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The maximum signless Laplacian spectral radius of graphs with forbidden subgraphs

Dandan Chen[a], Xiaoling Ma[a]

[a]*College of Mathematics and System Sciences, Xinjiang University, Urumqi Xinjiang 830017, P. R. China*

Abstract. The Brualdi-Solheid-Turán type problem has attracted wide attention recently. Let $\mathcal{F}$ be a set of graphs. A graph G is $\mathcal{F}$-free if it does not contain any element of $\mathcal{F}$ as a subgraph. In this paper, when $\mathcal{F}_1 = \{W_5, C_6\}$ and $\mathcal{F}_2 = \{F_5\}$, we determine the maximum signless Laplacian spectral radius of $\mathcal{F}_i$-free graph with n vertices, respectively and depict the corresponding extremal graph for $i = 1, 2$.

1. Introduction

Let $G = (V(G), E(G))$ be a simple undirected graph with vertex set $V(G)$ and edge set $E(G)$, where $n = |V(G)|$ is the *order* of G. For $v \in V(G)$, the *neighbour set* of vertex v is denoted by $N(v) = \{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighbour set* of vertex v is defined as $N[v] = N(v) \cup \{v\}$. $d(v) = |N(v)|$ is called the *degree* of vertex v. Let $\Delta(G)$ denote the maximum degree of the graph G.

The *adjacency matrix* of G is defined as the matrix $A(G) = (a_{s,t})_{s,t \in V(G)}$ with $a_{s,t} = 1$ if s, t are adjacent in G, and $a_{s,t} = 0$ otherwise. Let $D(G)$ denote the *diagonal matrix* of vertex degrees of G. The *signless Laplacian matrix* of G is defined as $Q(G) = D(G) + A(G)$. Moreover, we call the largest eigenvalue of $A(G)$ or $Q(G)$ is the *spectral radius* or *signless Laplacian spectral radius* of G, denoted by $\rho(G)$ or $q(G)$, respectively. Note that $Q(G)$ is a non-negative matrix. Using Perron-Frobenius theorem, there exists a non-negative unit eigenvector X of $Q(G)$ corresponding to $q(G)$. Such eigenvector is called the *Perron vector* of $Q(G)$. Particularly, if G is connected, then $Q(G)$ is irreducible, and thus its Perron vector is a positive vector.

Since the signless Laplacian spectra perform better in comparison to spectra of other commonly used graph matrices, van Dam and Haemers [8] stated that among matrices associated with a graph, the signless Laplacian seems to be the most convenient for use in studying graph properties. Therefore, on the studying of the signless Laplacian spectral theory has been widely concerned by researchers [5–7].

For any $S, T \subseteq V(G)$ with $S \cap T = \emptyset$, let $e(S, T)$ denote the number of the edges of G with one end vertex in S and the other in T. Particularly, let $e(S) := e(S, S)$. Let $G[S]$ denote the subgraph of G induced by S. Let $G - v$ and $G - uv$ denote the graphs obtained from G by deleting the vertex $v \in V(G)$ and the edge $uv \in E(G)$, respectively. Given two graphs G and H, let $G \nabla H$ denote the graph obtained from $G \cup H$ by joining each vertex of G to each vertex of H. As usual, K_n, P_n and C_n denote respectively the complete graph, the path

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Email addresses: 1398649520@qq.com (Dandan Chen), mxling2018@163.com (Xiaoling Ma)

and the cycle on n vertices. A *double star* $S(a, b)$ is the graph obtained by joining the center of a star with a leaves to a center of a star with b leaves by an edge. A *fan* F_n is the graph obtained by joining P_{n-1} with an additional vertex and a *wheel* W_n is the graph obtained by joining C_{n-1} with an additional vertex.

The Turán-type problem arises in extremal graph theory of determining $ex(n, F)$ and characterizing extremal graph. Let $\mathcal{F}$ be a set of graphs. A graph G is $\mathcal{F}$-free if it does not contain any graph of $\mathcal{F}$ as a subgraph. When the set $\mathcal{F}$ is a singleton, say $\{F\}$, we say that G is F-free. The *Turán number* of a graph F, denoted by $ex(n, F)$, is the maximum number of edges in a F-free graph of order n. A F-free graph of order n with $ex(n, F)$ edges is said to be an *extremal graph* with respect to F.

In 2010, Nikiforov [17] proposed a Brualdi-Solheid-Turán type problem: determine the maximal spectral radius of a F-free graph of order n, and characterize those graphs which attain the maximum spectral radius. For the adjacent spectral radius, the Brualdi-Solheid-Turán type problem has been investigated for various kinds of F, such as $\{\cup_{i=1}^{k} P_i\}$ [3], $\{K_r\}$ [15], $\{K_{s,t}\}$ [18], $\{C_6\}$ [23] and so on. For the signless Laplacian spectral radius, the Brualdi-Solheid-Turán type problem has also been studied for a variety of F, such as $\{P_k\}$ [19], $\{C_{2k}\}$ [20], $\{C_{2k+1}\}$ [22], $\{F_k\}$ [24] and so on.

Let $S_{n,k}$ be the graph obtained by joining each vertex of K_k to each vertex of an independent set of order $n-k$, i.e., $S_{n,k} = K_k \triangledown (n-k)K_1$. Also, let $S^+_{n,k}$ be the graph obtained by adding an edge to $S_{n,k}$, where $n-k \geq 2$.

In 2015, Nikiforov and Yuan studied the maximum signless Laplacian spectral radius of forbidden even cycles with n vertices, and gave the following result.

