Filomat 37:24 (2023), 8361–8376
https://doi.org/10.2298/FIL2324361E

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Variational inequality problem over the solution set of split monotone variational inclusion problem with application to bilevel programming problem

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Abstract. The purpose of this paper is to study variational inequality problem over the solution set of multiple-set split monotone variational inclusion problem. We propose an iterative algorithm with inertial method for finding an approximate solution of this problem in real Hilbert spaces. Strong convergence of the sequence of iterates generated from the proposed method is obtained under some mild assumptions.The iterative scheme does not require prior knowledge of operator norm. Also we present some applications of our main result to solve the bilevel programming problem, the bilevel monotone variational inequalities, the split minimization problem, the multiple-set split feasibility problem and the multiple set split variational inequality problem.

1. Introduction

The variational inclusion problems are being used as mathematical models for the study of several optimization problems arising in finance, economics, network, transportation and engineering. For a real Hilbert space $\mathcal{H}$, the monotone inclusion problem is formulated as follows:

$$\text{Find an element} \quad x^\star \in \mathcal{H} \quad \text{such that} \quad 0 \in (A+B)x^*, \tag{1}$$

where $B : \mathcal{H} \to 2^{\mathcal{H}}$ is a maximal monotone operator and $A : \mathcal{H} \to \mathcal{H}$ is a Lipschitz continuous monotone operator. The set of solutions of the problem (1) is denoted by $(A+B)^{-1}(0)$.

Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$ and $F : \mathcal{H} \to \mathcal{H}$ be an operator. The classical variational inequality problem (VIP) is formulated as follows:

$$\text{Find an element} \quad x^\star \in C \quad \text{such that} \quad \langle Fx^\star, y - x^\star\rangle \geq 0, \qquad \forall y \in C. \tag{2}$$

The set of solutions of this problem is denoted by $VI(C,F)$. Several researches used different approaches to develop iterative algorithms for solving various classes of variational inequality and variational inclusion problems. For details see [1, 17, 28, 31, 34–36, 43, 48, 50, 56] and the references therein.

2020 *Mathematics Subject Classification*. Primary 47H05; Secondary 47H09; 49J53; 90C25
Keywords. Split monotone variational inclusion problem, bilevel programming problem, inertial algorithm
Received: 25 August 2022; Revised: 19 January 2023; Accepted: 10 April 2023
Communicated by Adrian Petrusel
This research was in part supported by a grant from IPM (No.1401470031).
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A popular method for solving problem (1) in real Hilbert spaces is the well-known forward-backward splitting method introduced by Passty [43] and Lions and Mercier [36]. The method is formulated as:

$$x_{n+1} = (I + \lambda_n B)^{-1}(I - \lambda_n A)x_n, \quad \lambda_n > 0, \tag{3}$$

under the condition that $Dom(B) \subset Dom(A)$. It was shown, see for example [17], that weak convergence of (3) requires quite restrictive assumptions on A and B, such that the inverse of A is strongly monotone or B is Lipschitz continuous and monotone and the operator $A + B$ is strongly monotone on $Dom(B)$. In [56], Tseng weakened these assumptions and included an extra step per each step of (3) (called Tseng's splitting algorithm) and obtained weak convergence result in real Hilbert spaces. Recently, Gibali and Thong [26], have obtained strong convergence result by modifying Tseng's splitting algorithm in real Hilbert spaces.

In the recent years, inertial terms have attracted the interest and research of scholars as a technique to accelerate the convergence speed of algorithms. A common feature of inertial-type algorithms is that the next iteration depends on the combination of the previous two iterations (see [2, 44] for more details). This small change greatly improves the computational efficiency of inertial-type algorithms. Recently, many researchers have constructed a large number of inertial-type algorithms to solve variational inclusion problems and other optimization problems; see, e.g., [38] and the references therein. The computational efficiency of these inertial-type algorithms was demonstrated by a number of computational tests and applications. Quite recently, Eslamian and Kamandi [24], proposed an iterative algorithm with inertial extrapolation step for finding a common element of the set of solutions of a system of monotone inclusion problems.

Let $\mathcal{H}$ and $\mathcal{K}$ be real Hilbert spaces, $T : \mathcal{H} \to \mathcal{K}$ be a bounded linear operator and let $\{C_i\}_{i=1}^{p}$ be a family of nonempty closed convex subsets in $\mathcal{H}$ and $\{Q_j\}_{j=1}^{r}$ be a family of nonempty closed convex subsets in $\mathcal{K}$. The multiple-set split feasibility problem was introduced by Censor et al. (2005) [14] and is formulated as finding a point $x^\star$ with the property:

$$x^\star \in \bigcap_{i=1}^{p} C_i \quad and \quad Tx^\star \in \bigcap_{j=1}^{r} Q_i.$$

The multiple-set split feasibility problem with $p = r = 1$ is known as the split feasibility problem [13].

In 2011, Moudafi [41] introduced the following split monotone variational inclusion problem:

$$\begin{cases} \text{Find} \quad x^\star \in \mathcal{H} \ \text{such that} \quad 0 \in (A_1 + B_1)x^*, \\ \text{and such that} \\ y^\star = Tx^\star \in \mathcal{K} \quad \text{solves} \quad 0 \in (A_2 + B_2)y^*, \end{cases} \tag{4}$$

where $B_1 : \mathcal{H} \to 2^{\mathcal{H}}$ is a multi-valued mapping on a Hilbert space $\mathcal{H}$, $B_2 : \mathcal{K} \to 2^{\mathcal{K}}$ is a multi-valued mapping on a Hilbert space $\mathcal{K}$, $T : \mathcal{H} \to \mathcal{K}$ is a bounded linear operator, $A_1 : \mathcal{H} \to \mathcal{H}$, $A_2 : \mathcal{K} \to \mathcal{K}$ are two given single-valued operators. In the case that A_1 and A_2 are inverse strongly monotone mapping, based on average operator technique, Moudafi [41] proposed an iterative method with weak convergence for solving it. Many mathematical problems such as split feasibility problem, split variational inequality problem, split zero problem, split equilibrium problem and split minimization problem [15, 40], are the special cases of the split monotone variational inclusion problem. These problems have been studied and applied to solving many real life problems such as in modelling intensity-modulated radiation therapy treatment planning, modelling of inverse problems arising from phase retrieval, in sensor networks in computerized tomography and data compression [9, 12, 14, 20]. Therefore split variational inclusion problem has drawn the attention of many mathematicians. In the case $A_1 = 0$ and $A_2 = 0$, Byrne et al. [10] studied the weak convergence of the following iterative method for the split variational inclusion problem:

$$x_{n+1} = J_\lambda^{B_1}(x_n + \tau T^*(J_\lambda^{B_2} - I))Tx_n \tag{5}$$

where $\lambda > 0$ and $\tau \in (0, \frac{2}{\|T\|^2})$. There have been many authors who modified of this method for solving the split variational inclusion problem in the several settings (see, e.g., [3, 16, 18, 21–23, 25, 45–47, 53]).

However, it is observed that the stepsizes of almost of the methods depend on the norm of a bounded linear operator. It is known that the norm of a bounded linear operator or matrix in the finite dimensional space is very difficult to compute (see [30]). To overcome this difficulty, López et al. [37], introduced a self-adaptive method for solving the split feasibility problem. The advantage of this method is the stepsize does not require the prior knowledge norm of a bounded linear operator. It is worth to interest the self-adaptive method because we can easily compute the stepsize. In recent years, there have been many authors who studied the modified methods such that the stepsizes do not depend on the norm (see, e.g., [19, 54, 58]).

