *Suppose that* $\psi : (\mathcal{B}, \varphi, \xi, \eta, g_{\mathcal{B}}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ *be an AIPR submersion with total manifold as a para-Kenmotsu manifold and base manifold as a Riemannian manifold. Then* $(ker\psi_*)$ *does not describe a totally geodesic foliation on* $\mathcal{B}$.

Proof. The proof is obtained directly using (8) and (22).

Theorem 3.11. *Suppose that* $\psi : (\mathcal{B}, \varphi, \xi, \eta, g_{\mathcal{B}}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ *be an AIPR submersion with total manifold as a para-Kenmotsu manifold and base manifold as a Riemannian manifold. Then* ψ *is not a totally geodesic map.*

Proof. We arrive at the proof using Theorem 3.10.

Theorem 3.12. *Suppose that* $\psi : (\mathcal{B}, \varphi, \xi, \eta, g_{\mathcal{B}}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ *be an AIPR submersion with total manifold as a para-Kenmotsu manifold and base manifold as a Riemannian manifold. Then* ψ *is not harmonic.*

Proof. Recall that ψ is harmonic if and only if has minimal fibers. Also, from the equation $H = \frac{1}{n}\sum_{j=1}^{n}\mathcal{T}_{E_j}E_j$, (8) and (14) we obtain

$$\begin{aligned} ng_{\mathcal{B}}(H,\xi) &= g_{\mathcal{B}}(\mathcal{T}_{E_1}E_1,\xi) + g_{\mathcal{B}}(\mathcal{T}_{E_2}E_2,\xi) + ... + g_{\mathcal{B}}(\mathcal{T}_{E_n}E_n,\xi) \\ &= -g_{\mathcal{B}}(\mathcal{T}_{E_1}\xi,E_1) - g_{\mathcal{B}}(\mathcal{T}_{E_2}\xi,E_2) - ... - g_{\mathcal{B}}(\mathcal{T}_{E_n}\xi,E_n) \\ &= -g_{\mathcal{B}}(E_1,E_1) - g_{\mathcal{B}}(E_2,E_2) - ... - g_{\mathcal{B}}(E_n,E_n) \\ &= -n \end{aligned}$$

Since $g_{\mathcal{B}}(H,\xi) = -1$, $ker\psi_*$ does not have minimal fibers. So, we can say that ψ is not harmonic.

4. Clairaut anti-invariant pseudo-Riemannian submersions

Suppose that $\psi : (\mathcal{B}, g_{\mathcal{B}}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a pseudo-Riemannian submersion with total manifold as a pseudo-Riemannian manifold and base manifold as a Riemannian manifold. Asssume that for any $\wp \in \mathcal{B}$, every horizontal vector field is space-like in $\mathcal{T}_{\wp}\mathcal{B}$. We easily see that $ker\psi_*$ is a time-like subspace and $(ker\psi_*)^{\perp}$ is a space-like of $\mathcal{T}_{\wp}\mathcal{B}$ [22].

Assume that ϖ is time-like geodesic in $(\mathcal{B}, g_{\mathcal{B}})$. For every V horizontal and every $\mathcal{Z}$ is time-like from the above expression $\varpi = F = \zeta + \mathcal{Z}$. The ζ space-like and time-like character of ϖ indicate that $\mathcal{Z}$ is time-like. At every point $\varpi(s)$, we describe $\beta(s)$ to be the hyperbolic angle between F and $\mathcal{Z}$, i.e., $\beta \geq 0$ is the number satisfying

$$g_{\mathcal{B}}(F,\mathcal{Z}) = -|F||\mathcal{Z}|\cosh\beta \tag{29}$$

where $|F|^2 = -g_{\mathcal{B}}(F,F)$ and $|\mathcal{Z}|^2 = -g_{\mathcal{B}}(\mathcal{Z},\mathcal{Z})$. Assuming β is the angle between the velocity vector of a time-like geodesic and a meridian, and Θ is the distance from the axis of a rotation surface. By the famous Clairaut's theorem, we know that $\Theta\cosh\beta$ is constant. [2] defined the idea of Clairaut submersion as follows.

