



On m -point homogeneous polytopes in Euclidean spaces

Valerii Nikolaevich Berestovskiĭ^a, Yuriĭ Gennadievich Nikonorov^b

^a*Sobolev Institute of Mathematics of the SB RAS, 4 Acad. Koptyug Ave., Novosibirsk 630090, Russian Federation*

^b*Southern Mathematical Institute of VSC RAS, 53 Vatutina St., Vladikavkaz, 362025, Russian Federation*

Abstract. This paper is devoted to the study the m -point homogeneity property and the point homogeneity degree for finite metric spaces. Since the vertex sets of regular polytopes, as well as of some their generalizations, are homogeneous, we pay much attention to the study of the homogeneity properties of the vertex sets of polytopes in Euclidean spaces. Among main results, there is a classification of polyhedra with all edges of equal length and with 2-point homogeneous vertex sets. In addition, a significant part of the paper is devoted to the development of methods and tools for studying the relevant objects.

1. Introduction

The main object of our study are homogeneous metric spaces with special properties. For instance, we interested in *homogeneous* finite metric spaces (M, d) which means that the isometry group $\text{Isom}(M)$ of (M, d) acts transitively on M . A finite homogeneous metric subspace of an Euclidean space represents the vertex set of a compact convex polytope with the isometry group that is transitive on the vertex set; in each case, all vertices lie on a sphere. In [6, 8, 9], the authors obtained the complete description of the metric properties of the vertex sets of regular and semiregular polytopes in Euclidean spaces from the point of view of the normal homogeneity and the Clifford — Wolf homogeneity, see also survey [10]. The corresponding classes of (generalized) normal homogeneous and Clifford — Wolf homogeneous Riemannian manifolds were studied earlier in [2–5, 7].

This paper is devoted to another one special property of metric spaces, which is a strengthening of the notion of homogeneity.

Definition 1.1. A metric space (M, d) is called m -point homogeneous, $m \in \mathbb{N}$, if for every m -tuples (A_1, A_2, \dots, A_m) and (B_1, B_2, \dots, B_m) of elements of M such that $d(A_i, A_j) = d(B_i, B_j)$, $i, j = 1, \dots, m$, there is an isometry $f \in \text{Isom}(M)$ with the following property: $f(A_i) = B_i$, $i = 1, \dots, m$.

From a formal point of view, there can be coinciding points among the points A_1, \dots, A_m (as well as among the points B_1, \dots, B_m) in the above definition. This observation immediately implies the following result.

2020 *Mathematics Subject Classification.* Primary 52B15; Secondary 54E35, 20B05.

Keywords. Archimedean solid; Finite homogeneous metric space; Gosset polytope; m -point homogeneous metric space; Platonic solid; Point homogeneity degree; Regular polytope; Semiregular polytope.

Received: 08 July 2022; Accepted: 22 December 2022

Communicated by Mića S. Stanković and Zoran Rakić

The work of the first author was carried out within the framework of the state Contract to the IM SB RAS, project FWNF-2022-0006

Email addresses: valeraberestovskii@gmail.com (Valerii Nikolaevich Berestovskiĭ), nikonorov2006@mail.ru (Yuriĭ Gennadievich Nikonorov)

Proposition 1.1. *If a metric space (M, d) is m -point homogeneous for some $m \geq 2$, then it is k -point homogeneous for any $k = 1, \dots, m - 1$. If the set M is finite, l is its cardinality, and (M, d) is l -point homogeneous, then (M, d) is m -point homogeneous for all $m \geq 1$.*

The most interesting results of this paper related to finite subsets of Euclidean space \mathbb{R}^n with some degree of homogeneity. It is assumed that any such set M is supplied with the metric d induced from \mathbb{R}^n , hence, is a finite metric space itself.

Since the barycenter of a finite system of material points (with one and the same mass) in any Euclidean space is preserved for any bijection (in particular, any isometry) of this system, we have the following result.

Proposition 1.2 ([6]). *Let $M = \{x_1, \dots, x_q\}$, $q \geq n + 1$, be a finite homogeneous metric subspace of Euclidean space \mathbb{R}^n , $n \geq 2$. Then M is the vertex set of a convex polytope P , that is situated in some sphere in \mathbb{R}^n with radius $r > 0$ and center $x_0 = \frac{1}{q} \cdot \sum_{k=1}^q x_k$. In particular, $\text{Isom}(M, d) \subset O(n)$.*

This result shows that the theory of convex polytopes is very important for the study of finite homogeneous subspaces of Euclidean spaces. For a more detailed acquaintance with the theory of convex polytopes, we recommend [11, 19, 20, 26].

Let P be a non-degenerate convex polytope in \mathbb{R}^n with the barycenter in the origin $O = (0, 0, \dots, 0) \in \mathbb{R}^n$ (the property to be non-degenerate means that $\text{Lin}(P) = \mathbb{R}^n$ or, equivalently, O is in the interior of P). The symmetry group $\text{Symm}(P)$ of P is the group of isometries of \mathbb{R}^n that preserve P . It is clear that each $\psi \in \text{Symm}(P)$ is an orthogonal transformation of \mathbb{R}^n (obviously, $\psi(O) = O$ for any symmetry ψ of P).

If M is the vertex set of a polytope P supplied with the metric d induced from the Euclidean one in \mathbb{R}^n , then the isometry group $\text{Isom}(M, d)$ of the metric space (M, d) is the same as $\text{Symm}(P)$.

Recall that a polytope P in \mathbb{R}^n is *homogeneous* (or *vertex-transitive*) if its symmetry (isometry) group acts transitively on the set of its vertices. The following definition is natural.

Definition 1.2. *A convex polytope P in \mathbb{R}^n is called m -point homogeneous if its vertex set (with induced metric d from \mathbb{R}^n) is m -point homogeneous.*

In particular, the property to be 1-point homogeneous means just the property to be homogeneous. Recall that regular as well as semiregular polytopes in Euclidean spaces are homogeneous (see details in Section 3).

There are many equivalent definitions of regular (convex) polytopes [28]. One of this is as follows: A polytope P is regular if its symmetry group acts transitively on flags of P . Recall that a flag in an n -dimensional polytope P is a sequence of its faces F_0, F_1, \dots, F_{n-1} , where F_k is a k -dimensional face and $F_i \subset F_{i+1}$, $i = 0, 1, \dots, n - 2$. See discussion of this property e. g. in [22, Chapter 4].

The above definition leads to the conjecture that the vertex set of any n -dimensional regular polytope is n -point homogeneous. But this conjecture is false (see details in Section 5). Therefore, the following problem is natural:

Problem 1.1. *Classify all convex polytopes P in \mathbb{R}^n whose vertex sets are m -point homogeneous, where $m \geq 2$.*

One can check that every regular polyhedron in \mathbb{R}^3 is 2-point homogeneous (Proposition 3.2). The same property has the cuboctahedron (Proposition 6.1). The main results of this paper are the following theorems.

Theorem 1.1. *The regular tetrahedron, cube, octahedron, icosahedron are m -point homogeneous polyhedra for every natural m , while the regular dodecahedron is 2-point homogeneous but is not 3-point homogeneous.*

Theorem 1.2. *If all edges of a given 2-point homogeneous polyhedron $P \subset \mathbb{R}^3$ have the same length, then P is either the cuboctahedron or a regular polyhedron.*

We obtain also several interesting results on m -point homogeneous finite subsets in Euclidean spaces \mathbb{R}^n for an arbitrary n . In particular, we get a couple of results on the m -point homogeneity degree (see Definition 5.1) for some particular types of polytopes. In order to do this, we developed some special tools and approaches that could be useful in the future studies.

As a rule, we use standard notation. When we deal with some metric space (M, d) , for given $c \in M$ and $r \geq 0$, we consider $S(c, r) = \{x \in M \mid d(x, c) = r\}$, $U(c, r) = \{x \in M \mid d(x, c) < r\}$, and $B(c, r) = \{x \in M \mid d(x, c) \leq r\}$, respectively the sphere, the open ball, and the closed ball with the center c and radius r . For $x, y \in \mathbb{R}^n$, the symbol $[x, y]$ means the closed interval in \mathbb{R}^n with the ends x and y .

The paper is organized as follows. In Section 2 we give some important information on regular and semiregular polytopes in Euclidean spaces. In Section 3 we prove some important results of general nature. In Section 4 we study the m -point homogeneity property for some finite homogeneous metric spaces, mainly the vertex sets of regular and semiregular polytopes. In Section 5 we obtain some results on the point homogeneity degree for some important classes of polytopes. Section 6 is devoted to the study of 2-point homogeneous polyhedra with equal edge lengths. In particular, we prove Theorems 1.1 and 1.2. Finally, in Section 7, we study the m -point homogeneity for polyhedra with several edge lengths. In Conclusion, some information on 2-point-homogeneous Riemannian manifolds and some of their generalizations is given.

2. Regular and semiregular polytopes

In this section, we recall some important properties of regular and semiregular polytopes in Euclidean spaces. Faces of dimension $n - 1$ (hyperfaces) of a n -dimensional polytope commonly referred to as *facets*. Note that they are also called *cells* for $n = 4$.

A polytope P is called a *polytope with regular faces* (respectively, a *polytope with congruent faces*), if all its facets are regular (respectively, congruent) polytopes.

A one-dimensional polytope is a closed segment, bounded by two endpoints. It is regular by definition. Two-dimensional regular polytopes are convex regular polygons on Euclidean plane. For other dimensions, regular polytopes are defined inductively. A convex n -dimensional polytope for $n \geq 3$ is called *regular*, if it is homogeneous and all its facets are regular congruent to each other polytopes. This definition is equivalent to other definitions of regular convex polytopes (see [28]). A *vertex figure* of n -dimensional regular polytope, $n \geq 3$, is a $(n - 1)$ -dimensional polytope, which is the convex hull of the vertices, having a common edge with a given vertex and different from it.

We also recall the definition of a wider class of semiregular convex polytopes. For $n = 1$ and $n = 2$, semiregular polytopes are defined as regular. A convex n -dimensional polytope for $n \geq 3$ is called *semiregular* if it is homogeneous and all its facets are regular polytopes.

A generalization of the class of semiregular polytopes is the class of uniform polytopes. For $n \leq 2$, uniform polytopes are defined as regular. For other dimensions, uniform polytopes are defined inductively. A convex n -dimensional polytope for $n \geq 3$ is called *uniform* if it is homogeneous and all its facets are uniform polytopes. In particular, for $n = 3$, the classes of uniform and semiregular polytopes coincide, and for $n = 4$ the facets of the uniform polytope must be semiregular three-dimensional polytopes. This class of polytopes is far from complete classification, see known results in [24, 28].

The classification of regular polytopes of arbitrary dimension was first obtained by Ludwig Schläfli and is presented in his book [29], see also Harold Coxeter's book [19]. The list of semiregular polytopes of arbitrary dimension was first presented without proof in Thorold Gosset's paper [25]. Later this list appeared in the work of Emanuel Lodewijk Elte [23]. The proof of the completeness of this list was obtained much later by Gerd Blind and Rosvita Blind, see [14] and the references therein. Semiregular (non-regular) polytopes in \mathbb{R}^n for $n \geq 4$ are called *Gosset polytopes*. A lot of additional information can be found in [21].

We briefly recall the classification of regular and semiregular polytopes in Euclidean spaces.

An one-dimensional polytope (closed segment) is regular. Two-dimensional regular polytopes (polygons) have equal sides and are inscribed in a circle.