Theorem 1.1. *[20] Let $k \geq 2$, $n \geq 400k^2$, and let G be a graph of order n. If G has no cycle of length $2k+2$, then $q(G) < q(S^+_{n,k})$, unless $G = S^+_{n,k}$.*

Based on Theorem 1.1, it is easy to get that if G is C_6-free graph, then $q(G) \leq q(S^+_{n,2})$ for $n \geq 1600$. For $n < 1600$, whether the result holds? It is not easy to solve this problem. Therefore, in this paper, we focus on the maximum signless Laplacian spectral radius of $\{W_5, C_6\}$-free graph with n vertices, and prove that

Theorem 1.2. *Let G be a graph of order n. If $n \geq 10$ and G is $\{W_5, C_6\}$-free graph, then $q(G) \leq q(S^+_{n,2})$, equality if and only if $G \cong S^+_{n,2}$.*

Recently, Wang and Zhai ensured the maximum signless Laplacian spectral radius of F_{2k+1}-free with n vertices, and obtained the following result.

Theorem 1.3. *[21] Let $k \geq 2$ and $n \geq 36k^6$. If G is a F_{2k+1}-free graph of order n, then $q(G) \leq q(S_{n,k})$, with equality if and only if $G \cong S_{n,k}$.*

According to Theorem 1.3, when $k = 2$, the result holds for $n \geq 2304$. In this paper, we improve the result for $k = 2$.

Theorem 1.4. *If $n \geq 6$ and G is F_5-free graph, then $q(G) \leq \frac{n+2+\sqrt{n^2+4n-12}}{2}$, equality if and only if $G \cong S_{n,2}$.*

2. Preliminary Lemmas

In order to prove our main results, we state several known conclusions, all of which are used in following sections.

Lemma 2.1. *[2] If $3 \leq k \leq n-3$, then $H_{n,k}$, the graph obtained from the star $K_{1,n-1}$ by joining a vertex of degree 1 to $k+1$ other vertices of degree 1, is the unique connected graph that maximum signless Laplacian spectral radius over all connected graphs with n vertices and $n+k$ edges.*

We shall need the following classical lemma which has been developed by Erdős and Gallai.

Lemma 2.2. *[10] Let $k \geq 1$. If G is a graph of order n with no P_{k+2}, then $e(G) \leq \frac{kn}{2}$, with equality holding if and only if G is a union of disjoint copies of K_{k+1}.*

Lemma 2.3. *[16] Let $k \geq 1$ and let the vertices of a graph G be partitioned into two sets A and B. If*

$$2e(A) + e(A, B) > (2k-1) \mid A \mid + k \mid B \mid,$$

then there exists a path of order $2k+1$ with both end-vertices in A.

In this subsection, we introduce some known bounds on $q(G)$. The first one can be traced back to Merris [14], while the case of equality has been established in [11].

Lemma 2.4. *If G is a connected graph, then*

$$q(G) \leq \max\{d(u) + \frac{1}{d(u)} \sum_{v \in N(u)} d(v) : u \in V(G)\},$$

with equality if and only if G is either a regular graph or a semiregular bipartite graph.

Lemma 2.5. *[9] Let G be a graph with n vertices and m edges. Then*

$$q(G) \leq \frac{2m}{n-1} + n - 2,$$

with equality holding if and only if G is either complete, or is a star, or is a complete graph with one isolated vertex.

Lemma 2.6. *[4] Let G be a connected graph and G' be a proper subgraph of G. Then $q(G) > q(G')$.*

Lemma 2.7. *[13] Let G be a connected graph, X be the Perron vector of $Q(G)$ with x_i corresponding to the vertex $v_i \in V(G)$, and $\{v_1, \ldots, v_s\} \subseteq N_G(v) \setminus N_G(u)$ for some two vertices u, v of G. Let G^* be the graph obtained from G by deleting the edges vv_i and adding the edges uv_i for $1 \leq i \leq s$. If $x_u \geq x_v$, then $q(G^*) > q(G)$.*

Let M be a real matrix of order n, and let $N = \{1, 2, \ldots, n\}$. Given a partition π: $N = X_1 \cup X_2 \cup \cdots \cup X_k$, the matrix M can be accordingly partitioned as

$$M = \begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1k} \\ M_{21} & M_{22} & \cdots & M_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ M_{k1} & M_{k2} & \cdots & M_{kk} \end{bmatrix}.$$

The quotient matrix of M with respect to π is defined as the $k \times k$ matrix $B_\pi = (b_{ij})_{i,j=1}^{k}$, where b_{ij} is the average value of all row sums of M_{ij}. If each block M_{ij} of M has constant row sum b_{ij}, for all $i, j \in \{1, 2, \ldots, k\}$, then the partition π is called an *equitable partition* of M, and the matrix $B_\pi = (b_{ij})_{i,j=1}^{k}$ is called an *equitable quotient matrix* of M.

Lemma 2.8. *[1] Let M be a real symmetric matrix, and let B_π be an equitable quotient matrix of M. Then the eigenvalues of B_π are also eigenvalues of M. Furthermore, if M is nonnegative and irreducible, then*

$$\lambda_1(M) = \lambda_1(B_\pi),$$

where $\lambda_1(M)$ and $\lambda_1(B_\pi)$ are the largest eigenvalues of matrices M and B_π, respectively.

Lemma 2.9. *[12] If $n \geq 4$, then*

$$q(S_{n,2}) = \frac{n + 2 + \sqrt{n^2 + 4n - 12}}{2} > n + 2 - \frac{4}{n+1}.$$

Lemma 2.10. *Let G be a graph having the maximum signless Laplacian spectral radius among all $\mathcal{F}$-free graphs of order n, where $\mathcal{F} \in \{\{W_5, C_6\}, \{F_5\}\}$. Then G is connected.*

Proof. We only give the proof for $\mathcal{F} = \{W_5, C_6\}$, the rest cases can be proved by a similar discussion, whose procedures are omitted here.