$\psi : \mathcal{B} \to \tilde{\mathcal{B}}$ is called as the Clairaut submersion in case we can talk about the existence of a positive function such that for all time-like geodesic, generating β angles from space-like subspaces, $\Theta\cosh\beta$ is constant.

Theorem 4.1. [2] *Suppose that $\psi : (\mathcal{B}, g_{\mathcal{B}}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a pseudo-Riemannian submersion with connected fibers. Then ψ is a Clairaut submersion with $\Theta = e^{\wp}$ if and only if each fiber is totally umbilic and has the mean curvature vector field $H = -\nabla\wp$, where $\nabla\wp$ is the gradient of the function $\wp$ with respect to $g_{\mathcal{B}}$.*

By looking at the above explanations, the idea of Clairaut's submersion comes from the time-like geodesic on a surface. Hence, we will investigate necessary conditions for a curve on the total space to be time-like geodesic.

Theorem 4.2. *Suppose that $\psi : (\mathcal{B}, \varphi, \xi, \eta, g_{\mathcal{B}}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be an AIPR submersion with total manifold as a para-Kenmotsu manifold and base manifold as a Riemannian manifold. In case $\varpi : \mathcal{J} \subset \mathcal{R} \to \mathcal{B}$ is a regular curve and $\zeta(s)$ and $\mathcal{Z}(s)$ are the vertical and horizontal parts of the tangent vector field $\dot{\varpi}(s) = F$ of $\varpi(s)$, respectively, therefore ϖ is time-like geodesic if and only if through ϖ the following two equations:*

$$v\nabla_{\dot{\varpi}}E\mathcal{Z} + \mathcal{A}_{\mathcal{Z}}\varphi\zeta + (\mathcal{T}_{\zeta} + \mathcal{A}_{\mathcal{Z}})F\mathcal{Z} - \eta(\mathcal{Z})E\mathcal{Z} = 0. \tag{30}$$

$$h(\nabla_{\dot{\varpi}}\varphi\zeta + \nabla_{\dot{\varpi}}F\mathcal{Z}) + (\mathcal{T}_{\zeta} + \mathcal{A}_{\mathcal{Z}})E\mathcal{Z} - \eta(\mathcal{Z})(F\mathcal{Z} + \varphi\zeta) = 0. \tag{31}$$

Proof. Using (4), we get

$$\nabla_{\dot{\varpi}}\varphi\dot{\varpi} = \varphi\nabla_{\dot{\varpi}}\dot{\varpi} + g_{\mathcal{B}}(\dot{\varpi},\varphi\dot{\varpi})\xi + \eta(\dot{\varpi})\varphi\dot{\varpi}.$$

Since $\dot{\varpi} = \zeta + \mathcal{Z}$ and $\eta(\zeta) = 0$, we arrive at

$$\nabla_\zeta \varphi\zeta + \nabla_\zeta \varphi\mathcal{Z} + \nabla_{\mathcal{Z}} \varphi\zeta + \nabla_{\mathcal{Z}} \varphi\mathcal{Z} = \varphi\nabla_{\dot{\varpi}}\dot{\varpi} + \eta(\mathcal{Z})(\varphi\zeta + \varphi\mathcal{Z}).$$

From (8), (11) and (19) we arrive at

$$\begin{aligned} h(\nabla_{\dot{\varpi}}\varphi\zeta + \nabla_{\dot{\varpi}}F\mathcal{Z}) + (\mathcal{T}_\zeta + \mathcal{A}_{\mathcal{Z}})(E\mathcal{Z} + F\mathcal{Z}) + v\nabla_{\dot{\varpi}}E\mathcal{Z} + \mathcal{A}_{\mathcal{Z}}\varphi\zeta \\ = \varphi\nabla_{\dot{\varpi}}\dot{\varpi} + \eta(\mathcal{Z})(E\mathcal{Z} + F\mathcal{Z} + \varphi\zeta) \end{aligned}$$

