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Solution of an integral equation in $\mathscr{G}$-metric spaces

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Abstract. In this paper, we prove a fixed point result in $\mathscr{G}$-metric spaces satisfying generalized contractive condition with new auxiliary functions, which generalize the result of Bhardwaj and Kumar. The motivation of this paper is to observe the solution of an integral equation using the fixed point technique in $\mathscr{G}$-metric space.

1. Introduction

Many areas of pure and applied science, such as biology, medicine, physics, and computer science use metric spaces a lot. Mustafa and Sims [16] came up with G-metric spaces as an extension of the idea of metric spaces. They showed up with some fixed point theorems for mappings that fit different contractive conditions. See references [15, 17–19] for more fixed-point results on G-metric space.

Alghamdi and Karapinar [3] identified fixed point theorems for $G - \beta - \psi-$contractive type mappings. Ansari [4] showed that fixed point results could be noticed for φ-ψ mappings for contractive types. Aggarwal et al. [2] showed Suzuki-type fixed point results in G-metric spaces. Mustafa et al. [20] came up with two new ideas for complete G-metric spaces, such as (g-F) contractions and generalized Mizoguchi-Takahashi contractions. They also found some new coincidence points and fixed point theorems that are used often. Many mathematicians have taken the Banach contraction principle in different directions over the years (see [1, 5, 8–14, 21–23]). In 2019, the C-class function were used by Aydi et al.[6] to add the Q property to G-metric spaces. Bhardwaj and Kumar [7] solved the fixed point theorem for auxiliary functions in G-metric space.

In this paper, we were inspired and motivated by the work of Bhardwaj and Kumar [7]. We build on their main result by adding new auxiliary functions in $\mathscr{G}$-metric spaces through a generalized contractive condition.

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2. Preliminaries

Throughout this paper, $\mathbb{R}$ represents $(-\infty, +\infty)$, $\mathbb{R}_+$ is $(0, +\infty)$ and $\mathbb{R}_+^0$ represents $[0, +\infty)$, respectively. Mustafa and Sims [16] initiated the $\mathscr{G}$-metric space in 2006.

Definition 2.1. ([16]) *Let $\mathfrak{I}$ be a non-void set and a function $\mathscr{G} : \mathfrak{I}^3 \to \mathbb{R}_+$ satisfying the following hypothesis:*

1. *If $\varrho = \vartheta = ø$, then $\mathscr{G}(\varrho, \vartheta, ø) = 0$,*
2. *$0 < \mathscr{G}(\varrho, \varrho, \vartheta)$, $\forall \varrho, \vartheta \in \mathfrak{I}$, with $\varrho \neq \vartheta$,*
3. *$\mathscr{G}(\varrho, \varrho, \vartheta) \leq \mathscr{G}(\varrho, \vartheta, ø)$, $\forall \varrho, \vartheta, ø \in \mathfrak{I}$, with $\varrho \neq \vartheta$,*
4. *$\mathscr{G}(\varrho, \vartheta, ø) = \mathscr{G}(\varrho, ø, \vartheta) = \mathscr{G}(\vartheta, ø, \varrho) = \ldots$ (symmetry in all three variables),*
5. *$\mathscr{G}(\varrho, \vartheta, ø) \leq \mathscr{G}(\varrho, \kappa, \kappa) + \mathscr{G}(\kappa, \vartheta, ø)$, for all $\varrho = \vartheta = ø \in \mathfrak{I}$, (rectangular inequality).*

Then $(\mathfrak{I}, \mathscr{G})$ is said to be a $\mathscr{G}$-metric space.

Definition 2.2. ([16]) *Let $(\mathfrak{I}, \mathscr{G})$ be a $\mathscr{G}$-metric space, and a sequence $\{\tau_ø\}$ of $\mathfrak{I}$. Then, the sequence $(\tau_ø)$ is called $\mathscr{G}$-convergent to $\tau \in \mathfrak{I}$ if $\lim_{ø,\vartheta \to \infty} \mathscr{G}(\tau, \tau_ø, \tau_\vartheta) = 0$, i.e., for $\varepsilon > 0$, there exists $N \in \mathcal{N}$ such that $\mathscr{G}(\tau, \tau_ø, \tau_\vartheta) < \varepsilon$, $\forall\, ø, \vartheta > N$. τ is called the limit of the sequence, then $\tau_ø \to \tau$. Symbolically, $\lim_{ø \to \infty} \tau_ø = \tau$.*

Definition 2.3. ([16]) *Let $(\mathfrak{I}, \mathscr{G})$ be a $\mathscr{G}$-metric space. A sequence $\{\tau_ø\}$ is said to be a $\mathscr{G}$-Cauchy sequence if $\varepsilon > 0$, there is $N \in \mathcal{N}$ such that $\mathscr{G}(\tau_ø, \tau_\vartheta, \tau_\varrho) < \varepsilon$, $\forall\, ø, \vartheta, \varrho > N$, that is $\mathscr{G}(\tau_ø, \tau_\vartheta, \tau_\varrho) \to 0$ as $ø, \vartheta, \varrho \to +\infty$.*

Definition 2.4. ([16]) *We say that every $\mathscr{G}$-metric space is complete if every $\mathscr{G}$-Cauchy sequence is $\mathscr{G}$-convergent in $(\mathfrak{I}, \mathscr{G})$.*

Mustafa and Sims [15] introduced the notion of contraction on $\mathscr{G}$-metric space.