Table 1: Regular 3-dimensional polyhedra.

Polyhedron	V	E	F	α	Face
Tetrahedron	4	6	4	$2 \arcsin(1/\sqrt{3})$	\triangle
Cube (hexahedron)	8	12	6	$\pi/2$	\square
Octahedron	6	12	8	$2 \arcsin(\sqrt{2/3})$	\triangle
Dodecahedron	20	30	12	$2 \arcsin(\sqrt{\varphi}/\sqrt[4]{5})$	\square
Icosahedron	12	30	20	$2 \arcsin(\varphi/\sqrt{3})$	\triangle

For a regular three-dimensional polyhedron, the vertex figure is a regular polygon. It is well known that there are only five regular three-dimensional polyhedra: the tetrahedron, cube, octahedron, dodecahedron, and icosahedron. These polyhedra are traditionally called *Platonic solids*. Some important properties of these polyhedra could be found in Table 1, where V , E , and F mean respectively the numbers of vertices, edges, and faces; α is the dihedral angle; the number $\varphi := \frac{1+\sqrt{5}}{2}$ is known as *the golden ratio*.

There are 6 regular 4-dimensional polytopes: the hypertetrahedron (5-cell), hypercube (8-cell), hyperoctahedron (16-cell), 24-cell, 120-cell, and 600-cell. A detailed description of the structure of four-dimensional regular polytopes could be found e. g. in [8, Section 3].

For each dimension $n \geq 5$, there exists exactly three regular polytopes: the n -dimensional simplex, the hypercube (n -cube) and the hyperoctahedron (n -orthoplex).

We recall some information on the structure of these polytopes and their symmetry groups. In what follows, S_n is denoted the symmetric group of degree n , i. e. is the group of all permutations on n symbols. Recall that the order of S_n is $n!$. We will use also the cyclic groups \mathbb{Z}_k of order k .

Clearly, the symmetry group of the regular n -dimensional simplex P is the group S_{n+1} .

The symmetry group of the n -cube P is the group $\mathbb{Z}_2^n \rtimes S_n$. It is easy to check if we represent P as a convex hull of the points $(\pm a, \pm a, \dots, \pm a) \in \mathbb{R}^n$, where $a > 0$, and the sign could take value $-$ or $+$ on each place independently on other places. Here, S_n is the group of permutation of all coordinates, and the group \mathbb{Z}_2^n is generated by the maps $x \mapsto -x$ applied for each coordinate separately.

The symmetry group of the n -orthoplex P is also the group $\mathbb{Z}_2^n \rtimes S_n$. It is easy to check if we represent P as a convex hull of the points $\pm b e_i$, $i = 1, 2, \dots, n$, where $b > 0$ and e_i is the i -th basic vector in \mathbb{R}^n (it has all zero coordinates excepting 1 in the i -th place). This result could be obtained in more simple way. Since the n -orthoplex is dual to the n -cube, they have one and the same symmetry group. The duality means that the centers of all facets of the n -orthoplex (the n -cube) constitute the vertex set of the n -cube (respectively, the n -orthoplex).

We now proceed to a brief description of the semiregular (non-regular) polytopes.

In three-dimensional space (in addition to Platonic solids), there are the following semiregular polyhedra: 13 *Archimedean solids* and two infinite series of regular prisms and right antiprisms. A detailed description of Archimedean solids could be found in Sections 4 and 5 of [8].

A *right prism* is a polyhedron whose two faces (called bases) are congruent (equal) regular polygons, lying in parallel planes, while other faces (called lateral ones) are rectangles (perpendicular to the bases). If lateral faces are squares then the prism is said to be *regular*. In this case we get an infinite family of semiregular convex polyhedra.

A *right antiprism* is a semiregular polyhedron, whose two parallel faces (bases) are equal regular n -gons, while other $2n$ (lateral) faces are regular triangles. Note that the octahedron is an antiprism with triangular bases.

For $n = 4$, we have exactly three semiregular polytopes: the rectified 4-simplex, the rectified 600-cell, and the snub 24-cell. A detailed description of these polytopes could be found in Section 5 of [9].

The unique (up to similarity) semiregular Gosset polytope in \mathbb{R}^n for $n \in \{5, 6, 7, 8\}$ we denote by the symbol Goss_n . Detailed descriptions of these polytopes could be found in Sections 6, 7, 8, and 9 of [9] respectively.

Example 2.1. Let us consider a brief explicit description of Goss_6 . This polytope can be implemented in different ways. Let us set it with the coordinates of the vertices in \mathbb{R}^6 , as it is done in [23]. Let us put $a = \frac{\sqrt{2}}{4}$ and $b = \frac{\sqrt{6}}{12}$. We define the points $A_i \in \mathbb{R}^6$, $i = 1, \dots, 27$, as follows:

$$\begin{aligned} A_1 &= (0, 0, 0, 0, 0, 4b), & A_2 &= (a, a, a, a, a, b), & A_3 &= (-a, -a, a, a, a, b), \\ A_4 &= (-a, a, -a, a, a, b), & A_5 &= (-a, a, a, -a, a, b), & A_6 &= (-a, a, a, a, -a, b), \\ A_7 &= (a, -a, -a, a, a, b), & A_8 &= (a, -a, a, -a, a, b), & A_9 &= (a, -a, a, a, -a, b), \\ A_{10} &= (a, a, -a, -a, a, b), & A_{11} &= (a, a, -a, a, -a, b), & A_{12} &= (a, a, a, -a, -a, b), \\ A_{13} &= (-a, -a, -a, -a, a, b), & A_{14} &= (-a, -a, -a, a, -a, b), & A_{15} &= (-a, -a, a, -a, -a, b), \\ A_{16} &= (-a, a, -a, -a, -a, b), & A_{17} &= (a, -a, -a, -a, -a, b), & A_{18} &= (2a, 0, 0, 0, 0, -2b), \\ A_{19} &= (0, 2a, 0, 0, 0, -2b), & A_{20} &= (0, 0, 2a, 0, 0, -2b), & A_{21} &= (0, 0, 0, 2a, 0, -2b), \\ A_{22} &= (0, 0, 0, 0, 2a, -2b), & A_{23} &= (-2a, 0, 0, 0, 0, -2b), & A_{24} &= (0, -2a, 0, 0, 0, -2b), \\ A_{25} &= (0, 0, -2a, 0, 0, -2b), & A_{26} &= (0, 0, 0, -2a, 0, -2b), & A_{27} &= (0, 0, 0, 0, -2a, -2b). \end{aligned}$$

The Gosset polytope Goss_6 is the convex hull of these points. It is easy to check that $d(A_1, A_i) = 1$ for $2 \leq i \leq 17$ and $d(A_1, A_i) = \sqrt{2}$ for $18 \leq i \leq 27$.

It is clear that the points $A_2 - A_{17}$ are vertices of a five-dimensional demihypercube (the corresponding hypercube has 32 vertices of the form $(\pm a, \pm a, \pm a, \pm a, \pm a, b)$), and the points $A_{18} - A_{27}$ are the vertices of the five-dimensional hyperoctahedron (orthoplex), which is a facet of the polytope Goss_6 (lying in the hyperplane $x_6 = -2b$). The origin $O = (0, 0, 0, 0, 0, 0) \in \mathbb{R}^6$ is the center of the hypersphere described around Goss_6 with radius $4b = \sqrt{2/3}$.

3. General results

Now we will prove one useful result on inscribed polytopes in \mathbb{R}^n .

Theorem 3.1. Let P be a convex polytope in \mathbb{R}^n such that all its vertices lie in some hypersphere. Then for every vertex A of P , every nearest vertex B of P for A is adjacent to A . As a corollary, the minimal distance between distinct vertices of P is attained only on the pairs of adjacent vertices.

Proof. Without loss of generality we may suppose that all vertices of P are in $S(O, 1)$, where O is the origin in \mathbb{R}^n . Let us fix a vertex A of P and define for $X \in P \setminus \{A\}$ the function

$$f(X) = 2(\overrightarrow{AO}, \overrightarrow{AX}) \cdot |\overrightarrow{AX}|^{-1},$$

which is equal to $2 \cos \alpha$, where α is the angle between vectors \overrightarrow{AO} and \overrightarrow{AX} . It is clear that f is positive. It has yet another geometric sense: $f(X) = |AY|$, where $Y \neq A$ is the intersection point of the ray $r(X) := \{t \cdot \overrightarrow{AX}, t \geq 0\}$ with $S(O, 1)$. This is true if $X = Y \in S(O, 1)$, because we have $f(X) = 2 \cos \alpha$ for the isosceles triangle ΔAOX with lateral sides $|OA| = |OX| = 1$. In another case this is true because $f(X) = f(Y)$.

Since P is convex, then $C(A) := \{r(X), X \in P \setminus \{A\}\}$ is a convex cone with the vertex A . Let us choose a point C on (OA) such that the hyperplane $H(C)$, passing through C orthogonally to \overrightarrow{AO} , separates the vertex A from other vertices of P . It is clear that $\mathcal{P} = C(A) \cap H(C)$ is a convex $(n-1)$ -dimensional polytope, lying inside closed $(n-1)$ -dimensional ball $B^{n-1}(C, r) = B^n(O, 1) \cap H(C)$ with the center C and radius $r = \sqrt{1 - |OC|^2}$. There exists the smallest number $r_0 > 0$ such that

$$\mathcal{P} \subset B^{n-1}(C, r_0) = B^n(C, r_0) \cap H(C).$$

Then $r_0 < r$ and the set $M_0 := \mathcal{P} \cap S^{n-2}(C, r_0)$, where $S^{n-2}(C, r_0) = S^{n-1}(C, r_0) \cap H(C)$, is non-empty and consists only from some vertices of \mathcal{P} . If X_0 is any such vertex, then it is clear from the previous argument that $f(X_0) = \min\{f(X) : X \in P \setminus \{A\}\}$, the point X_0 necessarily lies on some edge e of P with the vertex A , and the length of e is equal to $f(X_0)$. The above considerations imply the statements of Theorem 3.1. \square

We get the following obvious

Corollary 3.1. *Let P be a homogeneous convex polytope in \mathbb{R}^n . Then the minimal distance between distinct vertices of P is attained on some pair of adjacent vertices.*

Remark 3.1. *It is easy to see that for the rhombus in \mathbb{R}^2 with the vertices $(-1,0)$, $(1,0)$, $(0,\alpha)$, and $(0,-\alpha)$, the minimal distance between distinct vertices is attained on the vertices $(0,\alpha)$ and $(0,-\alpha)$ (that are not adjacent) for sufficiently small α . Analogous example could be easily produced in \mathbb{R}^n , $n \geq 3$. This shows that the condition that a polyhedron is inscribed in a hypersphere is necessary in Theorem 3.1.*

Let us consider any homogeneous metric space (M, d) with the isometry group $\text{Isom}(M)$. For any $x \in M$, the group $I(x) = \{\psi \in \text{Isom}(M) \mid \psi(x) = x\}$ is called the *isotropy subgroup at the point x* . If $\eta \in \text{Isom}(M)$ is such that $y = \eta(x)$, then, obviously, $I(y) = \eta \circ I(x) \circ \eta^{-1}$. Hence, the isotropy subgroups for every two points of a given homogeneous finite metric M space are conjugate each to other in the isometry group. In particular, the cardinality of M is the quotient of the cardinality of $\text{Isom}(M)$ by the cardinality of $I(x)$ (for any $x \in M$).

