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On isometric immersions of sub-Riemannian manifolds

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Abstract. We study curvature invariants of a sub-Riemannian manifold (i.e., a manifold with a Riemannian metric on a non-holonomic distribution) related to mutual curvature of several pairwise orthogonal subspaces of the distribution, and prove geometrical inequalities for a sub-Riemannian submanifold. As applications, inequalities are proved for submanifolds with mutually orthogonal distributions that include scalar and mutual curvature. For compact submanifolds, inequalities are obtained that are supported by known integral formulas for almost-product manifolds.

Keywords: Sub-Riemannian manifold, isometric immersion, mutual curvature, mean curvature

Mathematics Subject Classifications (2010) 53C12; 53C15; 53C42

1. Introduction

Extrinsic geometry of Riemannian submanifolds deals with properties that can be expressed in terms of the second fundamental form and its invariants (e.g., principal curvatures). The recent development of the geometry of submanifolds was inspired by the embedding theorem of J.F. Nash, [7], and theorems that surfaces with positive curvature are easily embedded in 3D space (A.D. Aleksandrov and A.V. Pogorelov), while surfaces with negative curvature usually do not allow such an embedding (D. Gilbert and N.V. Efimov). This led to the following problem (see [3, Problem 2]): *find a simple optimal connection between intrinsic and extrinsic invariants of a Riemannian submanifold*. The difficulty was to understand smooth submanifolds (the problem is different for C^1-immersions, [8]) of large codimension using only a few known relationships (fundamental Gauss-Codazzi-Ricci equations) between intrinsic and extrinsic geometry. In 1968, S.S. Chern posed a question on other obstacles for a Riemannian manifold to admit an isometric minimal immersion in a Euclidean space. To study these questions, it is necessary to introduce new types of Riemannian invariants, and to find optimal relations between them and extrinsic invariants of submanifolds.

In 1990s, B.Y. Chen introduced the concept of δ-curvature invariants for a Riemannian manifold and proved the optimal inequality for a submanifold that involves these invariants and the square of mean curvature, e.g., [4], the equality case led to the notion of "ideal immersions" (isometric immersions of least possible tension). The δ-invariants are obtained from the scalar curvature (which is the "sum" of sectional curvatures) by discarding some of sectional curvatures. Similar scalar invariants are known for Kähler, contact and affine manifolds, warped products and submersions, see [4, 5]. For manifolds endowed with nonholonomic distributions or foliations, such curvature invariants have hardly been studied.

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Distributions on a manifold, i.e., sub-bundles of the tangent bundle, arise in differential geometry in terms of line fields, submersions, Lie groups actions, and almost product manifolds. A nonholonomic manifold, i.e., a pair $(M, \mathcal{D})$, where $\mathcal{D}$ is a distribution on a smooth manifold M, was introduced for the geometric interpretation of constrained systems in classical mechanics and thermodynamics. A sub-Riemannian manifold, that is $(M, \mathcal{D})$ equipped with a Riemannian metric g on $\mathcal{D}$, is a certain type of generalization of a Riemannian manifold. There are several lines of research in sub-Riemannian geometry based on optimal control methods, partial differential equations and constrains of other geometries, see [1, 2].

In [13], we introduced curvature invariants (different from δ-invariants by Chen) of a Riemannian manifold equipped with complementary orthogonal distributions, and proved the geometric inequality for submanifolds that includes our curvature invariants and the square of mean curvature. These curvature invariants are related with the mixed scalar curvature – a well-known curvature invariant of a Riemannian almost k-product manifold, in particular, (multiply) twisted or warped products, e.g., [12]. In [14] we introduced invariants of a Riemannian manifold more general than in [13], related to the mutual curvature of noncomplementary pairwise orthogonal subspaces of the tangent bundle. In the case of one-dimensional subspaces, the mutual curvature is equal to half the scalar curvature of the subspace spanned by them, and in the case of complementary subspaces, this is the mixed scalar curvature. Using these invariants, we proved inequalities for Riemannian submanifolds and gave applications for sub-Riemannian submanifolds.

In this article, we study curvature invariants (defined in [13, 14]) and also introduce Chen-type invariants for a sub-Riemannian manifold. We prove geometrical inequalities for submanifolds with mutually orthogonal distributions that include scalar and mutual curvature. In the case of compact submanifolds, we obtain the inequalities supported by known integral formulas for almost-product manifolds.

The article is organized as follows. In Section 2 (following the introductory Section 1), we recall some integral formulas containing scalar and mutual curvature for a sub-Riemannian manifold. In Section 3, we introduce and study scalar invariants based on this kind of curvature. In Section 4, we prove geometric inequalities for a sub-Riemannian submanifold equipped with distributions.

2. The mutual curvature of distributions

Here, we recall definitions of mutual curvature and mixed scalar curvature of distributions on a sub-Riemannian manifold and briefly discuss equalities with them and divergence of some vector fields, which lead to integral formulas on a compact manifold.

Let an n-dimensional Riemannian manifold (M, g) with the Levi-Civita connection ∇ be endowed with a d-dimensional distribution $\mathcal{D}$ (subbundle of the tangent bundle TM of rank d). The Riemannian curvature tensor is given by $R_{X,Y} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$, its contraction is the Ricci tensor $\mathrm{Ric}_{X,Y} = \mathrm{trace}(Z \mapsto R_{Z,X}\, Y)$, and the trace of Ricci tensor is the scalar curvature $\tau = \mathrm{trace}_g \mathrm{Ric}$, e.g., [9].