Suppose to the contrary that G is disconnected. Let $G_1, G_2, \ldots, G_r$ be the connected components of G. Then we can add $r-1$ edges to G so that the obtained graph G^* is connected and $\{W_5, C_6\}$-free. By using the Rayleigh quotient and the Perron-Frobenius theorem, we can deduce that $q(G^*) > q(G)$, contrary to the maximality of $q(G)$. □

3. Proof of Theorem 1.2

In this section, before proving the Theorem 1.2, we first formulate an auxiliary result that is very important case for proving the Theorem 1.2. In this lemma as well as in the sequel, we shall assume that X is the Perron vector of $Q(G)$ with coordinate x_w corresponding to vertex $w \in V(G)$.

Lemma 3.1. *Let G be a graph of order $n \geq 6$. If G is $\{W_5, C_6\}$-free graph and $\Delta(G) = n-1$, then $q(G) \leq q(S^+_{n,2})$, the equality holds if and only if $G = S^+_{n,2}$.*

Proof. Assume that G is a graph attaining the maximum signless Laplacian spectral radius among all $\{W_5, C_6\}$-free graphs of order $n \geq 6$ with $\Delta(G) = n-1$. In view of Lemma 2.10, we obtain that G is connected. Since $S_{n,2}$ is a proper subgraph of $S^+_{n,2}$ and by Lemma 2.6, we have

$$q(S_{n,2}) < q(S^+_{n,2}). \tag{1}$$

Since $\Delta(G) = n-1$, there exists a vertex $u \in V(G)$ such that $d(u) = n-1$. As $S^+_{n,2}$ is $\{W_5, C_6\}$-free graph and $\Delta(S^+_{n,2}) = n-1$, combining Lemma 2.5 with Lemma 2.9, we get

$$n+2-\frac{4}{n+1} < q(S^+_{n,2}) \leq q(G) \leq \frac{2e(G)}{n-1} + n - 2,$$

which implies

$$e(G) > 2n-4+\frac{4}{n+1}.$$

Thus, we can obtain

$$e(N(u)) = e(G) - d(u) > 2n-4+\frac{4}{n+1} - n + 1 = n-3+\frac{4}{n+1},$$

and so $e(N(u)) \geq n-2 = d(u)-1$, which implies that $G[N(u)]$ contains at most one tree component.

When $G[N(u)]$ contains no cycle, we can find that $G[N(u)]$ is a tree. Moreover, since G is C_6-free, the graph $G[N(u)]$ contains no P_5. And so $G[N(u)]$ is a star or double star. If $G[N(u)]$ is a star, then $G = S_{n,2}$. Therefore, we conclude that $q(G) < q(S^+_{n,2})$ by (1), contrary to our assumption. Thus $G[N(u)]$ is a double star $S(a,b)$. Let z and v be the center vertices of $S(a,b)$, $N_{G[N(u)]}(z) = \{v, z_1, z_2, \ldots, z_a\}$ and $N_{G[N(u)]}(v) = \{z, v_1, v_2, \ldots, v_b\}$, where $a, b \geq 1$. Without loss of generality, we may assume $x_z \geq x_v$. Let $G_1 = G + \{v_i z \mid i = 1, 2, \ldots, b\} - \{v_i v \mid i = 1, 2, \ldots, b\}$. Then $q(G) < q(G_1)$ from Lemma 2.7. On the other hand, note that $G_1 = S_{n,2}$. Then by (1), we derive that $q(G_1) < q(S^+_{n,2})$. Hence, we conclude that $q(G) < q(S^+_{n,2})$, which contradicts our hypothesis.

Therefore, the graph $G[N(u)]$ contains cycles. Since G is $\{W_5, C_6\}$-free graph, $G[N(u)]$ is $\{C_4, P_5\}$-free graph. Let $K_{1,r}+e$ be the graph obtained by adding an edge in an independent set of the star $K_{1,r}$. Therefore, $G[N(u)]$ must contains $K_{1,r}+e$ as a subgraph, where $r \geq 2$. Then $d(u)-1 \leq e(N(u)) \leq d(u)$. Now we proceed by considering the following two cases.

Case 1. $G[N(u)]$ contains a tree component T.

In this case, $G[N(u)]$ consists of a tree and some unicyclic graphs $K_{1,r_i}+e$ for $r_i \geq 2$. Hence, we will divide it into the following three subcases according to the type of T.

Subcase 1.1. T is an isolated vertex, say w.

Denote by u_0 the maximal degree vertex of $K_{1,r_i}+e$ for some $r_i \geq 2$. Let G_2 be the graph obtained from G by connecting w and u_0. Clearly, G_2 is also $\{W_5, C_6\}$-free graph and $\Delta(G_2) = n-1$. Note that G is a proper subgraph of G_2. Therefore, by Lemma 2.6, we attain $q(G) < q(G_2)$, contradicting the choice of G.

Subcase 1.2. T is a star $K_{1,t}$.

Let $\mathcal{H}$ be the family of components of $G[N(u)]$, each of which contains C_3 as a subgraph, and let $x_{u^*} = \max\{x_i \mid i \in V(\mathcal{H})\}$. Note that the vertex u^* is the maximum degree vertex in $\mathcal{H}$. Denote by $V(K_{1,t}) = \{y, y_1, \ldots, y_t\}$ with $d_{N(u)}(y) = t$. Notice that $x_y = \max\{x_i \mid i \in V(K_{1,t})\}$. If $x_{u^*} \geq x_y$, then let

$G_3 = G - \{yy_j \mid 1 \le j \le t\} + \{u^*y_j \mid 1 \le j \le t\}$. Clearly, G_3 is $\{W_5, C_6\}$-free and $\Delta(G_3) = n-1$. According to Lemma 2.7, we can see that $q(G_3) > q(G)$, a contradiction. If $x_{u^*} < x_y$, then let $G_4 = G - \{u^*w \mid w \in N(u^*)\backslash\{u\}\} + \{yw \mid w \in N(u^*)\backslash\{u\}\}$. Note that G_4 is $\{W_5, C_6\}$-free and $\Delta(G_4) = n-1$. Thus $q(G_4) > q(G)$ from Lemma 2.7, a contradiction.