Definition 2.5. ([15]) *Let $(\mathfrak{I}, \mathscr{G})$ be a $\mathscr{G}$-metric space and a self-map Υ on $\mathfrak{I}$ is called a contraction on $\mathfrak{I}$ if for each $\wedge \in [0, 1]$, the following condition is true:*

$$\mathscr{G}(\Upsilon\varrho, \Upsilon\vartheta, \Upsilon ø) \leq \wedge \mathscr{G}(\varrho, \vartheta, ø), \ \forall\ \varrho, \vartheta, ø \in \mathfrak{I}.$$

Example 2.6. *Let $\mathfrak{I} = \mathbb{R}^3$. Define a mapping $\mathscr{G} : \mathfrak{I}^3 \to \mathbb{R}_+$ by*

$$\mathscr{G}(\varrho, \vartheta, ø) = |\varrho_1 - \varrho_2| + |\varrho_2 - \varrho_3| + |\vartheta_1 - \vartheta_2| + |\vartheta_2 - \vartheta_3| + |ø_1 - ø_2| + |ø_2 - ø_3|, \ \forall \varrho, \vartheta, ø \in \mathfrak{I}$$

and define a function $\Upsilon : \mathbb{R}^3 \to \mathbb{R}^3$ by $\Upsilon\varrho = \frac{5}{8}\varrho$, $\forall \varrho \in \mathbb{R}^3$. Then Υ is a contraction on $\mathfrak{I}$ as

$$\mathscr{G}(\varrho, \vartheta, ø) = |\varrho_1 - \vartheta_1 - ø_1| + |\varrho_2 - \vartheta_2 - ø_2| + |\varrho_3 - \vartheta_3 - ø_3| = \frac{5}{8}\mathscr{G}(\varrho, \vartheta, ø).$$

In 2008, Mustafa and Sims [17] proved the below contraction theorem on $\mathscr{G}$-metric space.

Theorem 2.7. ([17]) *Let $(\mathfrak{I}, \mathscr{G})$ be a complete $\mathscr{G}$-metric space and let $\Upsilon : \mathfrak{I} \to \mathfrak{I}$ be a self-map satisfy the following*

$$\mathscr{G}(\Upsilon\varrho, \Upsilon\vartheta, \Upsilon ø) \leq \mu\mathscr{G}(\varrho, \Upsilon\varrho, \Upsilon\varrho) + \nu\mathscr{G}(\vartheta, \Upsilon\vartheta, \Upsilon\vartheta) + \eta\mathscr{G}(ø, \Upsilon ø, \Upsilon ø) + \omega\mathscr{G}(\varrho, \vartheta, ø),$$

$\forall \varrho, \vartheta, ø \in \mathfrak{I}$ and μ, ν, η, ω non-negative with $\mu + \nu + \eta + \omega < 1$. Then Υ has a unique fixed point in $\mathfrak{I}$.

Definition 2.8. ([3]) *Let Ψ be the class of all mappings $\Xi : \mathbb{R}_+^0 \to \mathbb{R}_+^0$ satisfying the following hypothesis:*

1. *$\sum_{£=1}^{+\infty} \Xi^{£}(\curlyvee) < +\infty$, $\curlyvee > 0$, where $\Xi^{£}$ is the $£^{th}$ iterate of Ξ,*
2. *Ξ is nondecreasing.*

Definition 2.9. ([4]) *A continuous function $C : (\mathbb{R}_+^0)^2 \to \mathbb{R}$ is called $\mathcal{D}$-class function if it satisfies the following hypothesis:*

1. $C(\tau,\rho) \leq \tau$ *for all* $\tau,\rho \in \mathbb{R}_+^0$;
2. *If* $C(\tau,\rho) = \tau$, *then either* $\tau = 0$ *or* $\rho = 0$;

Let us consider:

$\Phi_1 : \{\varphi_1 : \mathbb{R}_+^0 \to \mathbb{R}_+^0$ *is a non-decreasing continuous function such that* $\phi_1(\vartheta) = 0$ *iff* $\vartheta = 0\}$,

$\Phi_2 : \{\varphi_2 : \mathbb{R}_+^0 \to \mathbb{R}_+^0$ *is a continuous such that* $\phi_2(0) = 0$ *and* $\varphi_1(\vartheta) > 0$ *for* $\vartheta > 0\}$,

$\Phi_3 : \Big\{\varphi_3 : \mathbb{R}_+^0 \to \mathbb{R}_+^0$ *is a non-negative Lebesgue-integrable function, summable on each compact subset of* $\mathbb{R}_+$, *and for each* $\varepsilon > 0$ *such that* $\int_0^\varepsilon \varphi(\curlyvee)\partial\curlyvee > 0\Big\}$.

The aim of this paper is to introduce a new generalized contraction, in the context of $\mathscr{G}$-metric space. Consequently, we shall examine the integral equation of the fixed point for such mapping in the mentioned setting. In order to indicate the validity, an illustrative example is considered.

3. Main Results

We now prove the following main theorem of $\mathscr{G}$-metric space, as inspired by Bhardwaj and Kumar [7]:

Theorem 3.1. *Let* $(\Im,\mathscr{G})$ *be a* $\mathscr{G}$*-metric space and a mapping* $\hbar : \Im \to \Im$ *satisfying*

$$\varphi_1\Big[\int_0^{\mathscr{G}(\hbar\tau,\hbar\rho,\hbar\varpi)} \varphi(\curlyvee)\Big]\partial\curlyvee \leq C\Big[\varphi_1\Big(\int_0^{Q(\tau,\rho,\varpi)} \varphi(\curlyvee)\partial\curlyvee\Big), \varphi_2\Big(\int_0^{Q(\tau,\rho,\varpi)} \varphi(\curlyvee)\partial\curlyvee\Big)\Big], \tag{1}$$

where C *is a* $\mathcal{D}$*-class function* $\varphi_1 \in \Phi_1, \varphi_2 \in \Phi_2, \varphi \in \Phi_3$ *and*

$$Q(\tau,\rho,\varpi) = \max\{\mathscr{G}(\tau,\rho,\varpi), \mathscr{G}(\tau,\hbar\tau,\hbar\tau), \mathscr{G}(\rho,\hbar\rho,\hbar\rho), \mathscr{G}(\varpi,\hbar\varpi,\hbar\varpi), \frac{\mathscr{G}(\hbar\tau,\rho,\varpi) + \mathscr{G}(\tau,\hbar\rho,\hbar\varpi)}{4}\}. \tag{2}$$

