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Screen generic lightlike submanifolds of indefinite cosymplectic manifolds

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Abstract. We introduce screen generic lightlike submanifolds of indefinite cosymplectic manifolds. We investigate the integrability of various distributions and prove a characterization theorem of such lightlike submanifolds in a cosymplectic space form. We study contact totally umbilical screen generic lightlike submanifolds and minimal screen generic lightlike submanifolds. We also give examples.

1. Introduction

It is well known that the intersection of the normal bundle and the tangent bundle of a submanifold of a semi-Riemannian manifold may be not trivial, it is more difficult and interesting to study the geometry of lightlike submanifolds than non-degenerate submanifolds. The two standard methods to deal with the above difficulties were developed by Kupeli [17], Duggal-Bejancu and Duggal-Şahin [4, 8] respectively.

Duggal and Şahin [6] introduced contact CR-lightlike submanifolds of indefinite Sasakian manifolds. But CR-lightlike submanifolds exclude the complex and totally real submanifolds as subcases. Then, Duggal and Şahin introduced contact SCR-lightlike submanifolds of indefinite Sasakian manifolds [6]. But there is no inclusion relation between screen Cauchy-Riemann and CR submanifolds, so Duggal and Şahin introduced a new class called GCR-lightlike submanifolds of indefinite Sasakian manifolds [7] which is an umbrella for all these types of submanifolds. Gupta, Upadhyay and Sharfuddin studied generalised Cauchy-Riemann (GCR) lightlike submanifold of an indefinite cosymplectic manifold [13]. These types of submanifolds have been studied in various manifolds by many authors [5, 10, 16, 19].

Since CR-lightlike, screen CR-lightlike and generalized CR-lightlike do not contain real lightlike curves, then Şahin introduced screen transversal lightlike submanifolds of indefinite Kaehler manifolds and show that such submanifolds contain lightlike real curves [20]. Yıldırım and Şahin introduced screen transversal lightlike submanifolds of indefinite Sasakian manifold [21] and Gupta and Sharfuddin introduced screen transversal lightlike submanifolds of indefinite cosymplectic manifold [11]. Doğan, Şahin and Yaşar introduced a new class of lightlike submanifolds for indefinite Kaehler manifolds which particularly contain invariant lightlike, screen real lightlike and generic lightlike submanifolds and they called this submanifolds as screen generic lightlike submanifolds [3]. After, Gupta introduced screen generic lightlike submanifolds of indefinite Sasakian manifolds [9].

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In this paper, we introduce screen generic lightlike submanifolds of indefinite cosymplectic manifolds. We investigate the integrability of various distributions and prove a characterization theorem of such lightlike submanifolds in a cosymplectic space form. We study contact totally umbilical screen generic lightlike submanifolds and minimal screen generic lightlike submanifolds. We also give examples.

2. Preliminaries

An odd-dimensional semi-Riemannian manifold $\bar{M}$ is said to be an indefinite almost contact metric manifold if there exist structure tensors $(\phi, V, \eta, \bar{g})$, where ϕ is a $(1,1)$ tensor field, V is a vector field called structure vector field, η is a 1-form and $\bar{g}$ is the semi-Riemannian metric on $\bar{M}$ satisfying

$$\phi^2 X = -X + \eta(X)V, \eta(X) = \bar{g}(X, V), \tag{1}$$

and

$$\bar{g}(\phi X, \phi Y) = \bar{g}(X, Y) - \eta(X)\eta(Y), \eta \circ \phi = 0, \phi V = 0, \eta(V) = 1, \tag{2}$$

for any $X, Y \in \Gamma(T\bar{M})$, where $T\bar{M}$ denotes the Lie algebra of vector fields on $\bar{M}$.

An indefinite almost contact metric manifold $\bar{M}$ is called an indefinite cosymplectic manifold if [2]

$$\bar{\nabla}_X \phi = 0, \tag{3}$$
$$\bar{\nabla}_X V = 0, \tag{4}$$

for any $X, Y \in \Gamma(T\bar{M})$, where $\bar{\nabla}$ denote the Levi-Civita connection on $\bar{M}$.

A plane section Π in $T_x\bar{M}$ of an indefinite cosympletic manifold $\bar{M}$ is called a $\phi-$section if it is spanned by a unit vector X orthogonal to V and ϕX, where X is non-null vector field on $\bar{M}$. The sectional curvature $K(\Pi)$ with respect to Π determined by X is called a $\phi-$sectional curvature. If $\bar{M}$ has a $\phi-$sectional curvature c which does not depend on the $\phi-$section at each point, then c is constant in $\bar{M}$. Then, $\bar{M}$ is called an indefinite cosympletic space form and is denoted by $\bar{M}(c)$. The curvature tensor $\bar{R}$ of $\bar{M}(c)$ is given by [18]

$$\begin{aligned}\bar{R}(X, Y)Z = \frac{c}{4}\{&\bar{g}(Y, Z)X - \bar{g}(X, Z)Y + \eta(X)\eta(Z)Y \\ &- \eta(Y)\eta(Z)X + \eta(Y)\bar{g}(X, Z)V - \eta(X)\bar{g}(Y, Z)V \\ &+ \bar{g}(\phi X, Z)\phi Y + \bar{g}(\phi Y, Z)\phi X + 2\bar{g}(X, \phi Y)\phi Z\}\end{aligned} \tag{5}$$

for any X, Y and Z vector fields on $\bar{M}$.

Let consider an m-dimensional submanifold (M, g) of a $(m+n)$-dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$. If the induced metric g on M is degenerate and the rank of the radical distribution $Rad(TM)$ of TM is r, $1 \leq r \leq m$, then (M, g) is called a lightlike submanifold of $(\bar{M}, \bar{g})$. While the normal bundle $TM^\perp$ of the tangent bundle TM is defined as

$$TM^\perp = \cup_{x\in M}\{u \in T_x\bar{M} \mid \bar{g}(u, W) = 0, \quad \forall W \in T_x M\}, \tag{6}$$

the radical distribution $Rad(TM)$ of TM is defined as

$$Rad(TM) = \cup_{x\in M}\{\xi \in T_x M \mid g(u, \xi) = 0, \quad \forall u \in T_x M, \quad \xi \neq 0\}. \tag{7}$$

It is clear that $Rad(TM) = TM \cap TM^\perp$. On the other hand we know that both TM and $TM^\perp$ are degenerate vector subbundles. So, there exist complementary non-degenerate *screen distribution* $S(TM)$ and *co-screen distribution (or screen transversal bundle)* $S(TM^\perp)$ of $Rad(TM)$ in TM and $TM^\perp$, respectively. Then we can write the following decompositions:

$$TM = S(TM) \perp Rad(TM), \tag{8}$$
$$TM^\perp = S(TM^\perp) \perp Rad(TM). \tag{9}$$

Similarly, $S(TM)$ has an orthogonal complementary bundle $S(TM)^{\perp}$ in $T\bar{M}$ such that

$$S(TM)^{\perp} = S(TM^{\perp}) \perp S(TM^{\perp})^{\perp}, \tag{10}$$

where $S(TM^{\perp})^{\perp}$ is the orthogonal complementary to $S(TM^{\perp})$ in $S(TM)^{\perp}$.

Theorem 2.1. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a r-lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then, there exists a complementary vector bundle $ltr(TM)$ called a lightlike transversal bundle of $Rad(TM)$ in $S(TM^{\perp})^{\perp}$ and a basis of $\Gamma(ltr(TM)\mid_U)$ consists of smooth sections $\{N_1, ..., N_r\}$ of $S(TM^{\perp})^{\perp}\mid_U$ such that*

$$\bar{g}(\xi_i, N_T) = \delta_{iT}\,, \quad \bar{g}(N_i, N_T) = 0\,, \quad i, T = 1, .., r,$$

where $\{\xi_1, ..., \xi_r\}$ is a basis of $\Gamma(Rad(TM))$ [4, page 144].