We will discuss some properties of 2-point homogeneous metric spaces. According to Definition 1.1, a metric space (M, d) is *two-point homogeneous* (2-point homogeneous), if for every pairs (A_1, A_2) and (B_1, B_2) of elements of M such that $d(A_1, A_2) = d(B_1, B_2)$, there is an isometry $f \in \text{Isom}(M)$ with the following property: $f(A_i) = B_i, i = 1, 2$.

Remark 3.2. *There are some publications, where some types of two-point homogeneity for finite sets with additional structures are studied. For instance, we refer to [31], where two-point homogeneous quandles are studied. On the other hand, we are focused on the study of metric spaces.*

For a given $x \in M$ and $r > 0$ we consider the sphere $S(x, r) = \{y \in M \mid d(x, y) = r\}$ with the center x and radius r .

Remark 3.3. *If (M, d) is a finite homogeneous subset in Euclidean space \mathbb{R}^n , then every sphere $S(x, r) \subset (M, d)$ lies in some $(n - 1)$ -dimensional Euclidean subspace, since all points of this sphere are also on one and the same distance from the barycenter of $M \subset \mathbb{R}^n$ (recall that $\|x - a\| = C$ and $\|x - b\| = D$ for $x, a, b \in \mathbb{R}^n$ implies $2(x, a - b) = \|a\|^2 - \|b\|^2 - C^2 + D^2$).*

It is easy to prove the following helpful result.

Proposition 3.1. *A homogeneous metric space (M, d) is two-point homogeneous if and only if for any point $x \in M$ the following property holds: for every $r > 0$ and every $y, z \in S(x, r)$, there is $f \in I(x)$ such that $f(y) = z$. In other words, a homogeneous metric space (M, d) is two-point homogeneous if and only if for any point $x \in M$, the isotropy subgroup $I(x)$ acts transitively on every sphere $S(x, r)$, $r > 0$.*

In particular, Proposition 3.1 implies

Proposition 3.2. *The vertex set of any regular polytope in \mathbb{R}^3 is 2-point homogeneous.*

The following result on the m -point homogeneity could be useful in some explicit calculations.

Proposition 3.3. *If a metric space (M, d) is m -point homogeneous, $m \geq 2$, then for any point $x \in M$ the following property holds: for every $r > 0$ and every $(m - 1)$ -tuples $(A_1, A_2, \dots, A_{m-1})$ and $(B_1, B_2, \dots, B_{m-1})$ of elements of $S(x, r)$ such that $d(A_i, A_j) = d(B_i, B_j)$, $i, j = 1, \dots, m - 1$, there is $f \in I(x)$ such that $f(A_i) = B_i$ for all $i = 1, \dots, m - 1$. In other words, if a homogeneous metric space (M, d) is m -point homogeneous, then for any point $x \in M$, the isotropy subgroup $I(x)$ acts $(m - 1)$ -point transitively on every sphere $S(x, r)$, $r > 0$ (in particular, $S(x, r)$ with the metric, induced by d , is $(m - 1)$ -point homogeneous).*

Proof. We can extend every $(m - 1)$ -tuples $(A_1, A_2, \dots, A_{m-1})$ and $(B_1, B_2, \dots, B_{m-1})$ of elements of $S(x, r)$ such that $d(A_i, A_j) = d(B_i, B_j)$, $i, j = 1, \dots, m - 1$, to m -tuples adding the points $A_m = B_m = x$. Since $d(x, A_i) = d(x, B_i) = r$, $i = 1, \dots, m - 1$, and (M, d) is m -point homogeneous, there is an isometry $f \in \text{Isom}(M)$ with the following property: $f(A_i) = B_i$, $i = 1, \dots, m$. Since $A_m = B_m = x$ then $f \in I(x)$, hence, f is an isometry of $S(x, r)$. \square

Remark 3.4. It should be noted that it could be some isometries of a given sphere $S(x, r)$ that are not generated by the elements of the isotropy subgroup $I(x)$.

Lemma 3.1. Let P be a n -dimensional convex centrally symmetric polytope with the vertex set M inscribed in the sphere $S^{n-1}(O, r)$ with the center in the origin of \mathbb{R}^n , M_1, M_2 be some ordered subsets in M , for which there exists an isometry f of the polytope P and the sphere $S(O, r)$ on themselves such that $f(M_1) = M_2$, $v \in M_1, w \in M_2$, $f(v) = w$. Then

$$f(M_1 \cup \{-v\}) = M_2 \cup \{-w\}, \quad f[(M_1 - \{v\}) \cup \{-v\}] = (M_2 - \{w\}) \cup \{-w\}.$$

Proof. Clearly, $f(-v) = -f(v) = -w$. It implies the first equality of the lemma. The second equality is a consequence of the first one. \square

Theorem 3.2. Let P be a polytope in \mathbb{R}^n , $n \geq 2$, such that its vertex set M is homogeneous, symmetric with respect to the center of P , and the distances between distinct vertices of P constitute a 3-element set, say $\{d_1, d_2, d_3\}$, where $0 < d_1 < d_2 < d_3$. If for some vertex $v \in M$ the isotropy subgroup $I(v)$ acts $(m - 1)$ -point transitively on the sphere $S(v, d_1)$, i. e. $S(v, d_1)$ is $(m - 1)$ -point homogeneous under the action of $I(v)$ for some $m \geq 2$, then M is m -point homogeneous.

Proof. Let M_1, M_2 be arbitrary ordered subsets in M consisting of m vertices such that the distance between two arbitrary points in M_1 is equal to the distance between corresponding two points in M_2 . Due to the homogeneity of M , we may assume that the first vertices in M_1 and M_2 are both equal to v .

If $M_1 \subset S(v, d_1) \cup \{v\}$, then the same is true for M_2 and the conditions of theorem imply that there is $f \in I(v)$ such that $f(M_1) = M_2$.

Otherwise, $M_1 = M'_1 \cup M''_1$ and $M_2 = M'_2 \cup M''_2$, where $M'_i \subset S(v, d_1) \cup \{v\}$ and $M''_i \subset S(v, d_2) \cup \{-v\}$, $i = 1, 2$. It is clear that the ordered sets $M'_1 \cup (-M''_1)$ and $M'_2 \cup (-M''_2)$ of vertices in M are mutually isometric and both are contained in $S(v, d_1) \cup \{v\}$. Conditions of the theorem imply that there is $f \in I(v)$ such that $f(M'_1 \cup (-M''_1)) = M'_2 \cup (-M''_2)$. Applying several times the second assertion of Lemma 3.1 to the subsets $-M''_1$ and $-M''_2$, we obtain $f(M_1) = M_2$. Hence, M is m -point homogeneous. \square

Remark 3.5. It is clear that the sphere $S(v, d_1)$ could be replaced by the sphere $S(v, d_2)$ in the statement of Theorem 3.2. Indeed, $S(v, d_1)$ is the vertex figure for the vertex v , while $S(v, d_2)$ is the vertex figure for the vertex $-v$.

Question 3.1. What is the classification of polytopes P in \mathbb{R}^n , $n \geq 2$, such that its vertex set M is homogeneous, symmetric with respect to the center of P , and the distances between distinct vertices of P constitute a 3-element set?

Lemma 3.2. For any subset of the sphere $\{v_1, \dots, v_n\} \subset S^{n-1}(0, 1) \subset \mathbb{R}^n$, where $n \geq 2$, lying in no great subsphere of the sphere $S^{n-1}(0, 1)$ of less dimension, every point $w \in S^{n-1}(0, 1)$ is uniquely determined by its distances to the points v_1, \dots, v_n .

Proof. Let us denote the spherical distances from the point w to the points v_i , $i = 1, \dots, n$, by φ_i . We consider the points v_1, \dots, v_n and w as unit vectors in \mathbb{R}^n . Due to the conditions of the lemma, the first n vectors are linearly independent. It is accepted to call the inner products $w_i := w \cdot v_i = \cos \varphi_i$, $i = 1, \dots, n$, as the *covariant* components of the vector w in the basis $v = (v_1, \dots, v_n)$. Then

$$(w_1, \dots, w_n)^t = Gr(v)(w^1, \dots, w^n)^t,$$

where the symbol t denotes the transposition of matrices, $w = \sum_{i=1}^n w^i v_i$, $Gr(v) = (v_i \cdot v_j)$ is nondegenerate Gram matrix of the basis v ,

$$(w^1, \dots, w^n)^t = Gr(v)^{-1}(w_1, \dots, w_n)^t = Gr(v)^{-1}(\cos \varphi_1, \dots, \cos \varphi_n)^t.$$

The lemma is proved. \square

Let us assume that the sphere $S^{n-1}(O, 1)$ of radius 1 centered at the origin $O \in \mathbb{R}^n$ is the circumscribed sphere for all homogeneous n -dimensional polyhedra under consideration, and the spherical distance is used as the distance.

Theorem 3.3. *Let $n \geq 3$ and P be n -dimensional convex $(n - 1)$ -point homogeneous polytope in \mathbb{R}^n with the vertex set M of cardinality $k \geq n + 1$. Then P is k -point homogeneous if and only if for any two isometric non-degenerate ordered n -tuples of vertices M_1 and M_2 in M with the first points v , there exists an isometry f from the isotropy subgroup $I(v)$ such that $f(M_1) = M_2$. If P is also centrally symmetric then M in this statement can be replaced by an arbitrary maximal by inclusion subset $N \subset M \cap B(v, \pi/2)$, containing no diametrically opposite vertices.*

Proof. We start with proving of the first statement. The necessity is evident. Let us prove the sufficiency.

Let M', M'' be two isometric ordered subsets in M of cardinality m , where $n \leq m \leq k$. There exists a smallest number $l \geq 2$ such that M' (respectively, M'') lies in some vector subspace V'_l (respectively, V''_l) of dimension l in \mathbb{R}^n .

Assume at first that $l \leq n - 1$. Then there exist an l -point ordered subset M'_l in M' and a subset M''_l in M'' with the same numbers as M'_l such that the radius-vectors of the vertices in every of these subsets constitute a basis in V'_l and V''_l respectively. Since P is l -point homogeneous, then there exists a self-isometry f of the polytope P such that $f(M'_l) = M''_l$. Hence, due to Lemma 3.2, applied to the spheres $S^{l-1}(0, 1) \subset V'_l$ and $S^{l-1}(0, 1) \subset V''_l$, containing respectively the sets M' and M'' , we obtain $f(M') = M''$.

Assume now that $l = n$. Then there exist an n -point ordered subset M'_n in M' and the subset M''_n in M'' with the same numbers as M'_n , such that the radius-vectors of the vertices of every such subset constitute bases in \mathbb{R}^n . By the homogeneity of P we may suppose that the first points of subsets M'_n and M''_n coincide with the vertex $v \in M$. By assumption, there exists a self-isometry $f \in I(v)$ of the polytope P such that $f(M'_n) = M''_n$. Then by Lemma 3.2, applied to the sphere $S^{n-1}(0, 1) \subset \mathbb{R}^n$, we get $f(M') = M''$.

Let the polytope P be centrally symmetric and $N \subset M \cap B(v, \pi/2)$, where $v \in M$, is some maximal by inclusion set containing no pair of diametrically opposite vertices. The triangle inequality imply that $M \cap U(v, \pi/2) \subset N$, and $M \cap U(v, \pi/2) = N$ if and only if $M \cap S(v, \pi/2) = \emptyset$.