Then $\hbar$ *has a unique fixed point.*

Proof. Let $\tau_0 \in \Im$. Choose a point $\tau_1 \in \Im$ such that $\tau_1 = \hbar\tau_0$. In this way, we construct $\tau_{\o+1}$ such that $\tau_{\o+1} = \hbar\tau_\o$ for $\o = 0, 1, 2, ...$ Assume that $\tau_\o \neq \tau_{\o+1}$ for each integer $\o > 1$, then by equation (1)

$$\varphi_1\Big[\int_0^{\mathscr{G}(\tau_\o,\tau_{\o+1},\tau_{\o+1}))} \varphi(\curlyvee)\Big]\partial\curlyvee \leq C\Big[\varphi_1\Big(\int_0^{Q(\tau_{\o-1},\tau_\o,\tau_\o)} \varphi(\curlyvee)\partial\curlyvee\Big), \varphi_2\Big(\int_0^{Q(\tau_{\o-1},\tau_\o,\tau_\o)} \varphi(\curlyvee)\partial\curlyvee\Big)\Big], \tag{3}$$

From (2), we get

$$\begin{aligned}
&Q(\tau_{\o-1},\tau_\o,\tau_\o)\\
&= \max\{\mathscr{G}(\tau_{\o-1},\tau_\o,\tau_\o), \mathscr{G}(\tau_{\o-1},\hbar\tau_{\o-1},\hbar\tau_{\o-1}), \mathscr{G}(\tau_\o,\hbar\tau_\o,\hbar\tau_\o), \mathscr{G}(\tau_\o,\hbar\tau_\o,\hbar\tau_\o),\\
&\qquad \frac{\mathscr{G}(\hbar\tau_{\o-1},\tau_\o,\tau_\o) + \mathscr{G}(\tau_{\o-1},\hbar\tau_\o,\hbar\tau_\o)}{4}\}\\
&= \max\{\mathscr{G}(\tau_{\o-1},\tau_\o,\tau_\o), \mathscr{G}(\tau_{\o-1},\tau_\o,\tau_\o), \mathscr{G}(\tau_\o,\tau_{\o+1},\tau_{\o+1}), \mathscr{G}(\tau_\o,\tau_{\o+1},\tau_{\o+1}),\\
&\qquad \frac{\mathscr{G}(\tau_\o,\tau_\o,\tau_\o) + \mathscr{G}(\tau_{\o-1},\tau_{\o+1},\tau_{\o+1})}{4}\}\\
&= \max\{\mathscr{G}(\tau_{\o-1},\tau_\o,\tau_\o), \mathscr{G}(\tau_\o,\tau_{\o+1},\tau_{\o+1}), \frac{\mathscr{G}(\tau_{\o-1},\tau_{\o+1},\tau_{\o+1})}{2}\}\\
&= \max\{\mathscr{G}(\tau_{\o-1},\tau_\o,\tau_\o), \mathscr{G}(\tau_\o,\tau_{\o+1},\tau_{\o+1}), \frac{\mathscr{G}(\tau_{\o-1},\tau_\o,\tau_\o) + \mathscr{G}(\tau_\o,\tau_{\o+1},\tau_{\o+1})}{2}\}.
\end{aligned} \tag{4}$$

If $\max\{\mathscr{G}(\tau_{ø-1},\tau_ø,\tau_ø),\mathscr{G}(\tau_ø,\tau_{ø+1},\tau_{ø+1})\}=\mathscr{G}(\tau_ø,\tau_{ø+1},\tau_{ø+1})$, from (3) and (4), we have

$$\varphi_1\Big[\int_0^{\mathscr{G}(\tau_ø,\tau_{ø+1},\tau_{ø+1})}\varphi(\curlyvee)\Big]\partial\curlyvee \le C\Big[\varphi_1\Big(\int_0^{\mathscr{G}(\tau_ø,\tau_{ø+1},\tau_{ø+1})}\varphi(\curlyvee)\partial\curlyvee\Big),\varphi_2\Big(\int_0^{\mathscr{G}(\tau_ø,\tau_{ø+1},\tau_{ø+1})}\varphi(\curlyvee)\partial\curlyvee\Big)\Big]. \tag{5}$$

By definition of $C\in\mathcal{D}$, then either

$$\varphi_1\Big(\int_0^{\mathscr{G}(\tau_ø,\tau_{ø+1},\tau_{ø+1})}\varphi(\curlyvee)\partial\curlyvee\Big)=0 \quad\text{or}\quad \varphi_2\Big(\int_0^{\mathscr{G}(\tau_ø,\tau_{ø+1},\tau_{ø+1})}\varphi(\curlyvee)\partial\curlyvee\Big)=0.$$

Only possible if $\int_0^{\mathscr{G}(\tau_ø,\tau_{ø+1},\tau_{ø+1})}\varphi(\curlyvee)\partial\curlyvee=0$. This is a contraction from definitions of φ_1 and φ_2. Thus, $\mathcal{Q}(\tau_{ø-1},\tau_ø,\tau_ø)=\mathscr{G}(\tau_{ø-1},\tau_ø,\tau_ø)$ and we have

$$\begin{aligned}\varphi_1\Big(\int_0^{\mathscr{G}(\tau_ø,\tau_{ø+1},\tau_{ø+1})}\varphi(\curlyvee)\Big)\partial\curlyvee &\le C\Big(\varphi_1\Big(\int_0^{\mathscr{G}(\tau_{ø-1},\tau_ø,\tau_ø)}\varphi(\curlyvee)\partial\curlyvee\Big),\varphi_2\Big(\int_0^{\mathscr{G}(\tau_{ø-1},\tau_ø,\tau_ø)}\varphi(\curlyvee)\partial\curlyvee\Big)\Big)\\ &\le \varphi_1\Big(\int_0^{\mathscr{G}(\tau_{ø-1},\tau_ø,\tau_ø)}\varphi(\curlyvee)\partial\curlyvee\Big).\end{aligned}$$

Since φ_1 is continuous and nondecreasing, therefore

$$\int_0^{\mathscr{G}(\tau_ø,\tau_{ø+1},\tau_{ø+1})}\varphi(\curlyvee)\partial\curlyvee\le\int_0^{\mathscr{G}(\tau_{ø-1},\tau_ø,\tau_ø)}\varphi(\curlyvee)\partial\curlyvee,$$

thus the sequence $\{\int_0^{\mathscr{G}(\tau_ø,\tau_{ø+1},\tau_{ø+1})}\varphi(\curlyvee)\partial\curlyvee\}$ is lower bounded monotone decreasing. Therefore, there exists $\pi\ge 0$ such that

$$\lim_{ø\to\infty}\int_0^{\mathscr{G}(\tau_ø,\tau_{ø+1},\tau_{ø+1})}\varphi(\curlyvee)\partial\curlyvee=\pi. \tag{6}$$

