Filomat 37:24 (2023), 8389–8404
https://doi.org/10.2298/FIL2324389N

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Approximation results for Beta Jakimovski-Leviatan type operators via q-analogue

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Abstract. We construct a new version of q-Jakimovski-Leviatan type integral operators and show that set of all continuous functions f defined on $[0, \infty)$ are uniformly approximated by our new operators. Finally we construct the Stancu type operators and obtain approximation properties in weighted spaces. Moreover, with the aid of modulus of continuity we discuss the rate of convergence, Lipschitz type maximal approximation and some direct theorems.

1. Preliminaries and introduction

In 1880, Appell investigated a class of polynomials which were named in his honor as Appell polynomials (see [11]). Much later, Jakimovski and Leviatan introduced and modified the Appell polynomials [15] in 1969. The tool for defining the new class is defined below:

$$R(u)e^{uy} = \sum_{k=0}^{\infty} \beta_k(y)u^k, \tag{1}$$

where $\beta_k(y) = \sum_{i=0}^{s} \alpha_i \frac{y^{s-i}}{(s-i)!}$ ($s \in \mathbb{N}$, the set of positive numbers) and $R(u) = \sum_{k=0}^{\infty} \alpha_k u^k,\ R(1) \neq 0$.

We recall some basic notations regarding the q-calculus (see [19–21]). For each non-negative integer s, the q-integer is defined as

$$[s]_q = \begin{cases} \frac{1-q^s}{1-q}, & q \neq 1 \\ s, & q = 1 \end{cases} \quad \text{for } s \in \mathbb{N} \text{ and } [0]_q = 0.$$

2020 *Mathematics Subject Classification*. Primary 41A25, 41A36; Secondary 33C45.
Keywords. Appell polynomials; q-Appell polynomials; Jakimovski-Leviatan operators; Korovkin's theorem; modulus of continuity.
Received: 06 December 2022; Revised: 21 February 2023; Accepted: 11 April 2023
Communicated by Miodrag Spalević
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For $|q| < 1$, the q-factorial $[s]_q!$ is defined by

$$[s]_q! = \begin{cases} 1 & (s = 0) \\ \prod_{k=1}^{s} [k]_q & (s \in \mathbb{N}). \end{cases} \tag{2}$$

In the standard approach the exponential functions for q-calculus:

$$e_q(y) = \sum_{k=0}^{\infty} \frac{y^k}{[k]_q!}. \tag{3}$$

The improper integral of function f is formally defined by as

$$\int_0^{\infty/A} f(y) d_q y = (1-q) \sum_{s \in \mathbb{N}} f\left(\frac{q^s}{A}\right) \frac{q^s}{A},\ A \in \mathbb{R} - \{0\}. \tag{4}$$

Al-Salam (see [6, 25]) introduced the family of q-Appell polynomials through the generating functions $R_q(t) = \sum_{s=0}^{\infty} R_{s,q} \frac{t^s}{[s]!},\ R_q(1) \neq 0$. We have

$$R_{r,q}(y) = \sum_{k=0}^{r} \begin{bmatrix} r \\ k \end{bmatrix}_q A_{r-k,q} y^k,\ (r \in \mathbb{N})$$

and q-differential, $D_{q,y}\Big(R_{r,q}(y)\Big) = [r] R_{r-1,q}(y),\ r = 1, 2, \ldots$. We $D_{q,y}\Big(R_{1,q}(y)\Big) = [1]_q R_{0,q}(y) = K$, where $R_{0,q}(y)$ is a nonzero constant. Moreover, we define $R_q(t) e_q(ty) = \sum_{s=0}^{\infty} R_{r,q}(y) \frac{t^r}{[r]!},\quad 0 < q < 1$.

In the recent years the Jakimovski-Leviatan type operators have attracted attention in the mathematical community and several papers have been published (for example, see [1–4, 7, 8, 10, 14, 22–24, 29, 31, 36–38, 40] and further recent works such as [5, 12, 13, 28, 30, 32–35, 43–45] on related topics).

Our aim is to construct the extended form in terms of the parametric Stancu variant of recent $q-$Jakimovski-Leviatan-Beta type integral operators and show that our newly positive linear operators are convergent to the identity acting on the space of continuous functions. Further, we obtain Korovkin type results, estimates on the rate of convergence, as well as direct theorems.

For all $y \in [0, \infty),\ R_{r,q}(y) \geq 0$ and $R_q(1) \neq 0$, the Jakimovski-Leviatan type operators suppose $\mathcal{T}_{s,q}$ are given by (see [9])

$$\mathcal{T}_{s,q}(f(t); y) = \frac{e_q(-[s]y)}{R_q(1)} \sum_{r=0}^{\infty} \frac{R_{r,q}([s]y)}{[r]_q!} \frac{\mathcal{K}(A, r+1)}{B_q(r+1, s)} \int_0^{\infty/A} \frac{t^r}{(1+t)_q^{r+s+1}} f(q^r t)\, d_q t, \tag{5}$$

where $s \in \mathbb{N}$, $0 < q < 1$ and $f \in C[0, \infty)$, the latter being the set of all continuous functions on $[0, \infty)$.

Lemma 1.1. *[9]*

$$\sum_{r=0}^{\infty} \frac{R_{r,q}([s]y)}{[r]!} = R_q(1) e_q([s]y),$$

$$\sum_{r=0}^{\infty} r \frac{R_{r,q}([s]y)}{[r]!} = \Big[[s] R_q(1) y + R'_q(1)\Big] e_q([s]y),$$

$$\sum_{r=0}^{\infty} r^2 \frac{R_{r,q}([s]y)}{[r]!} = \Big[[s]^2 R_q(1) y^2 + 2[s] R'_q(1) y + R''_q(1)\Big] e_q([s]y),$$

$$\sum_{r=0}^{\infty} r^3 \frac{R_{r,q}([s]y)}{[r]!} = \Big[[s]^3 R_q(1) y^3 + 3[s]^2 R_q'(1) y^2 + 3[s] R_q''(1) y + R_q'''(1)\Big] e_q([s]y),$$

$$\sum_{r=0}^{\infty} r^4 \frac{R_{r,q}([s]y)}{[r]!} = \Big[[s]^4 R_q(1) y^4 + 4[s]^3 R_q'(1) y^3 + 6[s]^2 R_q''(1) y^2 + 4[s] R_q'''(1) y + R_q^{(4)}(1)\Big] e_q([s]y).$$

Lemma 1.2. *[9] For the test function $f(t) = 1, t, t^2, t^3, t^4$ the following identities hold true for s large enough, where in the expression of $\mathcal{T}_{s,q}(f(t); y)$ the term t is the dummy variable and y the global one.*