Now, we capture the vertical and horizontal components from the last equation, we have

$$v\nabla_{\dot{\varpi}}E\mathcal{Z} + \mathcal{A}_{\mathcal{Z}}\varphi\zeta + (\mathcal{T}_\zeta + \mathcal{A}_{\mathcal{Z}})F\mathcal{Z} = v\varphi\nabla_{\dot{\varpi}}\dot{\varpi} + \eta(\mathcal{Z})E\mathcal{Z}. \tag{32}$$

$$h(\nabla_{\dot{\varpi}}\varphi\zeta + \nabla_{\dot{\varpi}}F\mathcal{Z}) + (\mathcal{T}_\zeta + \mathcal{A}_{\mathcal{Z}})E\mathcal{Z} = h\varphi\nabla_{\dot{\varpi}}\dot{\varpi} + \eta(\mathcal{Z})(F\mathcal{Z} + \varphi\zeta). \tag{33}$$

From (32) and (33), we can say that ϖ is geodesic if and only if (30) and (31) are valid.

Theorem 4.3. *Suppose that $\psi : (\mathcal{B}, \varphi, \xi, \eta, g_{\mathcal{B}}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be an AIPR submersion with total manifold as a para-Kenmotsu manifold and base manifold as a Riemannian manifold. Then ψ is a Clairaut submersion with $\Theta = e^{\wp}$ if and only if through ϖ*

$$g_{\mathcal{B}}(h\nabla_{\dot{\varpi}}F\mathcal{Z} + (\mathcal{T}_\zeta + \mathcal{A}_{\mathcal{Z}})E\mathcal{Z}, \varphi\zeta) = \{g_{\mathcal{B}}(\nabla\wp, \mathcal{Z}) + \eta(\mathcal{Z})\}\|\zeta\|^2. \tag{34}$$

holds, where $\zeta(s)$ and $\mathcal{Z}(s)$ are vertical and horizontal components of the tangent vector field $\dot{\varpi}(s)$ of the time-like geodesic $\varpi(s)$ at $\mathcal{B}$, in the same order.

Proof. Let $\varpi(s)$ be a time-like geodesic at $\mathcal{B}$, then we get

$$\|\dot{\varpi}(s)\|^2 = \tau$$

So,

$$g_{\mathcal{B}}(\zeta, \zeta) = \tau\cosh^2\beta \tag{35}$$

since ϖ is a time-like geodesic, $\tau = g_{\mathcal{B}}(F, F)$ is a negative constant. Moreover

$$g_{\mathcal{B}}(\mathcal{Z}, \mathcal{Z}) = -\tau\sinh^2\beta. \tag{36}$$

Differentiating (35), we obtain

$$g_{\mathcal{B}}(\nabla_{\dot{\varpi}(s)}\zeta(s), \zeta(s)) = -\tau\cosh\beta\sinh\beta\frac{\partial\beta}{\partial s}. \tag{37}$$

Using the para-Kenmotsu structure, we obtain

$$g_{\mathcal{B}}(\varphi\nabla_{\dot{\varpi}(s)}\zeta(s), \varphi\zeta(s)) = \tau\cosh\beta\sinh\beta\frac{\partial\beta}{\partial s}. \tag{38}$$

