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Analytical and numerical discussion for the quadratic integral equations

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Abstract. In this work, we study the existence of at least one solution of the Quadratic integral equation with Phase–lag term. Our proof depends on a suitable combination of the Darbo's fixed point principle and the technique of measures of noncompactness. Homotopy perturbation method is presented to obtain an approximate solution of Quadratic integral equation with Phase–lag term. Convergence and error estimate of Homotopy perturbation method are obtained. Homotopy perturbation method is a powerful device for solving a wide variety of problems. It gives excellent flexibility to the expression of the solution and how the solution is explicitly obtained, and provides great freedom in choosing the base functions of the desired solution and the corresponding auxiliary linear operator of homotopy. These methods produce the solutions in terms of convergent series without needing to restrictive assumptions, to illustrate the ability and credibility of the methods, we deal with two examples that show simplicity and effectiveness.

1. Introduction

Phase–lag has a very important role in our applied science and there are currently one, dual and three phases and each phase has a different applications. For example the three–Phase–lag model incorporates the microstructural interaction effect in the fast–transient process of heat transport. It describes the finite time required for the various microstructural interactions to take place, including the phonon–electron interaction in metals, the phonon scattering in dielectric crystals, insulators, and semiconductors, and the activation of molecules at extremely low temperature, by the resulting Phase–lag (time delay) in the process of heat transport see [12]. Integral equations with Phase–lag term are the mathematical model of many evolutionary in problems chemistry, engineering, quantum mechanics, biology, optimal control systems, mathematical physics and so on. For example, integral equations for the dual lag model of heat transfer.

The technique of special measures of noncompactness defined in [13] is often used in many branches of nonlinear analysis. Especially, that technique seems to be a really useful tool within the existence theory for many kinds of integral equations [9]. The aim of this paper is to use the technique of special measures of noncompactness and fixed point theorem of Darbo type to prove the existence theorem for a nonlinear integral equation in class $C[0,1]$ and nondecreasing on the interval $[0,1]$. The results conferred during

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this paper appear to be new and original. The result obtained within the paper generalizes a lot of results obtained earlier in many papers, see [10, 23].

We consider the Quadratic integral equation with Phase–lag term (in short QIEPLT),

$$\varphi(t+q) = g(t) + \varphi(t+q)\int_0^1 k(t,\tau)f(\tau,\varphi(\tau))\mathrm{d}\tau; \quad (q << 1), \tag{1}$$

where q is the Phase–lag constant, the function $\varphi(t)$ is unknown in the Banach space and continuous with their derivative with respect to time, where $[0,1]$ is the domain of integration with respect to the time $t \in [0,1]$. The kernel $k(t,\tau)$ is positive and continuous in $C[0,1]$ and the functions $g(t)$, $f(t)$ are continuous its derivatives with respect to time.
Using Taylor expansion after neglecting the second derivative in Eq. (1), we get

$$\varphi(t) + q\frac{\mathrm{d}\varphi}{\mathrm{d}t} = g(t) + (\varphi(t) + q\frac{\mathrm{d}\varphi}{\mathrm{d}t})\int_0^1 k(t,\tau)f(\tau,\varphi(\tau))\mathrm{d}\tau; \quad (q << 1), \tag{2}$$

with initial condition

$$\varphi(0) = \varphi_0. \tag{3}$$

Equation (2) with initial condition (3) is called integro–differential equation. The integro–differential equation (IDEs) is a kind of functional equation that has associate integral and derivatives of an unknown function. These equations were named after the leading mathematicians who have first studied them, such as Fredholm, Volterra. Fredholm and Volterra equations are the most encountered types.
Integrating Eq. (2) twice and using initial condition (3), we get

$$\begin{aligned} \varphi(t) &= a(t) - \frac{1}{q}\int_0^t \varphi(\tau)\mathrm{d}\tau + \frac{1}{q}\int_0^t (\varphi(\tau) + q\frac{\mathrm{d}\varphi}{\mathrm{d}\tau})\int_0^1 k(\tau,s)f(s,\varphi(s))\mathrm{d}s\mathrm{d}\tau \\ a(t) &= \varphi_0 + \frac{1}{q}\int_0^t g(\tau)\mathrm{d}\tau, \end{aligned} \tag{4}$$

integration by parts and applying the Leibniz's rule, we obtain

$$\begin{aligned} \varphi(t) = a(t) &+ \frac{1}{q}\int_0^t \varphi(\tau)\mathrm{d}\tau\left(\int_0^1 k(t,s)f(s,\varphi(s))\mathrm{d}s - 1\right) - \frac{1}{q}\int_0^t \varphi(\tau)\mathrm{d}\tau\int_0^t\int_0^1 k_\tau(\tau,s)f(s,\varphi(s))\mathrm{d}s\mathrm{d}\tau \\ &+ (\varphi(t) - \varphi_0)\left(\int_0^1 k(t,s)f(s,\varphi(s))\mathrm{d}s - \int_0^t\int_0^1 k_\tau(\tau,s)f(s,\varphi(s))\mathrm{d}s\mathrm{d}\tau\right). \end{aligned} \tag{5}$$

Equation (5) is called the Quadratic integral equation with Phase–lag term in time. Here we presented the Quadratic integral equation with Phase-lag term (1) that is difficult to work on analytically and numerically, but after mathematical attempts, it was converted into another form that is easy to deal with analytically and numerically and resulted in new ideas and modern discussions in the physical sciences and biology. We use a numerical method to obtain the solution of Eq. (5), where the existence of at least one solution of Eq. (5) can be discussed and proved using a combination of Darbo's fixed point principle and the technique of measures of noncompactness.

Beside the introduction, the distribution of the paper is as under: we give the preliminaries and auxiliary results about the measure of noncompactness and fixed point theorem needed in the subsequent part of the paper. In Sec. **3**, we give theorems on the existence of nondecreasing continuous solutions of a Quadratic integral equation of Volterra type (5). In Sec. **4**, we provide two examples concerning theorems on the existence of solutions of a Quadratic integral equation. Sec. **5** is assigned to a Homotopy perturbation method and convergence of the method. In Sec. **6**, Quadratic integral equations with Phase–lag term have been solved by the proposed method. To illustrate and show the efficiency of the method two examples are presented in Sec. **7**, and Sec. **8** includes the conclusion of the paper.

2. Notation and auxiliary results

In this section, we have a tendency to introduce some notations and preliminary facts that are used in the analytical part of the paper.

Definition 2.1. *A function $\mu : M_E \to R^+ = [0, +\infty)$ is said to be the measure of noncompactness in E, if it satisfies the following conditions:*

(1*) *The family ker $\mu = \{X \in M_E : \mu(X) = 0\}$ is nonempty and ker $\mu \subset N_E$;*

(2*) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$;

(3*) $\mu(\bar{X}) = \mu(Conv\ X) = \mu(X)$;

(4*) $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ *for* $0 \leq \lambda \leq 1$;

(5*) *If $\{X_n\}$ is a sequence of closed sets from M_E such that $X_{n+1} \subset X_n$ for $n = 1, 2, ...,$ and if $lim_{n\to\infty}\mu\{X_n\} = 0$, then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.*

For our purposes, we will only need the following fixed–point theorem of Darbo type. Let us suppose that M is a nonempty subset of a Banach space E and the operator $H : M \to E$ is continuous and transforms bounded sets onto bounded ones. We say that H satisfies the Darbo condition (with constant $\alpha \geq 0$) with respect to a measure of noncompactness μ if for any bounded subset X of M, we have

$$\mu(TX) \leq \alpha\mu(X).$$

If T satisfies the Darbo condition with $\alpha < 1$, then it is called a contraction with respect to μ.