A constrained optimization problem in which the constrained set is a solution set of another optimization problem is called a bilevel programming problem. Among the applications where bilevel programming problems play an important role we mention the modelling of Stackelberg games, the determination of Wardrop equilibria for network flows, convex feasibility problems, domain decomposition methods for PDEs, image processing problems and optimal control problems, etc, (see [4–7, 11, 39, 42]). If the first-level problem is a variational inequality problem and the second-level problem is a set of fixed points of a mapping, then the bilevel problem is called hierarchical variational inequality problem. Many important application problems, such as signal recovery, power-control, bandwidth allocation, optimal control, and beam-forming, are special cases of hierarchical variational inequality problem, see ([32, 33, 51, 57]). Yamada [57] considered the following hybrid steepest-decent iterative method for solving hierarchical variational inequality:

$$x_{n+1} = (I - \mu\alpha_n F)Tx_n,$$

where F is a Lipschitzian continuous and strongly monotone operator and T is a nonexpansive operator.

Let $\mathcal{H}$ and $\mathcal{K}$, be real Hilbert spaces and let $T_i : \mathcal{H} \to \mathcal{K}, (i = 1, 2, ..., m)$, be bounded linear operators such that $T_i \neq 0$. Let for $i \in \{1, 2, ..., m\}$, $G_i : \mathcal{K} \to 2^{\mathcal{K}}$, $B_i : \mathcal{H} \to 2^{\mathcal{H}}$ be maximal monotone operators and let $A_i : \mathcal{H} \to \mathcal{H}$ be monotone and Lipschitz continuous operator. Let F be a Lipschitzian continuous and strongly monotone operator. In this paper we study the following problem:

$$\begin{cases} \text{Find} \qquad x^\star \in \Omega = \bigcap_{i=1}^m (A_i + B_i)^{-1}(0) \cap T_i^{-1}(G_i^{-1}0), \\ \text{such that} \qquad\qquad x^\star \in VI(\Omega, F). \end{cases} \tag{6}$$

Inspired by the inertial algorithm, the hybrid steepest-descent method and Tseng's splitting algorithm, we introduce a new and efficient iterative method for solving the problem (6). The strong convergence of the proposed algorithm is proved without knowing any information of the Lipschitz and strongly monotone constants of the mappings. Moreover, the iterative scheme does not require prior knowledge of operator norm. Also we present some applications of our main results to solve the bilevel programming problem, the bilevel monotone variational inequalities, the split minimization problem, the multiple-set split feasibility problem and the multiple set split variational inequality problem. Our results improve and generalize the results of Anh et al. [3], Censor et al. [15], Thong et al. [55], and many others.

2. Preliminaries

We use the following notation in the sequel:
• $\rightharpoonup$ for weak convergence and $\to$ for strong convergence.

Given a nonempty closed convex subset C of a Hilbert space $\mathcal{H}$, the mapping that assigns every point $x \in \mathcal{H}$, to its unique nearest point in C is called the metric projection onto C and is denoted by P_C; i.e., $P_C(x) \in C$ and $\|x - P_C(x)\| = inf_{y \in C}\|x - y\|$. The metric projection P_C is characterized by the fact that $P_C(x) \in C$ and

$$\langle y - P_C(x), x - P_C(x)\rangle \leq 0, \qquad \forall x \in \mathcal{H}, y \in C.$$

We recall the following definitions concerning operator $F : \mathcal{H} \to \mathcal{H}$.

Definition 2.1. *The operator $F : \mathcal{H} \to \mathcal{H}$ is called*
• *Lipschitz continuous with constant $L > 0$ if*

$$\|F(x) - F(y)\| \leq L\|x - y\|, \qquad \forall x, y \in \mathcal{H}.$$

• *Contraction, if there exists a constant* $0 \leq k < 1$ *such that*

$$\|F(x) - F(y)\| \leq k\|x - y\|, \qquad \forall x, y \in \mathcal{H}.$$

• *Monotone if*

$$\langle F(x) - F(y), x - y\rangle \geq 0, \quad \forall x, y \in \mathcal{H}.$$

• *Strongly monotone with constant* $\beta > 0$, *if*

$$\langle F(x) - F(y), x - y\rangle \geq \beta\|x - y\|^2, \quad \forall x, y \in \mathcal{H}.$$

• *Inverse strongly monotone with constant* $\beta > 0$, ($\beta - ism$) *if*

$$\langle F(x) - F(y), x - y\rangle \geq \beta\|F(x) - F(y)\|^2, \quad \forall x, y \in \mathcal{H}.$$

• *Nonexpansive, if*

$$\|Fx - Fy\| \leq \|x - y\|, \qquad \forall x, y \in \mathcal{H}.$$

• *Firmly nonexpansive, if*

$$\|Fx - Fy\|^2 \leq \|x - y\|^2 - \|(x - Fx) - (y - Fy)\|^2, \qquad \forall x, y \in \mathcal{H}.$$

Definition 2.2. *Let C be a nonempty convex subset of a real Hilbert space $\mathcal{H}$. A mapping $F : C \to \mathcal{H}$ is said to be hemicontinuous if for any fixed $x, y \in C$, the mapping $t \to F(x + t(y - x))$ defined on $[0, 1]$ is continuous, that is, if F is continuous along the line segments in C.*

It is easy to see that every Lipschitz continuous mapping is hemicontinuous.

Lemma 2.3. *[35] Let C be a nonempty closed and convex subset of real Hilbert space $\mathcal{H}$ and $A : C \to \mathcal{H}$ be a strongly monotone and Lipschitz continuous mapping. Then $VI(C, A)$ consists only one point.*

Lemma 2.4. *[57] Let the operator $A : \mathcal{H} \to \mathcal{H}$ be l-Lipschitz continuous and δ-strongly monotone with constants $l > 0$, $\delta > 0$. Assume that $\gamma \in (0, \frac{2\delta}{l^2})$. For $\alpha \in (0, 1)$ define $T_\alpha = I - \alpha\gamma A$. Then for all $x, y \in \mathcal{H}$,*

$$\|T_\alpha x - T_\alpha y\| \leq (1 - \alpha\eta)\|x - y\|$$

holds, where $\eta = 1 - \sqrt{1 - \gamma(2\delta - \gamma l^2)} \in (0, 1)$.

Lemma 2.5. *[27] (Demiclosed Principle) Let C be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$ and $T : C \to \mathcal{H}$ a nonexpansive mapping. Then $I - T$ is demiclosed at zero, that is, if $\{x_n\}$ is a sequence in C converges weakly to x and $(I - T)x_n$ converges strongly to zero, then $(I - T)x = 0$.*

Let B be a mapping of $\mathcal{H}$ into $2^{\mathcal{H}}$. The effective domain of B is denoted by $Dom(B)$, that is, $Dom(B) = \{x \in \mathcal{H} : Bx \neq \emptyset\}$. A multi-valued mapping B on $\mathcal{H}$ is said to be monotone if $\langle x - y, u - v\rangle \geq 0$ for all $x, y \in Dom(B)$, $u \in Bx$ and $v \in By$. A monotone mapping B on $\mathcal{H}$ is said to be maximal if its graph is not properly contained in the graph of any other monotone mapping on $\mathcal{H}$. For a maximal monotone mapping B on $\mathcal{H}$ and $r > 0$, we may define a single-valued mapping $J_r^B = (I + rB)^{-1} : \mathcal{H} \to Dom(B)$, which is called the resolvent of B for r. Let B be a maximal monotone mapping on $\mathcal{H}$ and let $B^{-1}0 = \{x \in \mathcal{H} : 0 \in Bx\}$. It is known that the resolvent J_r^B is firmly nonexpansive and $B^{-1}0 = Fix(J_r^B)$ for all $r > 0$; see [52] for more details.