This result implies that there exists a complementary (but not orthogonal) vector bundle $tr(TM)$ to TM in $T\bar{M}|_M$, which called transversal vector bundle, such that the following decompositions hold:

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}) \tag{11}$$

and

$$S(TM^{\perp})^{\perp} = Rad(TM) \oplus ltr(TM). \tag{12}$$

Using the above equations we can write

$$T\tilde{M}\mid_M = TM \oplus tr\,(TM) = \{Rad(TM) \oplus ltr\,(TM)\} \perp S\,(TM) \perp S(TM^{\perp}). \tag{13}$$

There exist four cases for a lightlike submanifold $(M, g, S(TM), S(TM^{\perp}))$:
Case 1: M is called r - lightlike if $r < min\{m,\ n\}$.
Case 2: M is called co - isotropic if $r = n < m$, *i.e.*, $S(TM^{\perp}) = \{0\}$.
Case 3: M is called isotropic if $r = m < n$, *i.e.*, $S(TM) = \{0\}$.
Case 4: M is called totally lightlike if $r = m = n$, *i.e.*, $S(TM) = \{0\} = S(TM^{\perp})$.

The Gauss and Weingarten equations of M are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{14}$$

and

$$\bar{\nabla}_X U = -A_U X + \nabla^t_X U, \tag{15}$$

for any $X, Y \in \Gamma(TM)$ and $U \in \Gamma(tr(TM))$, where $\{\nabla_X Y, A_U X\}$ and $\{h(X, Y), \nabla^t_X U\}$ are belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively. ∇ and ∇^t are linear connections on M and on the vector bundle $tr(TM)$, respectively. The second fundamental form h is a symmetric $\mathcal{F}(M)$-bilinear form on $\Gamma(TM)$ with values in $\Gamma(tr(TM))$ and the shape operator A_U is a linear endomorphism of $\Gamma(TM)$.

According to (13), considering the projection morphisms L and S of $tr\,(TM)$ on $ltr\,(TM)$ and $S\,(TM^{\perp})$, respectively, (14) and (15) become

$$\begin{aligned}
\tilde{\nabla}_X Y &= \nabla_X Y + h^l(X, Y) + h^s(X, Y), && (16)\\
\tilde{\nabla}_X N &= -A_N X + \nabla^l_X N + D^s(X, N), && (17)\\
\tilde{\nabla}_X W &= -A_W X + \nabla^s_X W + D^l(X, W), && (18)
\end{aligned}$$

for any $X, Y \in \Gamma(TM)$, $N \in \Gamma(ltr\,(TM))$ and $W \in \Gamma(S\,(TM^{\perp}))$, where $h^l(X,Y) = Lh(X,Y)$, $h^s(X,Y) = Sh(X,Y)$, $\nabla_X Y, A_N X, A_W X \in \Gamma(TM)$, $\nabla^s_X W, D^s(X,N) \in \Gamma(S\,(TM^{\perp}))$ and $\nabla^l_X N, D^l(X,W) \in \Gamma(ltr\,(TM))$. Then, by using (16)-(18) and taking into account that $\tilde{\nabla}$ is a metric connection we obtain

$$\begin{aligned} g(h^s(X,Y),W) + g(Y,D^l(X,W)) &= g(A_W X,Y), && (19)\\ g(D^s(X,N),W) &= g(A_W X,N), && (20)\\ g(h^l(X,Y),\xi) + g(Y,h^l(X,\xi)) + g(Y,\nabla_X\xi) &= 0. && (21) \end{aligned}$$

Let $\bar{P}$ be a projection of TM on $S(TM)$. Thus, using (8) we can obtain

$$\begin{aligned} \nabla_X \bar{P}Y &= \nabla^*_X \bar{P}Y + h^*(X,\bar{P}Y)\xi, && (22)\\ \nabla_X \xi &= -A^*_\xi X - \nabla^{*t}_X \xi, && (23) \end{aligned}$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$, where $\{\nabla^*_X \bar{P}Y, A^*_\xi X\}$ and $\left\{h^*(X,\bar{P}Y), \nabla^{*t}_X \xi\right\}$ belong to $\Gamma(S\,(TM))$ and $\Gamma(Rad\,(TM))$, respectively.
Considering above equations, we derive

$$\begin{aligned} \bar{g}(h^l(X,\bar{P}Y),\xi) &= g(A^*_\xi X,\bar{P}Y), && (24)\\ \bar{g}(h^*(X,\bar{P}Y),N) &= g(A_N X,\bar{P}Y), && (25)\\ \bar{g}(h^l(X,\xi),\xi) &= 0, \quad A^*_\xi \xi = 0. && (26) \end{aligned}$$

We know that the induced connection ∇ on M, generally is not metric connection. If we consider that $\bar{\nabla}$ is a metric connection and use (16), we get

$$(\nabla_X g)(Y,Z) = \bar{g}(h^l(X,Y),Z) + \bar{g}(h^l(X,Z),Y), \tag{27}$$

i.e., ∇ is not a metric connection. However, it is important to note that $\nabla^\star$ is a metric connection on $S(TM)$.

Theorem 2.2. *Let M be an r-lightlike submanifold of a semi-Riemannian manifold $\bar{M}$. Then the induced connection ∇ is a metric connection iff $Rad(TM)$ is a parallel distribution with respect to ∇ [4].*

The curvature tensor $\bar{R}$ of $\bar{M}(c)$ is given

$$\begin{aligned} \bar{R}(X,Y)Z &= R(X,Y)Z + A_{h^l(X,Z)}Y - A_{h^l(Y,Z)} + A_{h^s(X,Z)}Y - A_{h^s(Y,Z)}X \\ &+ (\nabla_X h^l)(Y,Z) - (\nabla_Y h^l)(X,Z) + D^l(X,h^s(Y,Z)) - D^l(Y,h^s(X,Z)) \\ &+ (\nabla_X h^s)(Y,Z) - (\nabla_Y h^s)(X,Z) + D^s(X,h^l(Y,Z)) - D^s(Y,h^l(X,Z)) \end{aligned} \tag{28}$$

for any $X, Y, Z \in \Gamma(TM)$.

3. Screen Generic Lightlike Submanifolds

Definition 3.1. *Let M be a real r-lightlike submanifold of an indefinite cosymplectic manifold $\bar{M}$ such that V is tangent to M. Then we say that M is a screen generic lightlike submanifold if the following conditions are satisfied:*

(A) $Rad(TM)$ is invariant respect to ϕ, that is,

$$\phi(Rad(TM)) = Rad(TM). \tag{29}$$

(B) There exists a subbundle D_0 of $S(TM)$ such that

$$D_0 = \phi(S(TM)) \cap S(TM), \tag{30}$$

where D_0 is a non-degenerate distribution on M.

From definition of a screen generic lightlike submanifold, we obtain that there exists a complementary non-degenerate distribution $D^{'}$ to D_0 in $S(TM)$ such that,

$$S(TM) = D_0 \oplus D^{'},$$

where

$$\phi(D^{'}) \not\subseteq S(TM) \;\; and \;\; \phi(D^{'}) \not\subseteq S(TM^{\perp}).$$

Let P_0, P_1 and Q be the projection morphisms on D_0, $Rad(TM)$ and $D^{'}$, respectively. Then we have, for any $X \in \Gamma(TM)$,

$$X = P_0X + P_1X + QX + \eta(X)V \tag{31}$$
$$= PX + QX + \eta(X)V, \tag{32}$$

where $D = D_0 \perp Rad(TM)$, D is invariant and $PX \in \Gamma(D)$, $QX \in \Gamma(D^{'})$.
From (31) we get

$$\phi X = TX + \omega X, \tag{33}$$

where TX and ωX are tangential and transversal parts of ϕX, respectively. Besides, it is clear that $\phi(D^{'}) \neq D^{'}$. On the other hand, for a vector field $Y \in \Gamma(D^{'})$, we have

$$\phi Y = TY + \omega Y$$

such that, $TY \in \Gamma(D^{'})$ and $\omega Y \in \Gamma(S(TM^{\perp}))$.
Similarly, for any $W \in \Gamma(tr(TM))$, we get following decomposition

$$\phi W = BW + CW \tag{34}$$

where BW is tangential part and CW is transversal part of ϕW, respectively.
We say that M is a proper screen generic lightlike submanifold of an indefinite cosymplectic manifold $\bar{M}$ if $D_0 \neq \{0\}$ and $D^{'} \neq \{0\}$. For proper screen generic lightlike submanifold we note that the following features:

1. The condition (A) implies that $dim(Rad(TM)) = 2s \geq 2$.
2. The condition (B) implies $dim(D_0) = 2r \geq 2$.
3. $dim(D') = 2p \geq 2$. Thus $dim(M) \geq 7$ and $dim(\bar{M}) \geq 11$.
4. Any proper $7-$ dimensional screen generic lightlike submanifold must be $2-$lightlike.
5. (A) and cosymplectic manifold $\bar{M}$ imply that $index(\bar{M}) \geq 2$.