Let $\mathcal{D}^\perp$ be the orthogonal complement to $\mathcal{D}$ in TM, its rank is $d^\perp = n - d$. We call $(M, g, \mathcal{D}, \mathcal{D}^\perp)$ a Riemannian almost product manifold, see [6]. The second fundamental form h and integrability tensor T of $\mathcal{D}$ (and, similarly, tensors $h^\perp$ and $T^\perp$ of $\mathcal{D}^\perp$) are defined as follows:

$$h(X, Y) = \frac{1}{2}\,(\nabla_X Y + \nabla_Y X)^\perp, \quad T(X, Y) = \frac{1}{2}\,(\nabla_X Y - \nabla_Y X)^\perp.$$

If $\mathcal{D}$ is integrable (i.e., $T = 0$), then it is tangent to a foliation. Denote by $H = \mathrm{trace}_g\, h$ and $H^\perp = \mathrm{trace}\, h^\perp$ the mean curvature vectors of $\mathcal{D}$ and $\mathcal{D}^\perp$, respectively. We call $\mathcal{D}$ *totally geodesic* if $h = 0$, *harmonic* if $H = 0$ and *totally umbilical* if $h = (H/d)\, g$ (and similarly, for $\mathcal{D}^\perp$).

Let $\{e_i\}$ be an adapted local orthonormal frame, i.e., $\{e_1, \ldots, e_d\} \subset \mathcal{D}$ and $\{e_{d+1}, \ldots, e_n\} \subset \mathcal{D}^\perp$. The mixed scalar curvature $\mathrm{S}_{\rm mix}(\mathcal{D}, \mathcal{D}^\perp)$ is a function on M defined by

$$\mathrm{S}_{\rm mix}(\mathcal{D}, \mathcal{D}^\perp) = \sum\nolimits_{1\le a\le d,\ d<b\le n} K(e_a \wedge e_b),$$

where $K(e_a \wedge e_b) = g(R_{e_a,e_b}\, e_b, e_a)$ is the sectional curvature of the plane $e_a \wedge e_b$, and it does not depend on the choice of frames. For example, if $\mathcal{D}$ (or $\mathcal{D}^\perp$) is one-dimensional and locally spanned by a unit vector field

N, then $S_{\rm mix}(\mathcal{D},\mathcal{D}^\perp)={\rm Ric}_{N,N}$. The following formula for complementary orthogonal distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ on a Riemannian manifold (M,g) was proved in [16]:

$$\operatorname{div}(H+H^\perp)=S_{\rm mix}(\mathcal{D},\mathcal{D}^\perp)+\|h\|^2+\|h^\perp\|^2-\|H\|^2-\|H^\perp\|^2-\|T\|^2-\|T^\perp\|^2. \tag{1}$$

Example 2.1. Let $\mathcal{D}$ be tangent to a codimension one foliation $\mathcal{F}$, and N be a unit normal to the leaves of $\mathcal{F}$. The shape operator $A_N: T\mathcal{F}\to T\mathcal{F}$ is given by $A_N(X)=-\nabla_X N$, where ∇ is the Levi–Civita connection. The generalized mean curvatures $\sigma_r=\sigma_r(A_N)$ are functions on M defined as coefficients of the n-th degree polynomial $\det({\rm id}_{\mathcal{D}}+\,tA_N)$ in t. Thus, $\sigma_0=1,\ \sigma_1={\rm trace}\,A_N,\ \ldots,\ \sigma_n=\det A_N$. In this case, (1) reduces to

$$\operatorname{div}(\nabla_N N+\sigma_1 N)={\rm Ric}_{N,N}-2\,\sigma_2. \tag{2}$$

Next, let a Riemannian manifold (M,g) be endowed with three pairwise orthogonal n_i-dimensional distributions $\mathcal{D}_i$ $(i=1,2,3)$ such that $TM=\mathcal{D}_1\oplus\mathcal{D}_2\oplus\mathcal{D}_3$. We call $(M,g,\mathcal{D}_1,\mathcal{D}_2,\mathcal{D}_3)$ a *Riemannian almost 3-product manifold*. Denote by $\mathcal{D}_i^\perp$ the orthogonal complement to $\mathcal{D}_i$ in TM, its rank is $n_i^\perp=n-n_i$.

Remark 2.2. A Riemannian almost multi-product manifold is a Riemannian manifold equipped with $k\ge 2$ pairwise orthogonal complementary distributions $\mathcal{D}_1,\ldots,\mathcal{D}_k$. We meet this structure in such topics of differential geometry as multiply-warped (or twisted) products and the webs of foliations; see e.g., [11]. In particular, almost 3-product manifolds appear naturally among almost para-f-manifolds, lightlike manifolds, orientable 3-manifolds (since they admit 3 linearly independent vector fields), webs composed of 3 generic foliations, minimal hypersurfaces in space forms with 3 distinct principal curvatures, tubes over standard embeddings of a projective plane in a sphere, etc.

The second fundamental forms $h_i:\mathcal{D}_i\times\mathcal{D}_i\to\mathcal{D}_i^\perp$ and the integrability tensors $T_i:\mathcal{D}_i\times\mathcal{D}_i\to\mathcal{D}_i^\perp$ of $\mathcal{D}_i$ (and similarly, $h_i^\perp$ and $T_i^\perp$ of orthogonal distributions $\mathcal{D}_i^\perp$) are defined by

$$2\,h_i(X,Y)=(\nabla_X Y+\nabla_Y X)^\perp,\quad 2\,T_i(X,Y)=(\nabla_X Y-\nabla_Y X)^\perp=[X,Y]^\perp.$$

Then $H_i={\rm trace}_g\,h_i$ is called the mean curvature vector field of the distribution $\mathcal{D}_i$. A distribution $\mathcal{D}_i$ is integrable (or involutive) if $T_i=0$, and $\mathcal{D}_i$ is *totally umbilical*, *minimal*, or *totally geodesic*, if $h_i=(H_i/n_i)\,g$, $H_i=0$, or $h_i=0$, respectively.