Further,

$$-g_{\mathcal{B}}(h\varphi\nabla_{\dot{\varpi}(s)}\zeta(s), \varphi\zeta(s)) = \tau\cosh\beta\sinh\beta\frac{\partial\beta}{\partial s}. \tag{39}$$

since $\eta(\zeta) = 0$ and we know that $\varphi\zeta$ is horizontal. From (31), we obtain

$$g_{\mathcal{B}}(h\nabla_{\dot{\varpi}}F\mathcal{Z} + (\mathcal{T}_\zeta + \mathcal{A}_{\mathcal{Z}})E\mathcal{Z} - \eta(\mathcal{Z})\varphi\zeta, \varphi\zeta) = \tau\cosh\beta\sinh\beta\frac{\partial\beta}{\partial s}. \tag{40}$$

Taking into account that ψ is a Clairaut submersion with $\Theta = e^{\wp}$ if and only if

$$\frac{\partial}{\partial s}(e^{\wp}\cosh\beta) = 0 \iff e^{\wp}\Big(\frac{\partial\wp}{\partial s}\cosh\beta + \sinh\beta\frac{\partial\beta}{\partial s}\Big) = 0.$$

Now, let's product of above equation by the non-zero factor $\tau\cosh\beta$, we get

$$\frac{\partial\wp}{\partial s}\tau\cosh^2\beta + \tau\cosh\beta\sinh\beta\frac{\partial\beta}{\partial s} = 0. \tag{41}$$

Adopting equations (40) and (41), we obtain

$$g_{\mathcal{B}}(h\nabla_{\dot{\omega}}F\mathcal{Z} + (\mathcal{T}_\zeta + \mathcal{A}_{\mathcal{Z}})E\mathcal{Z}, \varphi\zeta) - \eta(\mathcal{Z})\|\zeta\|^2 = \frac{\partial\wp}{\partial s}(\varpi(s))\|\zeta\|^2. \tag{42}$$

Then, at the point, it is clear to view that $\frac{\partial\wp}{\partial s}(\varpi(s)) = \varpi[\wp] = g_{\mathcal{B}}(\nabla\wp, \mathcal{Z})$ the assertion (34) follows from (42).

Hence, the following corollary is dedicated:

Corollary 4.4. *Suppose that $\psi : (\mathcal{B}, \varphi, \xi, \eta, g_{\mathcal{B}}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a CAIPR submersion with total manifold as a para-Kenmotsu manifold and base manifold as a Riemannian manifold. Then, we infer*

$$g_{\mathcal{B}}(\nabla\wp, \xi) = 1. \tag{43}$$

Theorem 4.5. *Suppose that $\psi : (\mathcal{B}, \varphi, \xi, \eta, g_{\mathcal{B}}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a CAIPR submersion with total manifold as a para-Kenmotsu manifold and base manifold as a Riemannian manifold with $\Theta = e^{\wp}$. After, we obtain*

$$\mathcal{A}_{\varphi\mathcal{V}}\varphi\mathcal{W} = -\mathcal{W}(\wp)\mathcal{V} \tag{44}$$

for space-like vector field $\mathcal{W} \in \mu$ and time-like vector field $\mathcal{V} \in ker\psi_*$ such that $\varphi\mathcal{V}$ is fundamental.

Proof. Suppose that $\psi : (\mathcal{B}, \varphi, \xi, \eta, g_{\mathcal{B}}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a CAIPR submersion with total manifold as a para-Kenmotsu manifold and base manifold as a Riemannian manifold with $\Theta = e^{\wp}$. Using Theorem 4.1, we acquire

$$\mathcal{T}_\zeta\mathcal{Z} = -g_{\mathcal{B}}(\zeta, \mathcal{Z})\nabla\wp. \tag{45}$$

for time-like vector fields $\zeta, \mathcal{Z} \in ker\psi_*$. If we extended equation (45) with $\varphi\mathcal{V}$, time-like vector field $\mathcal{V} \in ker\psi_*$ such that $\varphi\mathcal{V}$ is fundamental and using (8), we find

$$g_{\mathcal{B}}(\nabla_\zeta\mathcal{Z}, \varphi\mathcal{V}) = -g_{\mathcal{B}}(\zeta, \mathcal{Z})g_{\mathcal{B}}(\nabla\wp, \varphi\mathcal{V}).$$

$$g_{\mathcal{B}}(\nabla_\zeta\varphi\mathcal{V}, \mathcal{Z}) = g_{\mathcal{B}}(\zeta, \mathcal{Z})g_{\mathcal{B}}(\nabla\wp, \varphi\mathcal{V}).$$