Theorem 2.2. *(Fixed point theorems)*
Fixed point theorems have always a major role in various fields, specially, in fields of differential, integral and functional equations. Fixed point theorems constitute a topological tool for the qualitative investigations of solution of linear and nonlinear equations. The theory of fixed point is concerned with the conditions which guarantee that a map $T : X \to X$ of a topological space X into itself admits one or more fixed points, that is, points x of X for which $x = Tx$. Here we give a brief history of fixed point theorems, see [24].

Theorem 2.3. *Let Q be a nonempty, bounded, closed and convex subset of the Banach space E and μ a measure of noncompactness in E. Let $H : Q \to Q$ be a contraction with respect to μ. Then H has at least one fixed point in the set Q.*

Let us fix a nonempty and bounded subset X of $C(I)$. For $x \in X$ and $\varepsilon \geq 0$ denoted by $\omega(x, \varepsilon)$, the modulus of continuity of x defined by

$$\omega(x, \varepsilon) = \sup\{|x(t) - x(s)| : t, s \in I, |t - s| \leq \varepsilon\}.$$

Further, let us put

$$\omega(X, \varepsilon) = \sup\{\omega(x, \varepsilon) : x \in X\}$$
$$\omega_0(X) = \sup_{\varepsilon\to 0}\{\omega(X, \varepsilon)\}.$$

Moreover, let us define the following quantities,

$$d(x) = \sup\{|x(s) - x(t)| - [x(s) - x(t)] : t, s \in I, t \leq s\}$$
$$d(X) = \sup\{d(x) : x \in X\}.$$

The quantity $d(X)$ measures the degree of decrease of functions from the set X.
Finally, let us define the function μ on the family $M_{C(I)}$ by the formula

$$\mu(X) = \omega_0(X) + d(X).$$

The function μ is a measure of noncompactness in the space $C[0, 1]$.

3. Existence of the solution

Equation (5) can be written in the following integral operator form:

$$\begin{aligned}(H\varphi)(t) =& a(t) + \frac{1}{q}(V\varphi)(t)\\ (V\varphi)(t) =& (T\varphi)(t)((KF)(t) - 1) + (T\varphi)(t)(K_\tau F)(t) + q(\varphi(t) - \varphi_0)((KF)(t) - (K_\tau F)(t)),\end{aligned} \tag{6}$$

where

$$(T\varphi)(t) = \int_0^t \varphi(\tau)\mathrm{d}\tau$$

$$(KF)(t) = \int_0^1 k(t,s)f(s,\varphi(s))\mathrm{d}s$$

$$(K_\tau F)(t) = -\int_0^t \int_0^1 \frac{\partial}{\partial\tau}k(\tau,s)f(s,\varphi(s))\mathrm{d}s\mathrm{d}\tau.$$

In order to discuss the existence of at least one solution of Eq. (5), we assume the following assumptions:

(i) $a : I \to R$ is a continuous, nondecreasing and nonnegative function on I.

(ii) $k : I \times I \to R$ is continuous and the functions $s \to k(t,s)$ and $t \to k(t,s)$ are nondecreasing on I for fixed $t \in I$ and $s \in I$, respectively. Also, $k_\tau = k_\tau(\tau,s) : [0,1] \times [0,1] \to R$ is continuous with respect to its both variables τ and s such that $|-k_\tau(\tau,s)| < c$, $\forall \tau \in I$, where c is a positive constant.

(iii) The operator $T : C(I) \to C(I)$ is continuous and satisfies the Darbo condition with a constant α for the measure of noncompactness μ, such that $|(T\varphi)(t)| \le \|\varphi\|$.

(iv) The function $f : I \times I \to R$ satisfies the conditions $(a) - (d)$, and there exists a nondecreasing function $m : R_+ \to R_+$, such that $|f(s,\varphi(s))| \le m(|\varphi|)$.

(v) The unknown function $\varphi(t)$ satisfies $|\varphi(t) - \varphi_0| \le \|\varphi\|$ in the space $C(I)$.

(vi) The inequality

$$\|a\| + rm(r)(k^* + c)(1 + \frac{1}{q}) \le r,$$

where $k^* = \max\{k(t,s) : t,s \in I\}$ and $\|a\|$ denotes the norm of the function $a(t)$ in the space $C(I)$. Moreover, $m(r_0)(k^* + c)(q + 1) < q$.

Now we can formulate the main existence theorem

Theorem 3.1. *Under the assumptions $(i) - (vi)$ Eq. (5) has at least one solution $\varphi = \varphi(t)$ which belongs to the space $C(I)$ and is nondecreasing and nonnegative on the interval I.*

Proof. Let us consider the operator H defined on the space $C(I)$ by the formula (6).

Taking into account assumptions $(i) - (v)$ and the properties of the superposition operator, we infer that the function $H\varphi$ is continuous on I for any function $\varphi \in C(I)$, i.e., the operator H transforms the space $C(I)$ into itself. Further, applying our assumptions, we derive the following estimate:

$$\begin{aligned}|(H\varphi)(t)| \le& |a(t)| + \frac{1}{q}|(T\varphi)(t)| \int_0^1 |k(t,s)||f(s,\varphi(s))|\mathrm{d}s + \frac{1}{q}|(T\varphi)(t)| \int_0^t \int_0^1 |-k_\tau(\tau,s)||f(s,\varphi(s))|\mathrm{d}s\mathrm{d}\tau\\ &+ |\varphi(t) - \varphi_0| \left(\int_0^1 |k(t,s)||f(s,\varphi(s))|\mathrm{d}s + \int_0^t \int_0^1 |-k_\tau(\tau,s)||f(s,\varphi(s))|\mathrm{d}s\mathrm{d}\tau \right)\\ \le& \|a\| + \frac{1}{q}\|\varphi\|k^* m(|\varphi|) + \frac{1}{q}c\|\varphi\|m(|\varphi|) + \|\varphi\|m(|\varphi|)(k^* + c).\end{aligned}$$

Hence, we obtain

$$|(H\varphi)(t)| \leq \|a\| + \|\varphi\| m(|\varphi|)(k^* + c)(1 + \frac{1}{q}).$$

From the above estimate and assumption (vi), we infer that there exists $r_0 > 0$ such that the operator H transforms the ball B_{r_0} into itself.

In what follows, we will consider the operator H on the subset $B^+_{r_0}$ of the ball B_{r_0} defined by:

$$B^+_{r_0} = \{\varphi \in B_{r_0} : \varphi(t) \geq 0, \quad \text{for} \quad t \in I\}.$$

Notice that the set $B^+_{r_0}$ is nonempty, bounded, closed, and convex. Hence and in view of assumptions $(i) - (v)$, we deduce easily that H transforms the set $B^+_{r_0}$ into itself.