Proposition 3.2. *A SCR-lightlike submanifold is a screen generic lightlike submanifold such that distribution $D^{'}$ is totally anti-invariant, that is,*

$$S(TM^{\perp}) = \omega D^{'} \oplus \mu$$

where μ is a non-degenerate invariant distribution.

Similar to Definition of generic lightlike submanifolds given by Jin-Lee [14], we have:

Definition 3.3. *Let M be a r-lightlike submanifold of an indefinite cosymplectic manifold $\bar{M}$. If there exists a screen distribution $S(TM)$ of M, such that*

$$\phi(S(TM^{\perp})) \subset S(TM)$$

then, M is a generic r-lightlike submanifold.

Proposition 3.4. *A generic r-lightlike submanifold is a screen generic lightlike submanifold with $\mu = \{0\}$.*

The tangent bundle TM of M have following decomposition:

$$TM = D \oplus D^{'} \perp Span\{V\}.$$

Proposition 3.5. *Any screen generic lightlike submanifold M of an indefinite cosymplectic manifold $\bar{M}$ is an invariant lightlike submanifold if $D^{'} = \{0\}$.*

The following construction will help in understanding the two examples of this paper. Consider $(R_q^{2m+1}, \phi_0, V, \eta, \bar{g})$ with its usual cosymplectic structure given by

$$\eta = dz, V = \partial z,$$
$$\bar{g} = \eta \otimes \eta - \sum_{i=1}^{\frac{q}{2}} dx^i \otimes dx^i + dy^i \otimes dy^i + \sum_{i=q+1}^{m} dx^i \otimes dx^i + dy^i \otimes dy^i),$$
$$\phi_0(\sum_{i=1}^{m}(X_i \partial x^i + Y_i \partial y^i) + Z\partial z) = \sum_{i=1}^{m}(Y_i \partial x^i - X_i \partial y^i),$$

where (x^i, y^i, z) are the Cartesian coordinates and q is considered an even number.

Example 3.6. *Let $\bar{M} = (\mathbb{R}_2^{11}, \bar{g})$ be a semi-Euclidean space, where $\bar{g}$ is of signature $(-,+,+,+,+,-,+,+,+,+,+)$ with respect to the canonical basis*

$$(x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3, y_4, y_5, z).$$

Consider a submanifold M of $\mathbb{R}_2^{11}$ defined by

$$\begin{aligned} x_1 &= u_1 \cos\alpha\,, \;\; x_2 = u_1 - u_2 \sin\alpha\,, \;\; x_3 = u_5\,, \;\; x_4 = \cos u_3 \cosh u_4, \\ x_5 &= u_5 \cos\alpha \; + u_6 \sin\alpha, \;\; y_1 = u_2 \cos\alpha\,, \;\; y_2 = u_1 \sin\alpha + u_2, \\ y_3 &= 0\,, \;\; y_4 = \sin u_3 \,\sinh u_4, \;\; y_5 = u_5 \sin\alpha, \; z = u_7. \end{aligned}$$

Then, TM is spanned by $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7\}$, where

$$\begin{aligned} Z_1 &= \cos\alpha\,\partial x_1 + \partial x_2 + \sin\alpha \partial y_2, \; Z_2 = -\sin\alpha \partial x_2 + \cos\alpha \partial y_1 + \partial y_2, \\ Z_3 &= -\sin u_3 \cosh u_4 \partial x_4 + \cos u_3 \sinh u_4 \partial y_4, \\ Z_4 &= \cos u_3 \sinh u_4 \partial x_4 + \sin u_3 \cosh u_4 \partial y_4, \\ Z_5 &= \partial x_3 + \cos\alpha \partial x_5 + \sin\alpha \partial y_5\,, \; Z_6 = \sin\alpha \partial x_5, \; Z_7 = V = \partial z. \end{aligned}$$

Hence M is a $2-$ lightlike submanifold of $\mathbb{R}_2^{11}$ with $Rad(TM) = Span\{Z_1, Z_2\}$, $D_0 = Span\{Z_3, Z_4\}$ and $D^{'} = Span\{Z_5, Z_6\}$. It is easy to see that $\phi_0 Z_1 = -Z_2$ and $\phi_0 Z_3 = Z_4$. By direct calculations, we get the lightlike transversal bundle spanned by

$$\begin{aligned} N_1 &= \frac{1}{2}(-\cos\alpha\,\partial x_1 + \partial x_2 + \sin\alpha \partial y_2), \\ N_2 &= \frac{1}{2}(-\sin\alpha \partial x_2 - \cos\alpha \partial y_1 + \partial y_2) \end{aligned}$$

and the screen transversal bundle spanned by

$$\begin{aligned} W_1 &= -\partial y_3 - \cos\alpha \partial y_5\,, \;\; W_2 = -\partial x_3 - \cos\alpha \partial x_5, \\ W_3 &= \cos\alpha \partial x_6 + \sin\alpha \partial y_6\,, \;\; W_4 = \sin\alpha \partial x_6 - \cos\alpha \partial y_6, \end{aligned}$$

where $\mu = Sp\{W_3, W_4\}$, $\phi_0 W_3 = W_4$ and $\phi_0 N_1 = -N_2$. Since

$$\begin{aligned} \phi_0 Z_5 &= Z_6 + W_1, \\ \phi_0 Z_6 &= -\sin\alpha \partial \; y_5 \; = -(Z_5 + W_2), \end{aligned}$$

then M is a screen generic lightlike submanifold.

Theorem 3.7. *There exist no coisotropic, isotropic or totally lightlike proper screen generic lightlike submanifold M of an indefinite cosymplectic manifold $\bar{M}$. Any screen generic isotropic, coisotropic or totally lightlike submanifold M is an invariant submanifold.*

Proof. Let M be a screen generic lightlike submanifold of an indefinite cosymplectic manifold $\bar{M}$. If M is isotropic, then $S(TM) = \{0\}$ which implies that $D_0 = \{0\}$ and $D' = \{0\}$. Therefore we derive $TM = Rad(TM) = \phi(Rad(TM))$, which is invariant respect to ϕ.
If M is coisotropic, then $S(TM^\perp) = \{0\}$ implies $\mu = \{0\}$ and the $\omega(D') = \{0\}$. Thus, $TM = D_0 \oplus \phi(D') \oplus Rad(TM)$ and M is invariant.
Finally, if M is totally lightlike, then $S(TM) = \{0\}$ and $S(TM^\perp) = \{0\}$. Hence, $TM = Rad(TM)$, which implies M is invariant.
So, it is clear that there exist no coisotropic, isotropic or totally lightlike proper screen generic lightlike submanifolds and the proof is completed. □

Theorem 3.8. *Let $(M, g, S(TM), S(TM^\perp))$ be a screen generic lightlike submanifold of an indefinite cosymplectic manifold $(\bar{M}, \bar{g})$. If ∇ is a metric connection, then $h^s(X, \phi Y)$ has no components in $\omega D'$. Conversely, the induced connection ∇ is a metric connection if*

$$\bar{g}(h^l(X, TU), \phi Y) = -\bar{g}(h^s(X, \phi Y), \omega U) \tag{35}$$

for any $X, Y \in \Gamma(Rad(TM))$ and $U \in \Gamma(S(TM))$.