Subcase 1.3. T is a double star $S(a,b)$.

Assume that $V(T) = \{z_1, \ldots, z_a, z, v, v_1, \ldots, v_b\}$, where $N_T(z) = \{z_1, \ldots, z_a, v\}$ and $N_T(v) = \{v_1, \ldots, v_b, z\}$. Without loss of generality, we may assume $x_z \ge x_v$. Let $G_5 = G + \{v_i z \mid i = 1, 2, \ldots, b\} - \{v_i v \mid i = 1, 2, \ldots, b\}$. Obviously, G_5 is $\{W_5, C_6\}$-free and $\Delta(G_5) = n-1$. From Lemma 2.7, we attain $q(G_5) > q(G)$, a contradiction.

Case 2. $G[N(u)]$ contains no tree component.

In this case, $G[N(u)]$ is a union of disjoint $K_{1,r_i} + e$, where $r_i \ge 2$. Now we assert that G must have vertex of degree two. Otherwise, each connected component of $G[N(u)]$ is a triangle, i.e., $G = K_1 \triangledown (\frac{n-1}{3})C_3$. According to the structure of graph G, $Q(G)$ has the equitable quotient matrix

$$B_\pi = \begin{pmatrix} n-1 & n-1 \\ 1 & 5 \end{pmatrix}.$$

By a simple calculation, the characteristic polynomial of B_π is equal to

$$f_{B_\pi}(x) = x^2 - (n+4)x + 4n - 4.$$

Thus, we obtain $q(G) = \lambda_1(B_\pi) = \frac{n+4+\sqrt{n^2-8n+32}}{2}$ by Lemma 2.8. For $n \ge 6$, it is easy to verify that

$$q(G) = \frac{n+4+\sqrt{n^2-8n+32}}{2} < \frac{n+2+\sqrt{n^2+4n-12}}{2} = q(S_{n,2}). \tag{2}$$

Thus, we have $q(G) < q(S_{n,2})$, which implies that $q(G) < q(S^+_{n,2})$, a contradiction. Therefore, the graph $G[N(u)]$ must contain $K_{1,r} + e$ as a connected component, where $r \ge 3$.

Next we conclude that the graph $G[N(u)]$ contains no triangle component. Suppose to the contrary that the graph $G[N(u)]$ contains C_3 as a component. Let $V(C_3) = \{w_1, w_2, w_3\}$. Note that $q(G)x_i = (Q(G)X)_i = d(i)x_i + \sum_{j \in N(i)} x_j$ for any vertex i. By using eigenvalue equations of $Q(G)$ on w_1 and w_2, we see that

$$q(G)x_{w_1} = 3x_{w_1} + x_{w_2} + x_{w_3} + x_u, \tag{3}$$
$$q(G)x_{w_2} = 3x_{w_2} + x_{w_1} + x_{w_3} + x_u. \tag{4}$$

Subtract (4) from (3) to get

$$(q(G) - 2)(x_{w_1} - x_{w_2}) = 0.$$

Recall that $q(G) > q(S_{n,2}) > n + 2 - \frac{4}{n+1} > 2$ for $n \ge 2$. Then we have $x_{w_1} = x_{w_2}$. Similarly, we can get $x_{w_1} = x_{w_2} = x_{w_3}$. Let $V(C_3) = \{u_1, u_2, u_3\}$ in some $K_{1,r_i} + e$ ($r_i \ge 3$), where u_1 is the dominating vertex. Notice that $x_{u_1} > x_{u_2} = x_{u_3}$. From the eigenvalue equations of $Q(G)$ with respect to w_1 and u_2, we can get

$$q(G)x_{w_1} = 5x_{w_1} + x_u,$$
$$q(G)x_{u_2} = 4x_{u_2} + x_{u_1} + x_u > 5x_{u_2} + x_u.$$

It is easy to verify that $(q(G) - 5)(x_{u_2} - x_{w_1}) > 0$. Since $q(G) > n + 2 - \frac{4}{n+1} > 5$ for $n \ge 4$, we obtain $x_{u_2} > x_{w_1}$. Thus $x_{u_1} > x_{w_1}$. Let $G_6 = G + \{u_1 w_i \mid i = 1, 2, 3\} - \{w_i w_j \mid 1 \le i < j \le 3\}$. Obviously, G_6 is $\{W_5, C_6\}$-free and $\Delta(G_6) = n-1$. On the other hand, we have

$$q(G_6) - q(G) \ge X^T Q(G_6) X - X^T Q(G) X = \sum_{i=1}^{3} (x_{u_1} + x_{w_i})^2 - \sum_{1 \le i < j \le 3} (x_{w_i} + x_{w_j})^2 > 0, \tag{5}$$

and so we get $q(G_6) > q(G)$, which contradicts the choice of G. This contradiction shows that $G[N(u)]$ contains components only the type $K_{1,r} + e$, where $r \ge 3$.

Finally, we show that $G[N(u)]$ has only one $K_{1,r}+e$ component, where $r \geq 3$. Otherwise, we assume that $H_1 = K_{1,r_1}+e_1$ and $H_2 = K_{1,r_2}+e_2$ are connected components of $G[N(u)]$, where $r_1, r_2 \geq 3$. Let u' and u'' be the dominating vertices in H_1 and H_2, respectively. Noting that $x_{u'} = \max\{x_w \mid w \in V(H_1)\}$ and $x_{u''} = \max\{x_w \mid w \in V(H_2)\}$. Without loss of generality, we assume that $x_{u'} \geq x_{u''}$. Let G_7 be the graph obtained from G by removing all edges of H_2 and joining all vertices of H_2 to u'. In this way H_1 and H_2 are replaced by a connected component $K_{1,r_1+r_2+1}+e_1$. Then by a similar analysis to (5), we gain $q(G_7) > q(G)$, contradicting the choice of G. Therefore, we obtain $G = S^+_{n,2}$.