Let $x\in M$ and $\{e_i\}$ be an adapted orthonormal frame on the subspace $\mathcal{D}_1(x)\oplus\mathcal{D}_2(x)$, i.e., $\{e_1,\ldots,e_{n_1}\}\subset\mathcal{D}_1(x)$, $\{e_{n_1+1},\ldots,e_d\}\subset\mathcal{D}_2(x)$. The *mutual curvature* of a pair $(\mathcal{D}_1,\mathcal{D}_2)$ is a function on M defined by

$$\mathrm{S}_{\rm m}(\mathcal{D}_1(x),\mathcal{D}_2(x))=\sum\nolimits_{a\le n_1,\ n_1<b\le d}K(e_a,e_b),$$

and it does not depend on the choice of frames, e.g., [15]. The mixed scalar curvature of the triple $(\mathcal{D}_1,\mathcal{D}_2,\mathcal{D}_3)$ is defined similarly as $\mathrm{S}_{\rm mix}(\mathcal{D},\mathcal{D}^\perp)$ for a pair $(\mathcal{D},\mathcal{D}^\perp)$, and it can be presented as follows, e.g., [11]:

$$\mathrm{S}_{\rm mix}(\mathcal{D}_1,\mathcal{D}_2,\mathcal{D}_3)=\mathrm{S}_{\rm m}(\mathcal{D}_1,\mathcal{D}_2)+\mathrm{S}_{\rm m}(\mathcal{D}_1,\mathcal{D}_3)+\mathrm{S}_{\rm m}(\mathcal{D}_2,\mathcal{D}_3). \tag{3}$$

Lemma 2.3. *The following formulas are true:*

$$2\,\mathrm{S}_{\rm mix}(\mathcal{D}_1,\mathcal{D}_2,\mathcal{D}_3)=\operatorname{div}(H_1+H_1^\perp+H_2+H_2^\perp+H_3+H_3^\perp)-Q_1-Q_2-Q_3, \tag{4}$$

$$S_{\rm m}(\mathcal{D}_1,\mathcal{D}_2)=\operatorname{div}(H_1+H_1^\perp+H_2+H_2^\perp-H_3-H_3^\perp)-Q_1-Q_2+Q_3, \tag{5}$$

where $Q_i=\|h_i\|^2+\|h_i^\perp\|^2-\|H_i\|^2-\|H_i^\perp\|^2-\|T_i\|^2-\|T_i^\perp\|^2$ *for* $i=1,2,3$.

Proof. We can write (3) in the form

$$2\,\mathrm{S}_{\rm mix}(\mathcal{D}_1,\mathcal{D}_2,\mathcal{D}_3)=\mathrm{S}_{\rm mix}(\mathcal{D}_1,\mathcal{D}_1^\perp)+\mathrm{S}_{\rm mix}(\mathcal{D}_2,\mathcal{D}_2^\perp)+\mathrm{S}_{\rm mix}(\mathcal{D}_3,\mathcal{D}_3^\perp),$$

or in the form (expressing mutual curvature in terms of mixed scalar curvature)

$$\begin{aligned}\mathrm{S}_{\rm m}(\mathcal{D}_1,\mathcal{D}_2)&=\mathrm{S}_{\rm mix}(\mathcal{D}_1,\mathcal{D}_2,\mathcal{D}_3)-2\,\mathrm{S}_{\rm mix}(\mathcal{D}_1\oplus\mathcal{D}_2,\mathcal{D}_3)\\&=\mathrm{S}_{\rm mix}(\mathcal{D}_1,\mathcal{D}_1^\perp)+\mathrm{S}_{\rm mix}(\mathcal{D}_2,\mathcal{D}_2^\perp)-\mathrm{S}_{\rm mix}(\mathcal{D}_3,\mathcal{D}_3^\perp).\end{aligned}$$

Thus, using (1), we get (4) and (5). $\square$

Example 2.4. Let (M^3, g) admit three pairwise orthogonal codimension-one foliations $\mathcal{F}_i$, and let N_i be unit vector fields orthogonal to $\mathcal{F}_i$. Writing down (2) for each N_i, summing for $i = 1, 2, 3$, and using the equality $\tau = \sum_{i=1}^3 \mathrm{Ric}_{N_i,N_i}$, where τ is the scalar curvature of (M, g), yields the formula

$$\operatorname{div} \sum\nolimits_{i=1}^3 (\nabla_{N_i} N_i + \sigma_1(A_{N_i})N_i) = 2 \sum\nolimits_{i=1}^3 \sigma_2(\mathcal{F}_i) - \tau\,.$$

Two consequences, using $2\,\sigma_2(A_{N_i}) = \mathrm{trace}(A_{N_i})^2 - (\mathrm{trace}\, A_{N_i})^2$:
- if $\tau < 0$ then each foliation $\mathcal{F}_i$ cannot be totally umbilical;
- if $\tau > 0$ then each foliation $\mathcal{F}_i$ cannot be harmonic.

Remark 2.5. Applying the Divergence Theorem to (1), (2), (4) and (5) on a compact Riemannian manifold gives well-known integral formulas. These formulas can be extended for distributions defined on the complement $M \setminus \Sigma$ of a union Σ of finitely many closed codimension $k \geq 2$ submanifolds of a manifold M. Namely, *if (M, g) is a closed oriented Riemannian manifold, X is a vector field on an open set $M \setminus \Sigma$, $(k-1)(p-1) \geq 1$ and $\|X\| \in L^p(M, g)$, then $\int_M (\operatorname{div} X)\, d\,\mathrm{vol}_g = 0$*, see [17] and [12, p. 75].

3. Invariants based on scalar and mutual curvature

Here, we introduce and study scalar invariants based on scalar and mutual curvature.

Given integer $k \geq 2$, let $V_1, \ldots, V_k$ be mutually orthogonal subspaces of $\mathcal{D}_x$ at a point $x \in M$ with $\dim V_i = n_i \geq 1$. Let $\{e_i\}$ be an adapted orthonormal basis of the subspace $V = \bigoplus_{i=1}^k V_i$, i.e., $\{e_1, \ldots, e_{n_1}\} \subset V_1, \ldots, \{e_{n_{k-1}+1}, \ldots, e_{n_k}\} \subset V_k$. Define the *mutual curvature* of the set $\{V_1, \ldots, V_k\}$ by

$$\mathrm{S_m}(V_1, \ldots, V_k) = \sum\nolimits_{i<j} \sum\nolimits_{n_{i-1}<a\leq n_i,\ n_{j-1}<b\leq n_j} K(e_a \wedge e_b).$$

Note that $\mathrm{S_m}(V_1, \ldots, V_k)$ does not depend on the choice of frames. We immediately have

$$\mathrm{S_m}(V_1, \ldots, V_k) = \sum\nolimits_{i<j} \mathrm{S_m}(V_i, V_j),$$

where $\mathrm{S_m}(V_i, V_j) = \sum_{n_{i-1}<a\leq n_i,\ n_{j-1}<b\leq n_j} K(e_a \wedge e_b)$.