Let M_1 and M_2 be two isometric nondegenerate ordered n -tuples of vertices in M . It is clear that every of sets M_1 and M_2 contains no pairs of diametrically opposite vertices. The vertices with the same number in M_1 and M_2 simultaneously belong or not belong to $M \cap B(v, \pi/2)$ and we will change every pair of vertices of the same name, which do not belong to this set, by the diametrically opposite pair. As a result, we obtain (possibly, new) isometric and nondegenerate n -tuples of vertices M'_1 and M'_2 of the polytope P , contained in $M \cap B(v, \pi/2)$.

If $M \cap S(v, \pi/2) = \emptyset$, then we set $M''_1 = M'_1$ and $M''_2 = M'_2$.

If not, then changing, if necessary, some vertices of the same name at M'_1 and M'_2 into diametrically opposite ones, we obtain isometric and non-degenerate n -tuples of vertices M''_1 and M''_2 , so that $M''_1 \subset N$.

Due to the assumption of the theorem, there exists a self-isometry $f \in I(v)$ of the polytope P such that $f(M_1) = M_2$. Moreover, due to Lemma 3.1, the last equality is equivalent to $f(M''_1) = M''_2$.

The above arguments give the proof of the second part of the theorem, regardless of whether M''_2 is contained in N or not; in the latter case, M''_2 can be ignored, leaving only $M''_1 \subset N$. □

Corollary 3.2. *Every n -dimensional convex n -point homogeneous polytope in \mathbb{R}^n which has the vertex set M with the cardinality $k \geq n + 1$, is k -point homogeneous.*

4. Examples

Let us start with a very simple example.

Example 4.1. *Any regular n -dimensional simplex S is $(n + 1)$ -point homogeneous. Indeed, the distances between any vertices S coincide and the symmetry group of S is the group S_{n+1} of all permutations of the vertex set.*

Example 4.2. *Any regular m -gon P in \mathbb{R}^2 , $m \geq 3$, is m -point homogeneous. Indeed, it is clear that P is 2-point homogeneous. Now the statement follows from Corollary 3.2.*

Example 4.3. The vertex set M of the 3-cube P is (at least) 4-point homogeneous. There are three possible distances between distinct vertices of P , $0 < d_1 < d_2 < d_3$, and P has the center of symmetry. Moreover, for any vertex v we see that $S(v, d_1)$ is the vertex set of a regular triangle. The isotropy subgroup $I(v)$ acts 3-point transitively on $S(v, d_1)$. Therefore, M is 4-point homogeneous by Theorem 3.2.

Example 4.4. The vertex set M of the icosahedron P is (at least) 6-point homogeneous. There are three possible distances between distinct vertices of P , $0 < d_1 < d_2 < d_3$, and P has the center of symmetry. Next, for any vertex v we see that $S(v, d_1)$ is the vertex set of a regular pentagon. The isotropy subgroup $I(v)$ contains all isometries of this pentagon, hence $I(v)$ acts 5-point transitively on $S(v, d_1)$ (see Example 4.2). Therefore, M is 6-point homogeneous by Theorem 3.2.

Corollary 3.2 together with Examples 4.3 and 4.4 imply

Corollary 4.1. The 3-dimensional cube is 8-point homogeneous. The icosahedron is 12-point homogeneous.

Corollary 4.2. The n -orthoplex in \mathbb{R}^n is $2n$ -point homogeneous for every $n \in \mathbb{N}$.

Proof. It follows from Theorem 3.2 and Corollary 3.2 applied to the n -orthoplex. \square

Example 4.5. Let us check that the convex hull of the union of regular n -dimensional simplex and the simplex which is centrally symmetric to it, is $2(n + 1)$ -point homogeneous.

As the first simplex S_1 , we can take the convex hull of $n + 1$ points $v_i \in \mathbb{R}^{n+1}$, $i = 1, \dots, n + 1$, where j -th coordinate of v_i is $v_{i,j} = -1$ if $j \neq i$ and $v_{i,i} = n$.

One can easily see that the convex hull P^n of $S_1 \cup (-S_1)$ is the image of the orthogonal projection (along $(1, \dots, 1) \in \mathbb{R}^{n+1}$) of the standard $(n + 1)$ -orthoplex Ort (in \mathbb{R}^{n+1}), multiplied by $n + 1$, onto n -dimensional hyperplane

$$H^n = \{x = (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1 + \dots + x_{n+1} = 0\}.$$

It is easy to see that P^n is homogeneous and satisfies conditions of Theorem 3.2 with $m - 1 = n$. Then by Theorem 3.2 and Corollary 3.2, P^n is $2(n + 1)$ -point homogeneous.

Remark 4.1. P^2 is the regular hexagon and P^3 is the cube.

Example 4.6. Let us check that the vertex set M of the polytope Goss_6 (see Example 2.1) is two-point homogeneous.

At first, we note that the spheres $S(A_1, r)$ are non-empty exactly for $r = 1$ and $r = \sqrt{2}$. Indeed, $d(A_1, A_i) = 1$ for $2 \leq i \leq 17$ and $d(A_1, A_i) = \sqrt{2}$ for $18 \leq i \leq 27$. The points $A_2 - A_{17}$ are vertices of a five-dimensional demihypercube (the corresponding hypercube has $32 = 2^5$ vertices of the form $(\pm a, \pm a, \pm a, \pm a, \pm a, b)$), and the points $A_{18} - A_{27}$ are the vertices of the five-dimensional hyperoctahedron (5-orthoplex), which is a facet of the polytope Goss_6 (lying in the hyperplane $x_6 = -2b$).

It is easy to check that the isotropy subgroup $I(A_1)$ of the group $\text{Isom}(M) = \text{Symm}(\text{Goss}_6)$ contains the orthogonal operators of the following two forms:

- 1) $B(x_1, x_2, x_3, x_4, x_5, x_6) = (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}, x_{\sigma(5)}, x_6)$, where $\sigma \in S_5$ is any permutation of $(1, 2, 3, 4, 5)$;
- 2) $B(x_1, x_2, x_3, x_4, x_5, x_6) = (\pm x_1, \pm x_2, \pm x_3, \pm x_4, \pm x_5, x_6)$, where the number of signs “-” is even.

Now, it is easy to see that a composition of some suitable maps of the above forms moves any given point in the sphere $S(A_1, 1)$ or in the sphere $S(A_1, \sqrt{2})$ to any other given point in the same sphere. By Proposition 3.1, M is two-point homogeneous.

Example 4.7. Let us consider a description of the Gosset polytope Goss_7 . Let us set it with the coordinates of the vertices in \mathbb{R}^7 , as it is done in [23]. In what follows, $\overline{a_1, a_2, \dots, a_m}$ means the set of all permutations of the elements a_1, a_2, \dots, a_m . Let us put $a = \frac{\sqrt{2}}{4}$, $b = \frac{\sqrt{6}}{12}$, and $c = \frac{\sqrt{3}}{6}$. We define the points $B_i \in \mathbb{R}^7$, $i = 1, \dots, 56$, as follows:

$$\begin{aligned} B_1 &= (0, 0, 0, 0, 0, 0, 3c), & B_2 &= (0, 0, 0, 0, 0, 4b, c), & B_3 &= (a, a, a, a, a, b, c), & B_4 - B_{13} &= (\overline{-a, -a, a, a, a}, b, c), \\ B_{14} - B_{18} &= (\overline{-a, -a, -a, -a, a}, b, c), & B_{19} - B_{23} &= (\overline{2a, 0, 0, 0, 0}, -2b, c), & B_{24} - B_{28} &= (\overline{-2a, 0, 0, 0, 0}, -2b, c), \\ B_{29} - B_{33} &= (\overline{-2a, 0, 0, 0, 0}, 2b, -c), & B_{34} - B_{38} &= (\overline{2a, 0, 0, 0, 0}, 2b, -c), & B_{39} - B_{48} &= (\overline{a, a, -a, -a, -a}, -b, -c), \\ B_{49} - B_{53} &= (\overline{a, a, a, a, -a}, -b, -c), & B_{54} &= (-a, -a, -a, -a, -a, -b, -c), & B_{55} &= (0, 0, 0, 0, 0, -4b, -c), \\ B_{56} &= (0, 0, 0, 0, 0, 0, -3c). \end{aligned}$$

The Gosset polytope $Goss_7$ is the convex hull of these points. It is clear that the last 28 points could be obtained from the first 28 point via the central symmetry with respect to the origin O . Hence, the circumradius of this polytope is $3c = \frac{\sqrt{3}}{2}$. It is clear that the convex hull of the points B_i for $2 \leq i \leq 28$ is the Gosset polytope $Goss_6$ (the “first” $Goss_6$, that is a vertex figure for the vertex $B_1 \in Goss_7$). The convex hull of the points B_i for $29 \leq i \leq 55$ is a polytope that is symmetric to the previous one (the “second” $Goss_6$).

It is easy to check that $d(B_1, B_i) = 1$ for $2 \leq i \leq 28$, $d(B_1, B_i) = \sqrt{2}$ for $29 \leq i \leq 55$, and $d(B_1, B_{56}) = \sqrt{3}$.

It is easy to see that the subgroup of the isometries of $Goss_7$ that preserve all points of the straight line Ox_7 , acts transitively on the vertices B_i for $2 \leq i \leq 28$, as well as for $29 \leq i \leq 55$. Indeed, any isometry of $Goss_6$ (that is identified with the convex hull of the points B_i for $2 \leq i \leq 28$) could be uniquely extended to the isometry of $Goss_7$, that fixes B_1 (as well as B_{56}). In particular, the isotropy group $I(B_1)$ acts transitively on all (non-empty) spheres $S(B_1, r)$. It is clear that there are exactly four non-empty spheres $S(B_1, r)$: for $r = 0$ and $r = 6c = \sqrt{3}$ we get one-point sets, whereas for $r = 1$ and $r = \sqrt{2}$ we have two copies of $Goss_6$. By Proposition 3.1, $Goss_7$ is 2-point homogeneous.

Moreover, $Goss_7$ is 3-point homogeneous by Theorem 3.2 and Example 4.6.

5. Some results on the point homogeneity degree

Definition 5.1. For a given n -dimensional convex polytope P , let q be a natural number such that P is q -point homogeneous, but is not $(q + 1)$ -point homogeneous. Then q is called the point homogeneity degree of the polytope P . If P is m -homogeneous for all natural m then we define its point homogeneity degree as ∞ .

It should be noted that there are polytopes with the point homogeneity degree ∞ (compare with Corollary 3.2). For instance, this property is inherent in the following polytopes: any regular m -gon P in \mathbb{R}^2 for $m \geq 3$ (Example 4.2), the n -dimensional regular simplex (Example 4.1) and the n -orthoplex for all $n \geq 1$ (Corollary 4.2); the n -cube for $n = 1, 2, 3$, the icosahedron (Corollary 4.1).

If the vertex set of a polytope P is homogeneous then the point homogeneity degree of P is ≥ 1 . Otherwise, it is equal to 0. We will show that the n -cube has the point homogeneity degree 3 for $n \geq 4$ (see Corollary 5.1 below), the dodecahedron has the point homogeneity degree 2 (Theorem 6.1 below). Note also that the cuboctahedron has the point homogeneity degree 2, while all other Archimedean polyhedra have the point homogeneity degree 1 (Theorems 1.1 and 6.1).