It completes the proof of Lemma 3.1. □

Now we give the proof of Theorem 1.2.

Proof of Theorem 1.2. Assume that G is a graph attaining the maximum signless Laplacian spectral radius among all $\{W_5, C_6\}$-free graphs of order $n \geq 10$. In view of Lemma 2.10, we know that G is connected. Based on Lemma 3.1, the result of Theorem 1.2 holds for $\Delta(G) = n-1$. So, in what follows, we can assume that $\Delta(G) \leq n-2$. By Lemma 2.9, we have

$$q(G) \geq q(S^+_{n,2}) > q(S_{n,2}) = \frac{n+2+\sqrt{n^2+4n-12}}{2} > n+2-\frac{4}{n+1}.$$

Let w be a vertex of G such that

$$d(w) + \frac{1}{d(w)}\sum_{v\in N(w)} d(v) = \max\{d(u) + \frac{1}{d(u)}\sum_{v\in N(u)} d(v) : u \in V(G)\}. \tag{6}$$

Take $A = N(w)$ and $B = V(G) \setminus N[w]$. Notice that $\mid A \mid = d(w)$ and $\mid B \mid = n - d(w) - 1$. We proceed by distinguishing the following three cases.

Case 1. $d(w) \leq 3$.

In this case, using Lemma 2.4, for $n \geq 4$, we obtain

$$q(G) \leq d(w) + \frac{1}{d(w)}\sum_{v\in A} d(v) \leq d(w) + \Delta(G) \leq n+1 < n+2-\frac{4}{n+1} < q(S^+_{n,2}),$$

a contradiction.

Case 2. $4 \leq d(w) \leq n-3$.

Let G_w be the graph induced by the set $V(G)\backslash\{w\}$. Since G is C_6-free, G_w is also C_6-free, which implies that G_w does not contain a path P_5 with both end-vertices in A. Hence, by Lemma 2.3, we get

$$2e(A) + e(A,B) \leq 3\mid A\mid + 2\mid B\mid = d(w) + 2(n-1). \tag{7}$$

Combining Lemma 2.4 with (7), we see that

$$\begin{aligned} q(G) &\leq d(w) + 1 + \frac{2e(A)+e(A,B)}{d(w)} \\ &\leq d(w) + 1 + \frac{d(w)+2(n-1)}{d(w)} \\ &= d(w) + 2 + \frac{2(n-1)}{d(w)}. \end{aligned}$$

Let $f(x) = x + \frac{2(n-1)}{x}$. It is easy to see that the function $f(x)$ is convex for $4 \leq x \leq n-3$ and its maximum in any closed interval is attained at one of the ends of this interval. Thus,

$$q(G) \leq \max\{f(4), f(n-3)\} + 2 < \frac{n+2+\sqrt{n^2+4n-12}}{2} < q(S^+_{n,2})$$

for $n \geq 10$, a contradiction.

Case 3. $d(w) = n - 2$.

In this case, $|B| = 1$. Let $B = \{u\}$. According to Lemma 2.4, we have

$$\begin{aligned} q(G) &\leq d(w) + 1 + \frac{2e(A) + e(A,B)}{d(w)} \\ &\leq d(w) + 1 + \frac{2e(A) + d(w)\,|B|}{d(w)} \\ &= n + \frac{2e(A)}{n-2}. \end{aligned}$$

Recall that $q(G) > n + 2 - \frac{4}{n+1}$. Then we get

$$n + 2 - \frac{4}{n+1} < n + \frac{2e(A)}{n-2},$$

and so $e(A) > n - 4 + \frac{6}{n+1}$. Thus, we can obtain $e(A) \geq n - 3 = d(w) - 1$, which indicates that $G[A]$ contains at most one tree component.

When $G[A]$ contains no cycle, we can find that $G[A]$ is a tree. Moreover, since G is C_6-free, the graph $G[A]$ contains no P_5. Then $G[A]$ is a star or double star. If $G[A]$ is a star $K_{1,r}$ for some $r \geq 7$, then $G[A]$ has exactly one vertex-disjoint copy of P_3. Denote by $V(K_{1,r}) = \{v, v_1, \ldots, v_r\}$ with $d_A(v) = r$. Let $V(P_3) = \{v, v_1, v_2\}$. In order to avoid W_5, u is adjacent to at most two vertices of $\{v, v_1, v_2\}$. In order to avoid C_6, u is adjacent to at most one vertex of $\{v_1, \ldots, v_r\}$. Therefore, $e(A, B) \leq 2$. By Lemma 2.4, we have

$$q(G) \leq d(w) + 1 + \frac{2(d(w)-1) + 2}{d(w)} \leq n + 1 < n + 2 - \frac{4}{n+1} < q(S^+_{n,2}), \tag{8}$$

a contradiction.

If $G[A]$ is a double star $S(a, b)$, since G is C_6-free, we see that u is adjacent only to one vertex of $S(a, b)$. Thus $e(A, B) = 1$. Based on Lemma 2.4, we get

$$q(G) \leq d(w) + 1 + \frac{2(d(w)-1) + 1}{d(w)} \leq n + 1 - \frac{1}{n-2} < n + 2 - \frac{4}{n+1} < q(S^+_{n,2}),$$

a contradiction.