For the scalar curvature $\tau(V) = \mathrm{trace}_g\,\mathrm{Ric}\,|_V$ (the trace of the Ricci tensor on a subspace $V = \bigoplus_{i=1}^k V_i$) and the scalar curvatures $\tau(V_i) = \mathrm{trace}_g\,\mathrm{Ric}\,|_{V_i}$ of subspaces V_i we get

$$\tau(V) = 2\,\mathrm{S_m}(V_1, \ldots, V_k) + \sum\nolimits_{i=1}^k \tau(V_i)\,. \tag{6}$$

For example, if all subspaces V_i are one-dimensional, then $2\,\mathrm{S_m}(V_1, \ldots, V_k) = \tau(V)$.

For an integer $k \geq 2$, denote by $S(d, k)$ the set of unordered k-tuples $(n_1, \ldots, n_k)$ of natural numbers satisfying $n_1 + \ldots + n_k \leq d$. Denote by $S(d)$ the set of all unordered k-tuples with $k \geq 2$ and $n_1 + \ldots + n_k \leq d$.

Definition 3.1 ([14]**).** For a k-tuple $(n_1, \ldots, n_k) \in S(d, k)$ the scalar invariants $\delta^{\pm}_{\mathrm{m},\mathcal{D}}(n_1, \ldots, n_k)$ are defined by

$$\delta^{+}_{\mathrm{m},\mathcal{D}}(n_1, \ldots, n_k)(x) = \max \mathrm{S_m}(V_1, \ldots, V_k), \quad \delta^{-}_{\mathrm{m},\mathcal{D}}(n_1, \ldots, n_k)(x) = \min \mathrm{S_m}(V_1, \ldots, V_k),$$

where $V_1, \ldots, V_k$ run over all k mutually orthogonal subspaces of $\mathcal{D}_x$ with $\dim V_i = n_i$ $(i = 1, \ldots, k)$. For $\mathcal{D} = TM$ we get invariants $\delta^{\pm}_{\mathrm{m}}(n_1, \ldots, n_k) = \delta^{\pm}_{\mathrm{m},TM}(n_1, \ldots, n_k)$, see also [13].

If the sectional curvature of (M, g) along $\mathcal{D}$ satisfies $c \leq K_{|\mathcal{D}} \leq C$ and $\sum_{i=1}^k n_i = s \leq d$, then

$$\frac{c}{2}\,(s^2 - \sum\nolimits_i n_i^2) = c \sum\nolimits_{i<j} n_i\, n_j \leq \delta^{-}_{\mathrm{m},\mathcal{D}}(n_1, \ldots n_k) \leq \delta^{+}_{\mathrm{m},\mathcal{D}}(n_1, \ldots n_k) \leq C \sum\nolimits_{i<j} n_i\, n_j = \frac{C}{2}\,(s^2 - \sum\nolimits_i n_i^2)\,.$$

Example 3.2. Recall that for a subspace V spanned by $q+1$ orthonormal vectors $\{e_0, e_1, \ldots, e_q\}$ of (M, g), the *q-th Ricci curvature* is $\mathrm{Ric}_q(V) = \sum_{i=1}^{q} K(E_0, E_i)$, e.g., [10]. For $k=2$ and $n_1 = 1$, using the intermediate Ricci curvature, we get $\delta^+_{\rm m}(1, n_2)(x) = \max \mathrm{Ric}_{n_2}(V)$ and $\delta^-_{\rm m}(1, n_2)(x) = \min \mathrm{Ric}_{n_2}(V)$, where $V = span(V_1, V_2)$ and V_1, V_2 run over all mutually orthogonal subspaces of $\mathcal{D}_x$ such that $\dim V_1 = 1$ and $\dim V_2 = n_2$.

For a k-tuple ($k \ge 0$) and $x \in M$, B.-Y Chen [4, Sect. 13.2] defined the following curvature invariants:

$$
\begin{aligned}
&2\,\delta(n_1, \ldots, n_k)(x) = \tau(x) - \min\{\tau(V_1) + \ldots + \tau(V_k)\},\\
&2\,\hat\delta(n_1, \ldots, n_k)(x) = \tau(x) - \max\{\tau(V_1) + \ldots + \tau(V_k)\},
\end{aligned} \tag{7}
$$

where $V_1, \ldots, V_k$ run over all k mutually orthogonal subspaces of T_xM with $\dim V_i = n_i$ $(i = 1, \ldots, k)$. The coefficient 2 in (7) is due to the definition of the scalar curvature in [4] as half of the "trace Ricci".

Definition 3.3 ([14]). For each k-tuple ($k \ge 0$) and $x \in M$, we define Chen-type $\delta_{\mathcal{D}}$-invariants of $(M, g; \mathcal{D})$ by

$$
\begin{aligned}
&2\,\delta_{\mathcal{D}}(n_1, \ldots, n_k)(x) = \tau(\mathcal{D}_x) - \min\{\tau(V_1) + \ldots + \tau(V_k)\},\\
&2\,\hat\delta_{\mathcal{D}}(n_1, \ldots, n_k)(x) = \tau(\mathcal{D}_x) - \max\{\tau(V_1) + \ldots + \tau(V_k)\},
\end{aligned} \tag{8}
$$

where $V_1, \ldots, V_k$ run over all k mutually orthogonal subspaces of $\mathcal{D}_x$ with $\dim V_i = n_i$ $(i = 1, \ldots, k)$.

The theory of $\delta_{\mathcal{D}}$-invariants (8) of a sub-Riemannian manifold can be developed similarly to the theory of Chen's δ-invariants of a Riemannian manifold.

The $\delta^\pm_{\rm m,\mathcal{D}}$-invariants are related with the curvature invariants in (8) by the following inequalities.