In fact $g_{\mathcal{B}}(\mathcal{Z}, \varphi\mathcal{V}) = 0$. Involving (2) and (4), we find

$$-g_{\mathcal{B}}(\nabla_\zeta\mathcal{V}, \varphi\mathcal{Z}) = g_{\mathcal{B}}(\zeta, \mathcal{Z})g_{\mathcal{B}}(\nabla\wp, \varphi\mathcal{V}).$$

Once again, using (8), we find

$$-g_{\mathcal{B}}(\mathcal{T}_\zeta\mathcal{V}, \varphi\mathcal{Z}) = g_{\mathcal{B}}(\zeta, \mathcal{Z})g_{\mathcal{B}}(\nabla\wp, \varphi\mathcal{V}).$$

Attained with equation (45)

$$g_{\mathcal{B}}(\zeta, \mathcal{V})g_{\mathcal{B}}(\nabla\wp, \varphi\mathcal{Z}) = g_{\mathcal{B}}(\zeta, \mathcal{Z})g_{\mathcal{B}}(\nabla\wp, \varphi\mathcal{V}). \tag{46}$$

Putting $\zeta = \mathcal{V}$ and exchange ζ with by $\mathcal{Z}$ in equation (46), we arrive at

$$\|\mathcal{Z}\|^2 g_{\mathcal{B}}(\nabla\wp, \varphi\zeta) = g_{\mathcal{B}}(\zeta, \mathcal{Z}) g_{\mathcal{B}}(\nabla\wp, \varphi\mathcal{Z}). \tag{47}$$

Accepting equation (46) with setting $\zeta = \mathcal{V}$, we obtain

$$g_{\mathcal{B}}(\nabla\wp, \varphi\zeta) = \frac{g^2_{\mathcal{B}}(\zeta, \mathcal{Z})}{\|\mathcal{Z}\|^2\|\zeta\|^2} g_{\mathcal{B}}(\nabla\wp, \varphi\zeta). \tag{48}$$

Then, using (4) and (2) we reveal

$$g_{\mathcal{B}}(\nabla_{\mathcal{Z}}\varphi\mathcal{V}, \varphi\mathcal{W}) = g_{\mathcal{B}}(\varphi\nabla_{\mathcal{Z}}\mathcal{V}, \varphi\mathcal{W}) = -g_{\mathcal{B}}(\nabla_{\mathcal{Z}}\mathcal{V}, \mathcal{W})$$

for space-like vector field $\mathcal{W} \in \mu$.
Attained with equation (8) and (45)

$$g_{\mathcal{B}}(\nabla_{\mathcal{Z}}\varphi\mathcal{V}, \varphi\mathcal{W}) = g_{\mathcal{B}}(\mathcal{Z}, \mathcal{V}) g_{\mathcal{B}}(\nabla\wp, \mathcal{W}) \tag{49}$$

Then, all $\varphi\mathcal{V}$ is fundamental and using the case that $h\nabla_{\mathcal{Z}}\varphi\mathcal{V} = \mathcal{A}_{\varphi\mathcal{V}}\mathcal{Z}$, we reveal

$$g_{\mathcal{B}}(\nabla_{\mathcal{Z}}\varphi\mathcal{V}, \varphi\mathcal{W}) = g_{\mathcal{B}}(\mathcal{A}_{\varphi\mathcal{V}}\mathcal{Z}, \varphi\mathcal{W}). \tag{50}$$

From (49) and (50) and the anti-symmetric nature of $\mathcal{A}$, we arrive at

$$g_{\mathcal{B}}(\mathcal{A}_{\varphi\mathcal{V}}\varphi\mathcal{W}, \mathcal{Z}) = -g_{\mathcal{B}}(\mathcal{Z}, \mathcal{V}) g_{\mathcal{B}}(\nabla\wp, \mathcal{W}.) \tag{51}$$

Because of $\mathcal{A}_{\varphi\mathcal{V}}\varphi\mathcal{W}, \mathcal{Z}$ and $\mathcal{V}$ are vertical and $\nabla\wp$ is horizontal, we discover statement (44).