Now, we show that H is continuous on the set $B^+_{r_0}$. To do this let us fix $\varepsilon > 0$ and choose $\delta > 0$ according to the continuity of H. Further, take arbitrarily $\varphi, \psi \in B^+_{r_0}$ such that $\|\varphi - \psi\| \leq \delta$. Then, for $t \in I$, we derive the following estimates:

$$\begin{aligned}
&|(H\varphi)(t) - (H\psi)(t)| \\
&\quad \leq \frac{1}{q}\|\varphi\| k^* l \int_0^1 \|\varphi(s) - \psi(s)\| \mathrm{d}s + \frac{1}{q}\|\varphi - \psi\| k^* \int_0^1 m(|\psi|)\mathrm{d}s \\
&\quad + \frac{1}{q}\|\varphi\| cl \int_0^t \int_0^1 \|\varphi(s) - \psi(s)\| \mathrm{d}s\mathrm{d}\tau + \frac{1}{q}\|\varphi - \psi\| c \int_0^t \int_0^1 m(|\psi|)\mathrm{d}s\mathrm{d}\tau \\
&\quad + \|\varphi\| \left| k^* l \int_0^1 \|\varphi(s) - \psi(s)\| \mathrm{d}s + cl \int_0^t \int_0^1 \|\varphi(s) - \psi(s)\| \mathrm{d}s\mathrm{d}\tau \right| \\
&\quad + \|\varphi - \psi\| \left| k^* \int_0^1 m(|\psi|)\mathrm{d}s + c \int_0^t \int_0^1 m(|\psi|)\mathrm{d}s\mathrm{d}\tau \right| \\
&\leq \frac{1}{q}(k^* l\|\varphi\| + k^* m(|\psi|) + c\, l\|\varphi\| + cm(|\psi|) + q\|\varphi\|(k^* l + c\, l) + qk^* m(|\psi|) + qcm(|\psi|))\|\varphi - \psi\| \\
&\leq \frac{1}{q}(1 + q)(k^* lr_0 + k^* m(r_0) + c\, lr_0 + cm(r_0))\delta.
\end{aligned}$$

The above estimate allows us to deduce that the operator H is continuous on the set $B^+_{r_0}$.

Now, let us take a nonempty set Φ, such that $\Phi \in B^+_{r_0}$. Further, fix arbitrarily $\varepsilon > 0$ and choose $\varphi \in \Phi$ and $t_1, t_2 \in I$ such that $|t_2 - t_1| \leq \varepsilon$. Then, keeping in mind our assumptions, we obtain

$$\begin{aligned}
&|(H\varphi)(t_2) - (H\varphi)(t_1)| \\
&\quad \leq |a(t_2) - a(t_1)| + \frac{1}{q}\left|(T\varphi)(t_2) - (T\varphi)(t_1)\right| \int_0^1 |k(t,s)||f(s,\varphi(s))|\mathrm{d}s \\
&\quad + \frac{1}{q}|(T\varphi)(t_2) - (T\varphi)(t_1)| \int_0^{t_1} \int_0^1 |-k_\tau(\tau,s)||f(s,\varphi(s))|\mathrm{d}s\mathrm{d}\tau \\
&\quad + \frac{1}{q}\left|(T\varphi)(t_2)\right| \int_{t_1}^{t_2} \int_0^1 |-k_\tau(\tau,s)||f(s,\psi(s))|\mathrm{d}s\mathrm{d}\tau \\
&\quad + |(\varphi)(t_2) - \varphi_0| \int_{t_1}^{t_2} \int_0^1 |-k_\tau(\tau,s)||f(s,\varphi(s))|\mathrm{d}s\mathrm{d}\tau \\
&\quad + |(\varphi)(t_2) - (\varphi)(t_1)| \left| \int_0^1 |k(t,s)||f(s,\varphi(s))|\mathrm{d}s + \int_0^{t_1} \int_0^1 |-k_\tau(\tau,s)||f(s,\varphi(s))|\mathrm{d}s\mathrm{d}\tau \right|
\end{aligned}$$

$$
\begin{aligned}
&|(H\varphi)(t_2)-(H\varphi)(t_1)| \\
&\quad \leq \omega(a,\varepsilon)+\frac{1}{q}\omega(T\varphi,\varepsilon)k^{*}m(r_0)+\frac{1}{q}(t_2-t_1)cm(r_0)r_0+\frac{1}{q}\omega(T\varphi,\varepsilon)cm(r_0) \\
&\qquad +(t_2-t_1)cm(r_0)r_0+\omega(\varphi,\varepsilon)(k^{*}m(r_0)+cm(r_0)) \\
&\quad \leq \omega(a,\varepsilon)+\frac{1}{q}(k^{*}m(r_0)+cm(r_0))\omega(T\varphi,\varepsilon)+cm(r_0)r_0\varepsilon(1+\frac{1}{q})+(k^{*}m(r_0)+cm(r_0))\omega(\varphi,\varepsilon).
\end{aligned}
$$

Hence, keeping in mind our assumptions and the above-established facts, we arrive at the following inequality

$$
\omega_0(H\Phi) \leq \frac{1}{q}m(r_0)(k^{*}+c)\omega_0(T\Phi)+m(r_0)(k^{*}+c)\omega_0(\phi). \tag{7}
$$

In what follows, fix arbitrary $\varphi \in \Phi$ and $t_1, t_2 \in I$ with $t_2 \geq t_1$. Then, taking into account our assumptions, we have

$$
\begin{aligned}
&|(H\varphi)(t_2)-(H\varphi)(t_1)|-[(H\varphi)(t_2)-(H\varphi)(t_1)] \\
&\leq \frac{1}{q}\{|(T\varphi)(t_2)-(T\varphi)(t_1)|-[(T\varphi)(t_2)-(T\varphi)(t_1)]\}\left|\int_0^1 k(t,s)f(s,\varphi(s))ds\right| \\
&\quad +\frac{1}{q}\{|(T\varphi)(t_2)-(T\varphi)(t_1)|-[(T\varphi)(t_2)-(T\varphi)(t_1)]\}\left|\int_0^{t_1}\int_0^1 |-k_\tau(\tau,s)|f(s,\varphi(s))dsd\tau\right| \\
&\quad +\frac{1}{q}\{|(T\varphi)(t_2)|-[(T\varphi)(t_2)]\}\int_{t_1}^{t_2}\int_0^1 |-k_\tau(\tau,s)|f(s,\psi(s))dsd\tau \\
&\quad +\{|(\varphi)(t_2)-\varphi_0|-[(\varphi)(t_2)-\varphi_0]\}\left|\int_{t_1}^{t_2}\int_0^1 |-k_\tau(\tau,s)|f(s,\varphi(s))dsd\tau\right| \\
&\quad +\{|(\varphi)(t_2)-(\varphi)(t_1)|-[(\varphi)(t_2)-(\varphi)(t_1)]\}\,|\int_0^1 k(t,s)f(s,\varphi(s))ds \\
&\qquad\qquad +\int_0^{t_1}\int_0^1 |-k_\tau(\tau,s)|f(s,\varphi(s))dsd\tau\Big| \\
&\leq \frac{1}{q}m(r_0)(k^{*}+c)d(T\varphi)+m(r_0)(k^{*}+c)d(\varphi).
\end{aligned}
$$

Hence, we get

$$
d(H\varphi) \leq \frac{1}{q}m(r_0)(k^{*}+c)d(T\varphi)+m(r_0)(k^{*}+c)d(\varphi),
$$

and consequently,

$$
d(H\Phi) \leq \frac{1}{q}m(r_0)(k^{*}+c)d(T\Phi)+m(r_0)(k^{*}+c)d(\phi). \tag{8}
$$