Proposition 3.4. *Let $k \ge 2$. If $n_1 + \ldots + n_k < d$, then the following inequalities are valid:*

$$
\begin{aligned}
&\delta^+_{\rm m,\mathcal{D}}(n_1, \ldots, n_k) \ge \delta_{\mathcal{D}}(n_1, \ldots, n_k) - \delta_{\mathcal{D}}(n_1 + \ldots + n_k),\\
&\delta^-_{\rm m,\mathcal{D}}(n_1, \ldots, n_k) \le \hat\delta_{\mathcal{D}}(n_1, \ldots, n_k) - \hat\delta_{\mathcal{D}}(n_1 + \ldots + n_k),
\end{aligned} \tag{9}
$$

and if $n_1 + \ldots + n_k = d$, then $\hat\delta_{\mathcal{D}}(n_1, \ldots, n_k) = \delta^-_{\rm m,\mathcal{D}}(n_1, \ldots, n_k) \le \delta^+_{\rm m,\mathcal{D}}(n_1, \ldots, n_k) = \delta_{\mathcal{D}}(n_1, \ldots, n_k)$.

In particular, if $n_1 + \ldots + n_k = d-1$, then

$$
\hat\delta_{\mathcal{D}}(n_1, \ldots, n_k) - \min \mathrm{Ric}_{d-1}(\mathcal{D}) \ge \delta^-_{\rm m,\mathcal{D}}(n_1, \ldots, n_k) \ge \delta^+_{\rm m,\mathcal{D}}(n_1, \ldots, n_k) \ge \delta_{\mathcal{D}}(n_1, \ldots, n_k) - \max \mathrm{Ric}_{d-1}(\mathcal{D}).
$$

Proof. Using (6) and the equality $-\min a = \max(-a)$, we get

$$
\begin{aligned}
&2\,\delta_{\mathcal{D}}(n_1, \ldots, n_k)(x) = \tau(V) - \min\{\tau(V_1) + \ldots + \tau(V_k)\}\\
&= \tau(\mathcal{D}_x) + \max(\tau_k(x) - (\tau(V_1) + \ldots + \tau(V_k)) - \tau_k(V))\\
&\le \tau(\mathcal{D}_x) - \min \tau_k(x) + 2 \max \mathrm{S}_{\rm m}(V_1, \ldots, V_k)\\
&= 2\,\delta_{\mathcal{D}}(n_1 + \ldots + n_k)(x) + 2\,\delta^+_{\rm m,\mathcal{D}}(n_1, \ldots, n_k)(x),
\end{aligned}
$$

hence, $(9)_1$ is valid. The proof of $(9)_2$ is similar. The case of $n_1 + \ldots + n_k = d$ follows from (9). The case of $n_1 + \ldots + n_k = d-1$ follows from $\delta_{\mathcal{D}}(d-1)(x) = \max \mathrm{Ric}_{d-1}(\mathcal{D}_x)$ and $\hat\delta_{\mathcal{D}}(d-1)(x) = \min \mathrm{Ric}_{d-1}(\mathcal{D}_x)$. □

Corollary 3.5. *If $(M, g; \mathcal{D})$ has nonnegative sectional curvature along $\mathcal{D}$ and $k \ge 2$, then*

$$
\hat\delta(n_1, \ldots, n_k) \le \delta^-_{\rm m,\mathcal{D}}(n_1, \ldots, n_k) \le \delta^+_{\rm m,\mathcal{D}}(n_1, \ldots, n_k) \le \delta(n_1, \ldots, n_k),
$$

and if $(M, g; \mathcal{D})$ has nonpositive sectional curvature along $\mathcal{D}$, then the inequalities are opposite.

4. Geometric inequalities for a submanifold equipped with distributions

First, we consider *adapted* isometric immersions $f : (M, g; \mathcal{D}) \to (\bar{M}, \bar{g}; \bar{\mathcal{D}})$ of sub-Riemannian manifolds, i.e., $f_*(\mathcal{D}) \subset \bar{\mathcal{D}}_{|f(M)}$. If $\mathcal{D}$ and $\bar{\mathcal{D}}$ are the sums of $s \ge 2$ mutually orthogonal distributions, i.e., $\mathcal{D} = \bigoplus_{i=1}^s \mathcal{D}_i$ and $\bar{\mathcal{D}} = \bigoplus_{i=1}^s \bar{\mathcal{D}}_i$, then we also require the following: $f_*(\mathcal{D}_i) \subset \bar{\mathcal{D}}_{i\,|f(M)}$ for all i. Below we assume $s = 2$.

Remark 4.1. A sub-Riemannian structure on a smooth manifold M can be obtained from a special immersion of M in $(\bar{M}, \bar{g}; \bar{\mathcal{D}})$. Namely, let $f_*(TM)$ intersects transversally with the distribution $\bar{\mathcal{D}}$ restricted to $f(M)$, then $f : M \to \bar{M}$ induces a required distribution $\mathcal{D} = f_*^{-1}(\bar{\mathcal{D}} \cap f(TM))$ on M with induced metric g.

We will identify M with its image $f(M)$ (since the induced metric on $f(M)$ is equal to g) and put a top "bar" for objects related to $\bar{M}$. Let $TM^\perp$ be the normal bundle of the submanifold $M \subset \bar{M}$ and $\bar{h} : TM \times TM \to TM^\perp$ be the second fundamental form of M. Recall the Gauss equation for an isometric immersion f, e.g., [4]:

$$\bar{g}(\bar{R}_{Y,Z}\, U, X) = g(R_{Y,Z}\, U, X) + \bar{g}(\bar{h}(Y, U), \bar{h}(Z, X)) - \bar{g}(\bar{h}(Z, U), \bar{h}(Y, X)), \quad U, X, Y, Z \in TM, \tag{10}$$

where $\bar{R}$ and R are the curvature tensors of $(\bar{M}, \bar{g})$ and (M, g), respectively. The mean curvature vector of a subspace $V \subset \mathcal{D}_x$ is given by $\bar{H}_V = \operatorname{trace} \bar{h}_{|V} = \sum_i \bar{h}(e_i, e_i)$, where e_i is an orthonormal basis of V. Thus, $\bar{H}_{\mathcal{D}} = \operatorname{trace}_g \bar{h}_{|\mathcal{D}}$ is the mean curvature vector of $\mathcal{D}$, and $\bar{H} = \operatorname{trace}_g \bar{h}$ is the mean curvature vector of M. An isometric immersion f with the property $\bar{H}_{\mathcal{D}} = 0$ is called $\mathcal{D}$-minimal (minimal if $\bar{H} = 0$). Set

$$\mathcal{H}_x(s) = \max\{\, \|\bar{H}_V\| \,:\, V \subset \mathcal{D}_x,\ \dim V = s > 0\}.$$

If $s = d$, then $\bar{H}_V = \bar{H}_{\mathcal{D}_x}$. For $s < d$ the condition $\mathcal{H}(s) = 0$ implies that $\bar{h}_{|\mathcal{D}} = 0$.

