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The generalized Khasminskii-type conditions in establishing existence, uniqueness and moment estimates of solution to neutral stochastic functional differential equations

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Abstract. The main objective of this paper involving neutral stochastic functional differential equations (NSFDEs), with finite or infinite history dependence, is to prove the existence and uniqueness of their global solutions by imposing conditions that hold for highly non-linear NSFDEs' coefficients. For that purpose, Yamada-Watanabe condition or the local Lipschitz condition for the drift and diffusion coefficients are imposed, together with contractivity condition for the neutral term. Also, instead of the linear growth condition for the drift and diffusion coefficients of the equations, generalized Khasminskii-type conditions are applied. The proof of the existence and uniqueness of the solution also leads us to estimates of the moments to the solution. Consequently, we discuss some asymptotic properties of the solution in terms of the generalized Lyapunov exponent. Additionally, we consider a class of neutral stochastic differential equations with state-dependent delay, as a special case of NSFDEs. The theoretical results are illustrated with two examples.

1. Introduction and Preliminary Results

The functional differential equations are introduced decades ago, in order to solve a modeling problems in many areas of science and engineering (see [16]). Adding the noise to these deterministic models makes these problems more realistic, so the development of stochastic functional differential equations is very significant (see [20]). The importance of neutral stochastic functional differential equations (NSFDEs), as a generalization of stochastic functional differential equations, is recognized and they are studied by many authors. Kolmanovskii and Nosov first introduced the term NSFDE in [17], studying the behavior of reactors in a chemical plant and mutual interaction between the forces that occur when an elastic body moves through a fluid and proved the existence and uniqueness of the solution (see [18]). Other authors also investigated the existence and uniqueness problem: in [9, 20] under the global Lipschitz and linear growth condition for drift and diffusion coefficients, while in [4, 7, 20] the global Lipschitz condition is

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replaced with the local one. Furthermore, uniform (see [15, 22, 28]) and local (see [22]) Yamada-Watanabe conditions are proposed. In [10] the drift and diffusion coefficients satisfy the local Lipschitz condition with the time-varying Lipschitz coefficient. Linear growth condition is left out, which allows drift and diffusion to be nonlinear (Lyapunov-Razumikhin technique is applied). On the other hand, in [30, 32] the linear growth condition is replaced with more general Khasminskii-type conditions. Also, the authors in [10, 20, 30, 32] consider NSDFEs with coefficients which depend on finite history of the state process, while in [4, 7, 9, 15, 22, 28] history is infinite. Motivated by the papers previously mentioned, in which neutral term satisfies the contractivity condition, and by the fact that many NSFDEs' coefficients are growing superlinearly, we propose Khasminskii-type conditions in order to show that NSFDEs have unique solutions with initial conditions defined on finite or infinite interval. These conditions should prevent the explosion of the solution in finite time. In order to assure the existence of the unique maximal local solution to NSFDE, with previously cited papers in mind, we impose Yamada-Watanabe condition and the local Lipschitz condition separately, depending on the boundedness of the initial condition of the equation. The question of the moment estimates of the solutions arises throughout the proof of the existence and uniqueness of the solution, which is used to obtain the generalized Lyapunov exponent.

Beside the idea to impose more general conditions then the linear growth condition, our work was remarkably influenced by trying to prove the existence and uniqueness problem for special type of NSFDE - neutral stochastic differential equations with delay that is both time and state dependent. Stochastic differential equations with this type of delay were investigated by only few authors ([3, 14, 26]) in oppose to numerous papers dealing with constant or time dependent delay (e. g. [11, 21, 23, 24]).

Let us consider a complete probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ satisfying the usual conditions and define m-dimensional Brownian motion $B(t) = (B_1(t), \dots, B_m(t))^T$, $t \geqslant 0$, on the given complete probability space. We denote by $|\cdot|$ the Euclidean norm in $\mathbb{R}^d$ and by Q^T the transpose of a vector or a matrix Q.

For the intervals $I = (-\infty, 0]$ or $I = [-\kappa, 0]$, where $\kappa \in (0, \infty)$, let us introduce notation of the following families of continuous functions:

- $C^\psi(I;\mathbb{R}^d)$ - functions $\varphi : I \to \mathbb{R}^d$, where $I = (-\infty, 0]$, such that $\frac{\varphi}{\psi}$ is uniformly continuous on I and $\sup_{\theta\in I} \frac{|\varphi(\theta)|}{\psi(\theta)} < \infty$, where $\psi : I \to [1, \infty)$ is a continuous, non-increasing function with $\psi(\theta) \to \infty$, as $\theta \to -\infty$ and $\psi(0) = 1$, with the norm $\|\varphi\| = \sup_{\theta\in I} \frac{|\varphi(\theta)|}{\psi(\theta)}$;
- $BC(I;\mathbb{R}^d)$ - bounded functions $\varphi_1 : I \to \mathbb{R}^d$, where $I = (-\infty, 0]$, with the norm $\|\varphi_1\| = \sup_{\theta\in I} |\varphi_1(\theta)|$;
- $C(I;\mathbb{R}^d) = \{\varphi_2 \,|\, \varphi_2 : I \to \mathbb{R}^d\}$, where $I = [-\kappa, 0]$, with the norm $\|\varphi_2\| = \sup_{\theta\in I} |\varphi_2(\theta)|$;
- $C([0,\infty);\mathbb{R}^d) = \{a \,|\, a : [0,\infty) \to \mathbb{R}^d\}$;
- $C([0,\infty); [0,\infty)) = \{a_1 \,|\, a_1 : [0,\infty) \to [0,\infty)\}$;
- $C(I \cup [0,\infty); [0,\infty)) = \{a_2 \,|\, a_2 : I \cup [0,\infty) \to [0,\infty)\}$;
- $C(\mathbb{R} \times \mathbb{R}^d; [0,\infty)) = \{a_3 \,|\, a_3 : \mathbb{R} \times \mathbb{R}^d \to [0,\infty)\}$;
- $C^{1,2}(I \cup [0,\infty) \times \mathbb{R}^d; [0,\infty)) = \{V \,|\, V : I \cup [0,\infty) \times \mathbb{R}^d \to [0,\infty)\}$, where $V = V(t,x)$ are continuously once differentiable in t and twice in x.

Also, $\mathcal{M}^p(I;\mathbb{R}^d)$ is the family of $\mathcal{F}_0$-measurable, $\mathbb{R}^d$-valued processes $\xi(t) = \xi(t,\omega)$, $t \in I$, with $E\int_I |\xi(t)|^p dt < \infty$, $C^\psi_{\mathcal{F}_0}(I;\mathbb{R}^d)$ is the family of $\mathcal{F}_0$-measurable $C^\psi(I;\mathbb{R}^d)$-valued random variables from $\mathcal{M}^p(I;\mathbb{R}^d)$ and $BC_{\mathcal{F}_0}(I;\mathbb{R}^d)$ is the family of $\mathcal{F}_0$-measurable $BC(I;\mathbb{R}^d)$-valued random variables from $\mathcal{M}^p(I;\mathbb{R}^d)$.

In this section, let W represent one of the spaces $C^\psi(I;\mathbb{R}^d)$, $BC(I;\mathbb{R}^d)$ or $C(I;\mathbb{R}^d)$ and according to this, $W_{\mathcal{F}_0}$ will denote $C^\psi_{\mathcal{F}_0}(I;\mathbb{R}^d)$, $BC_{\mathcal{F}_0}(I;\mathbb{R}^d)$ or $C_{\mathcal{F}_0}(I;\mathbb{R}^d)$. Considering Borel-measurable functionals $u : [0,\infty) \times W \to$

$\mathbb{R}^d$, $f : [0,\infty)\times\mathbb{R}^d\times W\to\mathbb{R}^d$ and $g : [0,\infty)\times\mathbb{R}^d\times W\to\mathbb{R}^{d\times m}$, we observe the d-dimensional neutral stochastic functional differential equation

$$d[x(t)-u(t,x_t)] = f(t,x(t),x_t)\,dt + g(t,x(t),x_t)\,dB(t),\quad t\geqslant 0, \tag{1}$$

with initial condition

$$x_0=\varphi=\{\varphi(\theta):\theta\in I\}\in W_{\mathcal{F}_0}, \tag{2}$$

where $x_t=\{x(t+\theta):\theta\in I\}$ is W-valued stochastic process.

For $V\in C^{1,2}(I\cup[0,\infty)\times\mathbb{R}^d;[0,\infty))$, we define an operator $LV:[0,\infty)\times\mathbb{R}^d\times W\to\mathbb{R}$ by

$$LV(t,x,y)=V_t(t,x-u(t,y))+V_x(t,x-u(t,y))f(t,x,y)+\frac{1}{2}\text{trace}[g^T(t,x,y)V_{xx}(t,x-u(t,y))g(t,x,y)]. \tag{3}$$

Necessary assumptions for proving the existence and uniqueness of the solution of Eq. (1), that will be introduced in the sequel, and the form of the equation itself require the application of the following lemma. It should be emphasized that it represents a useful generalization of the Bellman-Gronwall inequality (see [25]).