Lemma 2.6. *[8] Let $B : \mathcal{H} \to 2^{\mathcal{H}}$ be a maximal monotone mapping and $A : \mathcal{H} \to \mathcal{H}$ be a Lipschitz continuous and monotone mapping. Then the mapping $A + B$ is a maximal monotone mapping.*

Lemma 2.7. *[26] Let $B : \mathcal{H} \to 2^{\mathcal{H}}$ be a maximal monotone operator and $A : \mathcal{H} \to \mathcal{H}$ be a mapping on $\mathcal{H}$. Define $T_\lambda := (I + \lambda B)^{-1}(I - \lambda A), (\lambda > 0)$.Then we have*

$$Fix(T_\lambda) = (A + B)^{-1}(0).$$

Lemma 2.8. *[8](The Resolvent Identity) For $\lambda, \mu > 0$, there holds the identity:*

$$J_{\lambda}^{T}x = J_{\mu}^{T}(\frac{\mu}{\lambda}x + (1-\frac{\mu}{\lambda})J_{\lambda}^{T}x), \qquad x \in \mathcal{H}.$$

Lemma 2.9. *([29]) Assume $\{s_n\}$ is a sequence of nonnegative real numbers such that*

$$\begin{cases} s_{n+1} \leq (1-\eta_n)s_n + \eta_n\delta_n, & n \geq 0, \\ s_{n+1} \leq s_n - \varrho_n + \zeta_n, & n \geq 0, \end{cases}$$

where $\{\eta_n\}$ is a sequence in $(0,1)$, $\{\varrho_n\}$ is a sequence of nonnegative real numbers and $\{\delta_n\}$ and $\{\zeta_n\}$ are two sequences in $\mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty}\eta_n = \infty$,
(ii) $\lim_{n\to\infty}\zeta_n = 0$
(iii) $\lim_{k\to\infty}\varrho_{n_k} = 0$, *implies* $\limsup_{k\to\infty}\delta_{n_k} \leq 0$ *for any subsequence* $\{n_k\} \subset \{n\}$.

Then $\lim_{n\to\infty} s_n = 0$.

3. The algorithm and its convergence

In this section, we first present the new algorithm for solving the problem (6) and then analyze its convergence.

Theorem 3.1. *Let $\mathcal{H}$ and $\mathcal{K}$, be real Hilbert spaces and let $T_i : \mathcal{H} \to \mathcal{K}$, $(i = 1,2,...,m)$, be bounded linear operators such that $T_i \neq 0$. Let for $i \in \{1,2,...,m\}$, $G_i : \mathcal{K} \to 2^{\mathcal{K}}$, $B_i : \mathcal{H} \to 2^{\mathcal{H}}$ be maximal monotone operators and let $A_i : \mathcal{H} \to \mathcal{H}$ be monotone and L_i-Lipschitz continuous operators. Suppose that $\Omega = \bigcap_{i=1}^{m}(A_i + B_i)^{-1}(0) \cap (T_i)^{-1}((G_i)^{-1}(0)) \neq \emptyset$. Let the operator $F : \mathcal{H} \to \mathcal{H}$ be l-Lipschitz continuous and δ-strongly monotone with constants $l > 0$, $\delta > 0$. Let $\alpha > 0$, $\gamma_i \in (0,1)$, $\lambda_{(1,i)} > 0$ and let $x_1, x_0 \in \mathcal{H}$ be two initial points. Let $\{x_n\}$ be a sequence defined by:*

$$\begin{cases} w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ z_{n,i} = (I - \tau_{n,i}T_i^*(I - J_{s_{n,i}}^{G_i})T_i)w_n \\ u_{n,i} = (I + \lambda_{n,i}B_i)^{-1}(z_{n,i} - \lambda_{n,i}A_i(z_{n,i})), \\ v_{n,i} = u_{n,i} + \lambda_{n,i}(A_i(z_{n,i}) - A_i(u_{n,i})), \quad i \in \{1,2,...,m\}, \\ y_n = \sum_{i=1}^{m} a_i v_{n,i}, \\ x_{n+1} = (I - \beta_n F)y_n, \qquad n \geq 1, \end{cases} \tag{7}$$

where $0 \leq \alpha_n \leq \overline{\alpha}_n$ such that

$$\overline{\alpha}_n = \begin{cases} \min\{\frac{\varepsilon_n}{\|x_n - x_{n-1}\|}, \alpha\}, & \text{if } \|x_n - x_{n-1}\| \neq 0, \\ \alpha, & \text{otherwise}, \end{cases} \tag{8}$$

and

$$\tau_{n,i} = \begin{cases} \frac{\rho_{n,i}\|(I - J_{s_{n,i}}^{G_i})T_i(w_n)\|^2}{\|(T_i^*(I - J_{s_{n,i}}^{G_i})T_i)(w_n)\|^2}, & \text{if } \|(T_i^*(I - J_{s_{n,i}}^{G_i})T_i)(w_n)\|^2 \neq 0, \\ 0, & \text{otherwise}, \end{cases} \tag{9}$$

and

$$\lambda_{(n+1,i)} = \begin{cases} \min\{\frac{\gamma_i\|z_{n,i} - u_{n,i}\|}{\|A_i(z_{n,i}) - A_i(u_{n,i})\|}, \lambda_{n,i}\}, & \text{if } \|A_i(z_{n,i}) - A_i(u_{n,i})\| \neq 0, \\ \lambda_{n,i}, & \text{otherwise}. \end{cases} \tag{10}$$

Assume that the sequences $\{\beta_n\}$, $\{a_i\}$, $\{s_{n,i}\}$, $\{\rho_{n,i}\}$ and $\{\varepsilon_n\}$ satisfying the following conditions:

(i) $\{\beta_n\} \subset (0,1)$, $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=0}^{\infty} \beta_n = \infty$,
(ii) $a_i > 0$, $\sum_{i=1}^{m} a_i = 1$, *for* $i = 1,2,...,m$,
(iii) $\liminf_n s_{n,i} > 0$ *for* $i = 1,2,...,m$,
(iv) $0 < \rho_{n,i} < 2$ *and* $\inf_n \rho_{n,i}(2-\rho_{n,i}) > 0$ *for* $i = 1,2,...,m$,
(v) $\varepsilon_n > 0$ *and* $\lim_{n\to\infty} \dfrac{\varepsilon_n}{\beta_n} = 0$.

Then, the sequence $\{x_n\}$ *converges strongly to the unique solution* $x^\star \in VI(\Omega, F)$.

Proof. First we show that $\{x_n\}$ is bounded. Since Ω is nonempty, closed and convex, from Lemma 2.3 we have $VI(\Omega, F)$ has a unique solution. We denote $x^\star \in \mathcal{H}$ the unique solution of $VI(\Omega, F)$.