$$
\begin{aligned}
(1)\quad \mathcal{T}_{s,q}(1; y) &= 1; \\
(2)\quad \mathcal{T}_{s,q}(t; y) &= \frac{1}{q[s-1]} + \frac{1}{[s-1]}\Big([s]_q y + \frac{R_q'(1)}{R_q(1)}\Big); \\
(3)\quad \mathcal{T}_{s,q}(t^2; y) &= \frac{(1+q)}{q^3[s-1][s-2]} + \frac{(1+2q)}{q^2[s-1][s-2]}\left(([s]y + \frac{R_q'(1)}{R_q(1)}\right) \\
&\quad + \frac{1}{[s-1][s-2]}\left([s]^2 y^2 + \frac{2[s]R_q'(1)}{R_q(1)} y + \frac{R_q''(1)}{R_q(1)}\right); \\
(4)\quad \mathcal{T}_{s,q}(t^3; y) &= \frac{1+2q+2q^2+q^3}{q^4[s-1][s-2][s-3]} + \frac{(1+3q+4q^2+3q^3}{q^3[s-1][s-2][s-3]}\left(([s]y + \frac{R_q'(1)}{R_q(1)}\right) \\
&\quad + \frac{(1+2q+3q^2)}{q[s-1][s-2][s-3]}\left([s]^2 y^2 + \frac{2[s]R_q'(1)}{R_q(1)} y + \frac{R_q''(1)}{R_q(1)}\right) \\
&\quad + \frac{q^2}{[s-1][s-2][s-3]}\left([s]^3 y^3 + 3[s]^2 \frac{R_q'(1)}{R_q(1)} y^2 + 3[s]\frac{R_q''(1)}{R_q(1)} y + \frac{R_q'''(1)}{R_q(1)}\right); \\
(5)\quad \mathcal{T}_{s,q}(t^4; y) &= \frac{1+3q+5q^2+6q^3+5q^4+3q^5+q^6}{q^5[s-1][s-2][s-3][s-4]} \\
&\quad + \frac{(1+5q+10q^2+13q^3+12q^4+7q^5+2q^6)}{q^3[s-1][s-2][s-3][s-4]}\Big([]_q y + \frac{R_q'(1)}{R_q(1)}\Big) \\
&\quad + \frac{(1+3q+7q^2+9q^3+9q^4+6q^5)}{q[s-1][s-2][s-3][s-4]}\left([s]^2 y^2 + \frac{2[s]R_q'(1)}{R_q(1)} y + \frac{R_q''(1)}{R_q(1)}\right) \\
&\quad + \frac{(q^2+2q^3+2q^4+2q^5+q^6+2q^7)}{[s-1][s-2][s-3][s-4]} \\
&\quad \times\left([s]^3 y^3 + 3[s]^2 \frac{R_q'(1)}{R_q(1)} y^2 + 3[s]\frac{R_q''(1)}{R_q(1)} y + \frac{R_q'''(1)}{R_q(1)}\right) \\
&\quad + \frac{q^6}{[s-1][s-2][s-3][s-4]} \\
&\quad \times\left([s]^4 y^4 + 4[s]^3 \frac{R_q'(1)}{R_q(1)} y^3 + 6[s]^2 \frac{R_q''(1)}{R_q(1)} y^2 + 4[s]\frac{R_q'''(1)}{R_q(1)} y + \frac{R_q^{(4)}(1)}{R_q(1)}\right).
\end{aligned}
$$

2. Operators and its associated Moments

We suppose $y \in [0, \infty)$, $R_{r,q}(y) \geq 0$, $R_q(1) \neq 0$ and $f \in C_\lambda[0, \infty) = \{f \in C[0, \infty) : f(t) = O(t^\lambda), \text{ as } t \to \infty\}$. Then for any $0 \leq \nu \leq \mu$ and $0 < q < 1$, we construct

$$\mathcal{W}_{s,q}(f; y) = \frac{e_q(-[s]y)}{R_q(1)} \sum_{r=0}^{\infty} \frac{R_{r,q}([s]y)}{[r]!} \frac{\mathcal{K}(A, r+1)}{B_q(r+1, s)} \int_0^{\infty/A} \frac{t^r}{(1+t)_q^{r+s+1}} f\left(\frac{q^r([s]t+\nu)}{[s]+\mu}\right) d_q t, \tag{6}$$

where $\lambda > s$, $s \in \mathbb{N}$, $[s] = [s]_q$, $[r] = [r]_q$ and

$$B_q(r,s) = \mathcal{K}(A,r)\int_0^{\infty/A} \frac{y^{r-1}}{(1+y)_q^{r+s}} d_q y = \frac{[r-1]}{[s]} B_q(r-1,s+1), \quad r > 1,\ s \geq 0,$$

with

$$\mathcal{K}(A,r+1) = q^r \mathcal{K}(A,r),$$

$$\mathcal{K}(A,r) = q^{\frac{r(r-1)}{2}},\ \mathcal{K}(A,0) = 1.$$

Lemma 2.1. *For all $y \geq 0$, $\mathcal{W}_{s,q}(1;y) = 1$ and the following identities hold true.*

$$\begin{aligned}
(1)\quad \mathcal{W}_{s,q}(t;y) &= \left(\frac{[s]}{[s]+\mu}\right)\mathcal{T}_{s,q}(t;y) + \left(\frac{\nu}{[s]+\mu}\right)\mathcal{T}_{s,q}(1;y),\\
(2)\quad \mathcal{W}_{s,q}(t^2;y) &= \left(\frac{[s]}{[s]+\mu}\right)^2 \mathcal{T}_{s,q}(t^2;y) + \frac{2\nu[s]}{([s]+\mu)^2}\mathcal{T}_{s,q}(t;y) + \left(\frac{\nu}{[s]+\mu}\right)^2 \mathcal{T}_{s,q}(1;y),\\
(3)\quad \mathcal{W}_{s,q}(t^3;y) &= \left(\frac{[s]}{[s]+\mu}\right)^3 \mathcal{T}_{s,q}(t^3;y) + \frac{3\nu[s]^2}{([s]+\mu)^3}\mathcal{T}_{s,q}(t^2;y)\\
&\quad + \frac{3\nu^2[s]}{([s]+\mu)^3}\mathcal{T}_{s,q}(t;y) + \left(\frac{\nu}{[s]+\mu}\right)^3 \mathcal{T}_{s,q}(1;y),\\
(4)\quad \mathcal{W}_{s,q}(t^4;y) &= \left(\frac{[s]}{[s]+\mu}\right)^4 \mathcal{T}_{s,q}(t^4;y) + \frac{4\nu[s]^3}{([s]+\mu)^4}\mathcal{T}_{s,q}(t^3;y) + \frac{6\nu^2[s]^2}{([s]+\mu)^4}\mathcal{T}_{s,q}(t^2;y)\\
&\quad + \frac{4\nu^3[s]}{([s]+\mu)^4}\mathcal{T}_{s,q}(t^2;y) + \left(\frac{\nu}{[s]+\mu}\right)^4 \mathcal{T}_{s,q}(1;y).
\end{aligned}$$

Lemma 2.2. *Take the test functions $f(t) = 1, t, t^2, t^3, t^4$. For any $s \in \mathbb{N} - \{1,2,3,4\}$, the operators in* (6) *satisfy the following identities.*

$$\begin{aligned}
1^\circ\quad \mathcal{W}_{s,q}(1;y) &= 1;\\
2^\circ\quad \mathcal{W}_{s,q}(t;y) &= \frac{[s]}{q([s]+\mu)[s-1]}\left(1 + q[s]y + \frac{qR_q'(1)}{R_q(1)}\right) + \frac{\nu}{[s]+\mu};\\
3^\circ\quad \mathcal{W}_{s,q}(t^2;y) &= \left(\frac{[s]}{[s]+\mu}\right)^2 \frac{1}{q^3[s-1][s-2]}\Bigg\{1 + q + (q+2q^2)\left(([s]y + \frac{R_q'(1)}{R_q(1)}\right)\\
&\quad + q^3\left([s]^2y^2 + \frac{2[s]R_q'(1)}{R_q(1)}y + \frac{R_q''(1)}{R_q(1)}\right)\Bigg\}\\
&\quad - \frac{2\nu[s]}{q[s-1]([s]+\mu)^2}\left(1 + q[s]y + \frac{qR_q'(1)}{R_q(1)}\right) + \left(\frac{\nu}{[s]+\mu}\right)^2;\\
4^\circ\quad \mathcal{W}_{s,q}(t^3;y) &= \left(\frac{[s]}{[s]+\mu}\right)^3 \frac{1}{q^4[s-1][s-2][s-3]}\Bigg\{(1 + 2q + 2q^2 + q^3)\\
&\quad + \left(q + 3q^2 + 4q^3 + 3q^4\right)\left([s]y + \frac{R_q'(1)}{R_q(1)}\right)\\
&\quad + \left(q^3 + 2q^4 + 3q^5\right)\left([s]^2y^2 + \frac{2[s]R_q'(1)}{R_q(1)}y + \frac{R_q''(1)}{R_q(1)}\right)
\end{aligned}$$