Proof. Suppose that ∇ is a metric connection. From (1) and (3) we have

$$\bar{\nabla}_X Y = -\phi(\bar{\nabla}_X \phi Y) \tag{36}$$

for any $X \in \Gamma(TM)$ and $Y \in \Gamma(Rad(TM))$. Using (16) and (23) we get

$$\bar{\nabla}_X Y = -\phi(-A^*_{\phi Y} X + \nabla^{*^t}_X \phi Y + h^l(X, \phi Y) + h^s(X, \phi Y)). \tag{37}$$

Considering (33) and (34) in (37) and taking the tangential parts of this equation, we obtain

$$\nabla_X Y = TA^*_{\phi Y} X - \nabla^{*^t}_X \phi Y - Bh^s(X, \phi Y). \tag{38}$$

From Theorem 2.2 we know that induced connection ∇ is a metric connection if and only if $Rad(TM)$ is a parallel distribution. Suppose that $Rad(TM)$ is parallel, then $g(\nabla_X Y, U) = 0$. From the above equation, we derive

$$g(\nabla_X Y, U) = \bar{g}(h^s(X, \phi Y), \omega U) \tag{39}$$

for any $X, Y \in \Gamma(Rad(TM))$ and $U \in \Gamma(S(TM))$. Therefore $h^s(X, \phi Y)$ has no components in $\omega D'$.
Conversely, we assume that

$$\bar{g}(h^l(X, TU), \phi Y) = -\bar{g}(h^s(X, \phi Y), \omega U) \tag{40}$$

for any $X, Y \in \Gamma(Rad(TM))$ and $U \in \Gamma(S(TM))$. On the other hand, from (2) and (3) we have

$$\bar{g}(\bar{\nabla}_X \phi Y, \phi U) = \bar{g}(\bar{\nabla}_X Y, U). \tag{41}$$

Using (16), (23), (33) and (34), we obtain

$$\begin{aligned} \bar{g}(\bar{\nabla}_X \phi Y, \phi U) &= \bar{g}(-A^*_{\phi Y} X + \nabla^{*^t}_X \phi Y + h^l(X, \phi Y) + h^s(X, \phi Y), \phi U) \\ &= -\bar{g}(A^*_{\phi Y} X, TU) + \bar{g}(h^s(X, \phi Y), \omega U) \end{aligned} \tag{42}$$

for any $X, Y \in \Gamma(Rad(TM))$ and $U \in \Gamma(S(TM))$.
Using (2), (3), (40), (41) and (42), we get

$$g(\nabla_X Y, U) = 0$$

i.e., $\nabla_X Y \in \Gamma(Rad(TM))$ which completes the proof. □

Theorem 3.9. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a screen generic lightlike submanifold of an indefinite cosymplectic manifold $(\bar{M}, \bar{g})$. Then $D_0 \perp \{V\}$ is integrable if and only if the following conditions hold:*

$$g(\nabla^*_X \phi Y - \nabla^*_Y \phi X, TZ) = g(B(h^s(X, \phi Y) - h^s(Y, \phi X)), Z), \tag{43}$$
$$h^*(X, \phi Y) = h^*(Y, \phi X), \tag{44}$$

for any $X, Y \in \Gamma(D_0 \perp \{V\})$, $Z \in \Gamma(D')$. Also, $D \perp \{V\}$ is integrable if and only if (43) holds.

Proof. From the definition of screen generic lightlike submanifold, $D_0 \perp \{V\}$ is integrable iff for any $X, Y \in \Gamma(D_0 \perp \{V\})$, $[X, Y] \in \Gamma(D_0 \perp \{V\})$, that is,

$$g([X, Y], Z) = \bar{g}([X, Y], N) = 0$$

$Z \in \Gamma(D')$ and $N \in \Gamma(ltr(TM))$.
Using that $\bar{\nabla}$ is a metric connection and (2), (3), (4), (16), (22), (33) and (34), we derive

$$g([X, Y], Z) = \bar{g}(\nabla^*_X \phi Y - \nabla^*_Y \phi X, TZ) - g(B(h^s(X, \phi Y) - h^s(Y, \phi X)), Z),$$

$$g([X, Y], N) = \bar{g}(h^*(X, \phi Y) - h^*(Y, \phi X), \phi N),$$

which hold (43) and (44). Hence, proof is completed. □

Theorem 3.10. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a screen generic lightlike submanifold of an indefinite cosymplectic manifold $(\bar{M}, \bar{g})$. Then, the distribution D' is integrable iff*

$$\nabla_Z TW - \nabla_W TZ - A_{\omega W} Z + A_{\omega Z} W \in \Gamma(D') \tag{45}$$

for any $Z, W \in \Gamma(D')$.

Proof. From the definition of screen generic lightlike submanifold, D' is integrable iff for any $Z, W \in \Gamma(D')$, $X \in \Gamma(D_0)$ and $N \in \Gamma(ltr(TM))$,

$$g([Z, W], X) = \bar{g}([Z, W], N) = g([Z, W], V) = 0.$$

Considering (2), (16), (18) and (33), we get

$$g([Z, W], X) = g(\nabla_Z TW - \nabla_W TZ - A_{\omega W} Z + A_{\omega Z} W, \phi X).$$

From last equation it is easy to see that

$$\nabla_Z TW - \nabla_W TZ - A_{\omega W} Z + A_{\omega Z} W \ \textit{ has no components on } \ \Gamma(D_0) \tag{46}$$

and using (2), (16), (18) and (33) we have

$$g([Z, W], N) = \bar{g}(\nabla_Z TW - \nabla_W TZ - A_{\omega W} Z + A_{\omega Z} W, \phi N).$$

Thus,

$$\nabla_Z TW - \nabla_W TZ - A_{\omega W} Z + A_{\omega Z} W \ \textit{ has no components on } \ \Gamma(Rad(TM)). \tag{47}$$

Moreover,

$$g([Z, W], V) = -g(W, \bar{\nabla}_Z V) + g(Z, \bar{\nabla}_W V) = 0 \tag{48}$$

is obtained and from (46), (47) and (48), it is clear that D' is integrable iff $\nabla_Z TW - \nabla_W TZ - A_{\omega W} Z + A_{\omega Z} W \in \Gamma(D')$. □

Theorem 3.11. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a screen generic lightlike submanifold of an indefinite cosymplectic manifold $(\bar{M}, \bar{g})$. Then the distribution $D \perp \{V\}$ is parallel iff*

$$\nabla^*_X TZ - A_{\omega Z}X \ \ has \ \ no \ \ components \ \ on \ \ \Gamma(D_0), \tag{49}$$

$$h^l(X, TZ) = -D^l(X, \omega Z), \tag{50}$$

for any $X \in \Gamma(D \perp \{V\})$, $Z \in \Gamma(D^{'})$.

Proof. From the definition of screen generic lightlike submanifold, $D \perp \{V\}$ is parallel iff for any $X, Y \in \Gamma(D \perp \{V\})$ and $Z \in \Gamma(D^{'})$,

$$g(\nabla_X Y, Z) = 0.$$

Using that $\bar{\nabla}$ is a metric connection and (2), (4), (16), (18) and (33), we derive

$$g(\nabla_X Y, Z) = -\bar{g}(\nabla_X TZ + h^l(X, TZ) - A_{\omega Z}X + D^l(X, \omega Z), \phi Y).$$

From this, the proof is completed. □

Theorem 3.12. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a screen generic lightlike submanifold of an indefinite cosymplectic manifold $(\bar{M}, \bar{g})$. Then $D^{'}$ is parallel if and only if for any $Z, W \in \Gamma(D^{'})$,*

$$\nabla^*_Z TW - A_{\omega W}Z \in \Gamma(D^{'}). \tag{51}$$

Proof. We assume that $D^{'}$ is a parallel distribution. Then, for any $Z, W \in \Gamma(D^{'})$, $\nabla_Z W \in \Gamma(D^{'})$. In the other words, for any $X \in \Gamma(D_0)$ and $N \in \Gamma(ltr(TM))$,

$$g(\nabla_Z W, X) = \bar{g}(\nabla_Z W, N) = g(\nabla_Z W, V) = 0.$$

Using (1), (16), (18) and (33), we obtain

$$g(\nabla_Z W, X) = \bar{g}(\nabla_Z TW - A_{\omega W}Z, \phi X)$$

and then

$$\nabla_Z TW - A_{\omega W}Z \ \ has \ \ no \ \ components \ \ on \ \ \Gamma(D_0). \tag{52}$$