Lemma 1.1. ***(Dhongade-Deo)*** *Let functions $l(t)$, $m(t):(0,\infty)\to(0,\infty)$, function $n(t):(0,\infty)\to(0,\infty)$ is monotonic non-decreasing, $p(t):(0,\infty)\to[1,\infty)$ and all functions are continuous on $(0,\infty)$. If*

$$l(t)\leqslant n(t)+p(t)\int_0^t m(s)l(s)ds,\quad t>0, \tag{4}$$

then

$$l(t)\leqslant n(t)p(t)\exp\left(\int_0^t m(s)p(s)ds\right),\quad t>0.$$

The paper is organized as follows. In Section 2 the main assumptions are imposed for the existence and uniqueness of the solution of NSFDEs to be proven and moment estimates of the solutions are established, considering coefficients from different phase spaces. Also, the upper bound of the generalized Lyapunov exponent is determined. In Section 3, an insight into the neutral stochastic differential equations with state dependent delay is provided. In Section 4 two examples that illustrate the theory are given.

2. The Existence, Uniqueness and Moment Estimates of the Solution of NSFDEs

The starting point for our proof of the existence and uniqueness of the global solution to Eq. (1) is the existence of the unique maximal local solution. Knowing that it exists, we can show that the explosion time is not finite almost surely, which can be assured by proposing Khasminskii-type conditions.

Definition 2.1. *A continuous $\mathcal{F}_t$-adapted $\mathbb{R}^d$-valued process $x(t)$, $t\in I\cup[0,\sigma_e)$, with a stopping time σ_e, is a local solution of Eq.* (1) *with initial condition* (2)*, if for every $t\geqslant 0$,*

$$x(\sigma_l\wedge t)=\varphi(0)+u(\sigma_l\wedge t,x_{\sigma_l\wedge t})-u(0,\varphi)+\int_0^{\sigma_l\wedge t}f(s,x(s),x_s)\,ds+\int_0^{\sigma_l\wedge t}g(s,x(s),x_s)\,dB(s)$$

holds for any $l\geqslant 1$, where $\{\sigma_l\}_{l\geqslant 1}$ is a non-decreasing sequence of finite stopping times, such that $\sigma_l\uparrow\sigma_e$ a.s. as $l\to\infty$. Moreover, if $\limsup_{t\to\sigma_e}|x(t)|=\infty$ a.s. whenever $\sigma_e<\infty$, then $x(t)$, $t\in I\cup[0,\sigma_e)$, is a maximal local solution and σ_e is called the explosion time. A maximal local solution is unique if for every other maximal local solution $y(t)$, $t\in I\cup[0,\sigma_E)$, $\sigma_e=\sigma_E$ a.s. and $x(t)=y(t)$ for every $t\in I\cup[0,\sigma_e)$ a.s.

Let us introduce the following assumptions.

P1. (Yamada-Watanabe condition) For any integer $n \geqslant 1$, there exists a function $K_n : [0, \infty) \to [0, \infty)$, such that, for all $x_1, x_2 \in \mathbb{R}^d$, $y_1, y_2 \in C^\psi(I; \mathbb{R}^d)$ (or $y_1, y_2 \in C(I; \mathbb{R}^d)$), with $|x_1| \vee |x_2| \vee \|y_1\| \vee \|y_2\| \leqslant n$ and all $t \geqslant 0$,

$$|f(t, x_1, y_1) - f(t, x_2, y_2)|^2 \vee |g(t, x_1, y_1) - g(t, x_2, y_2)|^2 \leqslant K_n(|x_1 - x_2|^2 + \|y_1 - y_2\|^2). \tag{5}$$

Functions K_n are continuous, non-decreasing, concave, $K_n(0) = 0$ and $\int_{0+}^{\infty} \frac{ds}{K_n(s)} = \infty$. Also, for every $t \geqslant 0$, there exists a constant $K > 0$, such that

$$|f(t, 0, 0)|^2 \vee |g(t, 0, 0)|^2 \leqslant K. \tag{6}$$

P1'. (Local Lipschitz condition) For any integer $n \geqslant 1$, there exists a positive constant $\overline{K}_n$, such that, for all $x_1, x_2 \in \mathbb{R}^d$, $y_1, y_2 \in BC(I; \mathbb{R}^d)$ (or $y_1, y_2 \in C(I; \mathbb{R}^d)$), with $|x_1| \vee |x_2| \vee \|y_1\| \vee \|y_2\| \leqslant n$ and all $t \geqslant 0$,

$$|f(t, x_1, y_1) - f(t, x_2, y_2)|^2 \vee |g(t, x_1, y_1) - g(t, x_2, y_2)|^2 \leqslant \overline{K}_n(|x_1 - x_2|^2 + \|y_1 - y_2\|^2).$$

P2. (Khasminskii-type conditions) There exist a function $V \in C^{1,2}(I \cup [0, \infty) \times \mathbb{R}^d; [0, \infty))$, positive constant C_1, nonnegative constant c_2 and functions C_2 and h in $C([0, \infty); [0, \infty))$, such that, for every $(t, x) \in [0, \infty) \times \mathbb{R}^d$ and $p > 0$,

$$C_1|x|^p \leqslant V(t, x) \leqslant c_2 h(t) + C_2(t)|x|^p \tag{7}$$

and there exists the function $D \in C([0, \infty); [0, \infty))$, such that

$$LV(t, x, y) \leqslant D(t)\Big(1 + \sup_{s \in I \cup [0,t]} V(s, x)\Big), \tag{8}$$

for every $(t, x, y) \in [0, \infty) \times \mathbb{R}^d \times W$.

P3. (Contractivity condition) There exists a constant $\alpha \in (0, 1)$, such that, for all $x, y \in W$ and $t \geqslant 0$,

$$|u(t, x) - u(t, y)| \leqslant \alpha \|x - y\|.$$

Let us denote $u(t, 0) = r(t)$, for every $t \geqslant 0$, where $r \in C([0, \infty); \mathbb{R}^d)$. Then, from the last inequality we have that $|u(t, y)| \leqslant \alpha\|y\| + |r(t)|$, for every $(t, y) \in [0, \infty) \times W$.

Remark 2.2. *Obviously, the condition* (5) *is weaker than the local Lipschitz condition. If $K_n(x) \equiv \overline{K}_n x$, for $x \geqslant 0$ and positive constants $\overline{K}_n$, **P1'** is satisfied. However, since the functions from $C^\psi(I; \mathbb{R}^d)$ are not necessarily bounded, condition* (6) *is required. If we consider $BC(I; \mathbb{R}^d)$ instead of $C^\psi(I; \mathbb{R}^d)$,* (6) *is no longer needed. This is the reason we propose both **P1** and **P1'**, where only one of them is required, according to the equation coefficients and their domain, for the existence of the unique maximal local solution.*

Also, the condition (7) *is weaker than the condition*

$$C_1|x|^p \leqslant V(t, x) \leqslant C_2|x|^p, \quad C_1, C_2 \in (0, \infty), \tag{9}$$

which is used in [30, 32]. This allows us to expand the family of the coefficients of Eq. (1) *for which there are a unique global solutions, by enlarging the number of choices for a function V for which the estimate* (8) *can be obtained.*

Theorem 2.3. *If one of the assumptions **P1** and **P1'** hold, together with **P2**, **P3** and the condition*

$$|V_x(t, x - u(t, y))g(t, x, y)| \leqslant D(t)\Big(1 + \sup_{s \in I \cup [0,t]} V(s, x)\Big), \tag{10}$$

then, for any initial condition (2)*, there exists a unique global solution $x(t)$, $t \in I \cup [0, \infty)$, of Eq.* (1)*. Moreover, for every $t \geqslant 0$*

$$E \sup_{z \in I \cup [0,t]} V(z, x(z)) \leqslant A(t)B(t)e^{B(t)t}, \tag{11}$$

where

$$\begin{aligned}A(t) &= 2 + 2E\sup_{\theta\in I} V(\theta,\varphi(\theta)) + 2c_2 \sup_{z\in[0,t]} h(z) + 2\sup_{z\in[0,t]} C_2(z)\left(d_1(p)E\|\varphi\|^p + d_2(p)\sup_{z\in[0,t]}|r(z)|^p + \frac{\widetilde{V}d_3(p)}{C_1}\right),\\ B(t) &= \frac{d_3(p)}{C_1^2}\sup_{z\in[0,t]} C_2(z)\sup_{z\in[0,t]} D(z)\left(16\sup_{z\in[0,t]} C_2(s)\sup_{z\in[0,t]} D(z) + \frac{C_1}{d_3(p)}\right)\vee 1,\end{aligned} \tag{12}$$

for $\widetilde{V} = EV(0,\varphi(0) - u(0,\varphi))$ *and*

$$d_1(p) = \begin{cases}\dfrac{\alpha^p}{1-\alpha^p}, & p\in(0,1],\\ \dfrac{\alpha}{1-\alpha}, & p\in(1,\infty),\end{cases}\qquad d_2(p) = \begin{cases}\dfrac{1}{1-\alpha^p}, & p\in(0,1],\\ \dfrac{(\sqrt{\alpha}-\alpha)^{1-p}}{1-\alpha}, & p\in(1,\infty),\end{cases}\qquad d_3(p) = \begin{cases}\dfrac{1}{1-\alpha^p}, & p\in(0,1],\\ \dfrac{(1-\sqrt{\alpha})^{1-p}}{1-\alpha}, & p\in(1,\infty).\end{cases} \tag{13}$$

In particular, if $\sup_{t\geqslant 0} h(t) = H$ *and* $\sup_{t\geqslant 0} C_2(t) = C_2$ *in condition* ***P2****, as well as* $\sup_{t\geqslant 0}|r(t)| = R$ *in condition* ***P3****, then*

$$E\sup_{z\in I\cup[0,t]} V(z,x(z)) \leqslant \widetilde{A}e^{\widetilde{B}(t)t},\quad t\geqslant 0, \tag{14}$$

where

$$\begin{aligned}\widetilde{A} &= 2 + 2E\sup_{\theta\in I} V(\theta,\varphi(\theta)) + 2c_2H + 2C_2\left(d_1(p)E\|\varphi\|^p + d_2(p)R^p + \frac{\widetilde{V}d_3(p)}{C_1}\right),\\ \widetilde{B}(t) &= \frac{2C_2d_3^2(p)}{C_1^2}\sup_{z\in[0,t]} D(z)\left(16C_2\sup_{z\in[0,t]} D(z) + \frac{C_1}{d_3(p)}\right).\end{aligned} \tag{15}$$

Proof. The existence of the unique maximal local solution $x(t)$, $t\in I\cup[0,\sigma_e)$, of Eq. (1) is assured under the assumptions **P1** and **P3** if $W = C^{\psi}(I;\mathbb{R}^d)$ or $W = C(I;\mathbb{R}^d)$ (see [22], Corollary 3.1), or the assumptions **P1′** and **P3** if $W = BC(I;\mathbb{R}^d)$ or $W = C(I;\mathbb{R}^d)$ (see [31], Theorem 3.1), for any given initial condition $\varphi\in W_{\mathcal{F}_0}$, where σ_e is the explosion time.