Now suppose that $G[A]$ contains cycles. Since G is $\{W_5, C_6\}$-free graph, $G[A]$ is $\{C_4, P_5\}$-free graph. Hence, we have $d(w) - 1 \leq e(A) \leq d(w)$ and $G[A]$ has $K_{1,r} + e$ as a subgraph, where $r \geq 2$. Particularly, let k be the number of vertex-disjoint copies of P_3 in $G[A]$. If $e(A) = d(w) - 1$, then $G[A]$ contains a tree component T, which implies that $k = 1$. Otherwise, suppose to the contrary that $k \geq 2$. Since G is a W_5-free graph, u is adjacent to at most two vertices of any P_3 of $G[A]$. Thus, by Lemma 2.4 we have

$$\begin{aligned} q(G) &\leq d(w) + 1 + \frac{2(d(w)-1) + (d(w)-k)\,|B|}{d(w)} \\ &\leq d(w) + 1 + \frac{2(d(w)-1) + d(w) - 2}{d(w)} \\ &= d(w) + 4 - \frac{4}{d(w)} \\ &= n + 2 - \frac{4}{n-2} < n + 2 - \frac{4}{n+1} < q(S^+_{n,2}), \end{aligned}$$

which is impossible by above arguments. Since $G[A]$ has exactly one vertex-disjoint copy of P_3, $G[A] = K_{1,r} + e \cup T$, where T is either a isolated vertex or a path P_2. To avoid $\{W_5, C_6\}$, we have $e(A, B) \leq 2$. Hence, by (8), we obtain

$$q(G) < n + 2 - \frac{4}{n+1} < q(S^+_{n,2}),$$

which contradicts the hypothesis.

Hence, we have $e(A) = d(w)$. Then $G[A]$ is a union of disjoint $K_{1,r_i} + e$, where $r_i \geq 2$. Similarly, to avoid $\{W_5, C_6\}$, we obtain $e(A, B) \leq 2$. By Lemma 2.4, we have

$$q(G) \leq d(w) + 1 + \frac{2d(w) + 2}{d(w)} \leq n + 1 + \frac{2}{n-2} < n + 2 - \frac{4}{n+1} < q(S_{n,2}^{+})$$

for $n \geq 7$, a contradiction.

This completes the proof of Theorem 1.2. □

4. Proof of Theorem 1.4

Before proving the Theorem 1.4, it is enough to get the following lemma which show a crucial case of Theorem 1.4.

Lemma 4.1. *Let G be a F_5-free graph of order $n \geq 6$. If $\Delta(G) = n - 1$, then $q(G) \leq q(S_{n,2})$, the equality holds if and only if $G \cong S_{n,2}$.*

Proof. Since $\Delta(G) = n - 1$, there exists a vertex $u \in V(G)$ such that $d(u) = n - 1$. If $4 \leq e(N(u)) \leq n - 2$, according to Lemma 2.1, then $H_{n,e(N(u))-1}$ is the unique connected graph attaining the maximum signless Laplacian spectral radius among all connected graphs with n vertices and $n + e(N(u)) - 1$ edges. Note that $S_{n,2}$ is F_5-free. Since $H_{n,e(N(u))-1}$ is a proper subgraph of $S_{n,2}$ for $4 \leq e(N(u)) \leq n - 3$ and $H_{n,e(N(u))-1} = S_{n,2}$ for $e(N(u)) = n - 2$, we get that $S_{n,2}$ is the unique connected graph that maximum signless Laplacian spectral radius over all the F_5-free graphs of order $n \geq 6$.

Next, let us consider the two cases $e(N(u)) \leq 3$ and $e(N(u)) \geq n - 1$.

Case 1. $e(N(u)) \leq 3$.

Notice that $e(G) = n - 1 + e(N(u)) \leq n + 2$. By Lemmas 2.5 and 2.9 we have

$$q(G) \leq \frac{2(n+2)}{n-1} + n - 2 < n + 2 - \frac{4}{n+1} < q(S_{n,2})$$

for $n \geq 6$. So $q(G) < q(S_{n,2})$.

Case 2. $e(N(u)) \geq n - 1$.

Since G is F_5-free graph, the graph $G[N(u)]$ contains no P_4. Using Lemma 2.2, we can get $e(N(u)) \leq n - 1$, which implies that $e(N(u)) = n - 1$. Then the equality of Lemma 2.2 holds. So applying Lemma 2.2, we deduce that $G[N(u)]$ is a union of disjoint triangles which shows that $G = K_1 \triangledown (\frac{n-1}{3})C_3$. By (2), we get $q(G) < q(S_{n,2})$ for $n \geq 6$. Therefore, it completes the proof of Lemma 4.1. □

In this section, we discuss the proof of Theorem 1.4.

Proof of Theorem 1.4. Assume that G is a graph attaining the maximum signless Laplacian spectral radius among all F_5-free graphs of order $n \geq 6$. According to Lemma 2.10, we obtain that G is connected. If $\Delta(G) = n - 1$, then we can get the result from Lemma 4.1. Thus, suppose that $\Delta(G) \leq n - 2$ for the rest of the proof.

Let $w \in V(G)$ be a vertex which satisfies (6). Take $A = N(w)$ and $B = V(G) \setminus N[w]$. Since $S_{n,2}$ is F_5-free, according to Lemma 2.9 we have

$$q(G) \geq q(S_{n,2}) = \frac{n + 2 + \sqrt{n^2 + 4n - 12}}{2} > n + 2 - \frac{4}{n+1}. \tag{9}$$

Notice that $\sum_{v \in A} d(v) = d(w) + 2e(A) + e(A, B)$, then by Lemma 2.4 we have

$$q(G) \leq d(w) + \frac{1}{d(w)} \sum_{v \in A} d(v) = d(w) + 1 + \frac{2e(A)}{d(w)} + \frac{e(A, B)}{d(w)}. \tag{10}$$

Together with (9) and (10), we get

$$d(w) + \frac{1}{d(w)} \sum_{v \in A} d(v) > n + 2 - \frac{4}{n+1}. \tag{11}$$

Note that $| A |= d(w)$ and $| B |= n - d(w) - 1$. Thus

$$\frac{e(A, B)}{d(w)} \le \frac{| A || B |}{| A |} =| B |= n - d(w) - 1,$$

which implies that

$$d(w) + \frac{1}{d(w)} \sum_{v \in A} d(v) \le d(w) + 1 + \frac{2e(A)}{d(w)} + n - d(w) - 1 = n + \frac{2e(A)}{d(w)}. \tag{12}$$

Therefore, combining (11), (12) with $d(w) \le n - 2$, we obtain

$$e(A) > d(w) - \frac{2d(w)}{n+1} \ge d(w) - \frac{2(n-2)}{n+1} = d(w) - 2 + \frac{6}{n+1} > d(w) - 2,$$

and so

$$e(A) \ge d(w) - 1, \tag{13}$$

which implies that $G[A]$ contains at most one tree component.