Finally, from Eqs. (7) and (8) give us that

$$
\omega_0(H\Phi)+d(H\Phi) \leq \frac{1}{q}m(r_0)(k^{*}+c)(\omega_0(T\Phi)+d(T\Phi))+m(r_0)(k^{*}+c)(\omega_0(\Phi)+d(\Phi)).
$$

Following in mind the concepts of the measure of noncompactness μ in Sec. **2** , we get

$$
\mu(H\Phi) \leq \frac{1}{q}m(r_0)(k^{*}+c)\mu(T\Phi)+m(r_0)(k^{*}+c)\mu(\Phi).
$$

From assumption (iii), we obtain

$$\mu(H\Phi) \leq m(r_0)(k^* + c)(1 + \frac{1}{q})\mu(\Phi).$$

Now, keeping in mind the above inequality and the fact that $m(r_0)(k^* + c)(q + 1) < q$, in view of Theorem 3.1, then Eq. (5) has at least one solution $\varphi \in C(I)$. This completes the proof. □

4. Examples

In this section, we provide two examples concerning Theorem 3.1 and connected mainly with assumptions $(i) - (vi)$ of this theorem.

Example 4.1. *Consider the following Quadratic integral equation with Phase–lag term:*

$$\varphi(t + 0.01) = \frac{1}{20}t^{10} + \varphi(t + 0.01)\int_0^1 \frac{t + \tau}{51 + 9e^{2\tau}} \frac{\varphi(\tau)}{1 + 15e^{1+\varphi(\tau)}} d\tau; \quad (\varphi(0) = 0). \tag{9}$$

Using numerical treatment of the equation (9), we obtained

$$\begin{aligned}\varphi(t) = {} & \frac{5}{11}t^{11} + \frac{1}{0.01}\int_0^t \varphi(\tau)d\tau\left(\int_0^1 \frac{t + s}{51 + 9e^{2s}} \frac{\varphi(s)}{1 + 15e^{1+\varphi(s)}} ds - 1\right) \\ & - \frac{1}{0.01}\int_0^t \varphi(\tau)d\tau \int_0^t \int_0^1 \frac{1}{51 + 9e^{2s}} \frac{\varphi(s)}{1 + 15e^{1+\varphi(s)}} dsd\tau \\ & + \varphi(t)\left(\int_0^1 \frac{t + s}{51 + 9e^{2s}} \frac{\varphi(s)}{1 + 15e^{1+\varphi(s)}} ds - \int_0^t \int_0^1 \frac{1}{51 + 9e^{2s}} \frac{\varphi(s)}{1 + 15e^{1+\varphi(s)}} dsd\tau\right); \quad (t \in I = [0, 1]).\end{aligned}$$

In this example, comparing with equation (5), we have $a(t) = 5t^{11}/11$ *which is nonnegative and continuous on I with norm* $\|a(t)\| = \max_{t\in I} |5t^{11}/11| = 5/11$, *the operator T is defined as* $(T\varphi)(t) = \int_0^t \varphi(\tau)d\tau$ *which is also nonnegative and continuous with norm* $\|T\| = 1$, *the kernel* $k(t, \tau) = (t + \tau)/(51 + 9e^{2\tau})$, *which is continuous with respect to t and τ. Also, we have* $|k(t, \tau)| = \left|t + \tau/(51 + 9e^{2\tau})\right| \leq 1/4e^2$, $(k^* = 1/4e^2)$ *and* $k_\tau(\tau, s) = 1/(51 + 9e^{2s})$, *where* $|k_\tau(\tau, s)| = \left|1/(51 + 9e^{2s})\right| \leq 1/4e^2$, $(c = 1/4e^2)$, *the function* $f(\tau, \varphi(\tau)) = \varphi(\tau)/(1 + 15e^{1+\varphi(\tau)})$, *which satisfies the assumption (iv) with*
$|f(\tau, \varphi(\tau))| = |\varphi(\tau)/(1 + 15e^{1+\varphi(\tau)})| \leq |\varphi(\tau)|/15$. *Thus, we get* $m(r) = r/15$.
Further, let us consider the inequality

$$\frac{5}{11} + \frac{101}{30e^2}r^2 \leq r, \tag{10}$$

or equivalently,

$$\frac{30e^2}{101}r - r^2 \geq \frac{150e^2}{1111}. \tag{11}$$

Using the standard methods we can verify that the function $v(r) = (30e^2/101)r - r^2$ *attains its maximum at the point* $r_0 = 0.7$ *and* $v(r_0) = (30e^2/101)(0.7) - (0.7)^2 \geq 150e^2/1111$. *So, the number* r_0 *is a positive solution of the inequality (11) for which* $m(|r_0|)(k^* + c)(1.01) = 3 \times 10^{-3} < 0.01$.
Finally, taking into account all the above-established facts and Theorem 3.1, we conclude that the equation (9) has at least one solution $\varphi = \varphi(t)$ *defined, continuous and nondecreasing on the interval I. Moreover,* $\|\varphi\| \leq r_0 = 0.7$.

Example 4.2. *Consider the following QIE with Phase–lag term:*

$$\varphi(t + 0.1) = t^{25} + \varphi(t + 0.1)\int_0^1 e^{t+\tau} \frac{\tau\varphi(\tau)}{8 + 7e^{45\varphi(\tau)}} d\tau; \quad (\varphi(0) = 0.1). \tag{12}$$

Using numerical treatment of the equation (12), we obtained

$$\begin{aligned}\varphi(t) &= 0.1 + \frac{1}{0.1}\frac{t^{26}}{26} + \frac{1}{0.1}\int_0^t \varphi(\tau)\mathrm{d}\tau\left(\int_0^1 e^{t+s}\frac{s\varphi(s)}{8+7e^{45\varphi(s)}}\mathrm{d}s - 1\right)\\ &- \frac{1}{0.1}\int_0^t \varphi(\tau)\mathrm{d}\tau\int_0^t\int_0^1 e^{\tau+s}\frac{s\varphi(s)}{8+7e^{45\varphi(s)}}\mathrm{d}s\mathrm{d}\tau\\ &+ (\varphi(t)-0.1)\left(\int_0^1 e^{t+s}\frac{s\varphi(s)}{8+7e^{45\varphi(s)}}\mathrm{d}s - \int_0^t\int_0^1 e^{\tau+s}\frac{s\varphi(s)}{8+7e^{45\varphi(s)}}\mathrm{d}s\mathrm{d}\tau\right).\end{aligned}$$

In this example, comparing with equation (5), we have $\|a(t)\| = 63/130$, *the kernel* $k(t,\tau) = e^{t+\tau}$, *which is continuous with respect to* t *and* τ. *Also, we have* $|k(t,\tau)| = \left|e^{t+\tau}\right| \le e^2$, $(k^* = e^2)$ *and* $k_\tau(\tau,s) = e^{\tau+s}$, *where* $|k_\tau(\tau,s)| = |e^{\tau+s}| \le e^2$, $(c = e^2)$, *the function* $f(\tau,\varphi(\tau)) = \tau\varphi(\tau)/(8+7e^{45\varphi(\tau)})$, *which satisfies the assumption (iv) with* $|f(\tau,\varphi(\tau))| = |\tau\varphi(\tau)/(8+7e^{45\varphi(\tau)})| \le |\varphi(\tau)|/e^6$. *Thus, we get* $m(r) = r/e^6$.