Since $\varphi\in W_{\mathcal{F}_0}$, there exists sufficiently large $l_0>0$, such that $\|\varphi\|\leqslant l_0$. For each integer $l\geqslant l_0$, define the increasing sequence of the stopping times $\{\sigma_l\}_{l\geqslant l_0}$, with

$$\sigma_l = \inf\{t\in[0,\sigma_e) : |x(t)|\geqslant l\},$$

where $\inf\emptyset = \infty$. Let $\sigma_\infty = \lim_{l\to\infty}\sigma_l$. Obviously, $\sigma_\infty\leqslant\sigma_e$ a.s.

For arbitrary $\varepsilon>0$ and $p>1$, using inequality

$$|a+b|^p \leqslant \left(1+\varepsilon^{\frac{1}{p-1}}\right)^{p-1}\left(|a|^p + \frac{|b|^p}{\varepsilon}\right),\quad a,b\in\mathbb{R}, \tag{16}$$

(see [20], Lemma 4.1) and the assumption **P3**, we get

$$\begin{aligned}|x(\sigma_l\wedge t)|^p &\leqslant \left(1+\varepsilon^{\frac{1}{p-1}}\right)^{p-1}|x(\sigma_l\wedge t) - u(\sigma_l\wedge t, x_{\sigma_l\wedge t})|^p + \varepsilon^{-1}\left(1+\varepsilon^{\frac{1}{p-1}}\right)^{p-1}|u(\sigma_l\wedge t, x_{\sigma_l\wedge t})|^p\\ &\leqslant \left(1+\varepsilon^{\frac{1}{p-1}}\right)^{p-1}|x(\sigma_l\wedge t) - u(\sigma_l\wedge t, x_{\sigma_l\wedge t})|^p + \varepsilon^{-1}\left(1+\varepsilon^{\frac{1}{p-1}}\right)^{2(p-1)}\left(\varepsilon^{-1}\alpha^p\|x_{\sigma_l\wedge t}\|^p + \sup_{z\in[0,t]}|r(z)|^p\right).\end{aligned} \tag{17}$$

As, in case of $W = C^{\psi}(I;\mathbb{R}^d)$, for $\varphi\in C^{\psi}(I;\mathbb{R}^d)$, $\|\varphi\| = \sup_{\theta\in I}\frac{|\varphi(\theta)|}{\psi(\theta)}\leqslant\sup_{\theta\in I}|\varphi(\theta)|$, we find that

$$\begin{aligned}\sup_{z\in[0,t]}|x(\sigma_l\wedge z)|^p &\leqslant \left(1+\varepsilon^{\frac{1}{p-1}}\right)^{p-1}\sup_{z\in[0,t]}|x(\sigma_l\wedge z) - u(\sigma_l\wedge z, x_{\sigma_l\wedge z})|^p + \varepsilon^{-2}\alpha^p\left(1+\varepsilon^{\frac{1}{p-1}}\right)^{2(p-1)}\left(\|\varphi\|^p + \sup_{z\in[0,t]}|x(\sigma_l\wedge z)|^p\right)\\ &\quad + \varepsilon^{-1}\left(1+\varepsilon^{\frac{1}{p-1}}\right)^{2(p-1)}\sup_{z\in[0,t]}|r(z)|^p,\end{aligned}$$

which yields

$$\Big(1-\varepsilon^{-2}\alpha^p\Big(1+\varepsilon^{\frac{1}{p-1}}\Big)^{2(p-1)}\Big)\sup_{z\in[0,t]}|x(\sigma_l\wedge z)|^p$$
$$\leqslant \Big(1+\varepsilon^{\frac{1}{p-1}}\Big)^{p-1}\sup_{z\in[0,t]}|x(\sigma_l\wedge z)-u(\sigma_l\wedge z,x_{\sigma_l\wedge z})|^p+\varepsilon^{-1}\Big(1+\varepsilon^{\frac{1}{p-1}}\Big)^{2(p-1)}\Big(\varepsilon^{-1}\alpha^p\|\varphi\|^p+\sup_{z\in[0,t]}|r(z)|^p\Big). \qquad (18)$$

Choosing $\varepsilon=\left(\frac{\sqrt{\alpha}}{1-\sqrt{\alpha}}\right)^{p-1}$, we have $1-\varepsilon^{-2}\alpha^p\Big(1+\varepsilon^{\frac{1}{p-1}}\Big)^{2(p-1)}=1-\alpha>0$, and (18) becomes

$$\sup_{z\in[0,t]}|x(\sigma_l\wedge z)|^p\leqslant\frac{(1-\sqrt{\alpha})^{1-p}}{1-\alpha}\sup_{z\in[0,t]}|x(\sigma_l\wedge z)-u(\sigma_l\wedge z,x_{\sigma_l\wedge z})|^p+\frac{\alpha}{1-\alpha}\|\varphi\|^p+\frac{(\sqrt{\alpha}-\alpha)^{1-p}}{1-\alpha}\sup_{z\in[0,t]}|r(z)|^p. \qquad (19)$$

For $p\in(0,1]$, by applying inequality

$$|a+b|^p\leqslant|a|^p+|b|^p,$$

instead of (17), we obtain

$$\sup_{z\in[0,t]}|x(\sigma_l\wedge z)|^p\leqslant\sup_{z\in[0,t]}|x(\sigma_l\wedge z)-u(\sigma_l\wedge z,x_{\sigma_l\wedge z})|^p+\alpha^p\Big(\|\varphi\|^p+\sup_{z\in[0,t]}|x(\sigma_l\wedge z)|^p\Big)+\sup_{z\in[0,t]}|r(z)|^p,$$

which gives us

$$\sup_{z\in[0,t]}|x(\sigma_l\wedge z)|^p\leqslant\frac{1}{1-\alpha^p}\Big(\sup_{z\in[0,t]}|x(\sigma_l\wedge z)-u(\sigma_l\wedge z,x_{\sigma_l\wedge z})|^p+\alpha^p\|\varphi\|^p+\sup_{z\in[0,t]}|r(z)|^p\Big). \qquad (20)$$

From (19), (20), (7) and (13), we see that, for $p>0$,

$$\begin{aligned}\sup_{z\in[0,t]}V(\sigma_l\wedge z,x(\sigma_l\wedge z))&\leqslant c_2\sup_{z\in[0,t]}h(z)+\sup_{z\in[0,t]}C_2(z)\sup_{z\in[0,t]}|x(\sigma_l\wedge z)|^p\\&\leqslant c_2\sup_{z\in[0,t]}h(z)+\sup_{z\in[0,t]}C_2(z)\Big(d_1(p)\|\varphi\|^p+d_2(p)\sup_{z\in[0,t]}|r(z)|^p\Big)\\&\quad+d_3(p)\sup_{z\in[0,t]}C_2(z)\sup_{z\in[0,t]}|x(\sigma_l\wedge z)-u(\sigma_l\wedge z,x_{\sigma_l\wedge z})|^p\\&\leqslant c_2\sup_{z\in[0,t]}h(z)+\sup_{z\in[0,t]}C_2(z)\Big(d_1(p)\|\varphi\|^p+d_2(p)\sup_{z\in[0,t]}|r(z)|^p\Big)\\&\quad+\frac{d_3(p)}{C_1}\sup_{z\in[0,t]}C_2(z)\sup_{z\in[0,t]}V\Big(\sigma_l\wedge z,x(\sigma_l\wedge z)-u(\sigma_l\wedge z,x_{\sigma_l\wedge z})\Big).\end{aligned} \qquad (21)$$

If we denote $V_0=V(0,\varphi(0)-u(0,\varphi))$, by applying the generalized Itô formula (see [20], Theorem 6.4) and (8), we obtain

$$V(t,x(t)-u(t,x_t))\leqslant V_0+\int_0^t D(s)\Big(1+\sup_{z\in I\cup[0,s]}V(z,x(z))\Big)ds+M(t),$$

where

$$M(t)=\int_0^t V_x(s,x(s)-u(s,x_s))\,g(s,x(s),x_s)\,dB(s)$$

is a local martingale and $M(0) = 0$. Consequently, (21) yields

$$\begin{aligned}
&\sup_{z\in I\cup[0,t]} V(\sigma_l \wedge z, x(\sigma_l \wedge z)) \\
&\quad\leqslant \sup_{\theta\in I} V(\theta,\varphi(\theta)) + c_2 \sup_{z\in[0,t]} h(z) + \sup_{z\in[0,t]} C_2(z)\Big(d_1(p)\|\varphi\|^p + d_2(p)\sup_{z\in[0,t]}|r(z)|^p + \frac{V_0\, d_3(p)}{C_1}\Big) \\
&\qquad + \frac{d_3(p)}{C_1}\sup_{z\in[0,t]} C_2(z)\int_0^t D(s)\Big(1+\sup_{z\in I\cup[0,s]} V(\sigma_l\wedge z, x(\sigma_l\wedge z))\Big)ds + \frac{d_3(p)}{C_1}\sup_{z\in[0,t]} C_2(z)\sup_{z\in[0,t]} M(\sigma_l\wedge z). \qquad (22)
\end{aligned}$$