Assume that $\pi\ge 0$. In equation (5), letting $\lim_{ø\to\infty}$ on both sides and by (6), we have

$$\varphi_1(\pi)\le C(\varphi_1(\pi),\varphi_2(\pi)).$$

It can be seen that from definition of $C\in\mathcal{D}$, then either

$$\varphi_1(\pi)=0 \quad\text{or}\quad \varphi_2(\pi)=0.$$

From the above φ_1 and φ_2, we have $\pi=0$. Hence by (6), we get

$$\lim_{ø\to\infty}\int_0^{\mathscr{G}(\tau_ø,\tau_{ø+1},\tau_{ø+1})}\varphi(\curlyvee)\partial\curlyvee=0, \tag{7}$$

implies

$$\lim_{ø\to\infty}\mathscr{G}(\tau_ø,\tau_{ø+1},\tau_{ø+1})=0. \tag{8}$$

Now, we show that the sequence $\{\tau_ø\}$ is Cauchy. On contrary, for an $\varepsilon>0$, there exists two subsequences $\{\tau_{\vartheta_£}\},\{\tau_{ø_£}\}\in\{\tau_ø\}$ with $\vartheta(£)<ø(£)<\vartheta(£+1)$ such that

$$\mathscr{G}(\tau_{\vartheta(£)},\tau_{ø(£)},\tau_{ø(£)})\ge\varepsilon,\mathscr{G}(\tau_{\vartheta(£)},\tau_{ø(£)-1},\tau_{ø(£)-1})<\varepsilon. \tag{9}$$

Consider

$$\begin{aligned}\varphi_1\int_0^{\varepsilon}\varphi(\curlyvee)\partial\curlyvee &\le\varphi_1\Big(\int_0^{\mathscr{G}(\tau_{\vartheta(£)},\tau_{ø(£)},\tau_{ø(£)})}\varphi(\curlyvee)\partial\curlyvee\Big)\\ &\le C\Big\{\varphi_1\Big(\int_0^{\mathcal{Q}(\tau_{\vartheta(£)-1},\tau_{ø(£)-1},\tau_{ø(£)-1})}\varphi(\curlyvee)\partial\curlyvee\Big),\varphi_2\Big(\int_0^{\mathcal{Q}(\tau_{\vartheta(£)-1},\tau_{ø(£)-1},\tau_{ø(£)-1})}\varphi(\curlyvee)\partial\curlyvee\Big)\Big\}.\end{aligned} \tag{10}$$

Using (2),

$$
\begin{aligned}
&\mathcal{Q}(\tau_{\vartheta(£)-1}, \tau_{ø(£)-1}, \tau_{ø(£)-1}) \\
&= \max\Big\{\mathscr{G}(\tau_{\vartheta(£)-1}, \tau_{ø(£)-1}, \tau_{ø(£)-1}), \mathscr{G}(\tau_{\vartheta(£)-1}, \hbar\tau_{\vartheta(£)-1}, \hbar\tau_{\vartheta(£)-1}), \mathscr{G}(\tau_{ø(£)-1}, \hbar\tau_{ø(£)-1}, \hbar\tau_{ø(£)-1}), \\
&\qquad \mathscr{G}(\tau_{ø(£)-1}, \hbar\tau_{ø(£)-1}, \hbar\tau_{ø(£)-1}), \frac{\mathscr{G}(\hbar\tau_{\vartheta(£)-1}, \tau_{ø(£)-1}, \tau_{ø(£)-1}) + \mathscr{G}(\tau_{\vartheta(£)-1}, \hbar\tau_{ø(£)-1}, \hbar\tau_{ø(£)-1})}{4}\Big\} \\
&= \max\Big\{\mathscr{G}(\tau_{\vartheta(£)-1}, \tau_{ø(£)-1}, \tau_{ø(£)-1}), \mathscr{G}(\tau_{\vartheta(£)-1}, \tau_{\vartheta(£)}, \tau_{\vartheta(£)}), \mathscr{G}(\tau_{ø(£)-1}, \tau_{ø(£)}, \tau_{ø(£)}), \\
&\qquad \mathscr{G}(\tau_{ø(£)-1}, \tau_{ø(£)}, \tau_{ø(£)}), \frac{\mathscr{G}(\tau_{\vartheta(£)}, \tau_{ø(£)-1}, \tau_{ø(£)-1}) + \mathscr{G}(\tau_{\vartheta(£)-1}, \tau_{ø(£)}, \tau_{ø(£)})}{4}\Big\} \\
&= \max\Big\{\mathscr{G}(\tau_{\vartheta(£)-1}, \tau_{ø(£)-1}, \tau_{ø(£)-1}), \mathscr{G}(\tau_{\vartheta(£)-1}, \tau_{\vartheta(£)}, \tau_{\vartheta(£)}), \mathscr{G}(\tau_{ø(£)-1}, \tau_{ø(£)}, \tau_{ø(£)}), \\
&\qquad \frac{\mathscr{G}(\tau_{\vartheta(£)-1}, \tau_{\vartheta(£)}, \tau_{\vartheta(£)}) + \mathscr{G}(\tau_{\vartheta(£)}, \tau_{ø(£)-1}, \tau_{ø(£)-1}) + \mathscr{G}(\tau_{ø(£)-1}, \tau_{ø(£)}, \tau_{ø(£)})}{4}\Big\} \\
&= \max\Big\{\mathscr{G}(\tau_{\vartheta(£)-1}, \tau_{ø(£)-1}, \tau_{ø(£)-1}), \mathscr{G}(\tau_{\vartheta(£)-1}, \tau_{\vartheta(£)}, \tau_{\vartheta(£)}), \mathscr{G}(\tau_{ø(£)-1}, \tau_{ø(£)}, \tau_{ø(£)})\Big\}. \qquad (11)
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\int_0^{\mathcal{Q}(\tau_{\vartheta(£)-1}, \tau_{ø(£)-1}, \tau_{ø(£)-1})} \varphi(\curlyvee)\partial\curlyvee \\
&= \int_0^{\max\{\mathscr{G}(\tau_{\vartheta(£)-1}, \tau_{ø(£)-1}, \tau_{ø(£)-1}), \mathscr{G}(\tau_{\vartheta(£)-1}, \tau_{\vartheta(£)}, \tau_{\vartheta(£)}), \mathscr{G}(\tau_{ø(£)-1}, \tau_{ø(£)}, \tau_{ø(£)})\}} \varphi(\curlyvee)\partial\curlyvee \\
&= \max\Big\{\int_0^{\mathscr{G}(\tau_{\vartheta(£)-1}, \tau_{ø(£)-1}, \tau_{ø(£)-1})} \varphi(\curlyvee)\partial\curlyvee, \int_0^{\mathscr{G}(\tau_{\vartheta(£)-1}, \tau_{\vartheta(£)}, \tau_{\vartheta(£)})} \varphi(\curlyvee)\partial\curlyvee, \int_0^{\mathscr{G}(\tau_{ø(£)-1}, \tau_{ø(£)}, \tau_{ø(£)})} \varphi(\curlyvee)\partial\curlyvee\Big\}. \qquad (12)
\end{aligned}
$$