On the other hand, we do not know what is the point homogeneity degree for the 24-cell, 120-cell, 600-cell, as well as for Gosset polytopes (with the exception of the four-dimensional truncated simplex, whose point homogeneity degree is 2 due to Proposition 5.5).

Remark 5.1. It should be noted that the 24-cell, as well as the 120-cell and 600-cell, contains a 4-cube (some 16 vertices of any of these polytope are the vertices of a 4-cube). We know that the 4-cube is not 4-point homogeneous. It is possible that the 24-cell, 120-cell, and 600-cell have the same property.

Question 5.1. What is the point homogeneity degree of a given regular polytope P ?

Question 5.2. What is the point homogeneity degree of a given semiregular polytope P ?

We know that the 6-dimensional Gosset polytope $Goss_6$ has the point homogeneity degree ≥ 2 (Example 4.6) and the 7-dimensional Gosset polytope $Goss_7$ has the point homogeneity degree ≥ 3 (Example 4.7).

We are going to find the point homogeneity degree of the n -cube for $n \geq 4$.

Proposition 5.1. The vertex set of the n -cube, $n \geq 2$, is 3-point homogeneous.

Proof. For $n = 2$ and $n = 3$ it follows respectively from Example 4.2 and Corollary 4.1. Hence, we consider the standard n -cube C_n , $n \geq 4$. The vertices of C_n has the form $(\pm 1, \pm 1, \dots, \pm 1) \in \mathbb{R}^n$.

For any vertex A of C_n , we consider the set $N(A)$ of indices i , $1 \leq i \leq n$, such that i -th coordinates of A is -1 .

Let us consider three vertices A_0, A_1, A_2 of C_n . Without loss of generality we may suppose that $A_0 = (1, 1, 1, \dots, 1)$, but A_1 and A_2 are chosen arbitrarily (and all points are pairwise distinct). The distance between A_0 and A_1 (A_2) is determined by the cardinality of the set $N(A_1)$ (respectively, $N(A_2)$), but the distance between A_1 and A_2 are determined by the cardinality of the set $N(A_1) \cap N(A_2)$.

If we have some another triple of vertices A'_0, A'_1, A'_2 with the same pairwise distances, then there is an isometry of C_n that maps the first triple of the vertices to the second one. Indeed, since C_n is homogeneous, we may assume that $A'_0 = A_0$.

Now, the cardinality of the sets $N(A_1), N(A_2)$, and $N(A_1) \cap N(A_2)$ are the same as the cardinality of the sets $N(A'_1), N(A'_2)$, and $N(A'_1) \cap N(A'_2)$ respectively. Therefore, there is a substitution of coordinates, that maps the sets $N(A_1), N(A_2)$, and $N(A_1) \cap N(A_2)$ onto the sets $N(A'_1), N(A'_2)$, and $N(A'_1) \cap N(A'_2)$ respectively. The point $A_0 = A_0$ is fixed under any permutation of coordinates, hence the above permutation generates a desired isometry of C_n . \square

Proposition 5.2. *The vertex set of the n -cube, $n \geq 4$, is not 4-point homogeneous.*

Proof. At first, we consider the case $n = 4$. Let us consider the following two 4-tuples of vertices of the standard 4-cube:

$$\begin{aligned} & ((1, 1, 1, 1), (-1, -1, 1, 1), (-1, 1, -1, 1), (-1, 1, 1, -1)) \\ & ((1, 1, 1, 1), (-1, -1, 1, 1), (-1, 1, -1, 1), (1, -1, -1, 1)). \end{aligned}$$

Let us suppose that there is an isometry f of the standard 4-cube, that maps the first 4-tuple onto the second one.

It is easy to see that the distances between the corresponding pairs of the points in these 4-tuples coincide. We see also that the first three vertices in these 4-tuples coincide. Now, any isometry that preserves these first three vertices (in particular, f) should be the identical map that is impossible.

If $n > 4$, we can consider two 4-tuples of vertices of the standard n -cube, such that the first four coordinates are the same as the above 4-tuples for the standard 4-cube have, but all other coordinates are equal to 1. The same argument shows that the standard n -cube, $n > 4$, is not 4-homogeneous. \square

We see that the vertex set of the n -cube is 3-point homogeneous but is not 4-point homogeneous for all $n \geq 4$, hence, we get

Corollary 5.1. *The point homogeneity degree of the n -cube is 3 for all $n \geq 4$.*

Remark 5.2. *It should be noted that the vectors in both 4-tuples in the above example generate the following Hadamard matrices:*

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

The necessary information about Hadamard matrices and related results can be found, for example, in [27].

We are going to find the point homogeneity degree of the n -dimensional demihypercube. We consider the standard n -dimensional demihypercube DC_n , $n \geq 1$, as the convex hull of the points $(\pm 1, \pm 1, \dots, \pm 1) \in \mathbb{R}^n$, where the quantity of the signs “-” is even (if we consider the odd quantity of such signs, then we get another one n -dimensional demihypercube). It is clear that DC_n has the center of symmetry if and only if n is even. The symmetry group of DC_n consists of those symmetries of the standard n -cube that preserved DC_n (it contains all permutations of coordinates and all inversions of signs for even quantity of coordinates).

It is easy to see that DC_2 (that is a segment) is two-point homogeneous and DC_3 (that is a regular simplex) is 4-point homogeneous. Therefore, the point homogeneity degree of DC_n is ∞ for $n = 1, 2, 3$.

Proposition 5.3. *The vertex set of DC_n , $n \geq 4$, is 3-point homogeneous.*

Proof. The vertices of DC_n has the form $(\pm 1, \pm 1, \dots, \pm 1) \in \mathbb{R}^n$ with even quantity of the signs “-”.

For any vertex A of DC_n , we consider the set $N(A)$ of indices i , $1 \leq i \leq n$, such that i -th coordinates of A is -1 .

Let us consider three vertices A_0, A_1, A_2 of DC_n . Without loss of generality we may suppose that $A_0 = (1, 1, 1, \dots, 1)$, but A_1 and A_2 are chosen arbitrarily (and all points are pairwise distinct). The distance between A_0 and A_1 (A_2) is determined by the cardinality of the set $N(A_1)$ (respectively, $N(A_2)$), but the distance between A_1 and A_2 are determined by the cardinality of the set $N(A_1) \cap N(A_2)$.

If we have some other triple of vertices A'_0, A'_1, A'_2 with the same pairwise distances, then there is an isometry of DC_n that maps the first triple of the vertices to the second one. Indeed, since DC_n is homogeneous, we may assume that $A'_0 = A_0$.

Now, the cardinality of the sets $N(A_1), N(A_2)$, and $N(A_1) \cap N(A_2)$ are the same as the cardinality of the sets $N(A'_1), N(A'_2)$, and $N(A'_1) \cap N(A'_2)$ respectively. Therefore, there is a substitution of coordinates, that maps the sets $N(A_1), N(A_2)$, and $N(A_1) \cap N(A_2)$ onto the sets $N(A'_1), N(A'_2)$, and $N(A'_1) \cap N(A'_2)$ respectively. The point $A_0 = A_0$ is fixed under any permutation of coordinates, hence the above permutation generates a desired isometry of DC_n . \square

Proposition 5.4. *The vertex set of DC_n , $n \geq 4$, is not 4-point homogeneous.*

Proof. Let us consider the following two 4-tuples of vertices of DC_n :

$$\begin{aligned} & \left((1, 1, 1, 1, \dots), (-1, -1, 1, 1, \dots), (-1, 1, -1, 1, \dots), (-1, 1, 1, -1, \dots) \right) \\ & \left((1, 1, 1, 1, \dots), (-1, -1, 1, 1, \dots), (-1, 1, -1, 1, \dots), (1, -1, -1, 1, \dots) \right), \end{aligned}$$

where \dots means $n - 4$ elements 1. Let us suppose that there is an isometry f of DC_n , that maps the first 4-tuple onto the second one.

It is easy to see that the distances between the corresponding pairs of the points in these 4-tuples coincide. We see also that the first three vertices in these 4-tuples coincide. Now, the isometry that preserves the first two vertices (in particular, f) could only interchange the first two coordinates as well as the last two coordinates of all points. On the other hand, from the third vertices in the above 4-tuples, we see that neither the first two coordinates nor the last two coordinates could be interchanged by f . Hence, f should preserve the first four coordinates of any vertex that is impossible and DC_n is not 4-homogeneous $n \geq 4$. \square

We see that the n -dimensional demihypercube DC_n is 3-point homogeneous but is not 4-point homogeneous for all $n \geq 4$. Therefore, we get

Corollary 5.2. *The point homogeneity degree of the n -dimensional demihypercube is 3 for all $n \geq 4$.*

We shall find the point homogeneity degree of the fully truncated n -dimensional simplex.

The standard truncated n -dimensional simplex TS_n can be represented as the convex hull of points $B_{i,j} \in \mathbb{R}^{n+1}$, $i, j = 1, \dots, n+1$, $i < j$, such that all coordinates of the point $B_{i,j}$ are zero except for its i -th and j -th coordinates, which are equal to 1. It is clear that there are exactly $C_n^2 = n(n-1)/2$ such points. The distance between (different) points $B_{i,j}$ and $B_{s,t}$ is $\sqrt{2}$ for $i = s$ or $j = t$ and 2 for $\{i, j\} \cap \{s, t\} = \emptyset$. The isometry group of the corresponding simplex (the convex hull of points $A_i \in \mathbb{R}^{n+1}$, $i = 1, \dots, n+1$, such that all coordinates of A_i are zero except for its i -th coordinate, which is 2) acts transitively on the vertex set of the truncated simplex under consideration. Let M be the vertex set of TS_n . The above arguments show that M is a finite homogeneous two-distance set. Moreover, TS_n is uniform in every dimension [18].

Note that TS_1 is a one-point set, TS_2 is a regular triangle, TS_3 is an orthoplex (octahedron), TS_4 is a semiregular polytope.

The isometry group of TS_n is the symmetric group S_{n+1} for $n = 2$ and $n \geq 4$ (the group of permutations of the symbols $1, 2, \dots, n, n+1$), see Theorem 1 in [12]. The isometry group of TS_3 is the isometry group of the octahedron $\mathbb{Z}_2^3 \rtimes S_3 = \mathbb{Z}_2 \rtimes S_4$.

We will denote the vertex $B_{i,j}$ also by the symbol $\{i, j\}$ (the set of two indices). Let us fix the vertex $B_{1,2} = \{1, 2\}$. We see that the sphere $S(\{1, 2\}, \sqrt{2})$ (respectively, the sphere $S(\{1, 2\}, 2)$) consists of vertices $\{i, j\}$ such that the set $\{1, 2\} \cap \{i, j\}$ contains exactly one element (respectively $\{1, 2\} \cap \{i, j\} = \emptyset$). The isotropy group $I(\{1, 2\})$ is the direct product $S_2 \times S_{n-1}$ (independent permutations of the elements 1, 2 as well as the elements 3, 4, ..., $n + 1$). Obviously, $I(\{1, 2\})$ acts transitively on spheres $S(\{1, 2\}, \sqrt{2})$ and $S(\{1, 2\}, 2)$. Therefore M is 2-point homogeneous.