By the Burkholder-Davis-Gundy inequality (see [20], Theorem 7.3), (10) and elementary inequality $2ab \leqslant \varepsilon a^2 + \varepsilon^{-1}b^2$, for $\varepsilon > 0$, we derive

$$\begin{aligned}
&\sup_{z\in[0,t]} C_2(z) E \sup_{z\in[0,t]} M(\sigma_l\wedge z) \\
&\quad\leqslant \sqrt{32}\sup_{z\in[0,t]} C_2(z)\, E\Big[\int_0^t \Big(\sup_{z\in[0,s]} D(z)\Big)^2\Big(1+\sup_{z\in I\cup[0,s]} V(\sigma_l\wedge z, x(\sigma_l\wedge z))\Big)^2 ds\Big]^{\frac{1}{2}} \\
&\quad\leqslant \sqrt{32}\sup_{z\in[0,t]} C_2(z)\, E\Big[\Big(1+\sup_{z\in I\cup[0,t]} V(\sigma_l\wedge z, x(\sigma_l\wedge z))\Big)\int_0^t \Big(\sup_{z\in[0,s]} D(z)\Big)^2\Big(1+\sup_{z\in I\cup[0,s]} V(\sigma_l\wedge z, x(\sigma_l\wedge z))\Big)ds\Big]^{\frac{1}{2}} \\
&\quad\leqslant \frac{\varepsilon}{2} E\Big(1+\sup_{z\in I\cup[0,t]} V(\sigma_l\wedge z, x(\sigma_l\wedge z))\Big) + \frac{16}{\varepsilon}\Big(\sup_{z\in[0,t]} C_2(z)\sup_{s\in[0,t]} D(s)\Big)^2 E\int_0^t \Big(1+\sup_{z\in I\cup[0,s]} V(\sigma_l\wedge z, x(\sigma_l\wedge z))\Big)ds.
\end{aligned}$$

Substituting the previous relation into (22), where expectation is taken on both sides, for $\widetilde{V} = EV_0$, we have that

$$\begin{aligned}
&1 + E\sup_{z\in I\cup[0,t]} V(\sigma_l\wedge z, x(\sigma_l\wedge z)) \\
&\quad\leqslant 1 + E\sup_{\theta\in I} V(\theta,\varphi(\theta)) + c_2\sup_{z\in[0,t]} h(z) + \frac{\varepsilon d_3(p)}{2C_1} E\Big(1+\sup_{z\in I\cup[0,t]} V(\sigma_l\wedge z, x(\sigma_l\wedge z))\Big) \\
&\qquad + \sup_{z\in[0,t]} C_2(z)\Big(d_1(p)E\|\varphi\|^p + d_2(p)\sup_{z\in[0,t]}|r(z)|^p + \frac{\widetilde{V}d_3(p)}{C_1}\Big) \\
&\qquad + \frac{d_3(p)}{C_1}\sup_{z\in[0,t]} C_2(z)\sup_{z\in[0,t]} D(z)\Big(1+\frac{16}{\varepsilon}\sup_{z\in[0,t]} C_2(z)\sup_{z\in[0,t]} D(z)\Big)\int_0^t\Big(1+E\sup_{z\in I\cup[0,s]} V(\sigma_l\wedge z, x(\sigma_l\wedge z))\Big)ds. \qquad (23)
\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
&\Big(1-\frac{\varepsilon d_3(p)}{2C_1}\Big)\Big(1+E\sup_{z\in I\cup[0,t]} V(\sigma_l\wedge z, x(\sigma_l\wedge z))\Big) \\
&\quad\leqslant 1 + E\sup_{\theta\in I} V(\theta,\varphi(\theta)) + c_2\sup_{z\in[0,t]} h(z) + \sup_{z\in[0,t]} C_2(z)\Big(d_1(p)E\|\varphi\|^p + d_2(p)|r(z)|^p + \frac{\widetilde{V}d_3(p)}{C_1}\Big) \\
&\qquad + \frac{d_3(p)}{C_1}\sup_{z\in[0,t]} C_2(z)\sup_{z\in[0,t]} D(z)\Big(1+\frac{16}{\varepsilon}\sup_{z\in[0,t]} C_2(z)\sup_{z\in[0,t]} D(z)\Big)\int_0^t\Big(1+E\sup_{z\in I\cup[0,s]} V(\sigma_l\wedge z, x(\sigma_l\wedge z))\Big)ds.
\end{aligned}$$

Choosing $\varepsilon = \dfrac{C_1}{d_3(p)}$, we have

$$1 + E\sup_{z\in I\cup[0,t]} V(\sigma_l\wedge z, x(\sigma_l\wedge z)) \leqslant A(t) + B(t)\int_0^t \Big(1+E\sup_{z\in I\cup[0,s]} V(\sigma_l\wedge z, x(\sigma_l\wedge z))\Big)ds, \qquad (24)$$

where the functions $A(t)$ and $B(t)$ are given by (12). By Lemma 1.1, we conclude that

$$E \sup_{z\in I\cup[0,t]} V(\sigma_l \wedge z, x(\sigma_l \wedge z)) \leqslant A(t)B(t)e^{B(t)t}, \quad t > 0. \tag{25}$$

Obviously, for $t = 0$, from (24) we conclude that

$$E \sup_{\theta\in I} V(\theta, \varphi(\theta)) \leqslant A(0) - 1 \leqslant A(0)B(0). \tag{26}$$

Specially, if $\sup_{t\geqslant 0} h(t) = H$, $\sup_{t\geqslant 0} C_2(t) = C_2$ and $\sup_{t\geqslant 0} |r(t)| = R$, then (24) becomes

$$1 + E \sup_{z\in I\cup[0,t]} V(\sigma_l \wedge z, x(\sigma_l \wedge z)) \leqslant \widetilde{A} + \widetilde{B}(t) \int_0^t \Big(1 + E \sup_{z\in I\cup[0,s]} V(\sigma_l \wedge z, x(\sigma_l \wedge z))\Big) ds,$$

where $\widetilde{A}$ and $\widetilde{B}(t)$ have the form (15). By the Gronwall type inequality (Bellman, [5]) one sees that

$$\sup_{z\in I\cup[0,t]} EV(\sigma_l \wedge z, x(\sigma_l \wedge z)) \leqslant \widetilde{A}e^{\widetilde{B}(t)t}, \quad t \geqslant 0. \tag{27}$$

From (25) and (7), for any $t \geqslant 0$, we obtain

$$\begin{aligned} A(t)B(t)e^{B(t)t} &\geqslant E \sup_{z\in I\cup[0,t]} V(\sigma_l \wedge z, x(\sigma_l \wedge z)) \geqslant EV(\sigma_l \wedge t, x(\sigma_l \wedge t)) \geqslant C_1 E|x(\sigma_l \wedge t)|^p \geqslant C_1 E|x(\sigma_l)|^p I_{\{\sigma_l \leqslant t\}} \\ &\geqslant C_1 l^p E I_{\{\sigma_l \leqslant t\}} \geqslant C_1 l^p P\{\sigma_l \leqslant t\} \end{aligned}$$

and then

$$P\{\sigma_l \leqslant t\} \leqslant \frac{A(t)B(t)e^{B(t)t}}{C_1 l^p}.$$

Analogously, in the special case

$$P\{\sigma_l \leqslant t\} \leqslant \frac{\widetilde{A}e^{\widetilde{B}(t)t}}{C_1 l^p}.$$

Since $t \geqslant 0$ is arbitrary, by letting $l \to \infty$, in the last inequality we have $P\{\sigma_\infty = \infty\} = 1$, i.e. there exists unique global solution of Eq. (1). Then, from (25) and (26) the required assertion (11) follows.