Using (9) and triangle inequality, we get

$$
\begin{aligned}
\mathscr{G}(\tau_{\vartheta(£)-1}, \tau_{ø(£)-1}, \tau_{ø(£)-1}) &\leq \mathscr{G}(\tau_{\vartheta(£)-1}, \tau_{\vartheta(£)}, \tau_{\vartheta(£)}) + \mathscr{G}(\tau_{\vartheta(£)}, \tau_{ø(£)-1}, \tau_{ø(£)-1}) \\
&< \mathscr{G}(\tau_{\vartheta(£)-1}, \tau_{\vartheta(£)}, \tau_{\vartheta(£)}) + \varepsilon.
\end{aligned}
$$

Therefore,

$$
\lim_{£\to\infty} \int_0^{\mathscr{G}(\tau_{\vartheta(£)-1}, \tau_{ø(£)-1}, \tau_{ø(£)-1})} \varphi(\curlyvee)\partial\curlyvee \leq \int_0^{\varepsilon} \varphi(\curlyvee)\partial\curlyvee\,. \qquad (13)
$$

Taking $\lim_{£\to\infty}$ on both sides of (10) and by (11), (12), (13), we have

$$
\varphi_1 \int_0^{\varepsilon} \varphi(\curlyvee)\partial\curlyvee \leq C\Big\{\varphi_1\Big(\int_0^{\varepsilon} \varphi(\curlyvee)\partial\curlyvee\Big), \varphi_2\Big(\int_0^{\varepsilon} \varphi(\curlyvee)\partial\curlyvee\Big)\Big\}.
$$

Again from definition of $C \in \mathcal{D}$, we get either

$$
\varphi_1\Big(\int_0^{\varepsilon} \varphi(\curlyvee)\partial\curlyvee\Big) = 0 \quad \text{or} \quad \varphi_2\Big(\int_0^{\varepsilon} \varphi(\curlyvee)\partial\curlyvee\Big) = 0.
$$

It is possible only if $\int_0^{\varepsilon} \varphi(\curlyvee)\partial\curlyvee = 0$. This is a contraction by our assumption, therefore, $\aleph$ be the limit of the Cauchy sequence $\{\tau_ø\}$ such that

$$
\lim_{ø\to\infty} \hbar\tau_{ø-1} = \aleph. \qquad (14)
$$

Next, we show that the fixed point of map $\hbar$ is $\aleph$. That is $\hbar\aleph = \aleph$, on contrary. Then $\mathscr{G}(\hbar\aleph, \aleph, \aleph) > 0$. Let $\lambda = \mathscr{G}(\hbar\aleph, \aleph, \aleph)$. Now,

$$\varphi_1 \int_0^{\lambda} \varphi(\curlyvee)\partial\curlyvee = \varphi_1\Big(\int_0^{\mathscr{G}(\hbar\aleph,\aleph,\aleph)} \varphi(\curlyvee)\partial\curlyvee\Big)$$
$$\leq C\Big\{\varphi_1\Big(\int_0^{Q(\varepsilon,\tau_\o,\tau_\o)} \varphi(\curlyvee)\partial\curlyvee\Big), \phi_2\Big(\int_0^{Q(\varepsilon,\tau_\o,\tau_\o)} \varphi(\curlyvee)\partial\curlyvee\Big)\Big\}. \tag{15}$$

where

$$Q(\varepsilon, \tau_\o, \tau_\o) = \max\{\mathscr{G}(\varepsilon, \tau_\o, \tau_\o), \mathscr{G}(\varepsilon, \hbar\varepsilon, \hbar\varepsilon), \mathscr{G}(\tau_\o, \hbar\tau_\o, \hbar\tau_\o), \mathscr{G}(\tau_\o, \hbar\tau_\o, \hbar\tau_\o),$$
$$\frac{\mathscr{G}(\hbar\varepsilon, \tau_\o, \tau_\o) + \mathscr{G}(\varepsilon, \hbar\tau_\o, \hbar\tau_\o)}{4}\}. \tag{16}$$

Since,

$$\lim_{\o\to\infty} \mathscr{G}(\varepsilon, \tau_\o, \tau_\o) = \lim_{\o\to\infty} \mathscr{G}(\tau_\o, \tau_{\o+1}, \tau_{\o+1}) = 0. \tag{17}$$

Taking $\lim_{\o\to\infty}$ in (15) and by (14), (16), (17), we have

$$\varphi_1 \int_0^{\lambda} \varphi(\curlyvee)\partial\curlyvee \leq C\Big\{\varphi_1\Big(\int_0^{\max \mathscr{G}(\varepsilon,\hbar\varepsilon,\hbar\varepsilon)} \varphi(\curlyvee)\partial\curlyvee\Big), \varphi_2\Big(\int_0^{\max \mathscr{G}(\varepsilon,\hbar\varepsilon,\hbar\varepsilon)} \varphi(\curlyvee)\partial\curlyvee\Big)\Big\}$$
$$\leq C\Big\{\varphi_1\Big(\int_0^{\lambda} \varphi(\curlyvee)\partial\curlyvee\Big), \varphi_2\Big(\int_0^{\lambda} \varphi(\curlyvee)\partial\curlyvee\Big)\Big\}. \tag{18}$$

Thus, we obtain either

$$\varphi_1\Big(\int_0^{\lambda} \varphi(\curlyvee)\partial\curlyvee\Big) = 0 \quad \text{or} \quad \varphi_2\Big(\int_0^{\lambda} \varphi(\curlyvee)\partial\curlyvee\Big) = 0$$

that is $\int_0^{\lambda} \varphi(\curlyvee)\partial\curlyvee = 0$. Hence the fact that $\lambda = 0$ means that $\mathcal{P}(\hbar\aleph, \aleph, \aleph) = 0$. So, the fixed point of map $\hbar$ is $\aleph$. □

For application purposes, from our main results have been derived some useful corollaries. If we let $\varphi(\curlyvee) = \curlyvee$ in Theorem 3.1, we get a corollary.