Since $x^\star \in \Omega$, we have $0 \in G_i(T_i x^\star)$. Thus $T_i x^\star \in (G_i)^{-1}(0) = Fix(J^{G_i}_{s_{n,i}})$. Hence we have

$$\begin{aligned}\langle T_i^*(I - J^{G_i}_{s_{n,i}})T_i w_n, w_n - x^\star\rangle &= \langle (I - J^{G_i}_{s_{n,i}})T_i w_n - (I - J^{G_i}_{s_{n,i}})T_i x^\star, T_i w_n - T_i x^\star\rangle \\ &\geq \|(I - J^{G_i}_{s_{n,i}})T_i w_n\|^2. \end{aligned} \tag{11}$$

If for some $n \geq 1$ and $i \in \{1,2,...,m\}$, $\|(T_i^*(I - J^{G_i}_{s_{n,i}})T_i)(w_n)\| = 0$, then $\|z_{n,i} - x^\star\| = \|w_n - x^\star\|$. Otherwise, from (9) and inequality (11) we get

$$\begin{aligned}&\|z_{n,i} - x^\star\|^2 \\ &= \|(I - \tau_{n,i}T_i^*(I - J^{G_i}_{s_{n,i}})T_i)w_n - x^\star\|^2 \\ &= \|(w_n - x^\star) - \tau_{n,i}T_i^*(I - J^{G_i}_{s_{n,i}})T_i w_n\|^2 \\ &= \|w_n - x^\star\|^2 + (\tau_{n,i})^2\|T_i^*(I - J^{G_i}_{s_{n,i}})T_i w_n\|^2 - 2\tau_{n,i}\langle w_n - x^\star, T_i^*(I - J^{G_i}_{s_{n,i}})T_i w_n\rangle \\ &\leq \|w_n - x^\star\|^2 + (\tau_{n,i})^2\|T_i^*(I - J^{G_i}_{s_{n,i}})T_i w_n\|^2 - 2\tau_{n,i}\|(I - J^{G_i}_{s_{n,i}})T_i w_n\|^2 \\ &\leq \|w_n - x^\star\|^2 - \rho_{n,i}(2-\rho_{n,i})\frac{\|(I - J^{G_i}_{s_{n,i}})T_i(w_n)\|^4}{\|T_i^*(I - J^{G_i}_{s_{n,i}})T_i(w_n)\|^2}. \end{aligned} \tag{12}$$

Note that, if $\|T_i^*(I - J^{G_i}_{s_{n,i}})T_i(w_n)\| = 0$, it follows from $J^{G_i}_{s_{n,i}}(T_i(x^\star)) = T_i(x^\star)$ and the equation (11) that $\|(I - J^{G_i}_{s_{n,i}})T_i(w_n)\| = 0$.

From [26] we know that for each $i \in \{1,2,...,m\}$ the limit of $\{\lambda_{n,i}\}$ exists, and $\lim_{n\to\infty} \lambda_{n,i} = \lambda_i > 0$. For each $i \in \{1,2,...,m\}$, we have (see [26]):

$$\|v_{n,i} - x^\star\|^2 \leq \|z_{n,i} - x^\star\|^2 - (1 - (\frac{\gamma_i \lambda_{n,i}}{\lambda_{(n+1,i)}})^2)\|z_{n,i} - u_{n,i}\|^2, \tag{13}$$

and

$$\|v_{n,i} - u_{n,i}\| \leq \frac{\gamma_i \lambda_{n,i}}{\lambda_{(n+1,i)}}\|z_{n,i} - u_{n,i}\|. \tag{14}$$

From the inequalities (12) and (13) and convexity of $\|.\|^2$ we get

$$\begin{aligned}\|y_n - x^\star\|^2 = \|\sum_{i=1}^{m} a_i v_{n,i} - x^\star\|^2 &\leq \sum_{i=1}^{m} a_i\|v_{n,i} - x^\star\|^2 \\ &\leq \sum_{i=1}^{m} a_i\|z_{n,i} - x^\star\|^2 - \sum_{i=1}^{m} a_i(1 - (\frac{\gamma_i\lambda_{n,i}}{\lambda_{n+1,i}})^2)\|z_{n,i} - u_{n,i}\|^2 \\ &\leq \|w_n - x^\star\|^2 - \sum_{i=1}^{m} a_i(1 - (\frac{\gamma_i\lambda_{n,i}}{\lambda_{n+1,i}})^2)\|z_{n,i} - u_{n,i}\|^2 \\ &- \sum_{i=1}^{m} a_i\rho_{n,i}(2-\rho_{n,i})\frac{\|(I - J^{G_i}_{s_{n,i}})T_i(w_n)\|^4}{\|T_i^*(I - J^{G_i}_{s_{n,i}})T_i(w_n)\|^2}. \end{aligned} \tag{15}$$

We have $\alpha_n\|x_n - x_{n-1}\| \leq \varepsilon_n$ for all n, which together with $\lim_{n\to\infty} \frac{\varepsilon_n}{\beta_n} = 0$ implies that

$$\lim_{n\to\infty} \frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\| = 0.$$

It follows that there exists a constant $M_1 > 0$ such that

$$\frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\| \leq M_1.$$

From the definition of w_n, we get

$$\begin{aligned} \|w_n - x^\star\| &= \|x_n + \alpha_n(x_n - x_{n-1}) - x^\star\| \\ &\leq \|x_n - x^\star\| + \alpha_n\|x_n - x_{n-1}\| \\ &= \|x_n - x^\star\| + \beta_n \tfrac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\| \\ &\leq \|x_n - x^\star\| + \beta_n M_1. \end{aligned} \tag{16}$$

Hence, it follows from (15) and (16) that

$$\|y_n - x^\star\| \leq \|w_n - x^\star\| \leq \|x_n - x^\star\| + \beta_n M_1. \tag{17}$$

Take $\gamma \in (0, \frac{2\delta}{l^2})$. Since $\lim_{n\to\infty} \beta_n = 0$, there exist $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $\beta_n < \gamma$. Hence $\frac{\beta_n}{\gamma} \in (0, 1)$. From Lemma 2.4 for all $n > n_0$ we have

$$\begin{aligned} \|(I - \beta_n F)y_n - (I - \beta_n F)x^\star\| &= \|(I - \tfrac{\beta_n}{\gamma}\gamma F)y_n - (I - \tfrac{\beta_n}{\gamma}\gamma F)x^\star\| \\ &\leq (1 - \tfrac{\beta_n}{\gamma}\eta)\|y_n - x^\star\|, \end{aligned} \tag{18}$$

where $\eta = 1 - \sqrt{1 - \gamma(2\delta - \gamma l^2)} \in (0, 1)$. Utilizing the inequalities (17) and (18) we get that

$$\begin{aligned} \|x_{n+1} - x^\star\| &= \|y_n - \beta_n F y_n - x^\star\| \\ &= \|(I - \beta_n F)y_n - (I - \beta_n F)x^\star - \beta_n F x^\star\| \\ &\leq \|(I - \beta_n F)y_n - (I - \beta_n F)x^\star\| + \beta_n\|Fx^\star\| \\ &\leq (1 - \tfrac{\beta_n}{\gamma}\eta)\|y_n - x^\star\| + \beta_n\|Fx^\star\| \\ &\leq (1 - \tfrac{\beta_n}{\gamma}\eta)\|x_n - x^\star\| + \beta_n M_1 + \beta_n\|Fx^\star\| \\ &\leq (1 - \tfrac{\beta_n}{\gamma}\eta)\|x_n - x^\star\| + \tfrac{\beta_n}{\gamma}\eta[\tfrac{\gamma(M_1 + \|Fx^\star\|)}{\eta}] \\ &\leq \max\{\|x_n - x^\star\|, \tfrac{\gamma(M_1 + \|Fx^\star\|)}{\eta}\} \\ &\leq \cdots \leq \max\{\|x_{n_0} - x^\star\|, \tfrac{\gamma(M_1 + \|Fx^\star\|)}{\eta}\} \end{aligned}$$

This implies that $\{x_n\}$ is bounded. We also get $\{y_n\}$ and $\{w_n\}$ are bounded. We have

$$\begin{aligned} \|w_n - x^\star\|^2 &= \|x_n + \alpha_n(x_n - x_{n-1}) - x^\star\|^2 \\ &\leq \|x_n - x^\star\|^2 + (\alpha_n)^2\|x_n - x_{n-1}\|^2 + 2\alpha_n\langle x_n - x^\star, x_n - x_{n-1}\rangle \\ &\leq \|x_n - x^\star\|^2 + (\alpha_n)^2\|x_n - x_{n-1}\|^2 + 2\alpha_n\|x_n - x^\star\|\|x_n - x_{n-1}\|. \end{aligned}$$