In the case where $\sup_{t\geqslant 0} h(t) = H$, $\sup_{t\geqslant 0} C_2(t) = C_2$ and $\sup_{t\geqslant 0} |r(t)| = R$, similar way we conclude that (14) holds with $\widetilde{A}$ and $\widetilde{B}(t)$ defined by (15). □

Remark 2.4. *If the conditions of Theorem 2.3 are satisfied and* $r(t) \equiv 0$, $t \geqslant 0$, *under the assumption* ***P3***, *for* $p > 1$, (17) *has the form*

$$|x(\sigma_l \wedge t)|^p \leqslant \Big(1 + \varepsilon^{\frac{1}{p-1}}\Big)^{p-1} \Big(|x(\sigma_l \wedge t) - u(\sigma_l \wedge t, x_{\sigma_l \wedge t})|^p + \varepsilon^{-1}\alpha^p \|x_{\sigma_l\wedge t}\|^p\Big).$$

Choosing $\varepsilon = \Big(\dfrac{\alpha}{1-\alpha}\Big)^{p-1}$, *instead of* (19) *we get*

$$\sup_{z\in[0,t]} |x(\sigma_l \wedge z)|^p \leqslant \frac{1}{(1-\alpha)^p} \sup_{z\in[0,t]} |x(\sigma_l \wedge z) - u(\sigma_l \wedge z, x_{\sigma_l\wedge z})|^p + \frac{\alpha}{1-\alpha}\|\varphi\|^p.$$

Moreover, for $p \in (0, 1]$, *we have*

$$|x(\sigma_l \wedge t)|^p \leqslant |x(\sigma_l \wedge t) - u(\sigma_l \wedge t, x_{\sigma_l\wedge t})|^p + \alpha^p \|x_{\sigma_l\wedge t}\|^p,$$

which gives us

$$\sup_{z\in[0,t]}|x(\sigma_l\wedge z)|^p \leqslant \frac{1}{1-\alpha^p}\Big(\sup_{z\in[0,t]}|x(\sigma_l\wedge z)-u(\sigma_l\wedge z, x_{\sigma_l\wedge z})|^p+\alpha^p\|\varphi\|^p\Big).$$

So, the estimate (11) *holds, with*

$$A(t) = 2+2E\sup_{\theta\in I}V(\theta,\varphi(\theta))+2c_2\sup_{z\in[0,t]}h(z)+2\sup_{z\in[0,t]}C_2(z)\left(d_1(p)E\|\varphi\|^p+\frac{\widetilde{V}d_4(p)}{C_1}\right),$$

$$B(t) = \frac{2d_4^2(p)}{C_1^2}\sup_{z\in[0,t]}C_2(z)\sup_{z\in[0,t]}D(s)\left(16\sup_{z\in[0,t]}C_2(s)\sup_{z\in[0,t]}D(s)+\frac{C_1}{d_4(p)}\right)\vee 1,$$

where $d_1(p)$ *is defined in* (13) *and*

$$d_4(p) = \begin{cases}\dfrac{1}{1-\alpha^p}, & p\in(0,1],\\ \dfrac{1}{(1-\alpha)^p}, & p\in(1,\infty).\end{cases} \tag{28}$$

In the special case, (14) *holds with*

$$\widetilde{A} = 2+2E\sup_{\theta\in I}V(\theta,x(\theta))+2c_2H+2C_2\left(d_1(p)E\|\varphi\|^p+\frac{\widetilde{V}d_4(p)}{C_1}\right),$$

$$\widetilde{B}(t) = \frac{2C_2d_4^2(p)}{C_1^2}\sup_{z\in[0,t]}D(s)\left(16C_2\sup_{z\in[0,t]}D(s)+\frac{C_1}{d_4(p)}\right). \qquad \triangle$$

Furthermore, (7) leads us to pth moment estimate of the solution.

Corollary 2.5. *Let the conditions of Theorem 2.3 be satisfied. Then, we have that*

$$E\sup_{z\in I\cup[0,t]}|x(z)|^p \leqslant \frac{1}{C_1}A(t)B(t)e^{B(t)t}, \quad t\geqslant 0, \tag{29}$$

where $A(t)$ *and* $B(t)$ *are given by* (12). *If* $\sup_{t\geqslant 0}h(t)=H$ *and* $\sup_{t\geqslant 0}C_2(t)=C_2$ *in condition* ***P2****, as well as* $\sup_{t\geqslant 0}|r(t)|=R$ *in condition* ***P3****, then*

$$E\sup_{z\in I\cup[0,t]}|x(z)|^p \leqslant \frac{\widetilde{A}}{C_1}e^{\widetilde{B}(t)t}, \quad t\geqslant 0, \tag{30}$$

where $\widetilde{A}$ *and* $\widetilde{B}(t)$ *are of the form* (15).

By applying Corollary 2.5 we can determine the upper bound of the generalized Lyapunov exponent, which is defined as follows.

Definition 2.6. *([8]) Let* $x(t)$ *be the solution of Eq.* (1) *and, for sufficiently large* $T>0$*, let* $\mu:[T,\infty]\to[0,\infty)$ *be increasing function, with* $\mu(t)\to\infty$*, when* $t\to\infty$*. The number*

$$\limsup_{t\to\infty}\frac{\ln|x(t)|}{\ln\mu(t)}$$

is the generalized Lyapunov exponent of $x(t)$ *with respect to* $\mu(t)$*.*

Theorem 2.7. *Assume that the conditions of Theorem 2.3 are satisfied. Let $T > 0$ be sufficiently large, such that $t-1 \geqslant T$ and, for every $\varepsilon > 0$, there exists*

$$\limsup_{t\to\infty} \frac{\ln A(t+1) + \ln B(t+1) + (B(t+1)+\varepsilon)(t+1)}{p\ln\mu(t-1)} = \nu, \tag{31}$$

with $A(t)$ and $B(t)$ defined in (12). *Then ν is the upper bound of the generalized Lyapunov exponent with respect to $\mu(t)$. In particular, if the conditions of Theorem 2.3 are satisfied with $\sup_{t\geqslant 0} h(t) = H$, $\sup_{t\geqslant 0} C_2(t) = C_2$ and $\sup_{t\geqslant 0} |r(t)| = R$, as well as, for every $\varepsilon > 0$,*

$$\limsup_{t\to\infty} \frac{(\widetilde{B}(t+1)+\varepsilon)(t+1)}{p\ln\mu(t-1)} = \widetilde{\nu},$$

with $\widetilde{B}(t)$ defined in (15), *then $\widetilde{\nu}$ is the upper bound of the generalized Lyapunov exponent with respect to $\mu(t)$.*

Proof. For each t with $t-1 \geqslant T$ there exists a positive integer n, such that $n-1 \leqslant t < n$. From (29) we get

$$E \sup_{t\in[n-1,n)} |x(t)|^p \leqslant \frac{1}{C_1} A(n)B(n)e^{B(n)n}.$$

For arbitrary $\varepsilon > 0$, Markov inequality gives us

$$P\left\{ \sup_{t\in[n-1,n)} |x(t)|^p > A(n)B(n)e^{(B(n)+\varepsilon)n} \right\} \leqslant \frac{1}{C_1 e^{\varepsilon n}}.$$

Applying the Borel-Cantelli lemma, we have that, for almost every $\omega \in \Omega$, there exists positive integer n_0, such that, for every $n \geqslant n_0$,

$$\sup_{t\in[n-1,n)} |x(t)|^p \leqslant A(n)B(n)e^{(B(n)+\varepsilon)n},$$

so the assumption of the theorem yields

$$\begin{aligned}\limsup_{t\to\infty} \frac{\ln|x(t)|}{\ln\mu(t)} &\leqslant \limsup_{n\to\infty} \frac{\ln A(n) + \ln B(n) + (B(n)+\varepsilon)n}{p\ln\mu(n-1)} \\ &\leqslant \limsup_{t\to\infty} \frac{\ln A(t+1) + \ln B(t+1) + (B(t+1)+\varepsilon)(t+1)}{p\ln\mu(t-1)} = \nu.\end{aligned}$$

Similarly, for $\sup_{t\geqslant 0} h(t) = H$, $\sup_{t\geqslant 0} C_2(t) = C_2$ and $\sup_{t\geqslant 0} |r(t)| = R$, using (30) we get

$$\limsup_{t\to\infty} \frac{\ln|x(t)|}{\ln\mu(t)} \leqslant \limsup_{n\to\infty} \frac{(\widetilde{B}(n)+\varepsilon)n}{p\ln\mu(n-1)} \leqslant \limsup_{t\to\infty} \frac{(\widetilde{B}(t+1)+\varepsilon)(t+1)}{p\ln\mu(t-1)} = \widetilde{\nu},$$

which completes the proof. □

3. Neutral Stochastic Differential Equations with the State-dependent Delay

The topic that has actively been developed during the last decades is describing physical systems by differential equations with delay that is not only time dependent, but also depends on current state of the system (see [13, 19, 29]). Namely, it turned out that many phenomena, that can be modeled by differential delay equations, in the domain of population dynamics ([1, 2, 6]), physics ([27]), human physiology ([12]), etc., depend on some characteristic that is directly related to the state of the system itself. This fact indicates that it is natural for delay to be both time and state dependent, in general case. As these phenomena are subject to the random influences, the models based on stochastic differential equations are more realistic than those based on differential equations. There are a few papers about stochastic differential equations with state-dependent delay, for example [3, 14, 26]. Starting from [3], the existence and uniqueness of the global solution are proven for autonomous d-dimensional stochastic differential equation with state-dependent delay under the global Lipschitz condition for the drift and diffusion coefficients, with the delay function which is assumed to be Lipschitz continuous and bounded. On the other hand, our results for NSFDEs are proved under the local Yamada-Watanabe condition or the local Lipschitz condition as well as Khasminskii-type conditions, without imposing any additional assumptions on delay function. It should be emphasized that in this paper the delayed argument could be unbounded from below, which is not case in [23, 24].

As a special case of Eq. (1), we consider the d-dimensional neutral stochastic differential equation with state-dependent delay

$$d[x(t) - u(t, x(t - \delta(t, x(t))))] = f(t, x(t), x(t - \delta(t, x(t))))\,dt + g(t, x(t), x(t - \delta(t, x(t))))\,dB(t),\ \ t \geqslant 0, \tag{32}$$

where the equation coefficients are defined same way as in Eq. (1), but with $W = \mathbb{R}^d$, satisfying initial condition

$$x_0 = \varphi = \{\varphi(\theta) : \theta \in I\} \in W_{\mathcal{F}_0}. \tag{33}$$

Here, $W_{\mathcal{F}_0}$ is the family of $\mathcal{F}_0$-measurable $\mathbb{R}^d$-valued random variables from $\mathcal{M}^p(I; \mathbb{R}^d)$ and

- $I = (-\infty, 0]$ if $\sup\limits_{(t,x)\in[0,\infty)\times\mathbb{R}^d} (\delta(t, x) - t) = \infty$;
- $I = [-\kappa, 0]$ if $\sup\limits_{(t,x)\in[0,\infty)\times\mathbb{R}^d} (\delta(t, x) - t) = \kappa$.