$$
\begin{aligned}
&+q^6\left([s]^3y^3+3[s]^2\frac{R_q'(1)}{R_q(1)}y^2+3[s]\frac{R_q''(1)}{R_q(1)}y+\frac{R_q'''(1)}{R_q(1)}\right)\Bigg\}\\
&+\frac{3\nu[s]^2}{([s]+\mu)^3}\frac{1}{q^3[s-1][s-2]}\Bigg\{\Bigg(1+q+(q+2q^2)\left([s]y+\frac{R_q'(1)}{R_q(1)}\right)\\
&+q^3\left([s]^2y^2+\frac{2[s]R_q'(1)}{R_q(1)}y+\frac{R_q''(1)}{R_q(1)}\right)\Bigg)\\
&-\frac{2\nu[s]}{q[s-1]([s]+\mu)^2}\left(1+q[s]y+\frac{qP_q'(1)}{R_q(1)}\right)+\left(\frac{\nu}{[s]+\mu}\right)^2\Bigg\}\\
&+\frac{3\nu^2[s]}{([s]+\mu)^3}\frac{1}{q[s-1]}\left\{\left(1+q[s]y+\frac{qR_q'(1)}{R_q(1)}\right)+\frac{\nu}{[s]+\mu}\right\}+\left(\frac{\nu}{[s]+\mu}\right)^3;
\end{aligned}
$$

$$
\begin{aligned}
5^\circ\quad \mathcal{W}_{s,q}(t^4;y) = &\left(\frac{[s]}{[s]+\mu}\right)^4\frac{1}{q^5[s-1][s-2][s-3][s-4]}\\
&\times\Big\{(1+3q+5q^2+6q^3+5q^4+3q^5+q^6)\\
&+\left(q^2+5q^3+10q^4+13q^5+12q^6+7q^7+2q^8\right)\left([s]_qy+\frac{R_q'(1)}{R_q(1)}\right)\\
&+\left(q^4+3q^4+7q^6+9q^7+9q^8+6q^9\right)\left([s]^2y^2+\frac{2[s]R_q'(1)}{R_q(1)}y+\frac{R_q''(1)}{R_q(1)}\right)\\
&+\left(q^7+2q^8+2q^9+2q^{10}+q^{11}+2q^{12}\right)\\
&\times\left([s]^3y^3+3[s]^2\frac{R_q'(1)}{R_q(1)}y^2+3[s]\frac{R_q''(1)}{R_q(1)}y+\frac{R_q'''(1)}{R_q(1)}\right)\\
&+q^{11}\left([s]^4y^4+4[s]^3\frac{R_q'(1)}{R_q(1)}y^3+6[s]^2\frac{R_q''(1)}{R_q(1)}y^2+4[s]\frac{R_q'''(1)}{R_q(1)}y+\frac{R_q^{(4)}(1)}{R_q(1)}\right)\Big\}\\
&+\frac{4\nu[s]^3}{([s]+\mu)^4}\frac{1}{q^4[s-1][s-2][s-3]}\Big\{(1+2q+2q^2+q^3)\\
&+\left(q+3q^2+4q^3+3q^4\right)\left([s]y+\frac{R_q'(1)}{R_q(1)}\right)\\
&+\left(q^3+2q^4+3q^5\right)\left([s]^2y^2+\frac{2[s]R_q'(1)}{R_q(1)}y+\frac{R_q''(1)}{R_q(1)}\right)\\
&+q^6\left([s]^3y^3+3[s]^2\frac{R_q'(1)}{R_q(1)}y^2+3[s]\frac{R_q''(1)}{R_q(1)}y+\frac{R_q'''(1)}{R_q(1)}\right)\Big\}\\
&+\frac{3\nu[s]^2}{([s]+\mu)^3}\frac{1}{q^3[s-1][s-2]}\Bigg\{\Bigg(1+q+(q+2q^2)\left([s]y+\frac{R_q'(1)}{R_q(1)}\right)\\
&+q^3\left([s]^2y^2+\frac{2[s]R_q'(1)}{R_q(1)}y+\frac{R_q''(1)}{R_q(1)}\right)\Bigg)\\
&-\frac{2\nu[s]}{q[s-1]([s]+\mu)^2}\left(1+q[s]y+\frac{qP_q'(1)}{R_q(1)}\right)+\left(\frac{\nu}{[s]+\mu}\right)^2\Bigg\}\\
&+\frac{3\nu^2[s]}{([s]+\mu)^3}\frac{1}{q[s-1]}\left\{\left(1+q[s]y+\frac{qR_q'(1)}{R_q(1)}\right)+\frac{\nu}{[s]+\mu}\right\}+\left(\frac{\nu}{[s]+\mu}\right)^3\Bigg\}
\end{aligned}
$$

$$+\frac{6\nu^2[s]^2}{([s]+\mu)^4}\frac{1}{q^3[s-1][s-2]}\Bigg\{\Bigg(1+q+(q+2q^2)\Bigg(([s]y+\frac{R'_q(1)}{R_q(1)}\Bigg)$$
$$+q^3\Bigg([s]^2y^2+\frac{2[s]R'_q(1)}{R_q(1)}y+\frac{R''_q(1)}{R_q(1)}\Bigg)\Bigg)$$
$$-\frac{2\nu[s]}{q[s-1]([s]+\mu)^2}\Bigg(1+q[s]y+\frac{qP'_q(1)}{R_q(1)}\Bigg)+\Bigg(\frac{\nu}{[s]+\mu}\Bigg)^2\Bigg\}$$
$$+\frac{4\nu^3[s]}{([s]+\mu)^4}\frac{[s]}{q([s]+\mu)[s-1]}\Bigg(1+q[s]y+\frac{qR'_q(1)}{R_q(1)}\Bigg)+\frac{\nu}{[s]+\mu}+\Bigg(\frac{\nu}{[s]+\mu}\Bigg)^4.$$

Remark 2.3. *For the choices for $\nu=\mu=0$ then our new operators defined by (6) simply reduces to those studied in the recent published article [9] so that $\mathcal{W}_{s,q}(t^i;y)$ is the same as $\mathcal{T}_{s,q}(t^i;y)$ for all $i=0,1,2,3,4$.*

Lemma 2.4. *Take $\gamma_i(y)=(t-y)^i$ for $i=1,2$. Then for every $y\in[0,\infty)$, $0<q<1$, $R_{r,q}(y)\geq 0$ with $R_q(1)\neq 0$ if we put*

$$\left(\delta_{s,q}\right)^i=\mathcal{W}_{s,q}(\gamma_i(y);y)\text{ for } i=1,2,\ \ s>1,\ 2. \tag{7}$$

Furthermore the following equalities hold.

$$\left(\delta_{s,q}\right)^i=\begin{cases}\Bigg(\frac{[s]}{[s]+\mu}\Bigg)^2\frac{1}{q^3[s-1][s-2]}\Bigg\{1+q+(q+2q^2)\Bigg(([s]y+\frac{R'_q(1)}{R_q(1)}\Bigg)\\ +q^3\Bigg([s]^2y^2+\frac{2[s]R'_q(1)}{R_q(1)}y+\frac{R''_q(1)}{R_q(1)}\Bigg)\Bigg\}\\ +\frac{2\nu[s]}{q[s-1]([s]+\mu)^2}\Bigg(1+q[s]y+\frac{qP'_q(1)}{R_q(1)}\Bigg)+\Bigg(\frac{\nu}{[s]+\mu}\Bigg)^2\\ -\frac{2[s]y}{q([s]+\mu)[s-1]}\Bigg(1+q[s]y+\frac{qR'_q(1)}{R_q(1)}\Bigg)-\frac{2\nu y}{[s]+\mu}+y^2,\\ \quad \text{for } i=2,\ s>2,\\ \frac{[s]}{q([s]+\mu)[s-1]}\Bigg(1+q[s]y+\frac{qR'_q(1)}{R_q(1)}\Bigg)+\frac{\nu}{[s]+\mu}-y,\\ \quad\text{for } i=1,\ s>1.\end{cases}$$

3. Approximation in Weighted Space

Suppose that $C_b[0,\infty)$ be the set of all bounded and continuous functions on $[0,\infty)$ with the supremum norm:

$$\parallel f\parallel_{C_b}=\sup_{y\in[0,\infty)}|f(y)|.$$

Let

$$E:=\Bigg\{f:y\in[0,\infty),\ \frac{f(y)}{1+y^2}\ \text{ is convergent as }\ y\to\infty\Bigg\}.$$

We choose $q=q_s$ where $0<q_s<1$ such that

$$\lim_{s}q_s\to 1,\qquad \lim_{\tau}q_s^s\to\alpha \tag{8}$$

Theorem 3.1. *For any function $f \in C[0,\infty) \cap E$, we have*

$$\lim_{s\to\infty} \mathcal{W}_{s,q}(f;y) \to f(y)$$

uniformly convergent on each compact subset of $[0,\infty)$.