Utilizing inequality (18) and inequality $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \forall x, y \in \mathcal{H}$, we arrive at

$$
\begin{aligned}
\|x_{n+1} - x^\star\|^2 &= \|y_n - \beta_n F y_n - x^\star\|^2 \\
&= \|(I-\beta_n F)y_n - (I-\beta_n F)x^\star - \beta_n F x^\star\|^2 \\
&\leq \|(I-\beta_n F)y_n - (I-\beta_n F)x^\star\|^2 - 2\beta_n\langle F x^\star, x_{n+1} - x^\star\rangle \\
&\leq (1 - \tfrac{\beta_n}{\gamma}\eta)^2\|y_n - x^\star\|^2 + 2\beta_n\langle F x^\star, x^\star - x_{n+1}\rangle \\
&\leq (1 - \tfrac{\beta_n}{\gamma}\eta)\|w_n - x^\star\|^2 + 2\beta_n\langle F x^\star, x^\star - x_{n+1}\rangle \\
&\leq (1 - \tfrac{\beta_n}{\gamma}\eta)\|x_n - x^\star\|^2 + 2\beta_n\langle F x^\star, x^\star - x_{n+1}\rangle \\
&+ (\alpha_n)^2\|x_n - x_{n-1}\|^2 + 2\alpha_n\|x_n - x^\star\|\|x_n - x_{n-1}\| \\
&\leq (1 - \tfrac{\beta_n}{\gamma}\eta)\|x_n - x^\star\|^2 + (\tfrac{\beta_n}{\gamma}\eta)(\tfrac{2\gamma}{\eta})\langle F x^\star, x^\star - x_{n+1}\rangle \\
&+ \alpha_n\|x_n - x_{n-1}\|(\alpha_n\|x_n - x_{n-1}\| + 2\|x_n - x^\star\|) \\
&\leq (1 - \tfrac{\beta_n}{\gamma}\eta)\|x_n - x^\star\|^2 + (\tfrac{\beta_n}{\gamma}\eta)(\tfrac{2\gamma}{\eta})\langle F x^\star, x^\star - x_{n+1}\rangle \\
&+ 3\alpha_n\|x_n - x_{n-1}\|M \\
&= (1 - \tfrac{\beta_n}{\gamma}\eta)\|x_n - x^\star\|^2 \\
&+ \tfrac{\beta_n}{\gamma}\eta[\tfrac{2\gamma}{\eta}\langle F x^\star, x^\star - x_{n+1}\rangle + \tfrac{3\gamma\alpha_n}{\beta_n}\tfrac{M}{\eta}\|x_n - x_{n-1}\|] \\
&= (1-\sigma_n)\|x_n - x^\star\|^2 + \sigma_n\vartheta_n, \quad \forall n > n_0,
\end{aligned}
\tag{19}
$$

where

$$\sigma_n = \frac{\beta_n}{\gamma}\eta, \quad \vartheta_n = \frac{2\gamma}{\eta}\langle F x^\star, x^\star - x_{n+1}\rangle + \frac{3\gamma\alpha_n}{\beta_n}\frac{M}{\eta}\|x_n - x_{n-1}\|$$

and $M = \sup_{n\in\mathbb{N}}\{\|x_n - x^\star\|, \alpha_n\|x_n - x_{n-1}\|\}$. It is easy to see that $\sigma_n \to 0, \sum_{n=1}^{\infty}\sigma_n = \infty$.

Since $\{x_n\}$ is bounded, there exists a constant $M_2 > 0$ such that

$$2\gamma\langle F x^\star, x^\star - x_{n+1}\rangle \leq M_2.$$

From Algorithm 7 and inequality (18) we have

$$
\begin{aligned}
\|x_{n+1} - x^\star\|^2 &= \|y_n - \beta_n F y_n - x^\star\|^2 \\
&= \|(I-\beta_n F)y_n - (I-\beta_n F)x^\star - \beta_n F x^\star\|^2 \\
&\leq \|(I-\beta_n F)y_n - (I-\beta_n F)x^\star\|^2 - 2\beta_n\langle F x^\star, x_{n+1} - x^\star\rangle \\
&\leq (1 - \tfrac{\beta_n}{\gamma}\eta)^2\|y_n - x^\star\|^2 + 2\beta_n\langle F x^\star, x^\star - x_{n+1}\rangle \\
&\leq \|y_n - x^\star\|^2 + \beta_n M_2 \quad \forall n > n_0.
\end{aligned}
$$

From above inequality and inequality (15), for all $n > n_0$, we get

$$
\begin{aligned}
\|x_{n+1} - x^\star\|^2 &\leq \|w_n - x^\star\|^2 - \textstyle\sum_{i=1}^m a_i(1 - (\frac{\gamma_i\lambda_{n,i}}{\lambda_{n+1,i}})^2)\|z_{n,i} - u_{n,i}\|^2 \\
&- \textstyle\sum_{i=1}^m a_i\,\rho_{n,i}(2-\rho_{n,i})\dfrac{\|(I - J^{G_i}_{s_{n,i}})T_i(w_n)\|^4}{\|T_i^*(I - J^{G_i}_{s_{n,i}})T_i(w_n)\|^2} + \beta_n M_2.
\end{aligned}
\tag{20}
$$

Also we have

$$
\begin{aligned}
\|w_n - x^\star\|^2 &= (\|x_n - x^\star\| + \beta_n M_1)^2 \\
&= \|x_n - x^\star\|^2 + \beta_n(2M_1\|x_n - x^\star\| + \beta_n M_1^2) \\
&\le \|x_n - x^\star\|^2 + \beta_n M_3.
\end{aligned}
$$

for some constant $M_3 > 0$. From above inequalities we get that for all $n > n_0$,

$$
\begin{aligned}
\|x_{n+1} - x^\star\|^2 &\le \|x_n - x^\star\|^2 + \beta_n M_3 - \textstyle\sum_{i=1}^m a_i(1 - (\frac{\gamma_i\lambda_{n,i}}{\lambda_{n+1,i}})^2)\|z_{n,i} - u_{n,i}\|^2 \\
&\quad - \textstyle\sum_{i=1}^m a_i\,\rho_{n,i}(2-\rho_{n,i})\dfrac{\|(I - J^{G_i}_{s_{n,i}})T_i(w_n)\|^4}{\|T_i^*(I - J^{G_i}_{s_{n,i}})T_i(w_n)\|^2} + \beta_n M_2.
\end{aligned} \tag{21}
$$

Now we set

$$
\begin{aligned}
\xi_n &= \textstyle\sum_{i=1}^m a_i(1 - (\frac{\gamma_i\lambda_{n,i}}{\lambda_{n+1,i}})^2)\|z_{n,i} - u_{n,i}\|^2 \\
&\quad + \textstyle\sum_{i=1}^m a_i\,\rho_{n,i}(2-\rho_{n,i})\dfrac{\|(I - J^{G_i}_{s_{n,i}})T_i(w_n)\|^4}{\|T_i^*(I - J^{G_i}_{s_{n,i}})T_i(w_n)\|^2},
\end{aligned}
$$

and

$$
\zeta_n = \beta_n(M_2 + M_3), \quad \Gamma_n = \|x_n - x^\star\|^2. \tag{22}
$$

Hence the inequality (21) can be rewritten in the following form:

$$
\Gamma_{n+1} \le \Gamma_n - \xi_n + \zeta_n. \tag{23}
$$

In order to prove $\Gamma_n \to 0$, by Lemma 2.9, (considering inequalities (19) and (23)) it is sufficient to prove that for any subsequence $\{n_k\} \subset \{n\}$, if $\lim_{k\to\infty} \xi_{n_k} = 0$, then

$$
\limsup_{k\to\infty} \vartheta_{n_k} \le 0.
$$

We assume that $\lim_{k\to\infty} \xi_{n_k} = 0$. By our assumption we get

$$
\lim_{k\to\infty} \|u_{n_k,i} - z_{n_k,i}\| = 0, \qquad i = 1, 2, ..., m. \tag{24}
$$