State-dependent delay function δ is non-negative, Borel measurable, defined on the set $[0, \infty) \times \mathbb{R}^d$ and it can be bounded or unbounded.

For Eq. (32), assertions of Theorem 2.3 and Theorem 2.7 hold, so it has unique global solution and generalized Lyapunov exponent can be found. Also, by Corollary 2.5, the estimate of $E\sup_{z\in I\cup[0,t]} |x(z)|^p$ is obtained. Besides, replacing (8) with new condition can contribute to achieving different pth moment estimate of the solution, so we propose following assumption.

P2′. There exist a function $V \in C^{1,2}(I \cup [0, \infty) \times \mathbb{R}^d; [0, \infty))$, positive constant C_1, non-negative constant c_2 and functions C_2 and h in $C([0, \infty); [0, \infty))$, such that, for every $(t, x) \in [0, \infty) \times \mathbb{R}^d$ and $p > 0$, (7) holds and there exists a function $\overline{D} \in C([0, \infty); [0, \infty))$, such that

$$LV(t, x, y) \leqslant \overline{D}(t)\Big(1 + V(t, x) + V(t - \delta(t, x), y)\Big), \tag{34}$$

for every $(t, x, y) \in [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$.

Theorem 3.1. *If, for $W = \mathbb{R}^d$, one of the assumptions **P1** and **P1′** hold, together with **P2′** and **P3**, then, for any initial condition* (33)*, there exists a unique global solution $x(t)$, $t \in I \cup [0, \infty)$, of Eq.* (32)*. Moreover, for every $t \geqslant 0$*

$$\sup_{z\in I\cup[0,t]} EV(z, x(z)) \leqslant \frac{1}{2} a(t)b(t)e^{b(t)t}, \tag{35}$$

where

$$\begin{aligned} a(t) &= 1 + 2\sup_{\theta\in I} EV(\theta, \varphi(\theta)) + 2c_2 \sup_{z\in[0,t]} h(z) + 2\sup_{z\in[0,t]} C_2(z)\bigg(d_1(p)E\|\varphi\|^p + d_2(p)\sup_{z\in[0,t]} |r(z)|^p + \frac{\widetilde{V}d_3(p)}{C_1}\bigg), \\ b(t) &= \frac{d_3(p)}{C_1} \sup_{z\in[0,t]} C_2(z) \sup_{z\in[0,t]} \overline{D}(z) \vee 1, \end{aligned} \tag{36}$$

for $\widetilde{V} = EV\Big(0, \varphi(0) - u(0, \varphi(-\delta(0, \varphi(0))))\Big)$ *and* $d_1(p)$, $d_2(p)$, $d_3(p)$ *given by* (13).

Specially, if $\sup_{t\geqslant 0} h(t) = H$ *and* $\sup_{t\geqslant 0} C_2(t) = C_2$ *in condition* ***P2'***, *as well as* $\sup_{t\geqslant 0} |r(t)| = R$ *in condition* ***P3***, *then*

$$\sup_{z\in I\cup[0,t]} EV(z, x(z)) \leqslant \frac{\widetilde{a}}{2} e^{\widetilde{b}(t)t}, \quad t \geqslant 0, \tag{37}$$

where

$$\widetilde{a} = 1+2\sup_{\theta\in I} EV(\theta, \varphi(\theta))+2c_2H+2C_2\bigg(d_1(p)E\|\varphi\|^p+d_2(p)R^p+\frac{\widetilde{V}d_3(p)}{C_1}\bigg), \qquad \widetilde{b}(t) = \frac{2C_2d_3(p)}{C_1}\sup_{z\in[0,t]} \overline{D}(z). \tag{38}$$

Proof. If we define the sequence of stopping times $\{\sigma_l\}_{l\geqslant l_0}$ same way as in Theorem 2.3, similarly to (19) we can prove that, for $p > 1$,

$$\begin{aligned}\sup_{z\in[0,t]} E|x(\sigma_l \wedge z)|^p \leqslant{}& \frac{(1-\sqrt{\alpha})^{1-p}}{1-\alpha} \sup_{z\in[0,t]} E|x(\sigma_l \wedge z) - u(\sigma_l \wedge z, x(\sigma_l \wedge z - \delta(\sigma_l \wedge z, x(\sigma_l \wedge z))))|^p \\ &+ \frac{\alpha}{1-\alpha}E\|\varphi\|^p + \frac{(\sqrt{\alpha}-\alpha)^{1-p}}{1-\alpha}\sup_{z\in[0,t]}|r(z)|^p \end{aligned}\tag{39}$$

and for $p \in (0, 1]$, instead of (20), we obtain

$$\begin{aligned}&\sup_{z\in[0,t]} E|x(\sigma_l \wedge z)|^p \\ &\quad\leqslant \frac{1}{1-\alpha^p}\Big(\sup_{z\in[0,t]} E|x(\sigma_l \wedge z) - u(\sigma_l \wedge z, x(\sigma_l \wedge z - \delta(\sigma_l \wedge z, x(\sigma_l \wedge z))))|^p + \alpha^p E\|\varphi\|^p + \sup_{z\in[0,t]}|r(z)|^p\Big).\end{aligned}\tag{40}$$

The generalized Itô formula, (3) and (34) give us

$$\begin{aligned}dV\Big(t, x(t) - u(t, x(t-\delta(t, x(t))))\Big) &= LV\Big(t, x(t), x(t-\delta(t, x(t)))\Big)dt + dM(t) \\ &\leqslant D(t)\Big(1 + V(t, x(t)) + V(t-\delta(t, x(t)), x(t-\delta(t, x(t))))\Big)dt + dM_1(t),\end{aligned}\tag{41}$$

where

$$M_1(t) = \int_0^t V_x(s, x(s) - u(s, x(s-\delta(s, x(s)))))\, g(s, x(s), x(s-\delta(s, x(s))))\, dB(s)$$

is a local martingale and $M_1(0) = 0$. For any $l \geqslant l_0$, integrating and taking the expectation of both sides of (41), we obtain that, for every $t \geqslant 0$,

$$\begin{aligned}&EV\Big(\sigma_l \wedge t, x(\sigma_l \wedge t) - u(\sigma_l \wedge t, x(\sigma_l \wedge t - \delta(\sigma_l \wedge t, x(\sigma_l \wedge t))))\Big) - \widetilde{V} \\ &\quad\leqslant \int_0^t \sup_{z\in[0,s]} D(z)\Big(1+EV(\sigma_l \wedge s, x(\sigma_l \wedge s)) + EV(\sigma_l \wedge s - \delta(\sigma_l \wedge s, x(\sigma_l \wedge s)), x(\sigma_l \wedge s - \delta(\sigma_l \wedge s, x(\sigma_l \wedge s))))\Big)ds \\ &\quad\leqslant \sup_{z\in[0,t]} D(z)\int_0^t \Big(1+2\sup_{z\in I\cup[0,s]} EV(\sigma_l \wedge z, x(\sigma_l \wedge z))\Big)ds.\end{aligned}\tag{42}$$

Hence, for $p > 0$, using (7), we get

$$\sup_{z\in I\cup[0,t]} EV(\sigma_l \wedge z, x(\sigma_l \wedge z)) \leqslant \sup_{\theta\in I} EV(\theta, \varphi(\theta)) + c_2 \sup_{z\in[0,t]} h(z) + \sup_{z\in[0,t]} C_2(z) \sup_{z\in[0,t]} E|x(\sigma_l \wedge z)|^p,$$

so, from (39), (40) and (42), we derive

$$\begin{aligned}
\sup_{z\in I\cup[0,t]} & EV(\sigma_l\wedge z, x(\sigma_l\wedge z)) \\
&\leqslant \sup_{\theta\in I} EV(\theta,\varphi(\theta)) + c_2\sup_{z\in[0,t]} h(z) + \sup_{z\in[0,t]} C_2(z)\Big(d_1(p)E\|\varphi\|^p + d_2(p)\sup_{z\in[0,t]}|r(z)|^p\Big) \\
&\quad + \sup_{z\in[0,t]} C_2(z) d_3(p)\sup_{z\in[0,t]} E|x(\sigma_l\wedge z) - u(\sigma_l\wedge z, x(\sigma_l\wedge z - \delta(\sigma_l\wedge z, x(\sigma_l\wedge z))))|^p \\
&\leqslant \sup_{\theta\in I} EV(\theta,\varphi(\theta)) + c_2\sup_{z\in[0,t]} h(z) + \sup_{z\in[0,t]} C_2(z)\left(d_1(p)E\|\varphi\|^p + d_2(p)\sup_{z\in[0,t]}|r(z)|^p + \frac{\widetilde{V}d_3(p)}{C_1}\right) \\
&\quad + \frac{d_3(p)}{C_1}\sup_{z\in[0,t]} C_2(z)\sup_{z\in[0,t]} D(z)\int_0^t\Big(1+2\sup_{z\in I\cup[0,s]} EV(\sigma_l\wedge z, x(\sigma_l\wedge z))\Big)ds,
\end{aligned}$$

where $d_1(p)$, $d_2(p)$ and $d_3(p)$ are defined in (13). Then, we have that

$$1+2\sup_{z\in I\cup[0,t]} EV(\sigma_l\wedge z, x(\sigma_l\wedge z)) \leqslant a(t) + b(t)\int_0^t\Big(1+2\sup_{z\in I\cup[0,s]} EV(\sigma_l\wedge z, x(\sigma_l\wedge z))\Big)ds, \tag{43}$$

where $a(t)$ and $b(t)$ are given in (36). Since (43) has the form (4), by applying Lemma 1.1, we conclude that

$$\sup_{z\in I\cup[0,t]} EV(\sigma_l\wedge z, x(\sigma_l\wedge z)) \leqslant \frac{1}{2}a(t)b(t)e^{b(t)t}, \quad t>0. \tag{44}$$