Proof. We take in account the Korovkin theorem [26]. Indeed, as $s \to \infty$, the operators $\mathcal{W}_{s,q}$ converges to the identity operator on the set of all continuous functions defined on $[0,\infty)$, because

$$\lim_{s\to\infty} \mathcal{W}_{s,q}(t^j;y) = y^j, \quad j = 0,1,2.$$

When $s \to \infty$, from (8), it is immediate to verify that $\frac{1}{[s]_{q_s}} \to 0$ and $\frac{[s]_{q_s}}{[s-1]_{q_s}} \to 1$. Consequently

$$\lim_{s\to\infty} \mathcal{W}_{s,q}(1;y) = 1, \ \lim_{s\to\infty} \mathcal{W}_{s,q}(t;y) = y, \ \lim_{s\to\infty} \mathcal{W}^*_{s,q}(t^2;y) = y^2,$$

which completes the proof. □

Suppose that $\phi(y)$ is a continuous and strictly increasing function on $[0,\infty)$ and take $\varrho(y) = 1 + \phi^2(y), \lim_{y\to\infty} \varrho(y) \to \infty$ such that $\phi(y) \to y$. Let $B_\varrho[0,\infty)$ be a set of functions defined on $[0,\infty)$, such that $\left|f(y)\right| \leq M_f \varrho(y)$, where M_f is any positive constant. Furthermore, we denote the set all continuous functions on $[0,\infty)$ by $C[0,\infty)$ and its subsets $C_\varrho[0,\infty)$ with $C_\varrho[0,\infty) = B_\varrho[0,\infty) \cap C[0,\infty)$. It is well known for the sequence of linear positive operators $\{L_s\}_{s\geq 1}$ (see [17]) maps $C_\varrho[0,\infty) \to B_\varrho[0,\infty)$ if and only if

$$|L_s(\varrho;y)| \leq C\varrho(y),$$

where C is a positive constant. For $m \in \mathbb{N}$, let us denote

$$C_\varrho^m[0,\infty) = \left\{ f \in C_\varrho[0,\infty) : \lim_{y\to\infty} \frac{f(y)}{\varrho(y)} = c \in \mathbb{R} \right\}. \tag{9}$$

Theorem 3.2. *[17, 18] Suppose $\{L_s\}_{s\geq 1}$ the positive linear operators which acting from $C_\varrho[0,\infty) \to B_\varrho[0,\infty)$ for $j = 0,1,2$ such that $\lim_{s\to\infty} \|L_s(\phi_j) - y^j\|_{\varrho(y)} = 0$. Then, for every $f \in C_\varrho[0,\infty)$, it follows that*

$$\lim_{s\to\infty} \|L_s(f) - f\|_{\varrho(y)} = 0.$$

Theorem 3.3. *Take $\varphi \in C_\varrho[0,\infty)$ which acting from $C_\varrho[0,\infty) \to B_\varrho[0,\infty)$ for $j = 0,1,2$ then for all $y \in [0,\infty)$ and $\varphi \in C_\varrho[0,\infty)$ with $R_{r,q}(y) \geq 0, \quad R_q(1) \neq 0$ it follows that*

$$\lim_{s\to\infty} \|\mathcal{W}_{s,q}(\varphi) - \varphi\|_{\varrho(y)} = 0.$$

Proof. Supposing $\lambda^j = t^j$ for $j = 0,1,2$, then by the use of the Korovkin theorem (see [26]), we directly conclude that

$$\lim_{s\to\infty} \|\mathcal{W}_{s,q}(\varphi^j) - \varphi^j\|_{\varrho(y)} = 0, \quad j = 0,1,2.$$

By considering the Lemma 2.1, we obtain

$$\|\mathcal{W}_{s,q}(\lambda^0) - 1\|_{\varrho(y)} = \sup_{y\in[0,\infty)} \frac{|\mathcal{W}_{s,q}(1;y) - 1|}{\varrho(y)} = 0.$$

For $j = 1$, we can write

$$\begin{aligned}
\|\mathcal{W}_{s,q}(\lambda^1) - y\|_{\varrho(y)} &= \sup_{y\in[0,\infty)} \frac{\left|\mathcal{W}_{s,q}(\lambda^1; y) - y\right|}{\varrho(y)} \\
&= \sup_{y\in[0,\infty)} \frac{y}{\varrho(y)}\left|\left(\frac{[s]^2}{([s]+\mu)[s-1]} - 1\right)\right| \\
&\quad + \sup_{y\in[0,\infty)} \frac{1}{\varrho(y)}\left|\frac{[s]}{([s]+\mu)[s-1]}\frac{R_q'(1)}{R_q(1)} + \frac{\nu}{[s]+\mu}\right|.
\end{aligned}$$

If $s \to \infty$, then easily we get $\|\mathcal{W}_{s,q}(\lambda^1) - y\|_{\varrho(y)} \to 0$.

For $j = 2$, we get

$$\begin{aligned}
\|\mathcal{W}_{s,q}(\lambda^2) - y^2\|_{\varrho(y)} &= \sup_{y\in[0,\infty)} \frac{\left|\mathcal{W}_{s,q}(\lambda^2; y) - y^2\right|}{\varrho(y)} \\
&= \sup_{y\in[0,\infty)} \frac{y^2}{\varrho(y)}\left|\left(\frac{[s]^4}{([s]+\mu)^2[s-1][s-2]} - 1\right)\right| \\
&+ \sup_{y\in[0,\infty)} \frac{y}{\varrho(y)}\left|\frac{[s]^2}{q^3([s]+\mu)^2[s-1]}\left\{\frac{1+q+(q+2q^2)[s]}{[s-2]}\right.\right. \\
&\left.\left.+\frac{[s]}{[s-2]}\frac{2q^3R_q'(1)}{R_q(1)} - 2\nu q^2\right\}\right| \\
&+ \sup_{y\in[0,\infty)} \frac{1}{\varrho(y)}\left|\frac{[s]}{q^3([s]+\mu)^2[s-1]}\left\{\frac{(1+q)[s]}{[s-2]}\right.\right. \\
&\left.\left.+\frac{q(1+2q)[s]}{[s-2]}\frac{R_q'(1)}{R_q(1)} + \frac{[s]}{[s-2]}\frac{q^3R_q''(1)}{R_q(1)} - 2\nu q^2\left(1+\frac{qR_q'(1)}{R_q(1)}\right)\right\}\right|
\end{aligned}$$

Thus we easily infer $\|\mathcal{W}_{s,q}(\lambda^2) - y^2\|_{\varrho(y)} \to 0$, as $s \to \infty$. □

Theorem 3.4. *For all $\varphi \in C_\varrho^m[0,\infty)$, $m \in \mathbb{N}$ and any $\xi \in [0,\infty)$, the operators $\mathcal{W}_{s,q}$ are such that the following limit relationship is satisfied:*

$$\lim_{s\to\infty} \sup_{y\in[0,\infty)} \frac{|\mathcal{W}_{s,q}(\varphi; y) - \varphi(y)|}{(\varrho(y))^{\xi+1}} = 0.$$