Suppose that $G[A]$ contains no cycle. Then based on (13) we know that $G[A]$ is a tree and $e(A) = d(w)-1$. Since G is F_5-free, the graph $G[A]$ is P_4-free, and so it is a star $K_{1,r}$ for some integer $r \ge 0$, where $K_{1,0}$ is an isolated vertex.

If $r = 0$, then $d(w) = 1$. Recall that $\Delta(G) \le n - 2$. Thus $e(A, B) \le n - 3$. By (10) we get

$$q(G) \le 1 + 1 + n - 3 = n - 1 < n + 2 - \frac{4}{n+1} < q(S_{n,2}),$$

which contradicts (9).

If $r = 1$, then $d(w) = 2$, $e(A) = 1$. As $\Delta(G) \le n - 2$, $e(A, B) \le 2(n - 4)$. By (10) we have

$$q(G) \le 2 + 1 + \frac{2 + 2(n-4)}{2} = n < n + 2 - \frac{4}{n+1} < q(S_{n,2}),$$

which contradicts (9).

If $r = 2$, then $d(w) = 3$ and $e(A) = 2$. Since G is F_5-free, each vertex in B is either joined to the center vertex only, or joined to the other vertices except the center vertex. So $e(A, B) \le 2(n - 4)$. By (10) we obtain

$$q(G) \le 3 + 1 + \frac{4 + 2(n-4)}{3} = \frac{2n+8}{3} < n + 2 - \frac{4}{n+1} < q(S_{n,2})$$

for $n \ge 5$, which contradicts (9).

If $r \ge 3$, it implies that $d(w) \ge 4$. Since G is F_5-free, each vertex in B is either joined to the center vertex only, or joined to other vertices except the center vertex of $K_{1,r}$. Hence, we conclude that

$$e(A, B) \le (d(w) - 1)(n - d(w) - 1). \tag{14}$$

Depending on the value of $d(w)$, we divide the proof into the following two cases.

Case 1. $4 \le d(w) \le n - 3$.

Note that $e(A) = d(w) - 1$. Substituting (14) into (10), we derive that

$$\begin{aligned} q(G) &\le d(w) + 1 + \frac{2(d(w)-1) + (d(w)-1)(n-d(w)-1)}{d(w)} \\ &= n + 3 - \frac{n+1}{d(w)} \le n + 3 - \frac{n+1}{n-3} \\ &< \frac{n+2+\sqrt{n^2+4n-12}}{2} = q(S_{n,2}) \end{aligned}$$

for $n \ge 4$, a contradiction.

Case 2. $d(w) = n - 2$.

In this case, $|A| = n - 2$ and $|B| = 1$. According to (14), we have

$$e(A, B) \le d(w) - 1 = n - 3. \tag{15}$$

For $e(A, B) \le n - 4$, plugging (15) into (10), we have

$$\begin{aligned} q(G) &\le d(w) + 1 + \frac{2(d(w)-1) + n - 4}{d(w)} \\ &= n + 2 - \frac{4}{n-2} < n + 2 - \frac{4}{n+1} < q(S_{n,2}), \end{aligned}$$

a contradiction. It is worth noting that $|A| = n - 2$, $|B| = 1$ and $G[A] = K_{1,r}$, where $r \ge 3$. If $e(A, B) = n - 3$, then G is a graph as shown in Fig.1. Clearly, the partition $V(G) = \{w, u\} \cup A\backslash\{u\} \cup B$ is an equitable partition (see Fig.1). Thus, $Q(G)$ has the equitable quotient matrix

$$B_\pi = \begin{pmatrix} n-1 & n-3 & 0 \\ 2 & 3 & 1 \\ 0 & n-3 & n-3 \end{pmatrix}.$$

By a simple calculation, the characteristic polynomial of B_π is equal to

$$f_{B_\pi}(x) = \det(xI - B_\pi) = x^3 + (1 - 2n)x^2 + (n^2 - n)x + 12 - 4n.$$

Since $f_{B_\pi}(x) > 0$ for $x \ge \frac{n+2+\sqrt{n^2+4n-12}}{2}$, we get $\lambda_1(B_\pi) < \frac{n+2+\sqrt{n^2+4n-12}}{2} = q(S_{n,2})$, which gives that $q(G) = \lambda_1(B_\pi) < q(S_{n,2})$ by Lemma 2.8, a contradiction.

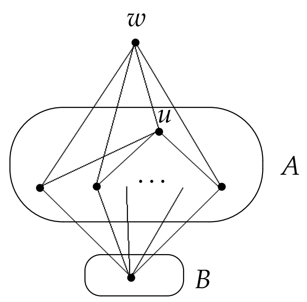

Figure 1: The graph G.