Further, let us consider the inequality

$$\frac{63}{130} + \frac{22}{e^4}r^2 \le r, \tag{13}$$

or equivalently,

$$\frac{e^4}{22}r - r^2 \ge \frac{63e^4}{2860}. \tag{14}$$

Using the standard methods we can verify that the function $v(r_0) = (e^4/22)(1.5) - (1.5)^2 \ge (63e^4/2860)$. *So, the number* r_0 *is a positive solution of the inequality (14) for which* $m(|r_0|)(k^* + c)(1.1) = 0.06044 < 0.1$.

Finally, taking into account all the above-established facts and Theorem 3.1, we conclude that the equation (12) has at least one solution $\varphi = \varphi(t)$ *defined, continuous and nondecreasing on the interval I. Moreover,* $\|\varphi\| \le r_0 = 1.5$.

5. Homotopy perturbation method and convergence of this method

Homotopy perturbation method (HPM) is an effective solution method for a wide class of problems [20]. It has been applied, for example, solving the ordinary differential equations and partial differential equations [31, 32]. Homotopy perturbation method has also been used for finding the exact and approximate solutions of the linear and nonlinear integral equations and their systems, see [5, 15], Quadratic integral equation [16], two–dimensional integral equations [4], as well as the integro–differential equations [4, 30].

The method has been used by many authors to handle a wide variety of scientific and engineering applications to solve various functional equations. In this method, the solution is considered as the sum of an infinite series, which converges rapidly to accurate solutions. Considerable research work has recently been conducted in applying this method to a class of linear and non–linear equations.

The convergence of the Homotopy perturbation method with so–called convex homotopy, for Fredholm and Volterra integral equations of the second kind, is discussed in the papers [17, 19]. The authors of the paper [11] prove the convergence and give the estimation of the errors for the piecewise Homotopy perturbation method used for solving the weakly singular Volterra integral equations. The convergence conditions for the Homotopy perturbation method for the Volterra–Fredholm integral equations of the second kind are formulated and proved in paper [18].

In this section, the application of the Homotopy perturbation method for solving the Quadratic integral equation with Phase–lag term of the second kind is presented. However, the main objective of this work is to give a condition of convergence for this method.

Homotopy perturbation method arose as a combination of two other methods: the homotopy technique from topology and the perturbation method. Inventor of this method is Chinese mathematician Ji–Huan He [21, 22]. Homotopy perturbation method enables to seek a solution of the following operator equation:

$$A(\varphi) = f(z), \qquad z \in \Omega, \tag{15}$$

where A denotes an operator, $f(z)$ is a known analytical function and φ is an unknown function. Operator A is presented in the form of sum:

$$A(\varphi) = L(\varphi) + N(\varphi), \tag{16}$$

where L defines the linear operator and N denotes the remaining part of operator A. Certainly, N can be a nonlinear operator as well. Thus, Eq. (15) can be written in the form:

$$L(\varphi) + N(\varphi) = f(z), \qquad z \in \Omega, \tag{17}$$

Let us define a new operator H, called the homotopy operator, in the following way

$$H(v, p) := (1 - p)(L(v) - L(\varphi_0)) + p(A(v) - f(z)), \tag{18}$$

where $p \in [0, 1]$ is the so-called homotopy parameter, $v(z, p) : \Omega \times [0, 1] \to R$, and φ_0 defines the initial approximation of a solution of Eq. (15). By using relation (16), we receive

$$H(v, p) = L(v) - L(\varphi_0) + pL(\varphi_0) + p(N(v) - f(z)). \tag{19}$$

Since $H(v, 0) = L(v) - L(\varphi_0)$, therefore for $p = 0$, the solution of operator equation $H(v, 0) = 0$ is equivalent to solution of a trivial problem $L(v) - L(\varphi_0) = 0$. Whereas, for $p = 1$, the solution of operator equation $H(v, 1) = 0$ is equivalent to the solution of Eq. (15). In this way, changing of parameter p from zero to one corresponds with the changing from the trivial equation $L(v) - L(\varphi_0) = 0$ to the given equation (which means, the changing of solution v from φ_0 to φ).

In next step of the method, we search for the solution of equation $H(v, p) = 0$ in the form of power series:

$$v = \sum_{j=0}^{\infty} p^j v_j. \tag{20}$$

If the above series possess the radius of convergence not smaller than one and the series $\sum_{j=0}^{\infty} v_j$, is absolutely convergent, then by Abel's Theorem, we obtain the solution of Eq. (15):

$$\varphi = \lim_{p \to 1^-} v = \sum_{j=0}^{\infty} v_j. \tag{21}$$

Sufficient conditions for the convergence of series (20), in case of the partial differential equations and systems of those, are given in papers [6, 8]. In many cases the series (20) is fast convergent, therefore by taking the partial sum of this series we obtain a very good approximation of the sought solution. If we take the first $n + 1$ components, we receive the so–called n–order approximate solution

$$\hat{\varphi}_n = \sum_{j=0}^{n} v_j. \tag{22}$$

In order to find the function v_j, we substitute relation (20) into equation $H(v, p) = 0$ and we compare expressions with the same powers of parameter p. In this way we obtain the sequence of operator equations which enables to determine the successive functions v_j. Thus, finding the solution of considered problem can be reduced to solving the sequence of problems, solutions of which are easy to find.

5.1. Convergence

Let us write Eq. (19) in the following form:

$$L(v) = L(\varphi_0) + p[f(z) - N(v) - L(\varphi_0)]. \tag{23}$$

Applying the inverse operator, L^{-1} to both sides of Eq. (23), we obtain

$$v = \varphi_0 + p[L^{-1}f(z) - L^{-1}N(v) - \varphi_0], \tag{24}$$

substituting (20) into the right–hand side of Eq. (24), we have Eq. (24) in the following form

$$v = \varphi_0 + p\left[L^{-1}f(z) - (L^{-1}N)\left(\sum_{j=0}^{\infty} p^j v_j\right) - \varphi_0\right]. \tag{25}$$

If $p \to 1^-$, the exact solution may be obtained by using

$$\begin{aligned}\varphi &= \lim_{p\to 1^-} v\\ &= L^{-1}f(z) - (L^{-1}N)\left(\sum_{j=0}^{\infty} v_j\right)\\ &= L^{-1}f(z) - \sum_{j=0}^{\infty}(L^{-1}N)(v_j).\end{aligned}$$

To study the convergence of the method let us state the following theorem.

Theorem 5.1. *(Sufficient Condition of Convergence). Suppose that X and Y are Banach spaces and $N : X \to Y$ is a contraction nonlinear mapping, that is*

$$\forall v, v^* \in X; \qquad \|N(v) - N(v^*)\|_Y \le \gamma\|v - v^*\|_X, \qquad 0 < \gamma < 1.$$

Which according to Banach's fixed point theorem, having the fixed point u, that is $N(\varphi) = \varphi$.