Let us show that M is not 3-point homogeneous. For this we consider the following triples of vertices:

$$(\{1, 2\}, \{1, 3\}, \{2, 3\}), \quad (\{1, 2\}, \{1, 3\}, \{1, 4\}).$$

If the isometry f maps the first triple onto the second one, then from $f(\{1, 2\}) = \{1, 2\}$ we see that f either interchanges the coordinates with numbers 1 and 2 or preserves both these coordinates. Next, from $f(\{1, 3\}) = \{1, 3\}$ we see that f preserves the coordinates with numbers 1, 2 and 3. Therefore, the equality $f(\{2, 3\}) = \{1, 4\}$ is impossible. In this construction, we used four distinct coordinates of the vertices, that is important.

Hence, we have proved the following result.

Proposition 5.5. *The point homogeneity degree of the fully truncated n -dimensional simplex is 2 for $n \geq 4$ and is ∞ for $n = 1, 2, 3$.*

6. On 2-point homogeneous polyhedra with equal edge lengths

Recall that any regular polytope in \mathbb{R}^3 is 2-point homogeneous (see Proposition 3.2). The main results of this section are the proofs of Theorems 1.1 and 1.2.

In the arguments below, we will use the spherical distance between the points. Every polyhedron is assumed to be inscribed into a sphere with center in the origin O and unit radius.

Theorem 6.1. *The point homogeneity degree of the dodecahedron D is 2 (i. e. M is 2-point homogeneous but not 3-point homogeneous).*

Proof. Let us recall some property of the dodecahedron D . We can assume that the vertices of D are the centers of the faces of an icosahedron I with the center O , so that the inscribed sphere for I is the circumscribed sphere for D , while the spherical distance d_1 between adjacent vertices for D is the angle between outer normals of adjacent faces for I .

The dihedral angle under the edge of the icosahedron is equal to $\alpha = 2 \arcsin(\varphi / \sqrt{3})$ (see Table 1 or [8]). Therefore

$$d_1 = 2\pi - (2\pi/2 + \alpha) = \pi - \alpha = 2(\pi/2 - \arcsin(\varphi / \sqrt{3})) = 2 \arccos(\varphi / \sqrt{3}),$$

$$\cos d_1 = \frac{2\varphi^2}{3} - 1 = \frac{2\varphi^2 - 3}{3} = \frac{3 + \sqrt{5} - 3}{3} = \frac{\sqrt{5}}{3}.$$

By the spherical cosine theorem, for the next spherical distance

$$\cos d_2 = \cos^2 d_1 + \cos\left(\frac{2\pi}{3}\right) \sin^2 d_1 = \cos^2 d_1 - \frac{1}{2}(1 - \cos^2 d_1) = \frac{3 \cos^2 d_1 - 1}{2} = \frac{1}{3}.$$

All possible spherical distances between vertices of the dodecahedron are equal to

$$0 < \arccos\left(\frac{\sqrt{5}}{3}\right) < \arccos\left(\frac{1}{3}\right) < \arccos\left(\frac{-1}{3}\right) < \arccos\left(\frac{-\sqrt{5}}{3}\right) < d_5 = \pi.$$

The isotropy subgroup $I(v)$ of the vertex $v \in M \subset D$ in the symmetry group of D consists of 6 isometries, three rotations around the axis $v(-v)$ by angles 0, $2\pi/3$, $4\pi/3$ and three mirror reflections relative to every

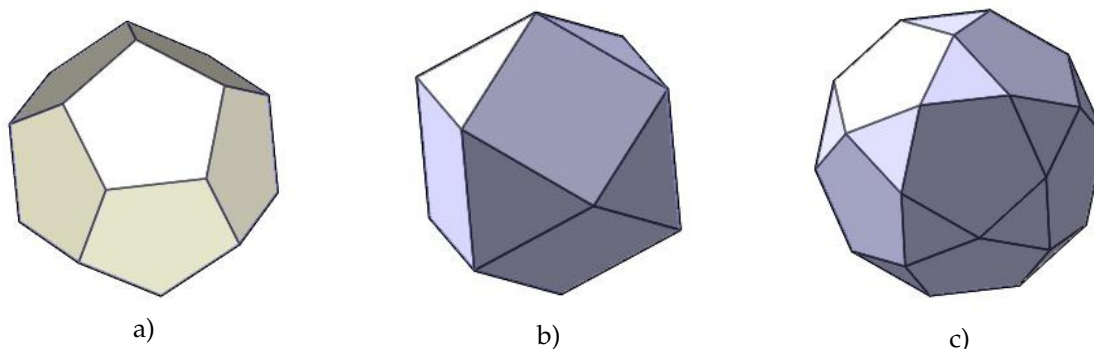


Figure 1: a) dodecahedron; b) cuboctahedron; c) icosidodecahedron.

plane, passing through O and one of the edges of the dodecahedron, which includes v . As a corollary, the group $\text{Isom}(D)$ of all isometries of the dodecahedron has order $6 \cdot 20 = 120$.

The spheres $S(v, d_1)$, $S(v, d_2)$, $S(v, d_3)$, $S(v, d_4)$ in $S^2(O, 1)$ contain respectively 3, 6, 6, 3 vertices of the dodecahedron D , while D has also the vertices v and $-v$. The isotropy group $I(v)$ acts twice transitively on sets $M \cap S(v, d_1)$ and $M \cap S(v, d_4)$ and simply transitively on sets $M \cap S(v, d_2)$ and $M \cap S(v, d_3)$.

We get that the vertex set M of D is 2-point homogeneous from the above arguments and the homogeneity of the dodecahedron D .

The set $M \cap S(v_1, d_2)$, where $v_1 \in M$, has vertices v_2, v_3, v'_3 with the spherical distances between any two distinct vertices in $\{v_2, v_3, v'_3\}$ equal to d_3 . Consequently, the ordered triples $M_1 = \{v_1, v_2, v_3\}$ and $M'_1 = \{v_1, v_2, v'_3\}$ are isometric. At the same time, there is no isometry of M and $S^2(0, 1)$ such that $f(M_1) = M'_1$ because the isotropy subgroup $I(v_1)$ acts simply transitively on $M \cap S(v_1, d_2)$ due to the above observations (see also Fig. 1 a)). \square

Proof. [Proof of Theorem 1.1] We know that the tetrahedron, cube, octahedron, icosahedron are at least 3-point homogeneous polyhedra (see Example 4.1, Corollary 4.2 and Corollary 4.1). Hence, it suffices to apply Corollary 3.2 to these four regular polyhedra. The corresponding result for the dodecahedron follows from Theorem 6.1. \square

Now, we are going to study semiregular polyhedra. Recall that all semiregular polytopes are (1-point) homogeneous.

Lemma 6.1. *If a semiregular polyhedron $P \in \mathbb{R}^3$ is 2-point homogeneous and is not regular, then P is possibly the cuboctahedron or the icosidodecahedron.*

Proof. All semiregular (right) prisms different from the cube are not 2-point homogeneous because there is no isometry of the polyhedron, moving the horizontal edge to the vertical one. Analogously, all semiregular antiprisms different from the octahedron are not 2-point homogeneous.

Among 13 Archimedean polyhedra, possibly only two, the cuboctahedron and the icosidodecahedron (see Fig. 1), are two-point homogeneous because all the other have different edges incident to nonisometric pairs of faces containing them. \square

The isotropy subgroups of all vertices of the cuboctahedron and the icosahedron coincide as subgroups of Euclidean isometry group: they are generated by mirror reflections with respect to two mutually perpendicular planes and include also only the identity mapping and the rotation by the angle π around straight

line that is the intersection of these planes. The cuboctahedron and icosidodecahedron are, respectively, the convex hulls of the set of midpoints of all edges of the cube (or octahedron) and of the dodecahedron (or icosahedron) and therefore have 12 and 30 vertices, respectively [8]. Therefore, their isometry groups have orders 48 and 120 which are equal to the orders of the isometry groups of the cube (or octahedron) and of the dodecahedron (or icosahedron).

Proposition 6.1. *The (semiregular) cuboctahedron is 12-point homogeneous (hence, its point homogeneity degree is ∞).*

Proof. The cuboctahedron, being centrally symmetric, has 4 possible nonzero spherical distances between the vertices: $\frac{\pi}{3} < \frac{\pi}{2} < \frac{2\pi}{3} < \pi$. For every vertex $v \in M$, the sets $M \cap S(v, \frac{\pi}{3})$, $M \cap S(v, \frac{\pi}{2})$, $M \cap S(v, \frac{2\pi}{3})$ have respectively 4, 2, 4 vertices; on these sets the isotropy subgroup $I(v)$ acts respectively simply transitively, twice transitively, and simply transitively. Consequently the cuboctahedron is two-point homogeneous.

There are two ordered pairs of vertices (v_1, v_2) and (w_1, w_2) in $M \cap S(v, \frac{\pi}{3})$ and one point $w \in M \cap S(v, \frac{\pi}{2})$ such that

$$d(v_1, v_2) = d(w_1, w_2) = \frac{\pi}{2}; \quad d(v_i, w_i) = \pi/3, \quad i = 1, 2; \quad d(v_1, w_2) = d(v_2, w_1) = 2\pi/3,$$

so that v is the midpoint of spherical segments $[v_1, w_2]$ and $[v_2, w_1]$. In addition,

$$d(w, w_i) = \frac{\pi}{3}, \quad d(w, v_i) = \frac{2\pi}{3}, \quad i = 1, 2,$$

so that w_i is the midpoint of spherical segments $[w, v_i]$, $i = 1, 2$. The set of all mentioned 6 vertices is a maximal by inclusion subset $N \subset M \cap B(v, \frac{\pi}{2})$, containing no pair of diametrically opposite points.

Next, we list all non-degenerate triangles with vertices in N and the chosen vertex v . There are only two isosceles triangles with base $[v, w]$ and lateral sides $\pi/3$ and two triangles with this base and other sides with lengths $\pi/3$ and $2\pi/3$. In addition, there are only two equilateral triangles with sides $\pi/3$ and two isosceles triangles with base length $\pi/2$ and lateral sides $\pi/3$ without the vertex w .

It is easy to check that for any two different isometric ordering of the vertices for these triangles with the first vertex v , there exists a self-isometry of the cuboctahedron from $I(v)$, moving one ordering to another one. In the first case (with the vertex w included) this is only the mirror reflection with respect to $[v, w]$, in the second case, either the same or a different mirror symmetry, or the central symmetry with respect to the vertex v .

As a consequence of this observation and Theorem 3.3, the cuboctahedron is 12-point homogeneous. \square

Proposition 6.2. *The icosidodecahedron is not 2-point homogeneous (hence, its point homogeneity degree is 1).*

Proof. We give two different proofs. Without loss of generality we can suppose that the origin O in \mathbb{R}^3 is the center of the icosidodecahedron P .

The first proof. We can represent the icosidodecahedron ID as a convex hull of the edge midpoints of a suitable dodecahedron D in \mathbb{R}^3 . Let us consider some vertex v of ID , i. e. a midpoint of some edge e of D . The isotropy subgroup $I(v)$ of ID is generated by two reflections in \mathbb{R}^3 with respect to planes through v and the origin, one of which contains e , and another one is orthogonal to it. In both these planes, there are two diametrically opposite midpoints of the edges of the dodecahedron at a spherical distance $\pi/2$ from v . This together with Proposition 3.1 imply the proposition, since there is no isometry from $I(v)$, sending one of the planes to the other.