Especially, by reason of $\nabla\wp \in \varphi(ker\psi_*)$, from (48) and the equality condition of Schwarz disparity, we arrive at that:

Corollary 4.6. *Suppose that $\psi : (\mathcal{B}, \varphi, \xi, \eta, g_{\mathcal{B}}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a CAIPR submersion with total manifold as a para-Kenmotsu manifold and base manifold as a Riemannian manifold with $\Theta = e^{\wp}$. By reason of $\nabla\wp \in \varphi(ker\psi_*)$ by then either $\wp$ is constant on $\varphi(ker\psi_*)$ or the fibers of ψ are one-dimensional.*

Additionlly, $\nabla\wp \equiv 0$ if and only if $\wp$ is constant.

Corollary 4.7. *Suppose that $\psi : (\mathcal{B}, \varphi, \xi, \eta, g_{\mathcal{B}}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a CAIPR submersion with total manifold as a para-Kenmotsu manifold and base manifold as a Riemannian manifold with $\Theta = e^{\wp}$ and $\nabla\wp \in \varphi(ker\psi_*)$. By reason of $Dim(ker\psi_*) > 1$ for time-like vector field $\mathcal{V} \in \varphi(ker\psi_*)$, $\varphi\mathcal{V}$ is fundamental and space-like vector field $\mathcal{W} \in \mu$, therefore fibers of ψ are totally geodesic if and only if $\mathcal{A}_{\varphi\mathcal{V}}\varphi\mathcal{W}, \mathcal{Z} = 0$.*
Additionally, We can say that $\mathcal{A}_{\varphi\mathcal{V}}\varphi\mathcal{W}, \mathcal{Z} = 0$ with μ is always zero or $\mu = span\{\xi\}$ if and only if $\psi : (\mathcal{B}, \varphi, \xi, \eta, g_{\mathcal{B}}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ in Theorem 4.5 is Lagrangian.

Corollary 4.8. *Suppose that $\psi : (\mathcal{B}, \varphi, \xi, \eta, g_{\mathcal{B}}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a CAIPR submersion with total manifold as a para-Kenmotsu manifold and base manifold as a Riemannian manifold with $\Theta = e^{\wp}$. As a result, either fibers of ψ can be one-dimensional or totally geodesic.*

Finally, we offer a non-trivial example of CAIPR submersion with total manifold as a para-Kenmotsu manifold accepting horizontal Reeb vector field.

Example 4.9. We consider $\mathcal{B} = \{(p, r, s) \in R^3_1\}$. We establish a para-Kenmotsu structure $(\varphi, \xi, \eta, g_{\mathcal{B}})$ at $\mathcal{B}$ given by

$$\eta = ds, \quad \xi = \frac{\partial}{\partial s} \quad g_{\mathcal{B}} = -e^{-2s}(dp)^2 + e^{-2s}(dr)^2 + (ds)^2, \quad s \neq 0$$

and

$$\varphi = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

an orthonormal φ -basis of this constructor is shown as

$$\Big\{E_1 = e^s \frac{\partial}{\partial p}, \quad E_2 = e^s \frac{\partial}{\partial r}, \quad E_3 = \xi = \frac{\partial}{\partial s}\Big\}.$$