Similary, we get

$$\bar{g}(\nabla_Z W, N) = \bar{g}(\nabla_Z TW - A_{\omega W}Z, \phi N)$$

and from this equation, it is clear that

$$\nabla_Z TW - A_{\omega W}Z \ \ has \ \ no \ \ components \ \ on \ \ \Gamma(Rad(TM)). \tag{53}$$

Moreover, we obtain

$$g(\nabla_Z W, V) = -g(W, \bar{\nabla}_Z V) = 0. \tag{54}$$

Hence, from (52), (53) and (54), we have that $D^{'}$ is a parallel iff $\nabla^*_Z TW - A_{\omega W}Z \in \Gamma(D^{'})$. □

Theorem 3.13. *Let M be a lightlike submanifold of an indefinite cosymplectic space form $\bar{M}(c)$. Then, M is a screen generic lightlike submanifold of $\bar{M}(c)$, with $c \neq 0$ iff the following conditions are satisfied:*

(i) $D = D_0 \perp Rad(TM)$ is the maximal complex distribution on M, where D_0 is a non-degenerate invariant distribution.

(ii) There exists a non-degenerate distribution $D^{'}$ on M such that

$$\bar{g}(\bar{R}(X,Y)Z,W) \neq 0, \quad \forall X,Y \in \Gamma(D_0), \ \ Z,W \in \Gamma(D^{'}).$$

(iii) There exists a non-degenerate distribution μ on $S(TM^{\perp})$ such that

$$\bar{g}(\bar{R}(X,Y)Z,W) \neq 0,$$

where for all $X,Y \in \Gamma(D_0)$, $Z \in \Gamma(D^{'})$, $W \in \Gamma(TM^{\perp})$, but $W \notin \Gamma(\mu)$.

Proof. Supoose that M is a screen generic lightlike submanifold of $\bar{M}(c)$, $c \neq 0$. Then

$$D = D_0 \perp Rad(TM)$$

is a maximal subspace and (i) is satisfied. If we use (5) for any $X,Y \in \Gamma(D_0)$ and $Z,W \in \Gamma(D^{'})$, we obtain

$$\bar{g}(\bar{R}(X,Y)Z,W) = -\frac{c}{2}\bar{g}(\phi X,Y)\bar{g}(\phi Z,W) \neq 0$$

then (ii) holds.
Similarly, for any $X,Y \in \Gamma(D_0)$, $Z \in \Gamma(D^{'}$, $W \in \Gamma(TM^{\perp})$, we have

$$\bar{g}(\bar{R}(X,Y)Z,W) = \frac{c}{2}\bar{g}(\phi X,Y)\bar{g}(\phi Z,W) \neq 0$$

then (iii) holds.
Conversely, we suppose that (i), (ii) and (iii) are holded. Then from (i), we see that $Rad(TM)$ is invariant on M. Thus, (A) of the Definition 3.1. is satisfied. Therefore, from (ii) we see that there exists a non-degenerate anti-invariant distribution $D^{'}$ on $S(TM)$ such that since $\bar{g}(\phi Z,W) \neq 0$, for any Z, $W \in \Gamma(D^{'})$, then

$$\Gamma(\phi(D^{'})) \subset \Gamma(D^{'}) \subset \Gamma(S(TM)). \tag{55}$$

On the other hand, from (iii), we have $\bar{g}(\phi Z,W) = -\bar{g}(Z,\phi W)$, for any $X,Y \in \Gamma(D_0)$, $Z \in \Gamma(D^{'})$, $W \in \Gamma(TM^{\perp})$, but $W \notin \Gamma(\mu)$. So, it is clear that

$$\Gamma(\phi(D^{'})) \subset \Gamma(S(TM^{\perp})). \tag{56}$$

Thus, from (55) and (56), we obtain neither $\Gamma(\phi(D^{'}))$ is in $\Gamma(D^{'})$ totally, nor $\Gamma(\phi(D^{'}))$ is in $\Gamma(S(TM^{\perp}))$ totally, which satisfies (B) of the Definition 3.1. This completes the proof. □

Definition 3.14. *We say that M is a $D \perp \{V\}$-geodesic screen generic lightlike submanifold if its second fundamental form h satisfies*

$$h(X,Y) = 0\,, \ \ \forall X,Y \in \Gamma(D \perp \{V\}). \tag{57}$$

It is easy to see that M is a $D \perp \{V\}$-geodesic screen generic lightlike submanifold if

$$h^{l}(X,Y) = h^{s}(X,Y) = 0 \tag{58}$$

for any $X,Y \in \Gamma(D \perp \{V\})$. On the other hand, if h satisfies

$$h(X,Y) = 0, \tag{59}$$

for any $X \in \Gamma(D)$, $Y \in \Gamma(D^{'} \perp \{V\})$, then M is called a mixed geodesic screen generic lightlike submanifold.

Proposition 3.15. *The distribution $D \perp \{V\}$ of a screen generic lightlike submanifold M of $\bar{M}$ is a totally geodesic foliation in $\bar{M}$ iff M is $D \perp \{V\}$-geodesic and $D \perp \{V\}$ is parallel respect to ∇ on M.*

Proof. Suppose that $D \perp \{V\}$ defines a totally geodesic foliation in $\bar{M}$, that is, $\bar{\nabla}_X Y \in \Gamma(D \perp \{V\})$, for any $X, Y \in \Gamma(D \perp \{V\})$. Then,

$$\bar{g}(\bar{\nabla}_X Y, \xi) = \bar{g}(\bar{\nabla}_X Y, W) = \bar{g}(\bar{\nabla}_X Y, Z) = 0$$

for any $\xi \in \Gamma(Rad(TM))$, $Z \in \Gamma(D^{'})$ and $W \in \Gamma(S(TM^{\perp}))$. Using (16), we obtain

$$\begin{aligned} \bar{g}(\bar{\nabla}_X Y, \xi) &= \bar{g}(h^l(X, Y), \xi), \\ \bar{g}(\bar{\nabla}_X Y, W) &= \bar{g}(h^s(X, Y), W) \end{aligned}$$

then, it is clear that, for any $X, Y \in \Gamma(D \perp \{V\})$, $h^l(X, Y) = h^s(X, Y) = 0$. In other words, M is $D \perp \{V\}$-geodesic and $D \perp \{V\}$ is parallel respect to ∇ on M.
Conversely, we assume that M is $D \perp \{V\}$-geodesic and $D \perp \{V\}$ is parallel respect to ∇ on M. Since $h^l(X, Y) = h^s(X, Y) = 0$, for any $X, Y \in \Gamma(D \perp \{V\})$, then $\bar{\nabla}_X Y \in \Gamma(TM)$. On the other hand, since $D \perp \{V\}$ is parallel on M, using (16), we have $\bar{\nabla}_X Y \in \Gamma(D \perp \{V\})$ which completes the proof. □

Theorem 3.16. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a screen generic lightlike submanifold of an indefinite cosymplectic manifold $(\bar{M}, \bar{g})$. Then M is mixed geodesic iff the following conditions hold:*

(i) $D^l(X, \omega Z) = -h^l(X, TZ)$,
(ii) $g(A_{\omega Z} X - \nabla_X TZ, BW) = \bar{g}(h^s(X, TZ) + \nabla^s_X \omega Z, CW)$,

for any $X \in \Gamma(D)$, $Z \in \Gamma(D^{'} \perp \{V\})$ and $W \in \Gamma(S(TM^{\perp}))$.