For $t = 0$, from (43) we conclude that

$$\sup_{\theta\in I} EV(\theta,\varphi(\theta)) \leqslant \frac{a(0)-1}{2} \leqslant \frac{a(0)b(0)}{2}.$$

Specially, if $\sup_{t\geqslant 0} h(t) = H$, $\sup_{t\geqslant 0} C_2(t) = C_2$ and $\sup_{t\geqslant 0}|r(t)| = R$, then (43) becomes

$$1+2\sup_{z\in I\cup[0,t]} EV(\sigma_l\wedge z, x(\sigma_l\wedge z)) \leqslant \widetilde{a} + \widetilde{b}(t)\int_0^t\Big(1+2\sup_{z\in I\cup[0,s]} EV(\sigma_l\wedge z, x(\sigma_l\wedge z))\Big)ds,$$

where $\widetilde{a}$ and $\widetilde{b}(t)$ have the form (38). By the Gronwall type inequality (Bellman, [5]) one derives

$$\sup_{z\in I\cup[0,t]} EV(\sigma_l\wedge z, x(\sigma_l\wedge z)) \leqslant \frac{\widetilde{a}}{2}e^{\widetilde{b}(t)t}, \quad t\geqslant 0.$$

From (44) and (7), for any $t \geqslant 0$, we obtain

$$\begin{aligned}
\frac{1}{2}a(t)b(t)e^{b(t)t} &\geqslant \sup_{z\in I\cup[0,t]} EV(\sigma_l\wedge z, x(\sigma_l\wedge z)) \geqslant EV(\sigma_l\wedge t, x(\sigma_l\wedge t)) \geqslant C_1E|x(\sigma_l\wedge t))|^p \geqslant C_1E|x(\sigma_l)|^p I_{\{\sigma_l\leqslant t\}} \\
&\geqslant C_1 l^p EI_{\{\sigma_l\leqslant t\}} \geqslant C_1 l^p P\{\sigma_l\leqslant t\}
\end{aligned}$$

and then

$$P\{\sigma_l\leqslant t\} \leqslant \frac{a(t)b(t)e^{b(t)t}}{2C_1 l^p}.$$

Analogously, in the special case

$$P\{\sigma_l\leqslant t\} \leqslant \frac{\widetilde{a}e^{\widetilde{b}(t)t}}{2C_1 l^p}.$$

Analogously to the proof of Theorem 2.3, we conclude that (35) and (37) hold. □

Remark 3.2. *Analogously to Remark 2.4, if the conditions of Theorem 3.1 are satisfied and* $r(t) \equiv 0$, $t \geqslant 0$, *in the assumption* ***P3***, *the assertion* (35) *holds with*

$$
\begin{aligned}
a(t) &= 1 + 2\sup_{\theta\in I} EV(\theta,\varphi(\theta)) + 2c_2 \sup_{z\in[0,t]} h(z) + 2\sup_{z\in[0,t]} C_2(z)\left(d_1(p)E\|\varphi\|^p + \frac{\widetilde{V}d_4(p)}{C_1}\right),\\
b(t) &= \frac{2d_4(p)}{C_1}\sup_{z\in[0,t]} C_2(z)\sup_{z\in[0,t]}\overline{D}(z)\vee 1,
\end{aligned}
\tag{45}
$$

where $d_1(p)$ *is defined in* (13) *and* $d_4(p)$ *in* (28). *Also, in the special case,* (37) *holds with*

$$
\widetilde{a} = 1 + 2\sup_{\theta\in I} EV(\theta,\varphi(\theta)) + 2c_2H + 2C_2\left(d_1(p)E\|\varphi\|^p + \frac{\widetilde{V}d_4(p)}{C_1}\right), \qquad \widetilde{b}(t) = \frac{2C_2d_4(p)}{C_1}\sup_{z\in[0,t]}\overline{D}(z). \qquad \triangle
$$

Corollary 3.3. *If the conditions of Theorem 3.1 are satisfied, then*

$$
\sup_{z\in I\cup[0,t]} E|x(z)|^p \leqslant \frac{1}{2C_1}a(t)b(t)e^{b(t)t}, \quad t\geqslant 0,
$$

where $a(t)$ *and* $b(t)$ *are defined by* (36). *If* $\sup_{t\geqslant 0} h(t) = H$ *and* $\sup_{t\geqslant 0} C_2(t) = C_2$ *in condition* ***P2′***, *as well as* $\sup_{t\geqslant 0}|r(t)| = R$ *in condition* ***P3***, *then*

$$
\sup_{z\in I\cup[0,t]} E|x(z)|^p \leqslant \frac{\widetilde{a}}{2C_1}e^{\widetilde{b}(t)t}, \quad t\geqslant 0,
$$

where $\widetilde{a}$ *and* $\widetilde{b}(t)$ *are given by* (38).

4. Examples

In order to illustrate the previous theory, let us impose two examples for stochastic differential equations with state-dependent delay. First, we consider the case where the drift and diffusion coefficients are highly nonlinear and the conditions of Theorem 3.1, i.e. Remark 3.2 are satisfied. In the second example the coefficients of the equation satisfy the linear growth condition and the assumptions of Theorem 2.3 hold. Then, by Theorem 2.7, the generalized Lyapunov exponent is obtained.

Example 4.1. *Let us consider one-dimensional neutral stochastic differential equation of the form* (32), *with*

$$
\begin{aligned}
&f:[0,\infty)\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}, \quad f(t,x,y) = \sqrt{1+t+x^2+y^2} - (x+ay\sin t)^{2k-1},\\
&g:[0,\infty)\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}, \quad g(t,x,y) = t + (x+ay\sin t)^2 + \frac{(x+ay\sin t)^{2m}}{\sqrt{1+t}},
\end{aligned}
$$

where $k \geqslant 2m-1$, $k\in\mathbb{N}$, $m\geqslant 1/2$, $a\in(0,1)$ *and*

$$
u:[0,\infty)\times\mathbb{R}\to\mathbb{R}, \quad u(t,y) = -ay\sin t,
$$

with the initial condition, given by $\varphi(\theta) = \theta+1$, $\theta\in[-1,0]$. *The delay function is defined with*

$$
\delta:[0,\infty)\times\mathbb{R}\to[0,\infty), \quad \delta(t,x) = \frac{x^2}{1+x^2} + t.
$$

It is clear that the drift and diffusion coefficients satisfy the condition ***P1′***, *as well as that*

$$
|u(t,y_1) - u(t,y_2)| = |ay_2\sin t - ay_1\sin t| \leqslant a|y_1-y_2|, \quad t\geqslant 0, \quad y_1,y_2\in\mathbb{R},
$$

*so with $u(t,0)=0$, assumption **P3** is satisfied. According to the drift and diffusion coefficients, to show that* (34) *holds, suitable choice for the function V is*

$$V:[-1,\infty)\times\mathbb{R}\to[0,\infty),\quad V(t,x)=\sqrt{1+t+x^2}.$$

It is obvious that estimates (9) *can not be obtained, but we have that*

$$|x|\leqslant V(t,x)\leqslant\sqrt{1+t}+|x|,\quad t\geqslant 0,\quad x\in\mathbb{R},$$

so, from (7)*, we see that $p=1$, $C_1=1$, $c_2=1$, $C_2(t)\equiv 1$ and $h(t)=\sqrt{1+t}$, for $t\geqslant 0$. Also*

$$V_t(t,x)=\frac{1}{2\sqrt{1+t+x^2}},\quad V_x(t,x)=\frac{x}{\sqrt{1+t+x^2}},\quad V_{xx}(t,x)=\frac{1+t}{\sqrt{(1+t+x^2)^3}}.$$