Also, we consider $\tilde{\mathcal{B}} = \{(t,s) \in R_0^2\}$ and its Riemannian metric $g_{\tilde{\mathcal{B}}} = e^{-2s}(dt)^2 + (ds)^2$. Let's specify map $\psi : R_1^3 \to R_0^2$

$$\psi(p,r,s) = (\sinh xp, \cosh xr, s)$$

Here by computations, we obtain

$$ker\psi_* = span\Big\{\mathcal{V} = -E_1 \cosh x + E_2 \sinh x\Big\}$$

and

$$(ker\psi_*)^{\perp} = span\Big\{\mathcal{W}_1 = -E_1 \sinh x + E_2 \cosh x, \ \mathcal{W}_2 = E_3 = \xi = \frac{\partial}{\partial s}\Big\}$$

We can easily seen that ψ is a pseudo-Riemannian submersion. In addition, we have $\varphi\mathcal{V} = -\mathcal{W}_1$. As a result, ψ is the AIPR submersion allowing horizontal Reeb vector field. Also, the fibres of ψ are one-dimensional, after all they are plainly totally umbilical.
Here, we shall provide that a $\wp \in C^{\infty}(\mathcal{B})$ satisfying $\mathcal{T}_{\mathcal{V}}\mathcal{V} = -g_{\mathcal{B}}(\mathcal{V},\mathcal{V})\nabla\wp$ for all $\mathcal{V} \in \Gamma(ker\psi_*)$. Since covariant derivative for vector fields $E = E_j \frac{\partial}{\partial x_j}, F = F_k \frac{\partial}{\partial x_k}$ on $\mathcal{B}$ is described as

$$\nabla^{\mathcal{B}}_E F = E_j F_k \nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_k} + E_j \frac{\partial F_k}{\partial x_j} \frac{\partial}{\partial x_k}, \tag{52}$$

where the covariant derivatives of basis vector fields $\frac{\partial}{\partial x_j}$ and $\frac{\partial}{\partial x_k}$ is described as

$$\nabla^{\mathcal{B}}_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_k} = \Gamma^m_{jk} \frac{\partial}{\partial x_m}, \tag{53}$$

Also, Christoffel symbols are described by

$$\Gamma^m_{jk} = \frac{1}{2} g^{m\ell}_{\mathcal{B}} \Big(\frac{\partial g_{\mathcal{B}k\ell}}{\partial x_j} + \frac{\partial g_{\mathcal{B}j\ell}}{\partial x_k} - \frac{\partial g_{\mathcal{B}jk}}{\partial x_\ell} \Big) \tag{54}$$

Then, we get

$$g_{\mathcal{B}jk} = \begin{pmatrix} -e^{-2s} & 0 & 0 \\ 0 & e^{-2s} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g^{jk}_{\mathcal{B}} = \begin{pmatrix} -e^{2s} & 0 & 0 \\ 0 & e^{2s} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

From (54) and the above expression, we find

$$\begin{aligned} \Gamma^1_{11} &= 0, \Gamma^2_{11} = 0, \Gamma^3_{11} = -e^{-2s} \\ \Gamma^1_{22} &= 0, \Gamma^2_{22} = 0, \Gamma^3_{22} = e^{-2s} \\ \Gamma^1_{12} &= 0, \Gamma^1_{21} = 0, \Gamma^2_{21} = 0, \Gamma^2_{12} = 0, \Gamma^3_{12} = 0, \Gamma^3_{21} = 0. \end{aligned} \tag{55}$$

From (52), (53) and (55), we find

$$\nabla^{\mathcal{B}}_{e_1}e_2 = \nabla^{\mathcal{B}}_{e_2}e_1 = 0,\ \nabla^{\mathcal{B}}_{e_1}e_1 = -\frac{\partial}{\partial s},\ \nabla^{\mathcal{B}}_{e_2}e_2 = \frac{\partial}{\partial s}$$