An isometric immersion $f : (M, g; \mathcal{D}) \to (\bar{M}, \bar{g})$ is called *mixed totally geodesic* on $V = \bigoplus_{i=1}^k V_i \subset \mathcal{D}$ if $\bar{h}(X, Y) = 0$ for all $X \in V_i$, $Y \in V_j$ and $i \ne j$.

Theorem 4.2. *Let $f : (M, g; \mathcal{D}) \to (\bar{M}, \bar{g}, \bar{\mathcal{D}})$ be an adapted isometric immersion, and $\sum_i n_i = s \le d$. Then*

$$\delta^+_{\mathrm{m},\mathcal{D}}(n_1, \ldots, n_k) \le \bar{\delta}^+_{\mathrm{m},\bar{\mathcal{D}}}(n_1, \ldots, n_k) + \frac{k-1}{2k} \begin{cases} \mathcal{H}_{\mathcal{D}}(s)^2, & \text{if } s < d, \\ \|\bar{H}_{\mathcal{D}}\|^2, & \text{if } s = d. \end{cases} \tag{11}$$

The equality in (11) *holds at a point $x \in M$ if and only if there exist mutually orthogonal subspaces $V_1, \ldots, V_k$ of $\mathcal{D}_x$ with $\sum_i n_i = s$ such that f is mixed totally geodesic on $V = \bigoplus_{i=1}^k V_i$, $\bar{H}_1 = \ldots = \bar{H}_k$, $\|\bar{H}_V\| = \mathcal{H}_{\mathcal{D}_x}(s)$ and $\bar{\mathrm{S}}_{\mathrm{m}}(V_1, \ldots, V_k) = \bar{\delta}^+_{\mathrm{m},\bar{\mathcal{D}}}(n_1, \ldots, n_k)(x)$.*

Proof. Taking trace of the Gauss equation (10) for the immersion f along V and V_i yields the equalities

$$\bar{\tau}(V) - \tau(V) = \|\bar{h}_V\|^2 - \|\bar{H}_V\|^2, \qquad \bar{\tau}(V_i) - \tau(V_i) = \|\bar{h}_i\|^2 - \|\bar{H}_i\|^2, \tag{12}$$

where $\bar{\tau}(V), \bar{\tau}(V_i)$ and $\tau(V), \tau(V_i)$ are the scalar curvatures of subspaces $V = \bigoplus_{i=1}^k V_i$ and V_i for the curvature tensors $\bar{R}$ and R, respectively, $\bar{h}_i$ and $\bar{H}_i$ are the second fundamental form and mean curvature vector of V_i.

Assume that $\bar{H}_V \ne 0$ is satisfied on an open set $U \subset M$ and complement over U an adapted local orthonormal frame $\{e_1, \ldots, e_n\}$ of (M, g) with vector e_{n+1} parallel to $\bar{H}_V$. Using $\bar{H}_V = \sum_{i=1}^k \bar{H}_i$ and the algebraic inequality $a_1^2 + \ldots + a_k^2 \ge \frac{1}{k}(a_1 + \ldots + a_k)^2$ for real $a_i = \bar{g}(\bar{H}_i, e_{n+1})$, we find

$$\sum\nolimits_i \|\bar{H}_i\|^2 \ge \sum\nolimits_i \bar{g}(\bar{H}_i, e_{n+1})^2 \ge \frac{1}{k}\|\bar{H}_V\|^2, \tag{13}$$

and the equality holds if and only if $\bar{H}_1 = \ldots = \bar{H}_k$. The above inequality is trivially satisfied for $\bar{H}_V = 0$, hence it is valid on M. Set $\|\bar{h}^{\mathrm{mix}}_{ij}\|^2 = \sum_{e_a \in V_i, e_b \in V_j} \|\bar{h}(e_a, e_b)\|^2$ for $i \ne j$ and note that

$$\|\bar{h}_V\|^2 = \sum\nolimits_i \|\bar{h}_i\|^2 + \sum\nolimits_{i<j} \|\bar{h}^{\mathrm{mix}}_{ij}\|^2 \ge \sum\nolimits_i \|\bar{h}_i\|^2, \tag{14}$$

and the equality holds if and only if $\|\bar h^{\rm mix}_{ij}\|^2=0$ ($\forall\, i<j$), i.e., f is mixed totally geodesic along V.

By (12), (13), (14) and the equalities

$$\bar\tau(V)=2\,\bar{\rm S}_{\rm m}(V_1,\ldots,V_k)+\sum\nolimits_i\bar\tau(V_i),\quad \tau(V)=2\,{\rm S}_{\rm m}(V_1,\ldots,V_k)+\sum\nolimits_i\tau(V_i),$$

see (6), we obtain

$$\begin{aligned}2\,{\rm S}_{\rm m}(V_1,\ldots,V_k)&=2\,\bar{\rm S}_{\rm m}(V_1,\ldots,V_k)+\sum\nolimits_i(\bar\tau(V_i)-\tau(V_i))+\|\bar H_V\|^2-\|\bar h_V\|^2\\ &\le 2\,\bar\delta^+_{{\rm m},\bar{\mathcal D}}(n_1,\ldots,n_k)-(\|\bar h_V\|^2-\sum\nolimits_i\|\bar h_i\|^2)+(\|\bar H_V\|^2-\sum\nolimits_i\|\bar H_i\|^2)\\ &\le 2\,\bar\delta^+_{{\rm m},\bar{\mathcal D}}(n_1,\ldots,n_k)+\frac{k-1}{k}\,\mathcal H(s)^2,\end{aligned}$$