Proof. If M is mixed geodesic, then from (59), $\bar{g}(h^l(X, Z), \xi) = 0$ and $\bar{g}(h^s(X, Z), W) = 0$ for any $X \in \Gamma(D)$, $Z \in \Gamma(D^{'} \perp \{V\})$, $\xi \in \Gamma(Rad(TM))$ and $W \in \Gamma(S(TM^{\perp}))$. Therefore from (16), we obtain

$$\bar{g}(\bar{\nabla}_X Z, \xi) = 0.$$

Since distribution $Rad(TM)$ is invariant, we can replace $\phi\xi$ with ξ. Then we obtain

$$\bar{g}(\bar{\nabla}_X Z, \phi\xi) = 0.$$

Using (16), (18) and (33) in the last equation

$$\bar{g}(h^l(X, TZ) + D^l(X, \omega Z), \xi) = 0$$

is obtained. Similarly, it is easy to get

$$\bar{g}(\bar{\nabla}_X \phi Z, \phi W) = 0$$

and from this, we have

$$g(\nabla_X TZ - A_{\omega Z} X, BW) + \bar{g}(h^s(X, TZ) + \nabla^s_X \omega Z, CW) = 0$$

which completes the proof. □

For any $Y \in \Gamma(TM)$, differentiating (33) and using (3), (16), (18), (33) and (34), we derive

$$\begin{aligned} & \nabla_X TY + h^l(X, TY) + h^s(X, TY) - A_{\omega Y} X + \nabla^s_X \omega Y + D^l(X, \omega Y) \\ = \; & T\nabla_X Y + \omega \nabla_X Y + Ch^l(X, Y) + Bh^s(X, Y) + Ch^s(X, Y). \end{aligned}$$

Taking tangential, lightlike transversal and screen transversal parts of this equation, we obtain

$$\nabla_X TY - A_{\omega Y} X = T\nabla_X Y + Bh^s(X, Y), \tag{60}$$

$$h^l(X, TY) + D^l(X, \omega Y) = Ch^l(X, Y), \tag{61}$$

$$h^s(X, TY) + \nabla^s_X \omega Y = \omega \nabla_X Y + Ch^s(X, Y). \tag{62}$$

Lemma 3.17. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a screen generic lightlike submanifold of an indefinite cosymplectic manifold $(\bar{M}, \bar{g})$. Then M is mixed geodesic iff*

(i) $D^l(X, \omega Z) = -h^l(X, TZ)$,
(ii) $\omega Q(A_{\omega Z}X - \nabla_X TZ) = C(h^s(X, TZ) + \nabla^s_X \omega Z)$,

for any $X \in \Gamma(D)$, $Z \in \Gamma(D' \perp \{V\})$.

Proof. Using (1), (16), (18) and (33), we obtain for any $X \in \Gamma(D)$, $Z \in \Gamma(D' \perp \{V\})$,

$$\begin{aligned} h(X, Z) &= -\phi(\nabla_X TZ + h^l(X, TZ) + h^s(X, TZ) \\ &\quad - A_{\omega Z}X + \nabla^s_X \omega Z + D^l(X, \omega Z)) - \nabla_X Z. \end{aligned}$$

Considering (31)-(34) and taking transversal part of this equation, we have

$$\begin{aligned} h(X, Z) &= \omega Q(A_{\omega Z}X - \nabla_X TZ) - C(h^l(X, TZ) + D^l(X, \omega Z)) \\ &\quad - C(h^s(X, TZ) + \nabla^s_X \omega Z) \end{aligned}$$

Hence, $h(X, Z) = 0 \Leftrightarrow$ (i) and (ii) hold. □

Lemma 3.18. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a screen generic lightlike submanifold of an indefinite cosymplectic manifold $(\bar{M}, \bar{g})$. Then for any $X \in \Gamma(D_0)$, $Z \in \Gamma(D')$ we have*

$$\nabla_X Z = TA_{\omega Z}X - T\nabla_X TZ - Bh^s(X, TZ) - B\nabla^s_X \omega Z - Bh^s(X, Z).$$

Proof. Using (1), (16), (18), (33) and (34), we can write for any $X \in \Gamma(D_0)$, $Z \in \Gamma(D')$,

$$\begin{aligned} \bar{\nabla}_X Z &= -T\nabla_X TZ - Bh^s(X, TZ) + TA_{\omega Z}X - B\nabla^s_X \omega Z \\ &\quad - \omega\nabla_X TZ - Ch^s(X, TZ) + \omega A_{\omega Z}X - C\nabla^s_X \omega Z \\ &\quad - Ch^l(X, TZ) - CD^l(X, \omega Z). \end{aligned}$$

If we take tangential parts of last equation, then we obtain

$$\nabla_X Z = TA_{\omega Z}X - T\nabla_X TZ - Bh^s(X, TZ) - B\nabla^s_X \omega Z - Bh^s(X, Z)$$

which proves our assertion. □

4. Contact Totally Umbilical Screen Generic Lightlike Submanifolds

In this section we study contact totally umbilical screen generic lightlike submanifolds.

Definition 4.1. *[12] A lightlike submanifold (M, g) of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. If the second fundamental form h of a submanifold M, tangent to the structure vector field V of an indefinite cosymplectic manifold $\bar{M}$ is of the form*

$$h(X, Y) = [g(X, Y) - \eta(X)\eta(Y)]\alpha \tag{63}$$

for any $X, Y \in \Gamma(TM)$, where α is a vector field transversal to M, then M is called contact totally umbilical submanifold and totally geodesic if $\alpha = 0$.

The above definition also holds for a lightlike submanifold M. For a contact totally umbilical M, we have

$$h^l(X, Y) = [g(X, Y) - \eta(X)\eta(Y)]\alpha_l, \tag{64}$$
$$h^s(X, Y) = [g(X, Y) - \eta(X)\eta(Y)]\alpha_s, \tag{65}$$

where $\alpha_l \in \Gamma(ltr(TM))$ and $\alpha_s \in \Gamma(S(TM^{\perp}))$.

Lemma 4.2. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a contact totally umbilical proper screen generic lightlike submanifold of an indefinite cosymplectic manifold $(\bar{M}, \bar{g})$. Then $\alpha_s \notin \Gamma(\mu)$.*

Proof. Let $(M, g, S(TM), S(TM^{\perp}))$ be a contact totally umbilical proper screen generic lightlike submanifold of an indefinite cosymplectic manifold $(\bar{M}, \bar{g})$. Then considering (62) we get

$$g(X, \phi Y)\alpha_s = g(X, Y)C\alpha_s + \omega\nabla_X Y - \nabla^s_X \omega Y\,,$$

for any $X, Y \in \Gamma(D_0)$. If we get $X = \phi Y$, then we obtain

$$g(X, X)\alpha_s = \omega\nabla_{\phi Y} Y.$$

Next from, we have $\alpha_s \notin \Gamma(\mu)$ and this completes the proof. □

Theorem 4.3. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a contact totally umbilical proper screen generic lightlike submanifold of an indefinite cosymplectic manifold $(\bar{M}, \bar{g})$. Then the induced connection ∇ is a metric connection.*

Proof. Let M be a contact totally umbilical proper screen generic lightlike submanifold of an indefinite cosymplectic manifold $\bar{M}$. Then using (61) and (64) we obtain

$$g(X, \phi Y)\alpha_l = g(X, Y)C\alpha_l$$

for any $X, Y \in \Gamma(D_0)$. From the last equation,

$$2g(X, \phi Y)\alpha_l = 0$$

is obtained. If we take $X = \phi Y$, then we have $\alpha_l = 0$ that is, from (64) $h^l = 0$. Hence from (27) the proof is completed. □

Theorem 4.4. *There exist no contact totally umbilical proper screen generic lightlike submanifold of an indefinite complex space form $\bar{M}(c)$, $c \neq 0$.*

Proof. Suppose that M is a contact totally umbilical proper screen generic lightlike submanifold of $\bar{M}(c)$, $c \neq 0$. Then, from (5) and (28) we get

$$\bar{R}(X, \phi X)Z = -\frac{c}{2} g(X, X)\phi Z \tag{66}$$

and

$$\bar{R}(X, \phi X)Z = (\nabla_X h^s)(\phi X, Z) - (\nabla_{\phi X} h^s)(X, Z), \tag{67}$$

for any $X \in \Gamma(D_0)$, $Z \in \Gamma(D^{'})$. Since M is contact totally umbilical, from (65) we have

$$(\nabla_X h^s)(\phi X, Z) = -g(\nabla_X \phi X, Z)\alpha_s - g(\phi X, \nabla_X Z)\alpha_s. \tag{68}$$

We know that

$$g(\phi X, Z) = 0.$$

From the last equation, we derive

$$(\bar{\nabla}_X g)(\phi X, Z) = g(\nabla_X \phi X, Z) + g(\phi X, \nabla_X Z) = 0. \tag{69}$$

Thus, from (68) and (69) we get

$$(\nabla_X h^s)(\phi X, Z) = 0. \tag{70}$$

Similarly, we have

$$(\nabla_{\phi X} h^s)(X, Z) = 0. \tag{71}$$

Hence, from (66), (67), (70) and (71)

$$\frac{c}{2} g(X, X)\phi Z = 0,$$

that is, $c = 0$ is obtained. This is a contradiction which completes the proof. □

5. Minimal Screen Generic Lightlike Submanifolds

Definition 5.1. *We say that a lightlike submanifold $(M, g, S(TM))$ isometrically immersed in a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is minimal if:*

(i) $h^s = 0$ *on* $Rad(TM)$ *and*

(ii) $traceh = 0$, *where trace is written with respect to* g *restricted to* $S(TM)$.