Corollary 3.2. *Let $(\Im, \mathscr{G})$ be a complete $\mathscr{G}$-metric space and a mapping $\hbar : \Im \to \Im$ such that $\forall\ \tau, \rho, \varpi \in \Im$,*

$$\varphi_1\Big(\int_0^{\mathscr{G}(\hbar\tau,\hbar\rho,\hbar\varpi)} \varphi(\curlyvee))\partial\curlyvee\Big) \leq C\Big(\Big(\int_0^{Q(\tau,\rho,\varpi)} \varphi(\curlyvee)\partial\curlyvee\Big), \varphi_2\Big(\int_0^{Q(\tau,\rho,\varpi)} \varphi(\curlyvee)\partial\curlyvee\Big)\Big),$$

where

$$Q(\tau, \rho, \varpi) = \max\{\mathscr{G}(\tau, \rho, \varpi), \mathscr{G}(\tau, \hbar\tau, \hbar\tau), \mathscr{G}(\rho, \hbar\rho, \hbar\rho), \mathscr{G}(\varpi, \hbar\varpi, \hbar\varpi), \frac{\mathscr{G}(\hbar\tau, \rho, \varpi) + \mathscr{G}(\tau, \hbar\rho, \hbar\varpi)}{4}\},$$

C is a $\mathcal{D}$-class function, $\varphi_2 \in \Phi_2, \varphi \in \Phi_3$.

Corollary 3.3. *Let $(\Im, \mathscr{G})$ be a complete $\mathscr{G}$-metric space and a mapping $\hbar : \Im \to \Im$ such that $\forall\ \tau, \rho, \varpi \in \Im$,*

$$\varphi_1\Big(\int_0^{\mathscr{G}(\hbar\tau,\hbar\rho,\hbar\varpi)} \varphi(\curlyvee)\partial\curlyvee\Big) \leq \wedge\varphi_1\Big(\int_0^{Q(\tau,\rho,\varpi)} \varphi(\curlyvee)\partial\curlyvee\Big), \tag{19}$$

where $Q(\tau, \rho, \varpi)$ is given in (2), $\wedge \in (0, 1), \varphi_1 \in \Phi_1, \varphi \in \Phi_3$. Then $\hbar$ has a unique fixed point.

Corollary 3.4. *Let $(\Im, \mathscr{G})$ be a complete $\mathscr{G}$-metric space and a mapping $\hbar : \Im \to \Im$ such that $\forall\ \tau, \rho, \varpi \in \Im$,*

$$\varphi_1\Big(\int_0^{\mathscr{G}(\hbar\tau,\hbar\rho,\hbar\varpi)} \varphi(\curlyvee)\partial\curlyvee\Big) \leq \varphi_1\Big(\int_0^{Q(\tau,\rho,\varpi)} \varphi(\curlyvee)\partial\curlyvee\Big) - \varphi_2\Big(\int_0^{Q(\tau,\rho,\varpi)} \varphi(\curlyvee)\partial\curlyvee\Big), \tag{20}$$

where $Q(\tau, \rho, \varpi)$ is given in (2), $\varphi_1 \in \Phi_1, \varphi_2 \in \Phi_2, \varphi \in \Phi_3$. Then $\hbar$ has a unique fixed point.

4. Application

In this section, we will use Corollary 3.3 to show that there is a solution to the integral equation:

$$\tau(\curlyvee) = \int_{\mathfrak{i}}^{\mathfrak{j}} \mathcal{U}(\curlyvee, \mathfrak{f})\mathcal{V}(\mathfrak{f}, \tau(\mathfrak{f}))\partial\mathfrak{f}, \curlyvee \in [\mathfrak{i}, \mathfrak{j}]. \tag{21}$$

Let $\mathfrak{I} = (\mathcal{D}[\mathfrak{i}, \mathfrak{j}], \mathbb{R})$ stand for the set of all functions that are continuous from $[\mathfrak{i}, \mathfrak{j}]$ to $\mathbb{R}$. Define a mapping $\hbar : \mathfrak{I} \to \mathfrak{I}$ by

$$\hbar\tau(\curlyvee) = \int_{\mathfrak{i}}^{\mathfrak{j}} \mathcal{U}(\curlyvee, \mathfrak{f})\mathcal{V}(\mathfrak{f}, \tau(\mathfrak{f}))\partial\mathfrak{f}, \curlyvee \in [\mathfrak{i}, \mathfrak{j}]. \tag{22}$$

Theorem 4.1. *Consider equation* (21) *and assume*

1. $\mathcal{U} : [\mathfrak{i}, \mathfrak{j}] \times [\mathfrak{i}, \mathfrak{j}] \to [0, 1)$ *is continuous mapping,*
2. $\mathcal{V} : [\mathfrak{i}, \mathfrak{j}] \times \mathbb{R} \to \mathbb{R}$, *where* $\mathcal{V}$ *is a continuous function,*
3. $\max_{\curlyvee\in[\mathfrak{i},\mathfrak{j}]} \int_{\mathfrak{i}}^{\mathfrak{j}} \mathcal{U}(\curlyvee, \mathfrak{f})\partial\mathfrak{f} < \wedge$ *for some* $\wedge \in (0, 1)$,
4. *For all* $\tau(\mathfrak{f}), \rho(\mathfrak{f}) \in \mathfrak{I}, \mathfrak{f} \in [\mathfrak{i}, \mathfrak{j}]$, *we have* $|\mathcal{V}(\mathfrak{f}, \tau(\mathfrak{f})) - \mathcal{V}(\mathfrak{f}, \rho(\mathfrak{f}))| \leq |\tau(\mathfrak{f}) - \rho(\mathfrak{f})|$.