From inequality (14) we get

$$
\lim_{k\to\infty} \|v_{n_k,i} - u_{n_k,i}\| = 0.
$$

Also we have

$$
\lim_{k\to\infty} \rho_{n_k,i}(2 - \rho_{n_k,i})\frac{\|(I - J^{G_i}_{s_{n_k,i}})T_i(w_{n_k})\|^4}{\|T_i^*(I - J^{G_i}_{s_{n_k,i}})T_i(w_{n_k})\|^2} = 0. \tag{25}
$$

By our assumption that $0 < \rho_{n,i} < 2$ and $\inf_n \rho_{n,i}(2 - \rho_{n,i}) > 0$, we get

$$
\lim_{k\to\infty} \|(I - J^{G_i}_{s_{n_k,i}})T_i(w_{n_k})\| = 0, \text{ for all } i = 1, 2, \ldots, m. \tag{26}
$$

Note that, if $\|T_i^*(I - J^{G_i}_{s_{n_k,i}})T_i(w_{n_k})\| = 0$, then $\|(I - J^{G_i}_{s_{n_k,i}})T_i(w_{n_k})\| = 0$.

From (26) we have,

$$
\lim_{k\to\infty} \|z_{n_k,i} - w_{n_k}\| = 0, \text{ for all } i = 1, 2, \ldots, m. \tag{27}
$$

Note that

$$\|x_n - w_n\| = \alpha_n\|x_n - x_{n-1}\| = \beta_n \frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\| \to 0.$$

From inequality

$$\|u_{n_k,i} - x_{n_k}\| \le \|u_{n_k,i} - z_{n_k,i}\| + \|z_{n_k,i} - w_{n_k}\| + \|w_{n_k} - x_{n_k}\|,$$

we arrive at

$$\lim_{k\to\infty} \|u_{n_k,i} - x_{n_k}\| = 0, \qquad i = 1, 2, ..., m.$$

Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ which converges weakly to $\widehat{x}$. Without loss of generality, we can assume that $x_{n_k} \rightharpoonup \widehat{x}$. Since $\lim_{k\to\infty} \|w_{n_k} - x_{n_k}\| = 0$, we have $w_{n_k} \rightharpoonup \widehat{x}$. Since T_i is a linear bounded operator, it yields that $T_i(w_{n_k}) \rightharpoonup T_i\widehat{x}$. Utilizing the resolvent identity for each $s > 0$ we have

$$\begin{aligned} \|(I - J_s^{G_i})T_i(w_{n_k})\| &\le \|(I - J_{s_{n_k,i}}^{G_i})T_i(w_{n_k})\| \\ &+ |1 - \frac{s}{s_{n_k,i}}|\|(I - J_{s_{n_k,i}}^{G_i})T_i(w_{n_k})\| \to 0,\ k \to \infty. \end{aligned} \tag{28}$$

Since $I - J_s^{G_i}$ is demiclosed at zero, we know that $T_i\widehat{x} \in Fix(J_s^{G_i}) = G_i^{-1}(0), i = 1, 2, ..., m$. Since $\lim_{k\to\infty} \|v_{n_k,i} - x_{n_k}\| = 0$, we have $v_{n_k,i} \rightharpoonup \widehat{x}$. Now by similar proof as Lemma 7 in [26], we obtain that $\widehat{x} \in \bigcap_{i=1}^m (A_i + B_i)^{-1}(0))$. Thus $\widehat{x} \in \Omega$. Now we show that

$$\limsup_{k\to\infty} \langle Fx^\star, x^\star - x_{n_k}\rangle \le 0. \tag{29}$$

To show this inequality, we choose a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that

$$\lim_{j\to\infty} \langle Fx^\star, x^\star - x_{n_{k_j}}\rangle = \limsup_{k\to\infty} \langle Fx^\star, x^\star - x_{n_k}\rangle.$$

Since $x^\star$ is the unique solution of $VI(\Omega, F)$ and $\{x_{n_{k_j}}\}$ converges weakly to $\widehat{x} \in \Omega$. we conclude that

$$\limsup_{k\to\infty} \langle Fx^\star, x^\star - x_{n_k}\rangle = \lim_{j\to\infty} \langle Fx^\star, x^\star - x_{n_{k_j}}\rangle = \langle Fx^\star, x^\star - \widehat{x}\rangle \le 0.$$

Therefore

$$\limsup_{k\to\infty} \vartheta_{n_k} \le 0.$$

Hence, all conditions of Lemma 2.9 are satisfied. Therefore, we immediately deduce that $\lim_{n\to\infty} \Gamma_n = \lim_{n\to\infty} \|x_n - x^\star\|^2 = 0$, that is $\{x_n\}$ converges strongly to $x^\star$ which is the unique solution of $VI(\Omega, F)$. □

Remark 3.2. *In our proposed algorithm, the step size $\{\tau_{n,i}\}, i = 1, 2, ..., m$ are independent of the norm of T_i. Also, we observe that the choice of the $\{\lambda_{n,i}\}, i = 1, 2, ..., m$ are independent of Lipschitz constants of the operators A_i. Moreover, we do not require any prior information regarding the Lipschitz constant of the mapping F and the modulus of strong monotonicity of F. In some applications, finding the norm T_i and Lipschitz constants of the operators A_i and F is a difficult task. Therefore, our proposed method is easier to implement than the methods in [3, 16, 19, 55].*

Remark 3.3. *Putting $F(x) = x - f(x)$ in Theorem 3.1, where the mapping $f : \mathcal{H} \to \mathcal{H}$ is ρ-contraction. It can be easily verified that the mapping $F : \mathcal{H} \to \mathcal{H}$ is $(1 + \rho)$-Lipschitz continuous and $(1 - \rho)$-strongly monotone. In this situation, we obtain a viscosity type algorithm for solving split monotone variational inclusion problem. Especially, when $F(x) = x$ for all $x \in \mathcal{H}$. Then F is 1-strongly monotone and 1-Lipschitz continuous on $\mathcal{H}$, and in this situation, the problem (6) becomes the problem of finding the minimum-norm solution of the split monotone variational inclusion problem.*

4. Application

In section, we present some special cases of our problem (6) and obtain some corollaries of Theorem 3.1.

4.1. Bilevel programming problem

A function, $\Phi : \mathcal{H} \to \mathbb{R}$ is said to be convex if, for any $x, y \in \mathcal{H}$ and for any $\lambda \in [0,1]$, $\Phi(\lambda x + (1-\lambda)y) \leq \lambda\Phi(x) + (1-\lambda)\Phi(y)$. In particular, a convex function $\Phi : \mathcal{H} \to \mathbb{R}$, is said to be strongly convex with $c > 0$ (c-strongly convex) if

$$\Phi(\lambda x + (1-\lambda)y) \leq \lambda\Phi(x) + (1-\lambda)\Phi(y) - \frac{c\lambda(1-\lambda)}{2}\|x-y\|^2,$$

for all $x, y \in \mathcal{H}$ and for all $\lambda \in [0,1]$. Let $\Phi : \mathcal{H} \to \mathbb{R}$ be a Fréchet differentiable function. If Φ is c-strongly convex, $\nabla\Phi$ is c-strongly monotone.