In Case 2, condition (i) is trivial. It has been shown in [1] that the above definition is independent of $S(TM)$ and $S(TM^{\perp})$, but it depends on $tr(TM)$.
Minimal lightlike submanifolds are investigated in detail in [8].

Theorem 5.2. *Let* $(M, g, S(TM), S(TM^{\perp}))$ *be a screen generic lightlike submanifold of an indefinite cosymplectic manifold* $(\bar{M}, \bar{g})$. *Then, the distribution* $D_0 \perp \{V\}$ *is minimal.*

Proof. From the definition, it is clear that $D_0 \perp \{V\}$ is minimal if and only if

$$\nabla_X X + \nabla_{\phi X}\phi X \in \Gamma(D_0 \perp \{V\}), \ \ \forall X \in \Gamma(D_0 \perp \{V\}).$$

Hence, from (2) and (18), we derive

$$g(\nabla_X X, \phi W) = -g(\phi X, A_W X)$$

and

$$g(\nabla_{\phi X}\phi X, \phi W) = g(X, A_W \phi X)$$

for any $X \in \Gamma(D_0 \perp \{V\})$ and $W \in \Gamma(S(TM^{\perp}))$. On the other hand, since the shape operator is symmetric on $S(TM)$, we get

$$g(\nabla_X X + \nabla_{\phi X}\phi X, \phi W) = 0.$$

The proof comes last equation. □

Example 5.3. *Let* $\bar{M} = (\mathbb{R}^{11}_4, \bar{g})$ *be a semi-Euclidean space, where* $\bar{g}$ *is signature* $(-,-,+,+,+,-,-,+,+,+,+)$ *with respect to the canonical basis*

$$(\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial z)$$

and M *be a submanifold of* $\mathbb{R}^{11}_4$ *given by*

$$\begin{aligned} x_1 &= u_1 \sinh\beta, x_2 = u_1 \cosh\beta - u_2, x_3 = \cos u_3 \cosh u_4, x_4 = \sin u_3 \sinh u_4, \\ x_5 &= \sin u_5 \sinh u_6, y_1 = u_2 \sinh\beta, y_2 = u_1 + u_2 \cosh\beta, \\ y_3 &= \sin u_3 \cosh u_4, y_4 = -\cos u_3 \sinh u_4, y_5 = -\cos u_5 \cosh u_6, z = u_7. \end{aligned}$$

Then, TM *is spanned by*

$$\begin{aligned} Z_1 &= \sinh\beta\partial x_1 + \cosh\beta\partial x_2 + \partial y_2, \\ Z_2 &= -\partial x_2 + \sinh\beta\partial y_1 + \cosh\beta\partial y_2, \\ Z_3 &= -\sin u_3 \cosh u_4 \partial x_3 + \cos u_3 \sinh u_4 \partial x_4 + \cos u_3 \cosh u_4 \partial y_3 + \sin u_3 \sinh u_4 \partial y_4, \\ Z_4 &= \cos u_3 \sinh u_4 \partial x_3 + \sin u_3 \cosh u_4 \partial x_4 + \sin u_3 \sinh u_4 \partial y_3 - \cos u_3 \cosh u_4 \partial y_4, \\ Z_5 &= \cos u_5 \sinh u_6 \ \partial x_5 + \sin u_5 \cosh u_6 \partial y_5, \\ Z_6 &= \sin u_5 \cosh u_6 \partial x_5 - \cos u_5 \sinh u_6 \,\partial y_5, Z_7 = V = \partial z, \end{aligned}$$

where $Rad(TM) = \{Z_1, Z_2\}$ *and* $D_0 = \{Z_5, Z_6\}$. *By direct calculation, we derive that* $ltr(TM)$ *is spanned by*

$$\begin{aligned} N_1 &= \frac{1}{2}(\sinh\beta\partial x_1 - \cosh\beta\partial x_2 - \partial y_2), \\ N_2 &= \frac{1}{2}(\partial x_2 + \sinh\beta\partial y_1 - \cosh\beta\partial y_2). \end{aligned}$$

Also, the screen transversal bundle is spanned by

$$\begin{aligned} W_1 &= \cos u_3\cosh u_4\partial x_3 - \sin u_3\sinh u_4\partial x_4 + \sin u_3\cosh u_4\partial y_3 + \cos u_3\sinh u_4\partial y_4, \\ W_2 &= \sin u_3\sinh u_4\partial x_3 + \cos u_3\cosh u_4\partial x_4 - \cos u_3\sinh u_4\partial y_3 + \sin u_3\cosh u_4\partial y_4. \end{aligned}$$

Since $\phi_0 W_1 \neq W_2$, *then it is easy to see that* $\mu = \{0\}$ *and*

$$\begin{aligned} \phi_0 Z_3 &= \frac{2\sinh u_4\cosh u_4}{\sinh^2 u_4 + \cosh^2 u_4}Z_4 + \frac{1}{\sinh^2 u_4 + \cosh^2 u_4}W_1, \\ \phi_0 Z_4 &= -\frac{2\sinh u_4\cosh u_4}{\sinh^2 u_4 + \cosh^2 u_4}Z_3 - \frac{1}{\sinh^2 u_4 + \cosh^2 u_4}W_2. \end{aligned}$$

Then, $D' = Sp\{Z_3, Z_4\}$ *and* M *is a screen generic lightlike submanifold of* $\mathbb{R}^{11}_4$. *Hence,* M *is a proper screen generic lightlike submanifold of* $\mathbb{R}^{11}_4$, *with a quasi-orthonormal basis of* $(\bar{M}, \bar{g})$ *along* M *is*

$$\begin{aligned} \xi_1 &= Z_1,\ \xi_2 = Z_2, \\ e_1 &= \frac{1}{\sqrt{\sinh^2 u_4 + \cosh^2 u_4}}Z_3,\ e_2 = \frac{1}{\sqrt{\sinh^2 u_4 + \cosh^2 u_4}}Z_4, \\ e_3 &= \frac{1}{\sqrt{\sin^2 u_5 + \sinh^2 u_6}}Z_5,\ e_4 = \frac{1}{\sqrt{\sin^2 u_5 + \sinh^2 u_6}}Z_6,\ V = Z_7, \\ e_5 &= \frac{1}{\sqrt{\sinh^2 u_4 + \cosh^2 u_4}}W_1,\ e_6 = \frac{1}{\sqrt{\sinh^2 u_4 + \cosh^2 u_4}}W_2, \\ &\quad N_1, N_2. \end{aligned}$$

On the other hand, by direct computations and using Gauss and Weingarten formulas, we obtain

$$\begin{aligned} h^s(X, \xi_1) &= h^s(X, \xi_2) = h^s(X, e_3) = h^s(X, e_4) = 0,\ h^l = 0,\ \forall X \in \Gamma(TM) \\ h^l(e_1, e_1) &= h^l(e_2, e_2) = 0, \\ h^s(e_1, e_1) &= -\frac{1}{\left(\sinh^2 u^4 + \cosh^2 u^4\right)^{\frac{3}{2}}}W_1, \\ h^s(e_2, e_2) &= \frac{1}{\left(\sinh^2 u^4 + \cosh^2 u^4\right)^{\frac{3}{2}}}W_1. \end{aligned}$$

Thus $h^s = 0$ *on* $Rad(TM)$ *and*

$$trace\ |_{S(TM)}\ h = 0.$$

Then, it is clear that M *is not totally geodesic and, but it is a minimal screen generic lightlike submanifold of* $\mathbb{R}^{11}_4$.