Proof. By the virtue of $|\varphi(y)| \leq \|(1+y^2)\varphi\|_{\varrho(y)}$ and for any positive number $y_0 \in \mathbb{R}$ (the real number), we easily obtain that

$$\begin{aligned}
\lim_{s\to\infty} \sup_{y\in[0,\infty)} \frac{|\mathcal{W}_{s,q}(\varphi; y) - \varphi(y)|}{(\varrho(y))^{\xi+1}} &\leq \sup_{y\leq y_0} \frac{|\mathcal{W}_{s,q}(\varphi; y) - \varphi(y)|}{(\varrho(y))^{\xi+1}} + \sup_{y\geq y_0} \frac{|\mathcal{W}_{s,q}(\varphi; y) - \varphi(y)|}{(\varrho(y))^{\xi+1}} \\
&\leq \|\mathcal{W}_{s,q}(\varphi; y) - \varphi(y)\|_{C[0,y_0]} \\
&+ \|\varphi\|_{\varrho(y)} \sup_{y\geq y_0} \frac{|\mathcal{W}_{s,q}(1+t^2; y) - \varphi(y)|}{(\varrho(y))^{\xi+1}} + \sup_{y\geq y_0} \frac{|\varphi(y)|}{(\varrho(y))^{\xi+1}} \\
&= \mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3, \text{(suppose)}.
\end{aligned}$$

Thus

$$\mathcal{M}_3 = \sup_{y\geq y_0} \frac{|\varphi(y)|}{(\varrho(y))^{\xi+1}} \leq \sup_{y\geq y_0} \frac{\|\varphi\|_{\varrho(y)}(1+y^2)}{(\varrho(y))^{\xi+1}} \leq \frac{\|\varphi\|_{\varrho(y)}}{(1+y_0^2)^{\xi}}. \tag{10}$$

In view of Lemma 2.2, we deduce that

$$\lim_{s\to\infty} \sup_{y\geq y_0} \frac{\mathcal{W}_{s,q}(1+t^2;y)}{\varrho(y)} = 1,$$

while, for any $\epsilon > 0$, there exists $s_1 \in \mathbb{N}$ such that $s \geq s_1$ satisfying

$$\sup_{y\geq y_0} \frac{\mathcal{W}_{s,q}(1+t^2;y)}{\varrho(y)} \leq \frac{(1+y_0^2)^{\xi}}{\|\varphi\|_{\varrho(y)}} \frac{\epsilon}{3} + 1.$$

Thus, for any $s \geq s_1$, we have

$$\mathcal{M}_2 = \|\varphi\|_{\varrho(y)} \sup_{y\geq y_0} \frac{\mathcal{W}_{s,q}(1+t^2;y)}{(\varrho(y))^{\xi+1}} \leq \frac{\|\varphi\|_{\varrho(y)}}{(1+y_0^2)^{\xi}} + \frac{\epsilon}{3}. \tag{11}$$

Taking into account the relations in (10) and (11), we infer

$$\mathcal{M}_2 + \mathcal{M}_3 \leq 2\frac{\|\varphi\|_{\varrho(y)}}{(1+y_0^2)^{\xi}} + \frac{\epsilon}{3}.$$

Choosing y_0 such that $\frac{\|\varphi\|_{\varrho(y)}}{(1+y_0^2)^{\xi}} \leq \frac{\epsilon}{6}$, we find

$$\mathcal{M}_2 + \mathcal{M}_3 \leq \frac{2\epsilon}{3}, \quad \text{for all } s \geq s_1, \tag{12}$$

and similarly, for $s_2 \geq s$ such that

$$\mathcal{M}_1 = \|\mathcal{W}_{s,q}(\varphi;y) - \varphi(y)\|_{C[0,y_0]} \leq \frac{\epsilon}{3}. \tag{13}$$

Next, choose $s_3 = \max(s_1, s_2)$. From (12) and (13), it is plain to deduce

$$\sup_{y\in[0,\infty)} \frac{|\mathcal{W}_{s,q}(\varphi;y) - \varphi(y)|}{(\varrho(y))^{\xi+1}} < \epsilon,$$

which completes the proof of Theorem 3.4. □

4. Order of Approximation

In the present section we study the order of convergence in terms of modulus of continuity functions in terms of Peetres K-functional and Lipschitz type maximal function. Suppose any $\delta > 0$, then for all $f \in C[a,b]$ (spaces of all continuous functions on $[a,b]$) the classical modulus of continuity is defined as

$$\omega(f;\delta) = \sup_{(y_1,y_2)\in[a,b],|y_1-y_2|\leq\delta} |f(y_1) - f(y_2)|.$$

Theorem 4.1. *[46] For any sequence of positive linear operators suppose* $\{L\}_{s\geq 1} : [y_1, y_2] \to C[u,v]$ *and* $[u,v] \subseteq [y_1, y_2]$*, we obtain*

1. *if* $f \in C[y_1, y_2]$ *and* $y \in [u, v]$, *then*
$$\begin{aligned} |L_s(f;y) - f(y)| &\leq |f(y)||L_s(1;y) - 1| \\ &+ \left\{L_s(1;y) + \frac{1}{\delta}\sqrt{L_s((t-y)^2;y)}\sqrt{L_s(1;y)}\right\}\omega(f;\delta). \end{aligned}$$
2. *Furthermore, for any* $\varphi' \in C[y_1, y_2]$, *for all* $y \in [u, v]$, *one has*
$$\begin{aligned} |L_s(\varphi;y) - \varphi(y)| &\leq |\varphi(y)||L_s(1;y) - 1| + |\varphi'(y)||L_s(t-y;y)| \\ &+ L_s((t-y)^2;y)\left\{\sqrt{L_s(1;y)} + \frac{1}{\delta}\sqrt{L_s((t-y)^2;y)}\right\}\omega(\varphi';\delta). \end{aligned}$$

Theorem 4.2. *Suppose* $f \in C_\varrho[0,\infty)$ *and* $y \in [0,\infty)$. *Then the inequality*
$$|\mathcal{W}_{s,q}(f;y) - f(y)| \leq 2\omega\left(f; \sqrt{\delta^*_{s,q}(y)}\right)$$
holds, with $\delta = \sqrt{\delta^*_{s,q}(y)} = \sqrt{\mathcal{W}_{s,q}((t-y)^2;y)}$.

Proof. By considering Lemma 2.2 and Theorem 4.1, we directly obtain
$$\begin{aligned} |\mathcal{W}_{s,q}(f;y) - f(y)| &\leq |f(y)||\mathcal{W}_{s,q}(1;y) - 1| + \left\{\mathcal{W}_{s,q}(1;y)\right. \\ &+ \left.\frac{1}{\delta}\sqrt{\mathcal{W}_{s,q}((t-y)^2;y)}\sqrt{\mathcal{W}_{s,q}(1;y)}\right\}\omega(f;\delta), \end{aligned}$$
on choosing $\delta = \sqrt{\delta^*_{s,q}(y)} = \sqrt{\mathcal{W}_{s,q}((t-y)^2;y)}$. The desired results follows now from a denumerability argument. □

Theorem 4.3. *For* $y \in [0,\infty)$, *for all* $\varphi' \in C_\varrho[0,\infty)$, *the inequality*
$$\begin{aligned} |\mathcal{W}_{s,q}(\varphi;y) - \varphi(y)| &\leq \left|\left(\frac{[s]^2}{([s]+\mu)[s-1]} - 1\right)y + \frac{[s]}{([s]+\mu)[s-1]}\frac{R'_q(1)}{R_q(1)} + \frac{\nu}{[s]+\mu}\right||\varphi'(y)| \\ &+2\delta^*_{s,q}(y)\omega\left(\varphi'; \sqrt{\delta^*_{s,q}(y)}\right) \end{aligned}$$
holds, where $\delta^*_{s,q}(y) = \mathcal{W}_{s,q}((t-y)^2;y)$.