Then, equation (21) *has a solution.*

Proof. Define the $\mathscr{G}$-metric on $\mathfrak{I}$ by

$$\mathscr{G}(\tau, \rho, \varpi) = \partial(\tau, \rho) + \partial(\rho, \varpi) + \partial(\tau, \varpi), \forall\, \tau, \rho, \varpi \in \mathfrak{I},$$

where

$$\partial(\tau, \rho) = sup_{\curlyvee\in[\mathfrak{i},\mathfrak{j}]}|\tau(\curlyvee) - \rho(\curlyvee)|.$$

Since $(\mathfrak{I}, \partial)$ is a complete metric space, it is easy to see that $(\mathfrak{I}, \mathscr{G})$ is also a complete $\mathscr{G}$-metric space. Let $\tau(\curlyvee), \rho(\curlyvee) \in \mathfrak{I}$. From (22), (3) and (4), we know that

$$\begin{aligned}
|\hbar\tau(\curlyvee) - \hbar\rho(\curlyvee)| &= |\int_{\mathfrak{i}}^{\mathfrak{j}} \mathcal{U}(\curlyvee, \mathfrak{f})\mathcal{V}(\mathfrak{f}, \tau(\mathfrak{f})) - \mathcal{V}(\mathfrak{f}, \rho(\mathfrak{f}))\partial\mathfrak{f}| \\
&\leq \int_{\mathfrak{i}}^{\mathfrak{j}} \mathcal{U}(\curlyvee, \mathfrak{f})|\mathcal{V}(\mathfrak{f}, \tau(\mathfrak{f})) - \mathcal{V}(\mathfrak{f}, \rho(\mathfrak{f}))|\partial\mathfrak{f} \\
&\leq \int_{\mathfrak{i}}^{\mathfrak{j}} \mathcal{U}(\curlyvee, \mathfrak{f})|\tau(\mathfrak{f}) - \rho(\mathfrak{f})|\partial\mathfrak{f} \\
&\leq \int_{\mathfrak{i}}^{\mathfrak{j}} \mathcal{U}(\curlyvee, \mathfrak{f})|\tau(\mathfrak{f}) - \rho(\mathfrak{f})|\partial\mathfrak{f} \\
&= |\tau(\curlyvee) - \rho(\curlyvee)| \int_{\mathfrak{i}}^{\mathfrak{j}} \mathcal{U}(\curlyvee, \mathfrak{f})\partial\mathfrak{f} \\
&\leq \wedge|\tau(\curlyvee) - \rho(\curlyvee)|.
\end{aligned}$$

Hence,

$$sup_{\curlyvee\in[\mathfrak{i},\mathfrak{j}]}|\hbar\tau(\curlyvee) - \hbar\rho(\curlyvee)| \leq \wedge sup_{\curlyvee\in[\mathfrak{i},\mathfrak{j}]}|\tau(\curlyvee) - \rho(\curlyvee)|. \tag{23}$$

Similarly, we have

$$sup_{\curlyvee\in[\mathfrak{i},\mathfrak{j}]}|\hbar\rho(\curlyvee) - \hbar\varpi(\curlyvee)| \leq \wedge sup_{\curlyvee\in[\mathfrak{i},\mathfrak{j}]}|\rho(\curlyvee) - \varpi(\curlyvee)| \tag{24}$$

and

$$sup_{\curlyvee\in[i,j]}|\hbar\tau(\curlyvee) - \hbar\varpi(\curlyvee)| \leq \wedge sup_{\curlyvee\in[i,j]}|\tau(\curlyvee) - \varpi(\curlyvee)|. \tag{25}$$

Therefore, from (23), (24), and (25), we have

$$\begin{aligned} &sup_{\curlyvee\in[i,j]}|\hbar\tau(\curlyvee) - \hbar\rho(\curlyvee)| + sup_{\curlyvee\in[i,j]}|\hbar\rho(\curlyvee) - \hbar\varpi(\curlyvee)| + sup_{\curlyvee\in[i,j]}|\hbar\tau(\curlyvee) - \hbar\varpi(\curlyvee)| \\ &\leq \wedge[sup_{\curlyvee\in[i,j]}|\tau(\curlyvee) - \rho(\curlyvee)| + sup_{\curlyvee\in[i,j]}|\rho(\curlyvee) - \varpi(\curlyvee)| + sup_{\curlyvee\in[i,j]}|\tau(\curlyvee) - \varpi(\curlyvee)|], \end{aligned} \tag{26}$$

which implies

$$\mathscr{G}(\hbar\tau, \hbar\rho, \hbar\varpi) \leq \wedge\mathscr{G}(\tau, \rho, \varpi).$$

Therefore, Corollary 3.3 is true. So, $\hbar$ has a fixed point in $\mathfrak{I}$ which is a solution of (21). □

Below example shows that the condition of Theorem 4.1 is true.

Example 4.2. *The integral equation*

$$\tau(\curlyvee) = \int_{In(2)}^{In(3)} cosh(\mathfrak{f}\curlyvee)\tau(\mathfrak{f})\partial\mathfrak{f}, \curlyvee \in [In(2), In(3)] \tag{27}$$

has a solution in $\mathfrak{I} = (\mathcal{D}[In(2), In(3)], \mathbb{R})$.