(and the equality holds in the second line if and only if $\bar{\rm S}_{\rm m}(V_1,\ldots,V_k)=\bar\delta^+_{{\rm m},\bar{\mathcal D}}(n_1,\ldots,n_k)$ and $\|\bar H_V\|=\mathcal H_x(s)$ at each point $x\in M$) that proves (11) for $s<d$. The case $\sum_i n_i=d$ of (11) was proved in [13]. □

Corollary 4.3. *For an adapted isometric immersion $f:(M,g;\mathcal D)\mapsto(\bar M,\bar g;\bar{\mathcal D})$ with sectional curvature along $\bar{\mathcal D}$ bounded above by c and $\sum_i n_i=s\le d$, from* (11) *we get the following inequality:*

$$\delta^+_{{\rm m},\mathcal D}(n_1,\ldots,n_k)\le\frac c2\,(d^2-\sum\nolimits_i n_i^2)+\frac{k-1}{2k}\begin{cases}\mathcal H_{\mathcal D}(s)^2, & \text{if } s<d,\\ \|\bar H_{\mathcal D}\|^2, & \text{if } s=d.\end{cases}$$

Corollary 4.4 (see [14, Corollary 4]). *A sub-Riemannnian manifold $(M,g;\mathcal D)$ with $\delta^+_{{\rm m},\mathcal D}(n_1,\ldots,n_k)>0$ for some $(n_1,\ldots,n_k)\in S(d,k)$ with $\sum_i n_i=d$ does not admit $\mathcal D$-minimal isometric immersions in a Euclidean space.*

Proof. This follows directly from (11). □

Corollary 4.5. *Let $f:(M,g;\mathcal D)\to(\bar M,\bar g)$ be an isometric immersion. If M is compact and $\mathcal D$ is defined on an open set $M\setminus\Sigma$, $(k-1)(p-1)\ge1$ and $\|H+H^\perp\|\in L^p(M,g)$ (see Remark 2.5), then*

$$\int_M(\|H\|^2+\|H^\perp\|^2+\|T\|^2+\|T^\perp\|^2-\|h\|^2-\|h^\perp\|^2)\,d\,{\rm vol}_g\le\frac14\int_M\|\bar H\|^2\,d\,{\rm vol}_g+\bar\delta^+_{\rm m}(d,d^\perp)\,{\rm Vol}(M,g).$$

Proof. Applying (1) to (11) and the Divergence-type theorem in Remark 2.5, proves the assertion. □

Remark 4.6. In conditions of Corollary 4.5, if distributions $\mathcal D$ and $\mathcal D^\perp$ are totally umbilical, i.e., $\|H\|^2-\|h\|^2=\frac{d-1}{d}\|H\|^2$ and $\|H^\perp\|^2-\|h^\perp\|^2=\frac{d^\perp-1}{d^\perp}\|H^\perp\|^2$, then such a compact (M,g) does not admit minimal isometric immersions ($\bar H=0$) in a Riemannian manifold $(\bar M,\bar g)$ with $\bar\delta^+_{\rm m}(d,d^\perp)<0$.

Corollary 4.7. *Let $f:(M,g;\mathcal D_1,\mathcal D_2,\mathcal D_3)\to(\bar M,\bar g)$ be an isometric immersion and $n_1+n_2+n_3=n$. If M is compact and all $\mathcal D_i$ are defined on an open set $M\setminus\Sigma$, $(k-1)(p-1)\ge1$ and $\|H_1+H_1^\perp+H_2+H_2^\perp+H_3+H_3^\perp\|\in L^p(M,g)$ then (for Q_i given in Lemma 2.3)*

$$-\frac12\int_M(Q_1+Q_2+Q_3)\,d\,{\rm vol}_g\le\frac13\int_M\|\bar H\|^2\,d\,{\rm vol}_g+\bar\delta^+_{\rm m}(n_1,n_2,n_3)\,{\rm Vol}(M,g).$$

Proof. This follows from (4), (15) and the Divergence-type theorem in Remark 2.5. □

Remark 4.8. In conditions of Corollary 4.7, if $h_i=h_i^\perp=0$ ($i=1,2,3$), then such a compact manifold (M,g) does not admit minimal isometric immersions in a Riemannian manifold $(\bar M,\bar g)$ with $\bar\delta^+_{\rm m}(n_1,n_2,n_3)<0$.

Example 4.9. Let $\mathcal D_i$ ($i=1,2,3$) be 1-dimensional distributions orthogonal to three pairwise orthogonal codimension-one foliations $\mathcal F_i$ on (M^3,g), see Example 2.4. If $f:(M^3,g;\mathcal D_1,\mathcal D_2,\mathcal D_3)\to(\bar M,\bar g)$ is an isometric immersion, then $\tau\le\frac13\|\bar H\|^2+\bar\delta^+_{\rm m}(1,1,1)$. Note that $\bar\delta^+_{\rm m}(1,1,1)(x)=\max\{\bar\tau(V):V\subset T_x\bar M,\ \dim V=3\}$. Moreover, if foliations $\mathcal F_i$ ($i=1,2,3$) are minimal and not totally geodesic, then (M^3,g) does not admit minimal isometric immersions in a Euclidean space.

Next, we consider the case when a distribution $\mathcal{D}$ is represented as the sum of two orthogonal distributions of ranks $n_i > 0$: $\mathcal{D} = \mathcal{D}_1 \oplus \mathcal{D}_2$, thus, $n_1 + n_2 = d$.