Assume that the sequence generated by Homotopy perturbation method can be written as

$$V_n = N(V_{n-1}), \qquad V_{n-1} = \sum_{j=0}^{n-1} v_j, \qquad n = 1, 2, 3\ldots,$$

and suppose that $V_0 = v_0 = \varphi_0 \in B_z(\varphi)$ where $B_z(\varphi) = \{\varphi^* \in X : \|\varphi - \varphi^*\| < z\}$, then we have the following statements:

(i) $\|V_n - \varphi\| \le \gamma^n\|v_0 - u\varphi\|$,

(ii) $V_n \in B_z(\varphi)$,

(iii) $\lim_{n\to\infty} V_n = \varphi$.

Proof. (i) By inductive approach, for $n = 1$, we have

$$\|V_1 - \varphi\| = \|N(V_0) - N(\varphi)\| \le \gamma\|v_0 - \varphi\|.$$

Assume that $\|V_{n-1} - \varphi\| = \|N(V_{n-2}) - N(\varphi)\| \le \gamma^{n-1}\|v_0 - \varphi\|$ as an induction hypothesis, then

$$\|V_n - \varphi\| = \|N(V_{n-1}) - N(\varphi)\| \le \gamma\|V_{n-1} - \varphi\| \le \gamma\gamma^{n-1}\|v_0 - \varphi\| = \gamma^n\|v_0 - \varphi\|.$$

(ii) Using statement (i), we have

$$\|V_n - \varphi\| \le \gamma^n\|v_0 - \varphi\| \le \gamma^n z < z \Rightarrow V_n \in B_z(\varphi).$$

(iii) Because of $\|V_n - \varphi\| \le \gamma^n\|v_0 - \varphi\|$, and $\lim_{n\to\infty}\gamma^n = 0$, we drive $\lim_{n\to\infty}\|V_n - \varphi\| = 0$, that is, $\lim_{n\to\infty} V_n = \varphi$. This completes the proof of theorem. □

6. Homotopy perturbation method for solving Quadratic integral equation with Phase–lag term

The numerical solution of integral equations has attracted researcher's attention to develop numerical method for approximating solution of these equations. Among these methods, we refer to the Degenerate kernel method [28], Separation of variables method [2, 29], and Iterative methods [1, 14, 25–27].
Equation (5) can be written in the following integral operator from:

$$\begin{aligned} L(v) =& v \\ N(v) =& -\frac{1}{q}\int_0^t v(\tau)\mathrm{d}\tau\left(\int_0^1 k(t,s)v(s)\mathrm{d}s-1\right)+\frac{1}{q}\int_0^t v(\tau)\mathrm{d}\tau\int_0^t\int_0^1 k_\tau(\tau,s)v(s)\mathrm{d}s\mathrm{d}\tau \\ &-(v(t)-v_0)\left(\int_0^1 k(t,s)v(s)\mathrm{d}s-\int_0^t\int_0^1 k_\tau(\tau,s)v(s)\mathrm{d}s\mathrm{d}\tau\right). \end{aligned} \tag{26}$$

By using the above definition and relation (19), we obtain the homotopy operator for the Quadratic integral equation with Phase–lag term in time

$$\begin{aligned} H(v,p) =& v(t)-\varphi_0(t) \\ &+p\left(\varphi_0(t)-\frac{1}{q}\int_0^t v(\tau)\mathrm{d}\tau\left(\int_0^1 k(t,s)v(s)\mathrm{d}s-1\right)+\frac{1}{q}\int_0^t v(\tau)\mathrm{d}\tau\int_0^t\int_0^1 k_\tau(\tau,s)v(s)\mathrm{d}s\mathrm{d}\tau\right. \\ &\left.-(v(t)-v_0)\left(\int_0^1 k(t,s)v(s)\mathrm{d}s-\int_0^t\int_0^1 k_\tau(\tau,s)v(s)\mathrm{d}s\mathrm{d}\tau\right)-a(t)\right). \end{aligned} \tag{27}$$

According to the method, in the next step we search for the solution of operator equation $H(v,p)=0$ in the form of power series

$$v(t)=\sum_{j=0}^{\infty}p^j v_j(t) \tag{28}$$

$$\begin{aligned} \sum_{j=0}^{\infty}p^j v_j(t) =& \varphi_0(t)+p(a(t)-\varphi_0(t))+\frac{1}{q}\int_0^t\sum_{j=1}^{\infty}p^j v_{j-1}(\tau)\mathrm{d}\tau\left(\int_0^1 k(t,s)\sum_{j=0}^{\infty}p^j v_j(s)\mathrm{d}s-1\right) \\ &-\frac{1}{q}\int_0^t\sum_{j=1}^{\infty}p^j v_{j-1}(\tau)\mathrm{d}\tau\int_0^t\int_0^1 k_\tau(\tau,s)\sum_{j=0}^{\infty}p^j v_j(s)\mathrm{d}s\mathrm{d}\tau+\left(\sum_{j=0}^{\infty}p^j v_j(t)-v_0\right) \\ &\times\left(\int_0^1 k(t,s)\sum_{j=1}^{\infty}p^j v_{j-1}(s)\mathrm{d}s+\int_0^t\int_0^1 k_\tau(\tau,s)\sum_{j=1}^{\infty}p^j v_{j-1}(s)\mathrm{d}s\mathrm{d}\tau\right). \end{aligned} \tag{29}$$

By comparing the expressions with the same powers of parameter p, we receive the relations

$$\begin{aligned} p^0: v_0(t) =& \varphi_0(t) \\ p^1: v_1(t) =& a(t)-\varphi_0(t)+\frac{1}{q}\int_0^t v_0(\tau)\mathrm{d}\tau\left(\int_0^1 k(t,s)v_0(s)\mathrm{d}s-1\right) \\ &-\frac{1}{q}\int_0^t v_0(\tau)\mathrm{d}\tau\int_0^t\int_0^1 k_\tau(\tau,s)v_0(s)\mathrm{d}s\mathrm{d}\tau+(v_0(t)-v_0) \\ &\left(\int_0^1 k(t,s)v_0(s)\mathrm{d}s-\int_0^t\int_0^1 k_\tau(\tau,s)v_0(s)\mathrm{d}s\mathrm{d}\tau\right) \\ &\vdots \end{aligned}$$

$$\begin{aligned}p^i: v_i(t) =&\frac{1}{q}\sum_{k=0}^{i-1}\left(\int_0^t v_k(\tau)\mathrm{d}\tau\int_0^1 k(t,s)v_{(i-k-1)}(s)\mathrm{d}s\right)-\frac{1}{q}\int_0^t v_{(i-1)}(\tau)\mathrm{d}\tau\\ &-\frac{1}{q}\sum_{k=0}^{i-1}\left(\int_0^t v_k(\tau)\mathrm{d}\tau\int_0^t\int_0^1 k_\tau(\tau,s)v_{(i-k-1)}(s)\mathrm{d}s\mathrm{d}\tau\right)\\ &+\sum_{k=0}^{i-1}\left(v_{(i-k-1)}(t)\left(\int_0^1 k(t,s)v_k(s)\mathrm{d}s-\int_0^t\int_0^1 k_\tau(\tau,s)v_k(s)\mathrm{d}s\mathrm{d}\tau\right)\right)\\ &-v_0\left(\int_0^1 k(t,s)v_{(i-1)}(s)\mathrm{d}s-\int_0^t\int_0^1 k_\tau(\tau,s)v_{(i-1)}(s)\mathrm{d}s\mathrm{d}\tau\right),\qquad i\geq 2.\end{aligned}$$

Remark 6.1. *Let us notice that if we take $\varphi_0(t) = 0$ or $\varphi_0(t) = a(t)$, then the considered Homotopy perturbation method is equivalent to the known method of successive approximations (in the first case, by omitting the first term which is identically equal to zero).*

Remark 6.2. *If we are not able to determine the sum of series (20) (for $p = 1$), then as an approximate solution of considered equation we can accept the partial sum of this series. If we take the first $n + 1$ components, we obtain the so-called n–order approximate solution*

$$\hat{\varphi}_n = \sum_{j=0}^{n} v_j. \tag{30}$$

7. Illustrative examples

In this section, we present the Homotopy perturbation method for solving Quadratic integral equations with Phase–lag term of the second kind.