Proof. By taking in account Lemma 2.2, Lemma 2.4 and Theorem 4.1, we deduce that
$$\begin{aligned} |\mathcal{W}_{s,q}(\varphi;y) - \varphi(y)| &\leq |\mathcal{W}_{s,q}(1;y) - 1||\varphi(y)| + |\varphi'(y)||\mathcal{W}_{s,q}(t-y;y)| \\ &+ \mathcal{W}_{s,q}((t-y)^2;y)\left\{\sqrt{\mathcal{W}_{s,q}(1;y)} + \frac{1}{\delta}\sqrt{\mathcal{W}_{s,q}((t-y)^2;y)}\right\}\omega(\varphi';\delta). \end{aligned}$$
In the light of Theorem 4.2, by using $\delta = \sqrt{\delta^*_{s,q}(y)} = \sqrt{\mathcal{W}_{s,q}((t-y)^2;y)}$, the claimed result follows. □

5. Approximation on Lipschitz space and Peetre's $K-$functional

In the present section we provide Lipschitz type maximal approximation results and a local direct theorem for our new operators (6). To this end, we recall the following facts regarding Lipschitz type maximal functions. Indeeed, for any choice of real parameters $\mu_1,\ \mu_2 > 0$ and $\chi \in (0,1]$, we can define (see [39])
$$Lip^{\chi}_{\mathcal{L}} = \left\{\varphi \in C_b[0,\infty) : |\varphi(t) - \varphi(y)| \leq \mathcal{L}\frac{|t-y|^{\chi}}{(\mu_1 y^2 + \mu_2 y + t)^{\frac{\chi}{2}}};\ y,\ t \in [0,\infty)\right\},$$
where $\mathcal{L}$ is denoted for Lipschitz constant.

Theorem 5.1. *For any $\varphi \in Lip_{\mathcal{L}}^{\chi}$ the operators $\mathcal{W}_{s,q}$ verify the inequality*

$$|\mathcal{W}_{s,q}(\varphi; y) - \varphi(y)| \leq \mathcal{L}\Big(\frac{\delta^*_{s,q}(y)}{(\mu_1 y^2 + \mu_2 y)}\Big)^{\frac{\chi}{2}},$$

where $\delta^*_{s,q}(y)$ is given by Theorem 4.2.

Proof. Take $\varphi \in Lip_{\mathcal{L}}^{\chi}$ for $0 < \chi \leq 1$. First we show that the result is true for $\chi = 1$. For any $\mu_1, \ \mu_2 \geq 0$ we know the inequality $(\mu_1 y^2 + \mu_2 y + t)^{-1/2} \leq (\mu_1 y^2 + \mu_2 y)^{-1/2}$, thus by use of Cauchy-Schwarz inequality it can be written as:

$$\begin{aligned}
|\mathcal{W}_{s,q}(\varphi; y) - \varphi(y)| &\leq |\mathcal{W}_{s,q}(|\varphi(t) - \varphi(y)|; y)| + \varphi(y)\ |(1; y) - 1| \\
&\leq \mathcal{W}_{s,q}\Big(\frac{|t-y|}{(\mu_1 y^2 + \mu_2 y + t)^{\frac{1}{2}}}; y\Big) \\
&\leq \mathcal{L}(\mu_1 y^2 + \mu_2 y)^{-1/2}\mathcal{W}_{s,q}(|t-y|; y) \\
&\leq \mathcal{L}(\mu_1 y^2 + \mu_2 y)^{-1/2}\mathcal{W}_{s,q}((t-y)^2; y)\Big|^{1/2}.
\end{aligned}$$

It can be concluded that the result is valid for $\chi = 1$. Now we verify the results for $0 < \chi < 1$. More precisely, we simply use the monotonicity property of the operators $\mathcal{W}_{s,q}$ and we use the Hölder's inequality. In this manner we have

$$\begin{aligned}
\Big|\mathcal{W}_{s,q}(\varphi; y) - \varphi(y)\Big| &\leq \mathcal{W}_{s,q}\Big(\Big|\varphi(t) - \varphi(y)\Big|; y\Big) \\
&\leq \Big(\mathcal{W}_{s,q}\Big(\Big|\varphi(t) - \varphi(y)\Big|^{\frac{2}{\chi}}; y\Big)\Big)^{\frac{\chi}{2}}\Big(\mathcal{W}_{s,q}(1; y)\Big)^{\frac{2-\chi}{2}} \\
&\leq \mathcal{L}\Big\{\frac{\mathcal{W}_{s,q}\Big(\Big(t-y\Big)^2; y\Big)}{\mu_1 y^2 + \mu_2 y + t}\Big\}^{\frac{\chi}{2}} \\
&\leq \mathcal{L}(\mu_1 y^2 + \mu_2 y)^{-\chi/2}\Big\{\mathcal{W}_{s,q}\Big(\Big(t-y\Big)^2; y\Big)\Big\}^{\frac{\chi}{2}} \\
&\leq \mathcal{L}(\mu_1 y^2 + \mu_2 y)^{-\chi/2}\Big(\mathcal{W}_{s,q}\Big(t-y\Big)^2; y)\Big)^{\frac{\chi}{2}} \\
&= \mathcal{L}\Big(\frac{\delta^*_{s,q}(y)}{(\mu_1 y^2 + \mu_2 y)}\Big)^{\frac{\chi}{2}},
\end{aligned}$$

which gives the desired proof. □

Here we also compute few other local approximation of $\mathcal{W}_{s,q}$ in terms of modulus of continuity of order one, by making use of Lipschitz maximal spaces. For any $\varphi \in C_b[0, \infty), \ 0 < \chi \leq 1$ and $t, y \in [0, \infty)$, we recall (see [27])

$$\omega_\chi(\varphi; y) = \sup_{t \neq y,\ t \in [0,\infty)} \frac{|\ \varphi(t) - \varphi(y)\ |}{|\ t - y\ |^{\chi}}, \tag{14}$$

where ω_χ denotes the modulus of continuity order one.

Theorem 5.2. *For any $\varphi \in C_b[0, \infty)$, we have*

$$\Big|\mathcal{W}_{s,q}(\varphi; y) - \varphi(y)\Big| \leq \Big(\delta^*_{s,q}(y)\Big)^{\frac{\chi}{2}}\ \omega_\chi(\varphi; y),$$

where $\omega_\chi(f; y)$ is defined by (14) *and $\delta^*_{s,q}(y)$ is given in Theorem 4.2.*

Proof. By exploiting the Hölder inequality, it is easy to deduce the following chain of relationships

$$\begin{aligned}\left|\mathcal{W}_{s,q}(\varphi;y)-\varphi(y)\right| &\leq \mathcal{W}_{s,q}\left(\left|\varphi(t)-\varphi(y)\right|;y\right)\\ &\leq \omega_{\chi}(\varphi;y)\mid \mathcal{W}_{s,q}\left(\left|t-y\right|^{\chi};y\right)\\ &\leq \omega_{\chi}(\varphi;y)\left(\mathcal{W}_{s,q}(1;y)\right)^{\frac{2-\chi}{2}}\left(\mathcal{W}_{s,q}(\left|t-y\right|^{2};y)\right)^{\frac{\chi}{2}}\\ &= \omega_{\chi}(\varphi;y)\left(\mathcal{W}_{s,q}\left((t-y)^{2};y\right)\right)^{\frac{\chi}{2}},\end{aligned}$$

so that the proof is concluded. □

Next, denoting by $C_b[0,\infty)$ the class of all continuous and bounded functions on the semi-axis $[0,\infty)$, we define

$$C_b^2[0,\infty)=\{\varphi\in C_b[0,\infty):\varphi',\varphi''\in C_b[0,\infty)\}, \tag{15}$$

$$\|\varphi\|_{C_b^2[0,\infty)}=\|\varphi\|_{C_b[0,\infty)}+\|\varphi'\|_{C_b[0,\infty)}+\|\varphi''\|_{C_b[0,\infty)}, \tag{16}$$

$$\|\varphi\|_{C_b[0,\infty)}=\sup_{y\in[0,\infty)}\mid\varphi(y)\mid. \tag{17}$$