Suppose that $\Phi : \mathcal{H} \to \mathbb{R}$ is δ-strongly convex and Fréchet differentiable, and $\nabla\Phi : \mathcal{H} \to \mathcal{H}$ is l-Lipschitz continuous. Let C be a nonempty closed convex subset of $\mathcal{H}$. Then, $VI(C, \nabla\Phi)$ can be characterized as the set of all minimizers of Φ over C.

$$VI(C, \nabla\Phi) = arg\min_{x\in C} \Phi(x) := \{x^* \in C : \quad \Phi(x^*) = \min_{x\in C} \Phi(x)\}.$$

Let $\mathcal{H}$ and $\mathcal{K}$ be two real Hilbert spaces. Let $f : \mathcal{H} \to (-\infty, \infty]$ and $g : \mathcal{K} \to (-\infty, \infty]$ be two proper, convex and lower semi-continuous functions, and $T : \mathcal{H} \to \mathcal{K}$ be a linear and bounded operators. The so-called split minimization problem (SMP) is the problem of finding

$$x^* \in \mathcal{H} \quad s.t., \quad f(x^*) = \min f(y)_{y\in\mathcal{H}} \quad and \quad g(Tx^*) = \min g(z)_{z\in\mathcal{K}}. \tag{30}$$

The subdifferential of f is the set-valued mapping $\partial f : \mathcal{H} \to 2^{\mathcal{H}}$ which is defined, for each $x \in \mathcal{H}$, by

$$\partial f(x) := \{z \in \mathcal{H} : f(y) - f(x) \geq \langle y - x, z\rangle \quad \forall y \in \mathcal{H}\}.$$

The proximity operator $Prox_f$ of f, is defined by

$$Prox_f(x) := arg\min_{y\in\mathcal{H}}\{f(y) + \frac{1}{2}\|y-x\|^2\}.$$

Equivalently, $Prox_f(x) = (I + \partial f)^{-1}x, \quad x \in \mathcal{H}$. It is known that ∂f is a maximal monotone operator and that $x_0 \in \arg\min_{x\in\mathcal{H}} f(x)$ if and only if $0 \in \partial f(x_0)$ (see[52] for details).

By Theorem 3.1, we obtain the following convergence result for solving minimization problem defined over the set of solutions of split minimization problem.

Theorem 4.1. *Let $\mathcal{H}$ and $\mathcal{K}$, be real Hilbert spaces and let $T_i : \mathcal{H} \to \mathcal{K}, (i = 1, 2, ..., m)$ be bounded linear operators such that $T_i \neq 0$. Let $f_i : \mathcal{H} \to (-\infty, \infty]$ and $g_i : \mathcal{K} \to (-\infty, \infty]$ be proper, lower semicontinuous and convex functions. Suppose that $\Omega = \{x^* \in \mathcal{H} \quad s.t., \quad f_i(x^*) = \min f_i(y)_{y\in\mathcal{H}} \quad and \quad g_i(T_i x^*) = \min g_i(z)_{z\in\mathcal{K}}, \quad (i = 1, 2, ..., m)\} \neq \emptyset$. Let the operator $\Phi : \mathcal{H} \to \mathbb{R}$ be a δ-strongly convex and Fréchet differentiable, and $\nabla\Phi$ be l-Lipschitz continuous. Let $\alpha > 0$, $\lambda_i > 0$ and let $x_1, x_0 \in \mathcal{H}$ be two initial points. Let $\{x_n\}$ be a sequence defined by:*

$$\begin{cases} w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n = \sum_{i=1}^m a_i J_{\lambda_i}^{\partial f_i}(I - \tau_{n,i} T_i^*(I - J_{s_{n,i}}^{\partial g_i})T_i)w_n \\ x_{n+1} = (I - \beta_n \nabla\Phi)y_n, \qquad n \geq 1, \end{cases} \tag{31}$$

where

$$\tau_{n,i} = \begin{cases} \dfrac{\rho_{n,i}\|(I - J_{s_{n,i}}^{\partial g_i})T_i(w_n)\|^2}{\|(T_i^*(I - J_{s_{n,i}}^{\partial g_i})T_i)(w_n)\|^2}, & if \ \|(T_i^*(I - J_{s_{n,i}}^{\partial g_i})T_i)(w_n)\|^2 \neq 0, \\ 0, & otherwise, \end{cases} \tag{32}$$

and $0 \leq \alpha_n \leq \overline{\alpha}_n$ *such that*

$$\overline{\alpha}_n = \begin{cases} \min\{\dfrac{\varepsilon_n}{\|x_n - x_{n-1}\|}, \alpha\}, & \text{if } \|x_n - x_{n-1}\| \neq 0, \\ \alpha, & \text{otherwise.} \end{cases} \tag{33}$$

Assume that the sequences $\{\beta_n\}$,$\{a_n\}$, $\{s_{n,i}\}$, $\{\rho_{n,i}\}$ *and* $\{\varepsilon_n\}$ *satisfying the following conditions:*

(i) $\{\beta_n\} \subset (0,1)$, $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=0}^{\infty} \beta_n = \infty$,

(ii) $a_i > 0$, $\sum_{i=1}^{m} a_i = 1$, *for* $i = 1, 2, ..., m$,

(iii) $\liminf_n s_{n,i} > 0$ *for* $i = 1, 2, ..., m$,

(iv) $0 < \rho_{n,i} < 2$ *and* $\inf_n \rho_{n,i}(2 - \rho_{n,i}) > 0$ *for* $i = 1, 2, ..., m$,

(v) $\varepsilon_n > 0$ *and* $\lim_{n\to\infty} \dfrac{\varepsilon_n}{\beta_n} = 0$.

Then, the sequence $\{x_n\}$ *converges strongly to the unique solution* $x^\star \in arg\min_{x\in\Omega} \Phi(x)$.

4.2. *Multiple-set split feasibility problem*

Let C be a nonempty, closed and convex subset of a real Hilbert space $\mathcal{H}$. Denote by i_C the indicator function of C, that is,

$$i_C := \begin{cases} 0, & \text{if } x \in C, \\ \infty, & \text{if } x \notin C, \end{cases}$$

It is not difficult to see that i_C is a proper, lower semicontinuous and convex function. Hence its subdifferential ∂i_C is a maximal monotone operator. It is known that

$$\partial i_C(u) = N(u, C) = \{f \in \mathcal{H} : \langle u - y, f\rangle \geq 0 \ \forall y \in C\},$$

where $N(u, C)$ is the normal cone of C at u.

We denote the resolvent operator of ∂i_C by J_r, where $r > 0$. Suppose $u = J_r x$ for $x \in \mathcal{H}$, that is,

$$\frac{x - u}{r} \in \partial i_C(u) = N(u, C).$$

Then we have

$$\langle x - u, u - y\rangle \geq 0$$

for all $y \in C$. Since this inequality characterizes the metric projection, it follows that $u = P_C x$.

Theorem 3.1 now yields the following result regarding an algorithm for solving the multiple-set split feasibility problem in Hilbert spaces.