The second proof. Let us consider two faces of the icosidodecahedron ID with a common edge l (one of them is triangular and the other is pentagonal), and let C be the midpoint of the edge l . In the triangular face and in the pentagonal face, take the vertices A and B respectively, that are the furthest vertices from the point C in their faces. It is clear that the distances from A and B to C are different. Therefore, the icosidodecahedron does not have an isometry that interchanges A and B . Indeed, such an isometry must preserve the midpoint of the segment $[A, B]$, and hence also the line passing through this midpoint and the origin. In particular, they will preserve the triangle ABC . It is clear that this is impossible. The proposition is proved. \square

Remark 6.1. Let assume that v is a vertex of the icosidodecahedron, i. e. is the midpoint of some edge e of the dodecahedron. Let us consider all 4 dodecahedron faces intersecting with the edge e and take the midpoint of all edges of these faces, i. e. the corresponding vertices of the icosidodecahedron. We shall obtain 15 such vertices plus two more its vertices on the distance $\frac{\pi}{2}$ from v outside these 4 faces. The corresponding picture of these vertices evidently implies the assertion in the first proof of Proposition 6.2 that $I(v)$ is generated by reflections in two mutually orthogonal planes.

It is possible to calculate that the spherical distances from v to another these 14 vertices are equal to $d_1 = \frac{\pi}{5} < d_2 = \frac{\pi}{3} < d_3 = \frac{\pi}{2}$. So other possible spherical distances between the vertices of the icosidodecahedron are $d_4 = \frac{2\pi}{3} < d_5 = \frac{4\pi}{5} < \pi$.

From the above, we can conclude that there are 6 geodesics (the great circles) of the circumsphere of the icosidodecahedron ID , passing through the vertices v and $-v$ of ID : 2 of them each contain 10 vertices, 2 others each contain 6 vertices, and the remaining 2 each contain 4 vertices from ID .

Proof. [Proof of Theorem 1.2] Since every edge of the polyhedron P is the intersection of two its adjacent faces, then it follows from the conditions of theorem that there exist no more than two face families such that the isometry group $I(P)$ acts transitively only on the faces from one family.

At first suppose that there are two such families, namely \mathcal{F}_1 and \mathcal{F}_2 . Take any face $F \in \mathcal{F}_1$ and vertices $A, B, C \in F$ on two adjacent edges of F . Since $d(A, B) = d(B, C)$ and P is 2-point homogeneous, then there exists an isometry $f \in I(P)$ such that $f(A) = B, f(B) = C$. Since f cannot map a face from \mathcal{F}_2 onto a face from \mathcal{F}_1 , then $f(F) = F$ and the inner angles of the face F by the vertices A and B coincide. This argument is also applied to any adjacent edges of the face F . Consequently the face F and all faces of the family \mathcal{F}_1 are regular pairwise congruent polygons. The same argument and statements are applied to the faces of the family \mathcal{F}_2 . Now in the consequence of 2-point homogeneity (hence homogeneity) of the polytope P , it is semiregular. On the base of the conducted investigation, in particular by Propositions 6.1 and 6.2, P is the cuboctahedron.

Now assume that $I(P)$ acts transitively on the family \mathcal{F} of all faces of the polytope P . Then one can easily deduce from condition of theorem that the sum σ of inner angles of two intersecting by joint edge faces under the same vertex of this edge depends nor an edge nor its vertex.

Let $F_1 \in \mathcal{F}$ and AB, BC be edges of the face F_1 . By conditions of theorem, there exists $f \in I(P)$ such that $f(A) = B, f(B) = A$. If moreover $f(F_1) = F_1$, then also $f(F_2) = F_2$, where F_2 is another face of the polyhedron P with the edge AB and f inverts the orientation of the space \mathbb{R}^3 . This and the transitivity of the group $I(P)$ on edges imply that for every face $F \in \mathcal{F}$ and every its edge e there is $g \in I(P)$ such that $g(F) = F$ and g transposes the vertices of the edge e . It follows from here that all faces of the polytope P are congruent regular polygons and P is regular polytope.

But (for now) a priori it may be that for any $f \in I(P)$ such that $f(A) = B, f(B) = A$, it must be $f(F_2) = F_1$ and $f(F_1) = F_2$. Then f preserves the orientation of the space \mathbb{R}^3 and in view of transitivity of the group $I(P)$ on edges, for any face $F \in \mathcal{F}$ and every its edge e there exists a $g \in I(P)$ such that g transposes the vertices of the edge e and the faces which include e . Then f (respectively, g) is the central symmetry of the sphere $S(0, 1)$ relative to the midpoint of the edge AB (respectively, e). Applying such procedures consecutively to the edges AB and BC , we obtain that inner angles of the face F_1 under vertices A and C are equal. Then, if the number of vertices of the face F is odd, then F is a regular polygon and P is a regular polyhedron.

Suppose that the number k of vertices of the face F is even. Then $k \geq 4$ and the sum of inner angles of the polygon F is equal to

$$\frac{k\sigma}{2} = (k - 2)\pi \implies \sigma = \frac{2(k - 2)\pi}{k} \geq \pi,$$

$$k \geq 6 \implies \sigma = \frac{2(k - 2)\pi}{k} \geq \frac{4\pi}{3}.$$

If the quality of faces of the polytope P with joint vertex $v \in M$ is $l = 4$, then the quality of pairs of adjacent faces (without joint faces) with common vertex v is equal to 2 and the sum of inner angles under v of all faces with the vertex v is equal to $2\sigma \geq 2\pi$, what is impossible. Hence $l = 3$ and since the sum of inner angles of two intersecting by joint edge faces under the same vertex v of this edge is equal to σ , then all these three inner angles are equal to $\sigma/2$. Consequently all faces and the polyhedron P are regular.

Furthermore $k \geq 6$ is impossible, otherwise in consequence of the second inequality, written above in the row, the sum of all inner angles of three polygons under the joint vertex v is equal to

$$\frac{3\sigma}{2} \geq \frac{3}{2} \cdot \frac{4\pi}{3} = 2\pi,$$

what is impossible. Thus, in the case under consideration it must be $k = 4, l = 3, \sigma = \pi$, and P is a (regular) cube. \square

7. On m -point homogeneous polyhedra with several edge lengths

Example 7.1. *The right three-dimensional (anti)prism with regular bases is three-point homogeneous if the length of the lateral edge is longer than the sides and diagonals of the bases.*

Example 7.2. *Let us consider a simplex S in \mathbb{R}^3 with given three edge lengths $0 < a < b < c, c^2 < a^2 + b^2$, such that every pair of edges without common vertex have equal Euclidean lengths and each of the number a, b, c is the length of edges for some pair. Hence we have a 3-parameter family of simplices in \mathbb{R}^3 . One can prove that every such simplex is 4-point homogeneous. Hence we have a 3-parameter family of 4-point homogeneous simplices (4-point sets) in \mathbb{R}^3 .*

Example 7.3. *If a finite metric space (M, d) is homogeneous and for some its point the distances to all other points are pairwise distinct, then (M, d) is k -point homogeneous for every $k \geq 1$. Indeed, it suffices to move only one point of the first k -tuple to the corresponding point of the second k -tuple with a suitable isometry and all other points will automatically be placed in the required places.*

Proposition 7.1. *Let P be a 3-dimensional polyhedron, isomorphic to the octahedron, with diametrically opposite pairs of vertices $A, B; C, D; E, F; d_1, d_2$ be positive numbers such that $d_1 < d_2, d_1 + d_2 = \pi$. Let us set distances*

$$d(A, B) = d(C, D) = d(E, F) = \pi;$$

$$d(A, C) = d(A, E) = d(B, D) = d(B, F) = d(C, E) = d(D, F) = d_1;$$

while the rest 6 distances to be equal to d_2 . Then (P, d) is realized as the three-dimensional convex centrally symmetric polyhedron in \mathbb{R}^3 with vertices on $S^2(0, 1)$ and with indicated spherical distances between the vertices. Moreover, (P, d) is a 6-point homogeneous polyhedron.

Proof. Let us consider the following matrix G with the value $\alpha = \cos d_1$:

$$G = \begin{pmatrix} 1 & \alpha & \alpha \\ \alpha & 1 & \alpha \\ \alpha & \alpha & 1 \end{pmatrix}.$$

The matrix G is positive definite. Therefore, there are linearly independent vectors $A, C, E \in S^2(0, 1) \subset \mathbb{R}^3$ with the Gram matrix G . For example, we can take the vectors whose components are lines of the following matrix

$$V = \frac{1}{\sqrt{1 + 2\beta^2}} \begin{pmatrix} 1 & \beta & \beta \\ \beta & 1 & \beta \\ \beta & \beta & 1 \end{pmatrix},$$

where $\beta > 0, \beta(2 + \beta)/(1 + 2\beta^2) = \alpha$. It is clear that the convex hull of the set $M = \{A, C, E, B = -A, D = -C, F = -E\}$ is a realization of the space (P, d) .

From the form of the matrix G it follows that every transposition of the ordered pairs $\{A, B\}, \{C, D\}, \{E, F\}$ and the central symmetry are isometries of the polyhedron P . Consequently it is two-point homogeneous. For the ordered pair $\{A, C\}$, there are only two non-degenerate triples $\{A, C, E\}$ and $\{A, C, F\}$, and they are not isometric. For the ordered pair $\{A, F\}$, there are only two non-degenerate triples $\{A, F, D\}$ and $\{A, F, C\}$ and they are not isometric. Consequently, (P, d) is 3-point and 6-point homogeneous by Theorem 3.3. \square

Let us consider some general idea how to disprove that a subset $M \subset \mathbb{R}^3$ is 3-point homogeneous.

Let us suppose that M is 3-point homogeneous, hence, it is 2-point homogeneous and 1-point homogeneous. Without loss of generality we may assume that $M \subset S(O, 1)$, where O is the origin in \mathbb{R}^3 . Now, let us take points $A, B \in M$, $A \neq B$, such that O is not on the straight line AB . Let E be the midpoint of the segment $[A, B]$ and L be a plane through E and orthogonal to AB .

Proposition 7.2. *In the above assumptions, let us suppose that points $C, D \in M$ are such that $d(A, C) = d(A, D)$, $d(B, C) = d(B, D)$, and $C \neq D$. Then the symmetry with respect to the plane OAB in \mathbb{R}^3 is an isometry of M that interchanges the points C and D .*

Proof. Let us consider the orthogonal projection $\psi : \mathbb{R}^3 \rightarrow L$ and $M' = \psi(M)$. It should be noted that $\psi(A) = \psi(B) = E$, $\psi(O) = O$ (since $A, B \in M \subset S(O, 1)$), and O is the barycenter of M' . It is clear that $C \neq D$ implies $\psi(C) \neq \psi(D)$ ($\psi(C) = \psi(D)$, $d(A, C) = d(A, D)$, and $d(B, C) = d(B, D)$ imply $C = D$). Since M is supposed to be 3-point homogeneous and (A, B, C) , (A, B, D) are triples of points from M with corresponding equal distances, then there is an isometry of M that sends the triple (A, B, C) to the triple (A, B, D) .

Applying ψ , we see that there is an isometry of M' that preserves E and sends $\psi(C)$ to $\psi(D)$. Since $E \neq O$ and O is the barycenter of M' , then the latter isometry should be the symmetry with respect to the straight line EO . From this we easily get that C and D are symmetric to each other with respect to the plane OAB (recall that $d(A, C) = d(A, D)$ and $d(B, C) = d(B, D)$). \square

Remark 7.1. *If M' in the above proof is not symmetric with respect to the straight line EO , then the equalities $d(A, C) = d(A, D)$ and $d(B, C) = d(B, D)$ for some points $C, D \in M$ imply that $C = D$.*

8. Conclusion

The notion of n -point homogeneity for a metric space was introduced by G. Birkhoff in [13] by the following sentence: “Any isometry between two (sub)sets of n points or fewer points can be extended to a self-isometry of space”. In the same paragraph he noticed that the Urysohn space [33] has n -point homogeneity for every finite order n .