Therefore, $G[A]$ contains cycles. Recall that $G[A]$ is P_4-free. Then we deduce that $G[A] = aC_3 \bigcup bK_{1,r}$, where a, b are positive integers and $a \ge 1$, $0 \le b \le 1$, $r \ge 0$. Let w_1, w_2, w_3 be the vertices of a triangle in

$G[A]$. Clearly, to avoid F_5, every vertex in B may be joined to at most one of the vertices w_1, w_2, w_3. This requirement leads to the bound on $e(A, B)$ as follows

$$e(A, B) \le d(w)(n - d(w) - 1) - 2(n - d(w) - 1).$$

Hence, by (10) we have

$$\begin{aligned} q(G) &\le d(w) + 1 + \frac{2e(A) + d(w)(n - d(w) - 1) - 2(n - d(w) - 1)}{d(w)} \\ &= n + 2 + \frac{2e(A)}{d(w)} - \frac{2(n-1)}{d(w)}. \end{aligned}$$

In view of (9), we get

$$n + 2 - \frac{4}{n+1} < n + 2 + \frac{2e(A)}{d(w)} - \frac{2(n-1)}{d(w)}.$$

Notice that $d(w) \le n - 2$. It is easy to verify that

$$e(A) > n - 1 - \frac{2d(w)}{n+1} \ge n - 1 - \frac{2(n-2)}{n+1} = n - 3 + \frac{6}{n+1},$$

which implies that $e(A) \ge n - 2$. Note that the graph $G[A]$ contains no P_4. Using Lemma 2.2, we can get $e(A) \le d(w)$, with equality holding if and only if $G[A]$ is a union of disjoint triangles. Then we see that $d(w) = n - 2$ and $e(A) = n - 2$, and so $G[A]$ is a union of disjoint triangles. At this time, $| B |= 1$. Let $B = \{v\}$. Notice that v may be joined to at most one vertex of each triangle of A, and so $e(A, B) \le \frac{n-2}{3}$. By (10) we obtain

$$q(G) \le n - 2 + 1 + \frac{2(n-2) + \frac{n-2}{3}}{n-2} = n + \frac{4}{3} < n + 2 - \frac{4}{n+1} < q(S_{n,2})$$

for $n \ge 6$, a contradiction.

It completes the proof of Theorem 1.4. $\square$

5. Conclusion

In this paper, we concentrate our attention on the signless Laplacian spectral extremal results with forbidding subgraphs. Though we also investigate the maxima of the signless Laplacian spectral radius of forbidding subgraphs, the whole proof process and conclusions are simple and objective, especially the number of vertices of graphs of considering is very different to that in [20] and [21]. Hence, the main conclusions of this paper are to supplements and complements the known results.

References

[1] A. E. Brouwer, W. H. Haemers, *Spectra of Graphs*, Springer, Berlin, 2011.
[2] T. C. Chang, B. S. Tam, *Connected graphs with maximal Q-index: The one-dominating-vertex case*, Linear Algebra Appl. **435** (2011), 2451–2461.
[3] M. Z. Chen, A. M. Liu, X. D. Zhang, *Spectral extremal results with forbidding linear forests*, Graphs Combin. **35** (2019), 335–351.
[4] D. Cvetković, M. Doob, H. Sachs, *Spectra of Graphs*, Academic Press, New York, 1980.
[5] D. Cvetković, S. K. Simić, *Towards a spectral theory of graphs based on the signless Laplacian I*, Publ. Inst. Math. (Beograd) **85** (2009), 19–33.
[6] D. Cvetković, S. K. Simić, *Towards a spectral theory of graphs based on the signless Laplacian II*, Linear Algebra Appl. **432** (2010), 2257–2272.
[7] D. Cvetković, S. K. Simić, *Towards a spectral theory of graphs based on the signless Laplacian III*, Appl. Anal. Discrete Math. **4** (2010), 156–166.
[8] E. R. van Dam, W. Haemers, *Which graphs are determined by their spectrum*? Linear Algebra Appl. **373** (2003), 241–272.
[9] K. Das, *Maximizing the sum of the squares of the degrees of a graph*, Discrete Math. **285** (2004), 57–66.

[10] P. Erdős, T. Gallai, *On maximal paths and circuits of graphs*, Acta Math. Acad. Sci. Hungar. **10** (1959), 337–356.
[11] L. Feng, G. Yu, *On three conjectures involving the signless laplacian spectral radius of graphs*, Publ. Inst. Math. (Beograd) (N.S.) **85** (2009), 35–38.
[12] M. A. A. de Freitas, V. Nikiforov, L. Patuzzi, *Maxima of the Q-index: forbidden* 4*-cycle and* 5*-cycle*, Electron. J. Linear Algebra. **26** (2013), 905–916.
[13] Y. Hong, X. D. Zhang, *Sharp upper and lower bounds for largest eigenvalue of the Laplacian matrices of trees*, Discrete Math. **296** (2005), 187–197.
[14] R. Merris, *A note on Laplacian graph eigenvalues*, Linear Algebra Appl. **285** (1998), 33–35.
[15] V. Nikiforov, *Bounds on graph eigenvalues II*, Linear Algebra Appl. **427** (2007), 183–189.
[16] V. Nikiforov, *Degree powers in graphs with forbidden even cycle*, Electronic J. Combin. **15** (2009), R107.
[17] V. Nikiforov, *The spectral radius of graphs without paths and cycles of specified length*, Linear Algebra Appl. **432** (2010), 2243–2256.
[18] V. Nikiforov, *A contribution to the Zarankiewicz problem*, Linear Algebra Appl. **432** (2010), 1405–1411.
[19] V. Nikiforov, X. Y. Yuan, *Maxima of the Q-index: graphs without long paths*, Electronic J. Linear Algebra. **27** (2014), 504–514.
[20] V. Nikiforov, X. Y. Yuan, *Maxima of the Q-index: forbidden even cycles*, Linear Algebra Appl. **471** (2015), 636–653.
[21] B. Wang, M. Q. Zhai, *Maxima of the Q-index: Forbidden a fan*, Discrete Mathematics. **346** (2023), 113264.
[22] X. Y. Yuan, *Maxima of the Q-index: forbidden odd cycles*, Linear Algebra Appl. **458** (2014), 207–216.
[23] M. Q. Zhai, H. Q. Lin, *Spectral extrema of graphs: forbidden hexagon*, Discrete Math. **343** (2020), 112028.
[24] Y. H. Zhao, X. Y. Huang, H. T. Guo, *The signless Laplacian spectral radius of graphs with no intersecting triangles*, Linear Algebra Appl. **618** (2021), 12–21.