Example 7.1. *Consider the following Quadratic integral equation with Phase–lag term of the form:*

$$\varphi(t+0.01) = (0.01-t)^2-\frac{1}{4}t^2(0.01-t)^2+\varphi(t+0.01)\int_0^1 t^2\tau\varphi(\tau)\mathrm{d}\tau;\quad (\varphi(0)=0), \tag{31}$$

with exact solution $\varphi(t) = t^2$.
Using numerical treatment of the equation (31), we obtained

$$\begin{aligned}\varphi(t) = a(t)&+\frac{1}{0.01}\int_0^t\varphi(\tau)\mathrm{d}\tau\left(\int_0^1 t^2 s\varphi(s)\mathrm{d}s-1\right)-\frac{2}{0.01}\int_0^t\varphi(\tau)\mathrm{d}\tau\int_0^t\int_0^1\tau s\varphi(s)\mathrm{d}s\mathrm{d}\tau\\ &+\varphi(t)\left(\int_0^1 t^2 s\varphi(s)\mathrm{d}s-2\int_0^t\int_0^1\tau s\varphi(s)\mathrm{d}s\mathrm{d}\tau\right);\quad (t\in I=[0,1]),\end{aligned} \tag{32}$$

where, $a(t) = 0.001t + t^2 + 0.08525t^3 - 0.015t^4 - 0.01t^5$. To solve Eq. (32), according to the Homotopy perturbation technique, we construct the homotopy and comparing coefficients of terms with identical powers of p, leads to:

$$\begin{aligned}p^0: v_0(t) =&\varphi_0(t)\\ p^1: v_1(t) =&a(t)-\varphi_0(t)+\frac{1}{0.01}\int_0^t v_0(\tau)\mathrm{d}\tau\left(\int_0^1 t^2 s v_0(s)\mathrm{d}s-1\right)\\ &-\frac{2}{0.01}\int_0^t v_0(\tau)\mathrm{d}\tau\int_0^t\int_0^1\tau s v_0(s)\mathrm{d}s\mathrm{d}\tau+(v_0(t)-v_0)\\ &\left(\int_0^1 t^2 s v_0(s)\mathrm{d}s-2\int_0^t\int_0^1\tau s v_0(s)\mathrm{d}s\mathrm{d}\tau\right)\\ &\vdots\end{aligned}$$

$$
\begin{aligned}
p^i : v_i(t) =& \frac{1}{0.01}\sum_{k=0}^{i-1}\left(\int_0^t v_k(\tau)\mathrm{d}\tau \int_0^1 t^2 s v_{(i-k-1)}(s)\mathrm{d}s\right) - \frac{1}{0.01}\int_0^t v_{(i-1)}(\tau)\mathrm{d}\tau \\
&- \frac{2}{0.01}\sum_{k=0}^{i-1}\left(\int_0^t v_k(\tau)\mathrm{d}\tau \int_0^t\int_0^1 \tau s v_{(i-k-1)}(s)\mathrm{d}s\mathrm{d}\tau\right) \\
&+ \sum_{k=0}^{i-1}\left(v_{(i-k-1)}(t)\left(\int_0^1 t^2 s v_k(s)\mathrm{d}s - 2\int_0^t\int_0^1 \tau s v_k(s)\mathrm{d}s\mathrm{d}\tau\right)\right) \\
&- v_0\left(\int_0^1 t^2 s v_{(i-1)}(s)\mathrm{d}s - 2\int_0^t\int_0^1 \tau s v_{(i-1)}(s)\mathrm{d}s\mathrm{d}\tau\right), \qquad i \geq 2.
\end{aligned}
$$

Let us set $\varphi_0(t) = 0$. Then, calculating the successive functions v_i determined by last relations, we receive successively

$$
\begin{aligned}
v_0(t) =& 0 \\
v_1(t) =& 0.001t + t^2 + 0.08525t^3 - 0.015t^4 - 0.01t^5 \\
v_2(t) =& \frac{1}{10}(0.0005t^2 + \frac{t^3}{3} + 0.0213125t^4 - 0.003t^5 - 0.00166667t^6), \\
&\vdots
\end{aligned}
\tag{33}
$$

We do not manage to find the general form of function v_j, but we could focus on the approximate solution $\hat{\varphi}_n$ determined by means of partial sums (30). Because of the existence of the exact solution, the accuracy of the nth–order approximate solutions can be evaluated.

In Table 1, we presented the absolute error $|\varphi - \hat{\varphi}_3|$, and relative errors ($\delta = |\varphi - \hat{\varphi}_3|/|\varphi|.100\%$) with which the nth–order approximate solutions reconstruct the exact solution. A plot of the error distribution in the entire interval $[0, 1]$ is displayed in Fig. 1. The results presented indicate that the method is rapidly convergent and that the calculation of just a few components of the series ensures a very good approximation of the exact solution.

Table 1. *Comparison of the numerical results with the exact solution $\varphi(t)$*

t	$\varphi(t)$	$\hat{\varphi}_3(t)$	$E_3 = \lvert\varphi(t) - \hat{\varphi}(t)\rvert$	$\delta(\%)$
0.1	0.01	0.01021611	2.16092×10^{-4}	2.160921
0.2	0.04	0.04010982	1.09816×10^{-4}	0.274544
0.3	0.09	0.09321643	3.21644×10^{-3}	3.573822
0.4	0.16	0.16700542	7.00469×10^{-3}	4.377931
0.5	0.25	0.24872761	1.27233×10^{-3}	0.508932
0.6	0.36	0.35795785	2.04220×10^{-3}	0.567277
0.7	0.49	0.51990334	2.99026×10^{-2}	6.102571
0.8	0.64	0.68068262	4.06821×10^{-2}	6.356578
0.9	0.81	0.86195612	5.19558×10^{-2}	6.414296
1.0	1.00	1.06256211	6.25621×10^{-2}	6.256212

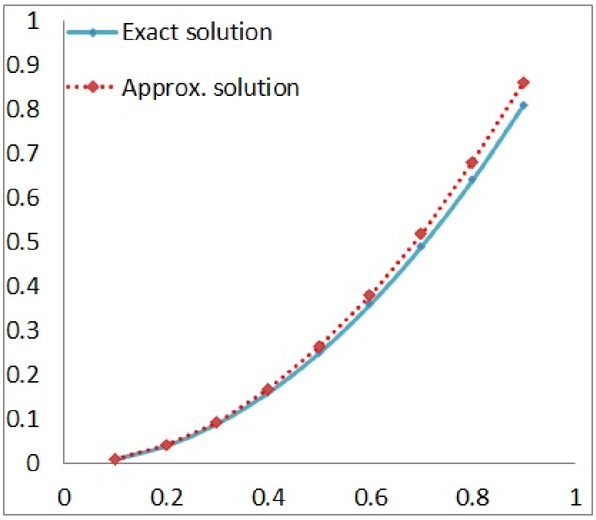

Figure 1. *Comparison of the approximate solution by HPM with the exact solution*