Theorem 5.3. *For all $\varphi\in C_b^2[0,\infty)$ the operators $\mathcal{W}_{s,q}(\cdot\,;\cdot)$ are such that*

$$\mid\mathcal{W}_{s,q}(\varphi;y)-\varphi(y)\mid\leq\left(2\Omega^*_{s,q}(y)+\delta^*_{s,q}(y)\right)\frac{\|\varphi\|_{C_b^2[0,\infty)}}{2},$$

where $\delta^*_{s,q}(y)$ is given in Theorem 4.2. Furthermore, we have

$$\Omega^*_{s,q}(y)=\mathcal{W}_{s,q}(t-y;y)=\left\{\frac{[s]}{q([s]+\mu)[s-1]}\left(1+q[s]y+\frac{qR_q'(1)}{R_q(1)}\right)+\frac{\nu}{[s]+\mu}-y\right\}.$$

Proof. We take $\varphi\in C_b^2[0,\infty)$. Then for any $\psi\in(y,t)$ the Taylor series expansions lead to

$$\varphi(t)=\varphi(y)+\varphi'(y)(t-y)+\varphi''(\psi)\frac{(t-y)^2}{2}.$$

Therefore

$$\mathcal{W}_{s,q}(\varphi;y)-\varphi(y)=\varphi'(y)\mathcal{W}_{s,q}\left(t-y;y\right)+\frac{\varphi''(\psi)}{2}\mathcal{W}_{s,q}\left((t-y)^2;y\right),$$

and hence

$$\mid\mathcal{W}_{s,q}(\varphi;y)-\varphi(y)\mid\ \leq\ \left\{\Omega^*_{s,q}(y)\right\}\|\varphi'\|_{C_b[0,\infty)}+\left\{\delta^*_{s,q}(y)\right\}\frac{\|\varphi''\|_{C_b[0,\infty)}}{2}.$$

Furthermore, by invoking Lemma 2.4, we deduce

$$\Omega^*_{s,q}(y)=\mathcal{W}_{s,q}(t-y;y)=\left\{\frac{[s]}{q([s]+\mu)[s-1]}\left(1+q[s]y+\frac{qR_q'(1)}{R_q(1)}\right)+\frac{\nu}{[s]+\mu}-y\right\}$$

and, by Theorem 4.2, the required value of $\delta^*_{s,q}(y)$ is obtained.

Because of (16), it is easy to conclude that $\|\varphi'\|_{C_b[0,\infty)}\leq\|\varphi\|_{C_b^2[0,\infty)}$ and $\|\varphi''\|_{C_b[0,\infty)}\leq\|\varphi\|_{C_b^2[0,\infty)}$. As a consequence we find

$$\mid\mathcal{W}_{s,q}(\varphi;y)-\varphi(y)\mid\ \leq\ \left\{\Omega^*_{s,q}(y)\right\}\|\varphi\|_{C_b^2[0,\infty)}+\left\{\delta^*_{s,q}(y)\right\}\frac{\|\varphi\|_{C_b^2[0,\infty)}}{2},$$

where the latter inequality completes the desired proof. □

In this section, we directly provide estimates in the form of direct theorems. For the considered type of approximation results we use Peetre's K-functional introduced by J. Peetre [42] in 1968 and defined as follows

$$K_{\varphi}(\varphi;\delta)=\inf\left\{\left(\|\varphi-\phi\|_{C_b[0,\infty)}+\delta\|\phi\|_{C_\beta^2[0,\infty)}\right):\phi\in C_b^2[0,\infty)\right\}. \tag{18}$$

Furthermore we consider Peetre's K-functional in relation with the modulus of continuity of order two ω_2. More precisely there exists a positive real number M for which

$$K_{\varphi}(\varphi;\delta)\leq M\omega_2(\varphi;\delta^{\frac{1}{2}}),\quad \delta>0, \tag{19}$$

where

$$\omega_2(\varphi;\delta)=\sup_{0<\lambda<\delta}\sup_{y\in[0,\infty)}|(\varphi(y+2\lambda)-2\varphi(y+\lambda)+\varphi(y)|. \tag{20}$$

Moreover, from [16], we recall that there exists an absolute constant $C>0$ satisfying

$$K_{\varphi}(\varphi;\delta)\leq C\{\omega_2(\varphi;\sqrt{\delta})+\min(1,\delta)\|\varphi\|\}. \tag{21}$$

Theorem 5.4. *Let $q=q_s$ be the sequences of positive numbers such that $0<q_s<1$. Then for any $\varphi\in C_b[0,\infty)$, there exists a positive constant C such that*

$$\begin{aligned}|\mathcal{W}_{s,q}(\varphi;y)-\varphi(y)| &\leq 2C\left\{\omega_2\left(\varphi;\sqrt{\frac{2\Omega^*_{s,q}(y)+\delta^*_{s,q}(y)}{2}}\right)\right.\\ &\quad\left.+\min\left(1,\frac{2\Omega^*_{s,q}(y)+\delta^*_{s,q}(y)}{2}\right)\|\varphi\|_{C_b[0,\infty)}\right\}.\end{aligned}$$

Proof. We use Theorem 5.3 by considering any $\phi\in C_b[0,\infty)$. In this way we $|\mathcal{W}_{s,q}(\varphi-\phi;y)|\leq\|\varphi-\phi\|_{C_b}$. Therefore, it is obvious that

$$\begin{aligned}|\mathcal{W}_{s,q}(\varphi;y)-\varphi(y)| &\leq |\mathcal{W}_{s,q}(\varphi-\phi;y)|+|\mathcal{W}_{s,q}(\phi;y)-\phi(y)|+|\varphi(y)-\phi(y)|\\ &\leq 2\|\varphi-\phi\|_{C_b[0,\infty)}+\left(2\Omega^*_{s,q}(y)+\delta^*_{s,q}(y)\right)\|\phi\|_{C_b^2[0,\infty)}\\ &= 2\left(\|\varphi-\phi\|_{C_b[0,\infty)}+\frac{2\Omega^*_{s,q}(y)+\delta^*_{s,q}(y)}{2}\|\phi\|_{C_b^2[0,\infty)}\right).\end{aligned}$$

By applying the infimum over all $\phi\in C_b^2[0,\infty)$ and by exploiting relations (18) and (19), we deduce

$$|\mathcal{W}_{s,q}(\varphi;y)-\varphi(y)|\leq 2K_{\varphi}\left(\varphi;\frac{2\Omega^*_{s,q}(y)+\delta^*_{s,q}(y)}{2}\right).$$

Hence, by virtue of (21), our desired results follow and the proof is complete. □

Theorem 5.5. *For any $\phi\in C_b^2[0,\infty)$, we define an auxiliary operator $\mathcal{X}_{s,q}$ such that*

$$\mathcal{X}_{s,q}(\phi;y)=\mathcal{W}_{s,q}(\phi;y)+\phi(y)-\phi\left\{\mathcal{W}_{s,q}(t;y)\right\}. \tag{22}$$

Then for all $\psi\in C_b^2[0,\infty)$ we get the inequality

$$\left|\mathcal{X}_{s,q}(\psi;y)-\psi(y)\right|\leq\left\{\delta^*_{s,q}(y)+\left(\mathcal{W}_{s,q}(t;y)-y\right)^2\right\}\|\psi''\|,$$

where $\mathcal{W}_{s,q}(t;y)=\frac{[s]}{q([s]+\mu)[s-1]}\left(1+q[s]y+\frac{qR'_q(1)}{R_q(1)}\right)+\frac{\nu}{[s]+\mu}$ defined by Lemma 2.2 and $\delta^*_{s,q}(y)$ is defined by Theorem 4.2.