This posthumous paper by P. Urysohn contains his construction of remarkable complete separable metric space U which besides the n -point homogeneity for every finite order n is universal in the sense that every separable metric space is isometric to a subspace of U . Moreover, any space with these properties is isometric to U [33].

Recently were appeared many interesting results on the space U . Let us mention only three of them. Any isometry between two compact subsets of the space U can be extended to an isometry of U onto itself [15]. The topological group $I(U)$ of all isometries of U is universal in the sense that it contains an isomorphic copy of every topological group with a countable base [34]. The space U is homeomorphic to the Hilbert space l_2 [35].

It should be noted that there is no sense to classify (even finite) m -point homogeneous metric spaces up to isometry or similarity without additional restrictions, because any such metric space (M, d) generates a class of m -point homogeneous metric spaces of the following type: $(M, \psi \circ d)$, where ψ is any increasing concave (convex up) function such that $\psi(0) = 0$. For instance, one can take $\psi(t) = \ln(1 + t)$, $t \geq 0$, or $\psi(t) = 2 \sin(t/2)$, $0 \leq t \leq \pi$.

H. Busemann considered 2-point homogeneous Busemann G -spaces (geodesic spaces) in 1942 [16]. The latter can be described as complete locally compact metric spaces (M, d) such that any two points in M can be joined by a *shortest arc*, i. e. an isometric embedding $p : [0, d(x, y)] \rightarrow (M, d)$; shortest arcs are locally extendable; any shortest arc has unique extension to a *geodesic*, i. e. locally isometric mapping $\gamma : \mathbb{R} \rightarrow (M, d)$.

It is well known that every 2-point homogeneous Riemannian manifold is isometric to Euclidean space or a Riemannian symmetric space of rank 1, see, e. g. Theorem 8.12.8 and Corollary 8.12.9 in [38]. In fact, this result holds for a more general situation, namely, for 2-point homogeneous Busemann G -spaces. This can be easily deduced from papers [36], [37] by H.C. Wang for compact spaces and [32] by J. Tits for non-compact case. Currently, there are various proofs of the above characterization of 2-point homogeneous

Riemannian manifolds. Especially important is very short paper [30] by Z.I. Szabó, where he showed also that the above 2-point homogeneous spaces are symmetric not using any classification, thereby solving a problem of Wolf from [38].

H. Busemann proved in [17] that if for every two geodesics of a compact Busemann G -space there exists an isometry of the space moving one of them to the other (this condition is much weaker than 2-point homogeneity of the space) then the space is isometric to one of compact Riemannian symmetric space of rank 1. V.N. Berestovskii in [1] proved that non-compact Busemann G -space with the isometry group transitive on the family of its geodesics is isometric to Euclidean space or non-compact Riemannian symmetric space of rank 1, thus giving a positive answer to one Busemann problem.

As a corollary of quoted results, every 3-point homogeneous Busemann G -space (in particular, Riemannian manifold) is isometric to Euclidean space, spherical space, elliptic space, or hyperbolic space with appropriate constant sectional curvature. This implies that in any such space every isometry of every two their subsets can be extended to an isometry of the entire space, which is extremely different from the case with the vertex sets of polytopes.

References

- [1] V. N. Berestovskii, Solution of a Busemann problem, *Sib. Mat. Zh.* 51(6) (2010) 1215–1227 (in Russian). English translation: *Siberian Math. J.* 51(6) (2010), 962–970. DOI: 10.1007/s11202-010-0095-3
- [2] V. N. Berestovskii, Yu. G. Nikonorov, Killing vector fields of constant length on Riemannian manifolds, *Sib. Mat. Zh.* 49(3) (2008) 497–514 (in Russian). English translation: *Siberian Math. J.* 49(3) (2008) 395–407. DOI: 10.1007/s11202-008-0039-3
- [3] V. N. Berestovskii, Yu. G. Nikonorov, On δ -homogeneous Riemannian manifolds, *Diff. Geom. Appl.* 26(5) (2008) 514–535. DOI: 10.1016/j.difgeo.2008.04.003
- [4] V. N. Berestovskii, Yu. G. Nikonorov, Clifford–Wolf homogeneous Riemannian manifolds, *J. Differ. Geom.* 82(3) (2009) 467–500. DOI: 10.4310/jdg/1251122544
- [5] V. N. Berestovskii, Yu. G. Nikonorov, Generalized normal homogeneous Riemannian metrics on spheres and projective spaces, *Ann. Global Anal. Geom.* 45(3) (2014) 167–196. DOI: 10.1007/s10455-013-9393-x
- [6] V. N. Berestovskii, Yu. G. Nikonorov, Finite homogeneous metric spaces, *Sib. Mat. Zh.* 60(5) (2019) 973–995 (in Russian). English translation: *Siberian Math. J.* 60(5) (2019) 757–773. DOI: 10.1134/S0037446619050021
- [7] V. N. Berestovskii, Yu. G. Nikonorov, *Riemannian Manifolds and Homogeneous Geodesics*, Springer Monographs in Mathematics. Cham: Springer, 2020. DOI: 10.1007/978-3-030-56658-6
- [8] V. N. Berestovskii, Yu. G. Nikonorov, Finite homogeneous subspaces of Euclidean spaces, *Mat. Trudy* 24(1) (2021) 3–34 (in Russian). English translation: *Siberian Advances in Mathematics* 31(3)(2021) 155–176. DOI: 10.1134/S1055134421030019
- [9] V. N. Berestovskii, Yu. G. Nikonorov, Semiregular Gosset polytopes, *Izv. Ross. Akad. Nauk, Ser. Mat.* 86(4) (2022) 51–84 (in Russian). English translation: *Izv. Math.*, 86(4) (2022) 667–698. DOI: 10.1070/IM9169, DOI: 10.4213/im9169e
- [10] V. N. Berestovskii, Yu. G. Nikonorov, On finite homogeneous metric spaces, *Vladikavkaz Mathematical Journal* 2(2) (2022) 51–61. DOI: 10.46698/h7670-4977-9928-z
- [11] M. Berger, *Geometry I*. Universitext, Springer-Verlag, Berlin, 2009.
- [12] L. W. Berman, B. Monson, D. Oliveros, G. I. Williams, Fully truncated simplices and their monodromy groups, *Adv. Geom.* 18(2) (2018) 193–206. DOI: 10.1515/advgeom-2017-0047
- [13] G. Birkhoff, Metric foundations of geometry I, *Trans. Amer. Math. Soc.* 55(3) (1944) 465–492.
- [14] G. Blind, R. Blind, The semiregular polytopes, *Comment. Math. Helv.* 66(1) (1991) 150–154. DOI: 10.1007/BF02566640
- [15] S. A. Bogatyri, Metrically homogeneous spaces, *Uspekhi Mat. Nauk* 57(2(344)) 2002 3–22 (in Russian). English translation: *Russian Math. Surveys* 57(2) (2002) 221–240. DOI: 10.4213/rm495
- [16] H. Busemann, *Metric methods in Finsler spaces and in the foundation of geometry*, *Annals of Mathematics Studies*, no. 8, Princeton University Press, Princeton, N.J., 1942.
- [17] H. Busemann, Groups of motions transitive on sets of geodesics, *Duke Math. J.* 24 (1956) 539–544. DOI: 10.1215/S0012-7094-56-02352-3
- [18] H. S. M. Coxeter, Wythoff’s construction for uniform polytopes, *Proc. London Math. Soc.* 38 (1934) 327–339. DOI: 10.1112/plms/s2-38.1.327
- [19] H. S. M. Coxeter, *Regular Polytopes*, (3rd edition), New York: Dover, 1973.
- [20] P. R. Cromwell, *Polyhedra*, Cambridge: Cambridge Univ. Press, 1997.
- [21] M. Dutour Sikirić, <http://mathieudutour.altervista.org/Regular/>, 24.12.2022.
- [22] P. Du Val, *Homographies, quaternions, and rotations*, Oxford Mathematical Monographs Clarendon Press, Oxford 1964.
- [23] E. L. Elte, *The Semiregular Polytopes of the Hyperspaces*, 1912, Groningen: University of Groningen, <https://quod.lib.umich.edu/u/umhistmath/ABR2632.0001.001>, 24.12.2022.
- [24] Four-Dimensional Euclidean Space, <http://eusebeia.dyndns.org/4d/index>, 24.12.2022.
- [25] Th. Gosset, On the regular and semi-regular figures in space of n dimensions, *Messenger Math.* 29 (1900) 43–48.
- [26] B. Grünbaum, *Convex Polytopes*, (2nd edition), Graduate Texts in Mathematics, 221. New York: Springer, 2003.
- [27] A. Hedayat, W. D. Wallis, Hadamard matrices and their applications, *Annals of Statistics* 6(6) (1978) 1184–1238. DOI:10.1214/aos/1176344370

- [28] H. Martini, A hierarchical classification of Euclidean polytopes with regularity properties, *Polytopes: Abstract, Convex and Computational / Proc. of the NATO Advanced Study Institute, Scarborough, Ontario, Canada, August 20–September 3, 1993* / Eds. T. Bisztriczky et al. Dordrecht: Kluwer Academic Publishers. NATO ASI Ser., Ser. C, Math. Phys. Sci. 440, 1994, 71–96.
- [29] L. Schläfli, *Theorie der vielfachen Kontinuität*, Hrsg. im Auftrage der Denkschriften-Kommission der schweizerischen naturforschenden Gesellschaft von J.H. Graf. Zürich, Basel: Georg & Co, 1901.
- [30] Z. I. Szabó, A short topological proof for the symmetry of 2 point homogeneous spaces, *Invent. Math.* 106(1) (1991) 61–64. DOI: 10.1007/BF01243903
- [31] H. Tamaru, Two-point homogeneous quandles with prime cardinality, *J. Math. Soc. Japan* 65(1) (2013) 1117–1134. DOI: 10.2969/jmsj/06541117
- [32] J. Tits, Sur certaines classes d'espaces homogenes de groupes de Lie, *Acad. Roy. Belg. Cl. Sci. Mém Coll.* 29(3) (1955), 268 P.
- [33] P. Urysohn, Sur un espace métrique universel, *Bull. Sci. Math.* 51 (1927) 43–64, 74–90.
- [34] V. Uspenskiy, On the group of isometries of the Urysohn universal metric space, *Comment. Math. Univ. Carolinae* 31(1) (1990) 181–182.
- [35] V. Uspenskiy, The Urysohn universal metric space is homeomorphic to a Hilbert space, *Topology and its Applications* 139(1) (2004) 145–149. DOI: 10.1016/j.topol.2003.09.008
- [36] H. C. Wang, Two theorems on metric spaces, *Pacific. J. Math.* 1 (1951) 473–480.
- [37] H. C. Wang, Two-point homogeneous spaces, *Ann. Math.* 55(1) (1952) 177–191. DOI: 10.2307/1969427
- [38] J. A. Wolf, *Spaces of constant curvature*, (6th edition), AMS Chelsea Publishing, Providence, RI, 2011.