For every $t\geqslant 0$ and $x,y\in\mathbb{R}$, by applying some elementary inequalities, we get the next estimate for the operator LV

$$\begin{aligned}
LV(t,x,y)&=\frac{1}{2\sqrt{1+t+(x+ay\sin t)^2}}+\frac{\big(x+ay\sin t\big)\Big(\sqrt{1+t+x^2+y^2}-(x+ay\sin t)^{2k-1}\Big)}{\sqrt{1+t+(x+ay\sin t)^2}}\\
&\quad+\frac{1+t}{2\sqrt{(1+t+(x+ay\sin t)^2)^3}}\left(t+(x+ay\sin t)^2+\frac{(x+ay\sin t)^{2m}}{\sqrt{1+t}}\right)^2\\
&\leqslant\frac{1}{2\sqrt{1+t}}+\sqrt{1+t+x^2+y^2}-\frac{\big(x+ay\sin t\big)^{2k}}{\sqrt{1+t+(x+ay\sin t)^2}}+\frac{(1+t)(1+t+(x+ay\sin t)^2)^2}{\sqrt{(1+t+(x+ay\sin t)^2)^3}}\\
&\quad+\frac{(x+ay\sin t)^{4m}}{\sqrt{(1+t+(x+ay\sin t)^2)^3}}\\
&\leqslant\frac{1}{2\sqrt{1+t}}+\sqrt{1+t+x^2+y^2}+(1+t)\sqrt{1+t+(x+ay\sin t)^2}+\frac{(x+ay\sin t)^{2(2m-1)}-\big(x+ay\sin t\big)^{2k}}{\sqrt{1+t+(x+ay\sin t)^2}}\\
&\leqslant\frac{1}{2\sqrt{1+t}}+\big(1+\sqrt{2}(1+t)\big)\Big(\sqrt{1+t+x^2}+\sqrt{1+t-\delta(t,x)+y^2}\Big)+\frac{(x+ay\sin t)^{2(2m-1)}-\big(x+ay\sin t\big)^{2k}}{\sqrt{1+t+(x+ay\sin t)^2}}\\
&\leqslant\frac{1}{2\sqrt{1+t}}+\big(1+\sqrt{2}(1+t)\big)\big(V(t,x)+V(t-\delta(t,x),y)\big)+\frac{K(k,m)}{\sqrt{1+t}},
\end{aligned}$$

where $K(k,m)=\frac{k-2m+1}{k}\left(\frac{2m-1}{k}\right)^{\frac{2m-1}{k-2m+1}}$. Hence,

$$LV(t,x,y)\leqslant\overline{D}(t)\big(1+V(t,x)+V(t-\delta(t,x),y)\big),$$

*which means that the assumption **P2′** is satisfied, with*

$$\overline{D}:[0,\infty)\to[0,\infty),\quad \overline{D}(t)=1+\sqrt{2}(1+t).$$

According to Theorem 3.1, there is the unique global solution of considered equation and

$$\sup_{z\in I\cup[0,t]}EV(z,x(z))\leqslant\frac{1}{2}a(t)b(t)e^{b(t)t},\quad t\geqslant 0,$$

where, from (45),

$$a(t)=1+2\sqrt{2}+2\sqrt{1+t}+\frac{2(a+\sqrt{2})}{1-a},\qquad b(t)=\frac{2}{1-a}\big(1+\sqrt{2}(1+t)\big).$$

Note that the coefficients given in the previous example do not satisfy the assumption (10) of Theorem 2.3. In the next example, the conditions of that theorem are satisfied.

Example 4.2. *Let the coefficients of Eq.* (32) *have the form:*

$$f : [0,\infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad f(t,x,y) = \sin x + \sqrt{1+t+x^2},$$
$$g : [0,\infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad g(t,x,y) = \ln\left(2 - \frac{x^2}{1+x^2} + (x+ay)^2\right),$$
$$u : [0,\infty) \times \mathbb{R} \to \mathbb{R}, \quad u(t,y) = -t - ay$$

and the delay function, as well as the initial condition, are given in the previous example. It is clear that

$$|u(t,y_1) - u(t,y_2)| = |ay_2 - ay_1| \leqslant a|y_1 - y_2|, \quad t \geqslant 0, \quad y_1, y_2 \in \mathbb{R},$$

*so with $u(t,0) = -t$, assumption **P3** is satisfied, as well as assumption **P1'**. Choosing the function V same way as in Example 4.1, for every $t \geqslant 0$ and $x, y \in \mathbb{R}$,*

$$\begin{aligned}
LV(t,x,y) &= \frac{1}{2\sqrt{1+t+(x+t+ay)^2}} + \frac{\big(x+t+ay\big)\big(\sin x + \sqrt{1+t+x^2}\big)}{\sqrt{1+t+(x+t+ay)^2}} \\
&\quad + \frac{\big(1+t\big)\Big(\ln\big(2+t-\delta(t,x)+(x+ay)^2\big)\Big)^2}{2\sqrt{(1+t+(x+t+ay)^2)^3}} \\
&\leqslant \frac{1}{2\sqrt{1+t}} + |\sin x| + V(t,x) \\
&\quad + \frac{\sqrt{2}\big(1+t\big)\big(1+t-\delta(t,x)+(x+t+ay)^2+t^2\big)^{\frac{3}{2}}}{\sqrt{(1+t+(x+t+ay)^2)^3}} \sqrt{1+t-\delta(t,x)+(x+ay)^2} \\
&\leqslant 1 + \frac{1}{2\sqrt{1+t}} + V(t,x) + \frac{2\big(1+t\big)\Big(\big(1+t+(x+t+ay)^2\big)^{\frac{3}{2}} + t^3\Big)}{\sqrt{(1+t+(x+t+ay)^2)^3}} \sqrt{1+t-\delta(t,x)+(x+ay)^2} \\
&\leqslant 1 + \frac{1}{2\sqrt{1+t}} + V(t,x) + 2\sqrt{2}(1+t)\left(1 + \frac{t^3}{\sqrt{(1+t)^3}}\right)\Big(\sqrt{1+t-\delta(t,x)+(ay)^2} + |x|\Big) \\
&\leqslant 1 + \frac{1}{2\sqrt{1+t}} + V(t,x) + 2\sqrt{2}\left(1+t+\frac{t^3}{\sqrt{1+t}}\right)\Big(V(t-\delta(t,x),y) + V(t,x)\Big) \\
&\leqslant D(t)\Big(1 + \sup_{s\in I\cup[0,t]} V(s,x)\Big),
\end{aligned}$$

which means that (8) *is satisfied, with*

$$D : [0,\infty) \to [0,\infty), \quad D(t) = 1 + 4\sqrt{2}\left(1+t+\frac{t^3}{\sqrt{1+t}}\right).$$

As

$$\begin{aligned}
|V_x(t,x-u(t,y))g(t,x,y)| &\leqslant \left|\frac{\big(x+t+ay\big)\ln\big(2+t-\delta(t,x)+(x+ay)^2\big)}{\sqrt{1+t+(x+t+ay)^2}}\right| \\
&\leqslant \ln\big(2+t-\delta(t,x)+(x+ay)^2\big) \\
&\leqslant 2\ln\sqrt{2+t-\delta(t,x)+(x+ay)^2} \\
&\leqslant 2\ln\Big(1+\sqrt{1+t-\delta(t,x)+(x+ay)^2}\Big),
\end{aligned}$$

we find that

$$\begin{aligned}|V_x(t,x-u(t,y))g(t,x,y)| &\leqslant 2\sqrt{1+t-\delta(t,x)+(x+ay)^2}\\ &\leqslant 2\sqrt{2}\Big(V(t-\delta(t,x),y)+V(t,x)\Big)\\ &\leqslant D(t)\Big(1+\sup_{s\in I\cup[0,t]}V(s,x)\Big),\end{aligned}$$

such that (10) *holds. According to Theorem 2.3, there exists the unique global solution of considered equation, and*

$$E\sup_{z\in I\cup[0,t]}V(z,x(z))\leqslant A(t)B(t)e^{B(t)t},\quad t\geqslant 0,$$

where, from (12),

$$A(t)=2\left(1+\sqrt{2}+\sqrt{1+t}+\frac{t+a+\sqrt{1+(1-\frac{a}{2})^2}}{1-a}\right),$$

$$B(t)=\frac{1}{(1-a)^2}\left(1+4\sqrt{2}\left(1+t+\frac{t^3}{\sqrt{1+t}}\right)\right)\left(17-a+64\sqrt{2}\left(1+t+\frac{t^3}{\sqrt{1+t}}\right)\right).$$

Since the conditions of Theorem 2.3 are satisfied, from Theorem 2.7 the upper bound of the generalized Lyapunov exponent with respect to $\mu(t)$ can be determined, where, for sufficiently large $T>0$, $\mu:[T,\infty]\to[0,\infty)$ is increasing function, with $\mu(t)\to\infty$, when $t\to\infty$. As, for $\varepsilon>0$,

$$\begin{aligned}&\ln A(t+1)+\ln B(t+1)+(B(t+1)+\varepsilon)(1+t)\\ &=\ln 2+\ln\frac{1}{(1-a)^2}+\ln\left(1+\sqrt{2}+\sqrt{2+t}+\frac{1+t+a+\sqrt{1+(1-\frac{a}{2})^2}}{1-a}\right)+\ln\left(1+4\sqrt{2}\left(2+t+\frac{(1+t)^3}{\sqrt{2+t}}\right)\right)\\ &+\ln\left(17-a+64\sqrt{2}\left(2+t+\frac{(1+t)^3}{\sqrt{2+t}}\right)\right)+\frac{1+t}{(1-a)^2}\left(1+4\sqrt{2}\left(2+t+\frac{(1+t)^3}{\sqrt{2+t}}\right)\right)\left(17-a+64\sqrt{2}\left(2+t+\frac{(1+t)^3}{\sqrt{2+t}}\right)\right)+\varepsilon(1+t),\end{aligned}$$

suitable choice of function $\mu(t)$ is made by having in mind (31). *It is clear that, for $\mu(t)=e^{t^6}$ and every $\varepsilon>0$*

$$\limsup_{t\to\infty}\frac{\ln A(t+1)+\ln B(t+1)+(B(t+1)+\varepsilon)(t+1)}{(t-1)^6}=\frac{512}{(1-a)^2},$$

so, the upper bound of the generalized Lyapunov exponent with respect to $\mu(t)=e^{t^6}$ is $\dfrac{512}{(1-a)^2}$.

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