Proof. For all $\psi \in C_b^2[0,\infty)$, it is obvious that $\mathcal{X}_{s,q}(1;y) = 1$ and

$$\mathcal{X}_{s,q}(t;y) = \mathcal{W}_{s,q}(t;y) + y - \left\{\frac{[s]}{q([s]+\mu)[s-1]}\left(1 + q[s]y + \frac{qR_q'(1)}{R_q(1)}\right) + \frac{\nu}{[s]+\mu}\right\} = y.$$

For any $\psi \in C_b^2[0,\infty)$ the Taylor series expansions lead to

$$\psi(t) = \psi(y) + (t-y)\psi'(y) + \int_y^t (t-\theta)\psi''(\theta)\mathrm{d}\theta.$$

Thus

$$\begin{aligned}
\mathcal{X}_{s,q}(\psi;y) - \psi(y) &= \psi'(y)\mathcal{X}_{s,q}(t-y;y) + \mathcal{X}_{s,q}\Big(\int_y^t (t-\theta)\psi''(\theta)\mathrm{d}\theta; y\Big) \\
&= \mathcal{X}_{s,q}\Big(\int_y^t (t-\theta)\psi''(\theta)\mathrm{d}\theta; y\Big) \\
&= \mathcal{W}_{s,q}\Big(\int_y^t (t-\theta)\psi''(\theta)\mathrm{d}\theta; y\Big) + \int_y^y (y-\theta)\psi''(\theta)\mathrm{d}\theta; y \\
&- \int_y^{\mathcal{W}_{s,q}(t;y)} \left\{\frac{[s]}{q([s]+\mu)[s-1]}\left(1 + q[s]y + \frac{qR_q'(1)}{R_q(1)}\right) + \frac{\nu}{[s]+\mu} - \theta\right\}\psi''(\theta)\mathrm{d}\theta;
\end{aligned}$$

$$\begin{aligned}
\mid \mathcal{X}_{s,q}(\psi;y) - \psi(y) \mid &\leq \left|\mathcal{W}_{s,q}\Big(\int_y^t (t-\theta)\psi''(\theta)\mathrm{d}\theta; y\Big)\right| \\
&+ \left|\int_y^{\mathcal{W}_{s,q}(t;y)-\theta} \psi''(\theta)\mathrm{d}\theta\right|.
\end{aligned}$$

In view of the known inequalities

$$\left|\int_y^t (t-\theta)\psi''(\theta)\mathrm{d}\theta\right| \leq (t-y)^2 \parallel \psi'' \parallel$$

and

$$\begin{aligned}
&\left|\int_y^{\mathcal{W}_{s,q}(t;y)} \left\{\frac{[s]}{q([s]+\mu)[s-1]}\left(1 + q[s]y + \frac{qR_q'(1)}{R_q(1)}\right) + \frac{\nu}{[s]+\mu} - \theta\right\}\psi''(\theta)\mathrm{d}\theta\right| \\
\leq\ & \left\{\frac{[s]}{q([s]+\mu)[s-1]}\left(1 + q[s]y + \frac{qR_q'(1)}{R_q(1)}\right) + \frac{\nu}{[s]+\mu} - y\right\}^2 \parallel \psi'' \parallel,
\end{aligned}$$

we deduce

$$\begin{aligned}
\mid \mathcal{X}_{s,q}(\psi;y) - \psi(y) \mid &\leq \Big\{\mathcal{W}_{s,q}\left((t-y)^2; x\right) \\
&+ \Big(\frac{[s]}{q([s]+\mu)[s-1]}\left(1 + q[s]y + \frac{qR_q'(1)}{R_q(1)}\right) + \frac{\nu}{[s]+\mu} - y\Big)^2\Big\} \parallel \psi'' \parallel .
\end{aligned}$$

As a consequence the proof is complete. □

Theorem 5.6. *Let $\phi \in C_b^2[0,\infty)$. Then for every $\varphi \in C_b[0,\infty)$, the operators $\mathcal{W}_{s,q}$ satisfy the relation*

$$\left|\mathcal{W}_{s,q}(\varphi;y) - \varphi(y)\right| \leq C\left[\omega_2\left\{\varphi; \frac{1}{2}\sqrt{\Theta_{s,q}^*(y)}\right\} + \min\left\{1; \frac{1}{4}\left(\Theta_{s,q}^*(y)\right)\right\}\|\varphi\|_{C_b[0,\infty)}\right]$$

$$+ \quad \omega_{\chi}\Big(\varphi; \Big|\Delta^*_{s,q}(y)\Big|\Big),$$

where $\delta^*_{s,q}(y)$ *is obtained in Theorem 4.2 and* $\Theta^*_{s,q}(y) = \delta^*_{s,q}(y) + \left(\frac{[s]}{q([s]+\mu)[s-1]}\left(1 + q[s]y + \frac{qR'_q(1)}{R_q(1)}\right) + \frac{\nu}{[s]+\mu}\right)^2$ *and* $\Delta^*_{s,q}(y) = \frac{[s]}{q([s]+\mu)[s-1]}\left(1 + q[s]y + \frac{qR'_q(1)}{R_q(1)}\right) + \frac{\nu}{[s]+\mu} - y.$

Proof. In order to prove the claimed thesis, we use the results from Theorem 5.5. Thus, take $\phi \in C^2_b[0,\infty)$. Hence for all $\varphi \in C_b[0,\infty)$, we deduce that

$$\begin{aligned}
|\mathcal{W}_{s,q}(\varphi;y) - \varphi(y)| &= \Big|\mathcal{X}_{s,q}(\varphi;y) - \varphi(y) + \varphi\Big(\mathcal{W}_{s,q}(t;y)\Big) - \varphi(y)\Big| \\
&\leq \Big|\mathcal{X}_{s,q}(\varphi - \phi;y)\Big| + \Big|\mathcal{X}_{s,q}(\phi;y) - \phi(y)\Big| \\
&+ \Big|\phi(y) - \varphi(y)\Big| + \Big|\varphi\Big(\mathcal{W}_{s,q}(t;y)\Big) - \varphi(y)\Big| \\
&\leq 4\parallel \varphi - \phi \parallel + \omega_{\chi}\Big(\varphi; \Big|\mathcal{W}_{s,q}(t;y) - y\Big|\Big) \\
&+ \left\{\delta^*_{s,q}(y) + \Big(\mathcal{W}_{s,q}(t;y)\Big)^2\right\} \parallel \phi'' \parallel .
\end{aligned}$$

Applying the infimum for all $\phi \in C^2_b[0,\infty)$, from equality (18), we conclude that

$$\begin{aligned}
\Big|\mathcal{W}_{s,q}(\varphi;y) - \varphi(y)\Big| &\leq 4K_{\varphi}\left[\varphi; \frac{1}{4}\left\{\delta^*_{s,q}(y) + \Big(\mathcal{W}_{s,q}(t;y)\Big)^2\right\}\right] \\
&+ \omega_{\chi}\Big(\varphi; \Big|\mathcal{W}_{s,q}(t;y) - y\Big|\Big) \\
&\leq C\left[\omega_2\left\{\varphi; \frac{1}{2}\sqrt{\delta^*_{s,q}(y) + \Big(\mathcal{W}_{s,q}(t;y)\Big)^2}\right\}\right. \\
&+ \left.\min\left\{1; \frac{1}{4}\Big(\delta^*_{s,q}(y) + \Big(\mathcal{W}_{s,q}(t;y)\Big)^2\Big)\right\}\|\varphi\|_{C_b[0,\infty)}\right] \\
&+ \omega_{\chi}\Big(\varphi; \Big|\mathcal{W}_{s,q}(t;y) - y\Big|\Big).
\end{aligned}$$

In this way the theorem is proven. □

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