Here by computations, we obtain

$$\nabla_{\mathcal{V}}\mathcal{V} = -\frac{\partial}{\partial s}$$

from (8), we find

$$\mathcal{T}_{\mathcal{V}}\mathcal{V} = -\frac{\partial}{\partial s}$$

Furthermore, for any $\wp \in C^{\infty}(\mathcal{B})$ the gradient of $\wp$ with respect to the metric $g_{\mathcal{B}}$ is

$$\nabla\wp = -e^{-2s}\frac{\partial\wp}{\partial p}\frac{\partial}{\partial p} + e^{-2s}\frac{\partial\wp}{\partial r}\frac{\partial}{\partial r} + \frac{\partial\wp}{\partial s}\frac{\partial}{\partial s}$$

In that case, it is easily seen that $\nabla\wp = \frac{\partial}{\partial s}$ with the function $\wp = s$. As a result, we arrive at

$$\mathcal{T}_{\mathcal{V}}\mathcal{V} = -\|\mathcal{V}\|^2\nabla\wp.$$

Hence, by Theorem 4.1, the submersion ψ is CAIPR.
Now, by Theorem 4.3, we can show that the submersion ψ is a CAIPR submersion from para-Kenmotsu manifold onto Riemannian manifold. Assume that $\mathcal{Z}$ is a space-like horizontal vector field orthogonal to ξ and ζ is a time-like vertical vector field. Thus, by applying (2), (26) and (25), we get

$$\begin{aligned} g_{\mathcal{B}}(\mathcal{T}_{\zeta}E\mathcal{Z},\varphi\zeta) &= g_{\mathcal{B}}(\mathcal{T}_{\zeta}\varphi\mathcal{Z},\varphi\zeta) = g_{\mathcal{B}}(\varphi\mathcal{T}_{\zeta}\mathcal{Z},\varphi\zeta) \\ &= -g_{\mathcal{B}}(\mathcal{T}_{\zeta}\mathcal{Z},\zeta) = g_{\mathcal{B}}(\mathcal{T}_{\zeta}\zeta,\mathcal{Z}) \end{aligned}$$

Then, we obtain

$$g_{\mathcal{B}}(\mathcal{T}_{\zeta}\varphi\mathcal{Z},\varphi\zeta) = g_{\mathcal{B}}(\nabla\wp,\mathcal{Z})\|\zeta\|^2. \tag{56}$$

In the same way, using (2), (27) and (25), we have

$$\begin{aligned} g_{\mathcal{B}}(\mathcal{A}_{\mathcal{Z}}E\mathcal{Z},\varphi\zeta) &= g_{\mathcal{B}}(\mathcal{A}_{\mathcal{Z}}\varphi\mathcal{Z},\varphi\zeta) = -g_{\mathcal{B}}(\mathcal{A}_{\mathcal{Z}}\varphi\zeta,\varphi\mathcal{Z}) \\ &= -g_{\mathcal{B}}(\varphi\mathcal{A}_{\mathcal{Z}}\zeta,\varphi\mathcal{Z}) = g_{\mathcal{B}}(\mathcal{A}_{\mathcal{Z}}\mathcal{Z},\zeta). \end{aligned}$$

Then, we obtain

$$g_{\mathcal{B}}(\mathcal{A}_{\mathcal{Z}}\varphi\mathcal{Z},\varphi\zeta) = 0, \tag{57}$$

since $\mathcal{A}_{\mathcal{Z}}\mathcal{Z} = 0$. Further, we get

$$h\nabla_{\dot{\omega}}F\mathcal{Z} = 0, \tag{58}$$

since ψ is $\varphi(ker\psi_*) \oplus \xi = (ker\psi_*)^{\perp}$. By applying, (56), (57) and (58), the condition (34) is fulfilled. Hence, by Theorem 4.3 the given submersion ψ is CAIPR.

Remark 4.10. We note that Example 4.9 satisfies the condition (43) in Corollary 4.4 and Theorem 4.5.

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