Theorem 5.4. *Let* $(M, g, S(TM), S(TM^\perp))$ *be a screen generic lightlike submanifold of an indefinite cosymplectic manifold* $(\bar{M}, \bar{g})$. *Then* M *is minimal iff*

$$traceA^*_{\xi_k}\ |_{S(TM)} = traceA_{W_T}\ |_{S(TM)} = 0$$

where $\dim(TM) = m$, $\dim(tr(TM)) = n$, $\dim(Rad(TM)) = r$ *and* $W_T \in \Gamma(S(TM^\perp))$.

Proof. Since $\bar{\nabla}_V V = 0$, from (4), we get $h^l(V,V) = h^s(V,V) = 0$.

Now, take an quasi orthonormal frame $\{\xi_1, ..., \xi_{2r}, e_1, ..., e_m, V, W_1, ..., W_n, N_1, ..., N_{2r}\}$ such that $\{e_1, ..., e_{2a}\}$ are tangent to D_0 and $\{e_{2a+1}, ..., e_m\}$ are tangent to $D^{'}$. First from (cf. [1], page 140), we know that $h^l = 0$ on $RadTM$.

From definition of minimal submanifold, we know that

$$\begin{aligned} traceh\mid_{S(TM)} &= traceh\mid_{D_0} + traceh\mid_{D'} \\ &= \sum_{i=1}^{a} h(Z_i, Z_i) + \sum_{T=1}^{b} h(U_j, U_j) \\ &= 0 \end{aligned}$$

and $h^s\mid_{Rad(TM)} = 0$. If we choose an orthonormal basis of $S(TM)$ as $\{e_i\}_{i=1}^{m-r}$, then we derive

$$\begin{aligned} traceh\mid_{S(TM)} &= \sum_{i=1}^{2a} \varepsilon_i \left[h^l(e_i, e_i) + h^s(e_i, e_i)\right] + \sum_{j=2a+1}^{m} \varepsilon_j \left[h^l(e_j, e_j) + h^s(e_j, e_j)\right] \\ &= \sum_{i=1}^{2a} \varepsilon_i [\frac{1}{2r} \sum_{k=1}^{r} \bar{g}(h^l(e_i, e_i), \xi_k) N_k + \frac{1}{n-2r} \sum_{T=1}^{n-2r} \bar{g}(h^s(e_i, e_i), W_T) W_T] \\ &\quad + \sum_{J=1}^{b} \varepsilon_T [\frac{1}{2r} \sum_{k=1}^{r} \bar{g}(h^l(e_j, e_j), \xi_k) N_k + \frac{1}{n-2r} \sum_{T=1}^{n-2r} \bar{g}(h^s(e_j, e_j), W_T) W_T]. \end{aligned}$$

On the other hand, since

$$\begin{aligned} \bar{g}(h^l(e_i, e_i), \xi_k) N_k &= g(A^*_{\xi_k} e_i, e_i) N_k \text{ and} \\ \bar{g}(h^s(e_i, e_i), W_T) W_T &= g(A_{W_T} e_i, e_i) W_T, \end{aligned}$$

we obtain

$$traceh\mid_{S(TM)} = traceA^*_{\xi_k}\mid_{D_0 \oplus D'} + traceA_{W_T}\mid_{D_0 \oplus D'} .$$

Hence, we get

$$traceA^*_{\xi_k}\mid_{D_0 \oplus D'} = 0 \quad and \quad traceA_{W_T}\mid_{D_0 \oplus D'} = 0.$$

This completes the proof. □

References

[1] C. L. Bejan and K. L. Duggal, *Global lightlike manifolds and harmonicity*, Kodai Math. J. **28** (2005), 131–145.

[2] D.E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds*, Progress in Mathematics, 203, Birkhäuser Boston, Inc., Boston, MA, 2002.

[3] B. Doğan, B. Şahin and E. Yaşar, *Screen generic lightlike submanifolds*, Mediterr. J. Math. **16**(4) (2019), 21.

[4] K. L. Duggal and A. Bejancu, *Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications*, Kluwer Academic Pub., The Netherlands, 1996.

[5] K. L. Duggal and B. Şahin, *Generalized Cauchy-Riemann lightlike submanifolds of Kaehler manifolds*, Acta Mathematica Hungarica, **112**(1-2) (2006), 107–130.

[6] K. L. Duggal and B. Şahin, *Lightlike submanifolds of indefinite Sasakian manifolds*, International Journal of Mathematics and Mathematical Sciences, **2007** (2007), Article ID 57585, 21 pages.

[7] K. L. Duggal and B. Şahin, *Generalized Cauchy-Riemann Lightlike Submanifolds of Indefinite Sasakian Manifolds*, Acta Math. Hungar. **122** (2009), 45–58, 2009.

[8] K. L. Duggal and B. Şahin, *Differential Geometry of Lightlike Submanifolds*, Birkhäuser Verlag AG., 2010.

[9] R. S. Gupta, *Screen Generic Lightlike Submanifolds of Indefinite Sasakian Manifolds*. Mediterranean Journal of Mathematics, **17**(5) (2020), 1-23.

[10] R. S. Gupta and A. Sharfuddin, *Generalized Cauchy-Riemann Lightlike Submanifolds of Indefinite Kenmotsu Manifolds*, Note di Mathematica. **30** (2010), 49–59.

[11] R. S. Gupta and A. Sharfuddin, *Screen transversal lightlike submanifolds of indefinite cosymplectic manifolds*, Rendiconti del Seminario Matematico della Università di Padova, **124** (2010), 145-156.
[12] R. S. Gupta, A. Upadhyay and A. Sharfuddin, Lightlike submanifolds of indefinite cosymplectic manifolds. An. Ştiinţ. Univ. "Al. I. Cuza" din Iaşi. Mat., **58**(1) (2012), 157-180.
[13] R. S. Gupta, A. Upadhyay and A. Sharfuddin, *Generalised Cauchy-Riemann lightlike submanifolds of indefinite cosymplectic manifolds*, An. Ştiinţ. Univ. "Al. I. Cuza" din Iaşi. Mat. (N.S.), **58**(2) (2012), 381-394.
[14] D. H. Jin and J. W. Lee, *Generic lightlike submanifolds of an indefinite Kähler manifold*, Int. J. Pure Appl. Math. **101**(4) (2015), 543–560.
[15] E. Kılıç, B. Şahin and S. Keleş, *Screen Semi Invariant Lightlike Submanifolds of Semi-Riemannian Product Manifolds*, International Electronic Journal of Geometry, **4**(2) (2011), 120-135.
[16] S. Kumar, R. Kumar and R. K. Nagaich, *GCR-lightlike submanifolds of a semi-Riemannian product manifold*, Bull. Korean Math. Soc. **51**(3) (2014), 883-899.
[17] D. N. Kupeli, *Singular semi-Riemannian geometry*, Kluwer Academic Publishers, Dordrecht, 366, 1996.
[18] G. D. Ludden, *Submanifolds of cosymplectic manifolds*, J. Differential Geometry, **4** (1970), 237–244.
[19] N. (Önen) Poyraz, Golden GCR-Lightlike Submanifolds of Golden Semi-Riemannian Manifolds. Mediterranean Journal of Mathematics, **17**(5) (2020), 1-16.
[20] B. Şahin, *Screen transversal lightlike submanifolds of Kähler manifolds*, Chaos, Solitons and Fractals, **38** (2008), 1439–1448.
[21] C. Yıldırım and B. Şahin, *Screen transversal lightlike submanifolds of indefinite sasakian manifolds*, An. Şt. Univ. Ovidius Constanta, **18**(2) (2010), 315-336.