Proof. Let $\hbar : \mathfrak{I} \to \mathfrak{I}$ be defined as

$$\hbar\tau(\curlyvee) = \int_{In(2)}^{In(3)} cosh(\mathfrak{f}\curlyvee)\tau(\mathfrak{f})\partial\mathfrak{f}, \curlyvee \in [In(2), In(3)].$$

Specifically $\mathcal{U}(\curlyvee, \mathfrak{f}) = cosh(\mathfrak{f}\curlyvee)$, $\mathcal{V}(\mathfrak{f}, \curlyvee) = \curlyvee$, and $\wedge \in (0, 1)$ in Theorem 4.1, we get

1. $\mathcal{U}(\curlyvee, \mathfrak{f})$ is continuous on $[In(2), In(3)] \times [In(2), In(3)]$,
2. $\mathcal{V}(\mathfrak{f}, \curlyvee)$ is continuous on $[In(2), In(3)] \times \mathbb{R}, \forall\mathfrak{f} \in [In(2), In(3)]$,
3. $$\begin{aligned} \max_{\curlyvee\in[In(2),In(3)]} \int_{In(2)}^{In(3)} cosh(\mathfrak{f}\curlyvee)\partial\mathfrak{f} &= max_{\curlyvee\in[In(2),In(3)]} \frac{sinh(In(3^{\curlyvee})) - sinh(In(2^{\curlyvee}))}{\curlyvee} \\ &= max_{\curlyvee\in[In(2),In(3)]} \frac{3^{\curlyvee} - 3^{-\curlyvee} - 2^{\curlyvee} + 2^{-\curlyvee}}{2\curlyvee} \\ &< 0.7 \leq \wedge, \end{aligned}$$
4. Clearly, $\forall\tau(\mathfrak{f}), \rho(\mathfrak{f}) \in \mathfrak{I}$, condition 4 of Theorem 4.1 is satisfied. Hence $\hbar$ has a fixed point in $\mathfrak{I}$, which is a solution to equation (27).

□

5. Conclusion

In this paper, we prove a fixed point results with the aid of new auxiliary functions for generalized contractive conditions in the setting of $\mathscr{G}$-metric spaces. An example and application are presented to strengthen our main results.

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References

[1] M.I. Abbas, M.A. Ragusa, Solvability of Langevin equations with two Hadamard fractional derivatives via Mittag–Leffler functions, Appl. Anal. 101(9) (2022) 3231–3245.

[2] M. Aggarwal, R. Chugh, R. Kamal, Suzuki-type fixed point results in G-metric spaces and applications, Int. J. Comput. Appl. 47(12) (2012) 14–17.

[3] M. A. Alghamdi, E. Karapinar, G-β-ψ-contractive type mappings and related fixed point theorems, J. Inequal. Appl. 70 (2013) 1–16.

[4] A.H. Ansari, Note on ϕ-ψ contractive type mappings and related fixed point, In: The 2nd Regional Conference on Mathematics and Applications, PNU (2014) 377–380.

[5] A.H. Ansari, O. Ege, S. Radenovic, Some fixed point results on complex valued G_b-metric spaces, RACSAM Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. 112(2) (2018) 463–472.

[6] H. Aydi, A.H. Ansari, B. Moeini, M.S.M. Noorani, H. Qawaqneh, Property Q on G-metric spaces via C-class functions, Int. J. Math. Comput. Sci. 14(3) (2019) 675–692.

[7] P. Bhardwaj, M. Kumar, Fixed point theorem in G-metric space for auxiliary functions, Commun. Math. Appl. 12(4) (2021) 795–802.

[8] V.S. Chary, G.S.M. Reddy, H. Isik, H. Aydi, D.S. Chary, Some fixed point theorems on α-β-G-complete G-metric spaces, Carpathian Math. Publ. 13(1) (2021) 58–67.

[9] O. Ege, I. Karaca, Common fixed point results on complex valued G_b-metric spaces, Thai J. Math. 16(3) (2018) 775–787.

[10] O. Ege, C. Park, A.H. Ansari, A different approach to complex valued G_b-metric spaces, Adv. Difference Equ. 152 (2020) 1–13.

[11] S. Etemad, M.M. Matar, M.A. Ragusa, S. Rezapour, Tripled fixed points and existence study to a tripled impulsive fractional differential system via measures of noncompactness, Mathematics, 10(1) (2022) 25.

[12] V. Gupta, O. Ege, R. Saini, M. de la Sen, Various fixed point results in complete G_b-metric spaces, Dyn. Sys. Appl. 30(2) (2021) 277–293.

[13] V. Gupta, O. Ege, R. Saini, Extended G_b-metric spaces and some fixed point results with an application to Fredholm integral equation, Electron. J. Math. Anal. Appl. 10(2) (2022) 219–227.

[14] A. Kaewcharoen, A. Kaewkhao, Common fixed points for single-valued and multi-valued mappings in G-metric spaces, Int. J. Math. Anal. 5 (2011) 1775–1790.

[15] Z. Mustafa, A new structure for generalized metric spaces with applications to fixed point theory, Ph.D. Thesis, The University of Newcastle, Australia (2005).

[16] Z. Mustafa, B. Sims, A new approach to generalized metric spaces, J. Nonlinear Convex Anal. 7 (2006) 289–297.

[17] Z. Mustafa, H. Obiedat, F. Awawdeh, Some fixed point theorems for mappings on complete G-metric spaces, Fixed Point Theory Appl. 189870 (2008) 1–12.

[18] Z. Mustafa, H. Aydi, E. Karapinar, Generalized Meir-Keeler type contractions on G-metric spaces, Appl. Math. Comput., 219 (2013) 10441–10447.

[19] Z. Mustafa, V. Parvaneh, M. Abbas, J. R. Roshan, Some coincidence point results for generalized $(\psi\ ,\ \phi)$-weakly contractive mappings in ordered G-metric spaces, Fixed Point Theory Appl. 326 (2013), 1–23.

[20] Z. Mustafa, M. Arshad, S.U. Khan, J. Ahmad, M.M.M. Jaradat, Common fixed points for multi valued mappings in G-metric spaces with applications, J. Nonlinear Sci. Appl. 10 (2017) 2550–2564.

[21] K.P.R. Rao, K. B. Lakshmi, Z. Mustafa, A unique common fixed point theorem for six maps in G-metric spaces, Int. J. Nonlinear Anal. Appl. 3 (2012) 17–23.

[22] N. Tahat, H. Aydi, E. Karapinar, W. Shatanawi, Common fixed points for single-valued and multi-valued maps satisfying a generalized contraction in G-metric spaces, Fixed Point Theory Appl. 48 (2012), 1–9.

[23] J. Yadav, M. Kumar, Reena, M. Imdad, S. Arora, Fixed point theorems for (ψ, β)-expansive mapping in G-metric space using control function, Nonlinear Funct. Anal. Appl. 26(5) (2021) 949–959.