Theorem 4.2. *Let* $\mathcal{H}$ *and* $\mathcal{K}$, *be real Hilbert spaces and let* $T_i : \mathcal{H} \to \mathcal{K}$ *be bounded linear operators such that* $T_i \neq 0$. *Let* $\{C_i\}_{i=1}^m$ *be a finite family of nonempty closed convex subsets of* $\mathcal{H}$ *and let* $\{Q_i\}_{i=1}^m$ *be a finite family of nonempty closed convex subsets of* $\mathcal{K}$. *Suppose that* $\Omega = \bigcap_{i=1}^m (C_i \cap T_i^{(-1)} Q_i) \neq \emptyset$. *Let the operator* $F : \mathcal{H} \to \mathcal{H}$ *be l-Lipschitz continuous and* δ*-strongly monotone with constants* $l > 0$, $\delta > 0$. *Let* $\alpha > 0$ *and let* $x_1, x_0 \in \mathcal{H}$ *be two initial points. Let* $\{x_n\}$ *be a sequence defined by:*

$$\begin{cases} w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n = \sum_{i=1}^m a_i P_{C_i}(I - \tau_{n,i} T_i^*(I - P_{Q_i})T_i)w_n, \\ x_{n+1} = (I - \beta_n F)y_n, \qquad n \geq 1, \end{cases} \tag{34}$$

where

$$\tau_{n,i} = \begin{cases} \dfrac{\rho_{n,i}\|(I - P_{Q_i})T_i(w_n)\|^2}{\|(T_i^*(I - P_{Q_i})T_i)(w_n)\|^2}, & \text{if } \|(T_i^*(I - P_{Q_i})T_i)(w_n)\|^2 \neq 0, \\ 0, & \text{otherwise}, \end{cases} \tag{35}$$

and $0 \leq \alpha_n \leq \overline{\alpha}_n$ *such that*

$$\overline{\alpha}_n = \begin{cases} \min\{\dfrac{\varepsilon_n}{\|x_n - x_{n-1}\|}, \alpha\}, & \text{if } \|x_n - x_{n-1}\| \neq 0, \\ \alpha, & \text{otherwise}. \end{cases} \tag{36}$$

Assume that the sequences $\{\beta_n\}$, $\{a_i\}$ $\{\rho_{n,i}\}$ *and* $\{\varepsilon_n\}$ *satisfying the following conditions:*

(i) $\{\beta_n\} \subset (0, 1)$, $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=0}^{\infty} \beta_n = \infty$,

(ii) $a_i > 0$, $\sum_{i=1}^{m} a_i = 1$, *for* $i = 1, 2, ..., m$,

(iii) $0 < \rho_{n,i} < 2$ *and* $\inf_n \rho_{n,i}(2 - \rho_{n,i}) > 0$ *for* $i = 1, 2, ..., m$,

(iv) $\varepsilon_n > 0$ *and* $\lim_{n\to\infty} \dfrac{\varepsilon_n}{\beta_n} = 0$.

Then, the sequence $\{x_n\}$ *converges strongly to the unique solution* $x^\star \in VI(\Omega, F)$.

4.3. Multiple Set Split Variational Inequality Problem

Let C be a nonempty convex subset of a real Hilbert space $\mathcal{H}$. Let $\Phi : \mathcal{H} \to \mathcal{H}$ be an operator. Further, the set valued mapping S^Φ related to the normal cone $N_C(x)$ is defined by

$$S^\Phi = \begin{cases} \Phi(x) + N_C(x), & x \in C, \\ \emptyset, & \text{otherwise}. \end{cases} \tag{37}$$

In the sense, if Φ is a is monotone and hemicontinuous operator, then S^Φ is a maximal monotone mapping. More importantly, $x \in VI(C, \Phi)$ if and only if $0 \in S^\Phi(x)$,(see [49] for details).

In [15], Censor et al. introduced the multiple set split variational inequality problem which is formulated as follows. Let $\mathcal{H}$ and $\mathcal{K}$ be two real Hilbert spaces. Given a bounded linear operator $T : \mathcal{H} \to \mathcal{K}$, functions $A_i : \mathcal{H} \to \mathcal{H}$, and $G_i : \mathcal{K} \to \mathcal{K}$ and nonempty, closed and convex subsets $C_i \subset \mathcal{H}$, $Q_i \subset \mathcal{K}$ for $i = 1, 2, ..., m$, the multiple set split variational inequality problem is formulated as follows:

$$\textit{Finding} \quad x^* \in \bigcap_{i=1}^{m} VI(C_i, A_i) : \quad \text{such that} \quad T(x^*) \in \bigcap_{i=1}^{m} VI(Q_i, G_i).$$

Censor et al.[15], present an algorithm with weak convergence for solving this problem for inverse strongly monotone operators.

Now, let $\Phi_i : \mathcal{K} \to \mathcal{K}$ be monotone and hemicontinuous operators and let $A_i : \mathcal{H} \to \mathcal{H}$ be monotone and L_i-Lipschitz continuous operators. Setting $G_i = S^{\Phi_i}$ and $B_i = \partial i_{C_i}$ in Theorem 3.1 we can apply our algorithm for solving the multiple set split variational inequality problem. Note that every inverse strongly monotone operators is monotone, hemicontinuous and Lipschitz continuous operator. Hence our result generalizes the result of Censor et al. [15].

4.4. Bilevel variational inequality problem

Finally, we utilize our algorithm for solving the Bilevel variational inequality problem in Hilbert spaces.

Theorem 4.3. *Let $\mathcal{H}$ be a Hilbert space. Let for each $i = 1, 2, .., m$, C_i be a nonempty closed convex subset of $\mathcal{H}$ and let $A_i : \mathcal{H} \to \mathcal{H}$ be a monotone and L_i- Lipschitz continuous operator. Suppose that $\Omega = \bigcap_{i=1}^{m} VI(C_i, A_i) \neq \emptyset$. Let the operator $F : \mathcal{H} \to \mathcal{H}$ be l-Lipschitz continuous and δ-strongly monotone with constants $l > 0$, $\delta > 0$. Let $\alpha > 0$, $\gamma_i \in (0, 1)$, $\lambda_{(1,i)} > 0$ and let $x_1, x_0 \in \mathcal{H}$ be two initial points. Let $\{x_n\}$ be a sequence defined by:*

$$\begin{cases} w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ u_{n,i} = P_{C_i}(w_n - \lambda_{n,i}A_i(w_n)), \\ v_{n,i} = u_{n,i} + \lambda_{n,i}(A_i(w_n) - A_i(u_{n,i})), \quad i \in \{1, 2, ..., m\}, \\ y_n = \sum_{i=1}^{m} a_i v_{n,i}, \\ x_{n+1} = (I - \beta_n F)y_n, \qquad n \geq 1, \end{cases} \tag{38}$$

where

$$\lambda_{(n+1,i)} = \begin{cases} min\{\frac{\gamma_i\|w_n - u_{n,i}\|}{\|A_i(w_n) - A_i(u_{n,i})\|}, \lambda_{n,i}\}, & \text{if } \|A_i(w_n) - A_i(u_{n,i})\| \neq 0, \\ \lambda_{n,i}, & \text{otherwise}, \end{cases} \tag{39}$$

and $0 \leq \alpha_n \leq \overline{\alpha}_n$ such that

$$\overline{\alpha}_n = \begin{cases} \min\{\dfrac{\varepsilon_n}{\|x_n - x_{n-1}\|}, \alpha\}, & \text{if } \|x_n - x_{n-1}\| \neq 0, \\ \alpha, & \text{otherwise}. \end{cases} \tag{40}$$

Assume that the sequences $\{\beta_n\}$, $\{a_i\}$ and $\{\varepsilon_n\}$ satisfying the following conditions:

(i) $\{\beta_n\} \subset (0, 1)$, $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=0}^{\infty} \beta_n = \infty$,

(ii) $\{a_i\} \subset (0, 1]$ *and* $\sum_{i=1}^{m} a_i = 1$, *for* $i = 1, 2, ..., m$,

(iii) $\varepsilon_n > 0$ *and* $\lim_{n\to\infty} \dfrac{\varepsilon_n}{\beta_n} = 0$.

Then, the sequence $\{x_n\}$ converges strongly to the unique solution $x^\star \in VI(\Omega, F)$.

Proof. We know that $J_r^{\partial i_{C_i}}(x) = P_{C_i}x$ for all $x \in \mathcal{H}$ and $r > 0$. Also, we know that

$$x \in (\partial i_{C_i} + A_i)^{-1}(0) \Leftrightarrow x \in VI(C_i, A_i).$$

Now putting $B_i = \partial i_{C_i}$, $\mathcal{K} = \mathcal{H}$, $T_i = I$, $G_i = 0$, $(i = 1, 2, ..., m)$, we obtain the desired result from Theorem 3.1. □

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