Example 7.2. *Consider the following QIEPLT:*

$$\varphi(t+0.001)=1.001+t-\frac{5}{6}t(1.001+t)+\varphi(t+0.001)\int_0^1 t\tau\varphi(\tau)\mathrm{d}\tau;\quad(\varphi(0)=1),\tag{34}$$

the exact solution is $\varphi(t)=t+1$.
Using numerical treatment of the equation (34), we obtained

$$\begin{aligned}\varphi(t)=a(t)&+\frac{1}{0.001}\int_0^t\varphi(\tau)\mathrm{d}\tau\left(\int_0^1 ts\varphi(s)\mathrm{d}s-1\right)-\frac{1}{0.001}\int_0^t\varphi(\tau)\mathrm{d}\tau\int_0^t\int_0^1 s\varphi(s)\mathrm{d}s\mathrm{d}\tau\\&+(\varphi(t)-1)\left(\int_0^1 ts\varphi(s)\mathrm{d}s-\int_0^t\int_0^1 s\varphi(s)\mathrm{d}s\mathrm{d}\tau\right);\quad(t\in I=[0,1]),\end{aligned}\tag{35}$$

where, $a(t)=1+1.001t+0.00229167t^2+(t^3)/100$. *To solve Eq. (35), according to the Homotopy perturbation technique, we construct the homotopy and comparing coefficients of terms with identical powers of* p, *leads to:*

$$\begin{aligned}p^0:v_0(t)=&\varphi_0(t)\\p^1:v_1(t)=&a(t)-\varphi_0(t)+\frac{1}{0.001}\int_0^t v_0(\tau)\mathrm{d}\tau\left(\int_0^1 tsv_0(s)\mathrm{d}s-1\right)\\&-\frac{1}{0.001}\int_0^t v_0(\tau)\mathrm{d}\tau\int_0^t\int_0^1 sv_0(s)\mathrm{d}s\mathrm{d}\tau+(v_0(t)-1)\\&\left(\int_0^1 tsv_0(s)\mathrm{d}s-\int_0^t\int_0^1 sv_0(s)\mathrm{d}s\mathrm{d}\tau\right)\\&\vdots\end{aligned}$$

$$
\begin{aligned}
p^i : v_i(t) =& \frac{1}{0.001}\sum_{k=0}^{i-1}\left(\int_0^t v_k(\tau)d\tau \int_0^1 tsv_{(i-k-1)}(s)ds\right) - \frac{1}{0.001}\int_0^t v_{(i-1)}(\tau)d\tau \\
&- \frac{1}{0.001}\sum_{k=0}^{i-1}\left(\int_0^t v_k(\tau)d\tau \int_0^t \int_0^1 sv_{(i-k-1)}(s)dsd\tau\right) \\
&+ \sum_{k=0}^{i-1}\left(v_{(i-k-1)}(t)\left(\int_0^1 tsv_k(s)ds - \int_0^t \int_0^1 sv_k(s)dsd\tau\right)\right) \\
&- \left(\int_0^1 tsv_{(i-1)}(s)ds - \int_0^t \int_0^1 sv_{(i-1)}(s)dsd\tau\right), \qquad i \geq 2.
\end{aligned}
$$

Let us set $\varphi_0(t) = 1$. *Then, calculating the successive functions* v_i *determined by last relations, we receive successively*

$$
\begin{aligned}
&v_0(t) = 0, \; v_1(t) = 1 + 1.001t + 0.00229167t^2 + \frac{2t^3}{45} \\
&v_2(t) = \frac{1}{1000}(-t - 0.5005t^2 - 0.000763889t^3 - \frac{t^4}{90}), \cdots .
\end{aligned} \tag{36}
$$

In Table 2, we presented the absolute error $|\varphi(t) - \hat{\varphi}_n(t)|$, *shows the approximate solution for* $n = 6$.

Table 2. *Absolute error of solution of Eq.* (34) *by using HPM with* $n = 6$.

t	$\varphi(t)$	$\hat{\varphi}_6(t)$	$E_6 = \lvert\varphi(t) - \hat{\varphi}_6(t)\rvert$	$\delta(\%)$
0.1	1.1	1.100003	2.79108×10^{-6}	0.000253728
0.2	1.2	1.200015	1.51636×10^{-5}	0.00126347
0.3	1.3	1.300043	4.31161×10^{-5}	0.00331552
0.4	1.4	1.400093	9.26464×10^{-5}	0.00661322
0.5	1.5	1.500170	1.69750×10^{-4}	0.01130432
0.6	1.6	1.600280	2.80429×10^{-4}	0.01749612
0.7	1.7	1.700431	4.30674×10^{-4}	0.02526971
0.8	1.8	1.800626	6.26482×10^{-4}	0.03468384
0.9	1.9	1.900874	8.73848×10^{-4}	0.04578146
1.0	2.0	2.001179	1.17877×10^{-3}	0.05859343

In addition, in Fig. 2, we presented a comparison between the exact solution and the approximate solution using the introduced numerical method with $n = 6$ *in the interval* $[0, 1]$.

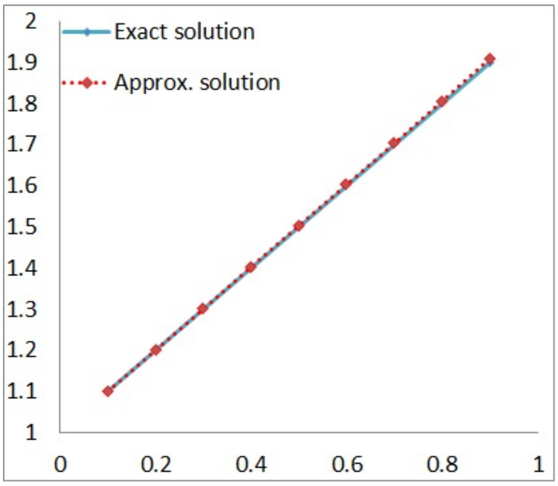

Figure 2. *Shows the exact solution* $\varphi(t)$ *and the approximate solution* $\hat{\varphi}(t)$.

8. Conclusion

In this work, from the above results and discussion, the following may be conclude, using numerical treatment of the equation (1), we obtained equation (5). The equation (5) has at least one solution $\varphi(t)$ in the space $C([0,1])$, under some conditions. Fixed point theorems are one of the best methods to prove the existence and uniqueness of solution of these equations. Eq. (5) is usually difficult to solve analytically. In many cases, it is required to obtain the approximate solution. For this purpose, the presented method can be proposed. HPM is used and analyzed for solving QIEPLT of the second kind. This method is more efficient and simpler. The results of the method exhibit excellent agreement with the exact solution. We have been observing that the accuracy can be improved by computing more n–terms off approximate solutions. If $q \to 0$, we find that the numerical solution quickly converges with the exact solution. The error takes the maximum value at $t = \pm 1$ and its values are decreasing as $t \to 0$.

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