An isometric immersion $f : (M, g; \mathcal{D}_1, \mathcal{D}_2) \to (\bar M, \bar g)$ is called *mixed totally geodesic* on $\mathcal{D}$ if

$$\bar h(X, Y) = 0 \quad \text{for all } \ X \in \mathcal{D}_1,\ Y \in \mathcal{D}_2\,.$$

Theorem 4.10. *Let $f : (M, g; \mathcal{D}) \to (\bar M, \bar g; \bar{\mathcal{D}})$ be an adapted isometric immersion and $\mathcal{D} = \mathcal{D}_1 \oplus \mathcal{D}_2$. Then*

$$\mathrm{S}_{\rm m}(\mathcal{D}_1, \mathcal{D}_2) \le \frac{1}{4}\,\|\bar H_{\mathcal{D}}\|^2 + \bar\delta^+_{{\rm m},\bar{\mathcal{D}}}(n_1, n_2)\,. \tag{15}$$

The equality in (15) *holds at a point $x \in M$ if and only if f is mixed totally geodesic on $\mathcal{D}_x$, $\bar H_1(x) = \bar H_2(x)$ (the mean curvature vectors of $\mathcal{D}_1$ and $\mathcal{D}_2$) and $\bar{\mathrm{S}}_{\rm m}(\mathcal{D}_1(x), \mathcal{D}_2(x)) = \bar\delta^+_{{\rm m},\bar{\mathcal{D}}}(n_1, n_2)(x)$.*

Proof. The proof of (15) is similar to the proof of Theorem 4.2. We take $V_i = \mathcal{D}_i(x)$. The proof of the second assertion follows directly from the cases of equality, as in the proof of Theorem 4.2. □

Remark 4.11. Let $f : (M, g; \mathcal{D}) \to (\bar M, \bar g; \bar{\mathcal{D}})$ be an adapted isometric immersion and $\mathcal{D} = \mathcal{D}_1 \oplus \mathcal{D}_2$. The following counterpart of (15) is a special case of [14, Eq. (19)]:

$$\mathrm{S}_{\rm mix}(\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}^\perp) \le \frac{1}{3}\,\|\bar H\|^2 + \bar\delta^+_{\rm m}(n_1, n_2, d^\perp)\,. \tag{16}$$

Corollary 4.12 (for (i) see [14, Corollary 6]). *Let $(M, g; \mathcal{D})$ be a sub-Riemannnian manifold with $\mathcal{D} = \mathcal{D}_1 \oplus \mathcal{D}_2$.*
(i) if $\mathrm{S}_{\rm m}(\mathcal{D}_1, \mathcal{D}_2) > 0$, then $(M, g; \mathcal{D})$ does not admit $\mathcal{D}$-minimal isometric immersions in a Euclidean space.
(ii) if $\mathrm{S}_{\rm mix}(\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}^\perp) > 0$, then $(M, g; \mathcal{D})$ does not admit minimal isometric immersions in a Euclidean space.

Proof. This follows directly from (15) for (i) and from (16) for (ii). □

Corollary 4.13. *In conditions of Theorem 4.10, let $\mathcal{D}_1$ be spanned by a unit vector field N. Then*

$$\mathrm{Ric}_{N,N} \le \frac{1}{4}\,\|\bar H_{\mathcal{D}}\|^2 + \bar r_{d-1\,|\,\bar{\mathcal{D}}}\,, \tag{17}$$

where $d = \dim\mathcal{D}$ and $\bar r_{d-1\,|\,\bar{\mathcal{D}}}$ is the supremum of the $(d-1)$-th Ricci curvature of $(\bar M, \bar g)$ along $\bar{\mathcal{D}}$. The equality in (17) *holds if and only if f is mixed totally geodesic along $\mathcal{D}$, $\bar H_1(x) = \bar H_2(x)$ and $\mathrm{Ric}_{N,N} = \bar r_{d-1}$ at each point $x \in M$.*

Applying (5) to (15) on a compact manifold M, gives the following

Corollary 4.14. *In conditions of Theorem 4.10, let $\mathcal{D}_3 = \mathcal{D}^\perp$. If M is compact and all $\mathcal{D}_i$ are defined on an open set $M \setminus \Sigma$, $(k-1)(p-1) \ge 1$ and $\|H_1 + H_1^\perp + H_2 + H_2^\perp - H_3 - H_3^\perp\| \in L^p(M, g)$, then (for Q_i given in Lemma 2.3),*

$$\int_M (Q_3 - Q_1 - Q_2)\, d\,{\rm vol}_g \le \frac{1}{4}\int_M \|\bar H_{\mathcal{D}}\|^2\, d\,{\rm vol}_g + \bar\delta^+_{{\rm m},\mathcal{D}}(n_1, n_2)\,{\rm Vol}(M, g).$$

Proof. This follows from (5), (15) and the Divergence-type theorem in Remark 2.5. □

Finally, we apply $\delta_{\mathcal{D}}$-invariants (8) to isometric immersions of sub-Riemannian manifolds.

Theorem 4.15. *Let $f : (M, g; \mathcal{D}) \to (\bar M, \bar g, \bar{\mathcal{D}})$ be an adapted isometric immersion. Then for eny k-tuple $(n_1, \ldots, n_k) \in S(d)$ we get the inequality*

$$\delta_{\mathcal{D}}(n_1, \ldots, n_k) \le \frac{d + k - 1 - \sum_i n_i}{2(d + k - \sum_i n_i)}\,\|\bar H_{\mathcal{D}}\|^2 + \frac{1}{2}\Big(d(d-1) - \sum\nolimits_i n_i(n_i - 1)\Big)\max \bar K_{|\bar{\mathcal{D}}}. \tag{18}$$

Proof. This is similar to the proof of [4, Theorem 13.3]. □

The case of equality in (18) is similar to [4, Theorem 13.3: (a), (b)]. Extremal immersions in Euclidean space in terms of $\delta_{\mathcal{D}}$-invariants are the sub-Riemannian analogue of Chen's "ideal immersions".

Corollary 4.16. *A sub-Riemannnian manifold $(M, g; \mathcal{D})$ with $\delta_{\mathcal{D}}(n_1, \ldots, n_k) > 0$ for some $(n_1, \ldots, n_k) \in S(d, k)$ does not admit $\mathcal{D}$-minimal isometric immersions in a Euclidean space.*

Proof. This follows directly from (18). □

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